Bridgeland Stability Conditions
in Algebra, Geometry and Physics

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Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik und Physik der
Albert-Ludwigs-Universität
Freiburg im Breisgau

2014

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Datum der Promotion:
21.7.2014

Vorveröffentlichungen

Teilergebnisse dieser Arbeit wurden von mir am 21. September 2012 unter dem Titel

"Bridgeland stability conditions on twisted Kummer surfaces"

in den Communications in Mathematical Physics und am 8. Mai 2013 unter dem Titel

"Tilting and Refined Donaldson-Thomas Invariants"

im Journal of Algebra zur Vorveröffentlichung eingereicht. Kapitel 8 ist eine überarbeitete und erweiterte Darstellung der Ergebnisse der Veröffentlichung [1]. Diese Vorveröffentlichungen wurden vom Fachvorsitzenden des Promotionsausschusses am 18. September 2012 und am 3. Mai 2013 genehmigt.

Acknowledgements

First of all I am grateful to my adviser Katrin Wendland for her manifold support of my PhD project. Her research group in mathematical physics at the University of Augsburg and then Freiburg was an excellent environment to further my studies. I benefited from the innumerable discussions with my colleagues: For this I thank Manfred Herbst, Emanuel Scheidegger und Dmytro Shklyarov. The comments of Wolfgang Soergel helped a lot to improve the first version of this thesis. Further I thank Marc Nieper-Wißkirchen for his interest in my work. I would like to express my sincere gratitude to Daniel Huybrechts, Heinrich Hartmann, Bernhard Keller and Emanuel Macrì for helpful discussions or correspondences. Special thanks go to Annette Huber-Klawitter.

My PhD studies were partially supported by the DFG-Graduiertenkolleg GRK 1821 "Cohomological Methods in Geometry" and the ERC Starting Independent Researcher Grant StG No. 204757-TQFT (Katrin Wendland, PI) at the University of Freiburg. I thank the Hausdorff Institute for Mathematics in Bonn and the Simons Center for Geometry and Physics in Stony Brook for hospitality.

Für Theodor Matthias Engenhorst (1935-2012)

Contents

Chapter 1

Introduction

The last two decades have seen extremely fruitful exchange of ideas between mathematics and theoretical physics (mirror symmetry, Gromov-Witten theory, dimer models, matrix factorizations, wall crossing,...). The subject of this work are Bridgeland stability conditions on triangulated categories and applications. In our geometrical example we study the stability manifold in an example, in the second project we use the structure of the space of stability conditions to study refined Donaldson-Thomas invariants. Physically we deal in the first case with the covering of the moduli space of superconformal field theories on a Kummer surface. In the second case we study BPS states for quiver gauge theories. Bridgeland stability conditions prove to be an effective tool here.

The derived category of coherent sheaves on a projective variety is a central object in algebraic geometry and string theory. Let us consider dimension one: a Bridgeland stability condition is defined by stable vector bundles on a complex projective curve. The so called stability function (or central charge) associates to every vector bundle a complex number in the upper half-plane, and in particular a phase in $(0,1]$. A vector bundle with some phase is (semi)stable if every subbundle has smaller (or equal) phase. We have a (Harder-Narasimhan) filtration of every coherent sheaf by semistable factors. In general, a Bridgeland stability condition on a triangulated category is now a stability function on an Abelian subcategory (the heart of a bounded t-structure) of a triangulated category that comes with a Harder-Narasimhan filtration.

It is possible to construct explicitly examples of stability conditions for any projective surface (but the Abelian subcategory will never be the category of coherent sheaves). K3 surfaces are compact complex surfaces X with trivial canonical bundle and $H^1(X,\mathscr{O}_X)=0$. They are Calabi-Yau manifolds what makes them interesting physically. Bridgeland described in [30] a connected component of the stability manifold for a projective K3 surface X that he denoted $Stab^{\dagger}(X)$.

An important conjecture in the case of K3 surfaces is the following

Conjecture (Bridgeland) The group of auto-equivalences of the derived category of coherent sheaves of a K3 surface preserves the connected component $Stab^{\dagger}(X)$. Moreover, $Stab^{\dagger}(X)$ is simply connected.

If true, this conjecture would give us a description of the group of cohomological trivial auto-equivalences as the fundamental group of some open subset of the complexified Mukai lattice $\mathcal{N} \otimes \mathbb{C}$ of the projective K3 surface X. This conjecture was recently proven in the case of K3 surfaces with Picard rank one by Bayer and Bridgeland [31].

The first important case of a K3 surface to look at is that of a Kummer surface $X = Km\ A$ associated to an Abelian surface A. We know that the stability manifold of an Abelian surface is indeed simply connected. One of the motivations to study Kummer surfaces is the hope to understand the topology of $Stab^{\dagger}(X)$ by its Abelian surface. In this direction we prove in section 7.5 the following

Theorem 1 Let $Stab^{\dagger}(A)$ be the (unique) maximal connected component of the space of stability conditions of an Abelian surface A and $Stab^{\dagger}(X)$ the distinguished connected component of $Stab(X)$ of the Kummer surface $X = Km\,A$. Then there is an embedding $Stab^{\dagger}(A) \hookrightarrow Stab^{\dagger}(X)$.

This theorem is based on our Proposition 7.4.1. Further we show that the group of deck transformations of $Stab^{\dagger}(A)$ (generated by the double shift) is isomorphic to a subgroup of the group of deck transformations of $Stab^{\dagger}(X)$ (Proposition 7.5.1). Bridgeland proved his results on the stability manifold of a K3 surface X studying the boundary of a distinguished subset $U(X) \subset Stab^{\dagger}(X)$. [30] We give examples of stability conditions on the boundary of $U(X)$ associated with orbifold ample classes of the Kummer surface X. These are the classes in the boundary of the ample cone of X whose intersection product with the 16 exceptional divisors of the Kummer surface vanishes.

Huybrechts, Macrì and Stellari have generalized Bridgeland's work to the case of twisted K3 surfaces and twisted sheaves that include a B-field in their data [32]. An embedding of $Stab^{\dagger}(A)$ into $Stab(X)$ using a functorial approach was given in [39]. The advantage of this method is that in principle you can explicitly derive the induced stability conditions. In contrast the relation to conformal field theory is explicit in our approach. Further our methods go through in the twisted case as well. We are the first to give an embedding in the twisted case (Theorem 7.5.5).

You can view Theorem 1 as the lift of the results on the embedding of the moduli space of SCFTs on complex tori into the moduli space of SCFTs on the associated Kummer surfaces given in [3, 5] to the space of stability conditions on K3 surfaces. Crucial for this work is the observation confirmed in [3, 5] that there are no ill-defined SCFTs coming from the complex torus. Rephrased in mathematical terms this is corollary 7.2.2. Bridgeland considers in [30] the open subset of the complexified Mukai lattice $\mathscr{N} \otimes \mathbb{C}$ consisting of vectors whose real and imaginary part span positive definite two-plans in $\mathscr{N} \otimes \mathbb{R}$ cut out by all hyperplanes orthogonal to (-2)-classes. This cut out is a necessary assumption to prove the covering map property in Theorem 7.4.3. So in a sense Bridgeland rederives the important role of ill-defined CFTs. This and Theorem 1 suggest that Bridgeland stability conditions are the right categorical framework for (orbifold) conformal field theories.

In principle, we could use similar arguments as in chapter 7 in the case of Borcea-Voisin threefolds. Unfortunately, in the case of Calabi-Yau threefolds it is not known if the space of stability conditions is empty or not. At least, there is a concrete conjecture depending on a generalization of the Bogomolov inequality [40].

For a class of supersymmetric quantum field theories the BPS states are encoded as stable representations of a quiver with potential. The pioneering work in this direction was [41]. An algorithm to derive the stable representations without going into linear algebra directly was developed in [46]. Unfortunately, it uses physics arguments. This mutation method is based on the idea that the BPS spectrum of a theory is also encoded in the Seiberg dual theory. The mathematically counter part of Seiberg duality [42] is mutation of quivers with potentials (Q,W) [43, 44]. An idea of Bridgeland is that mutation is modeled by tilting of hearts of t-structures of triangulated categories. Inspired from the mutation method in physics we study tilting for the heart of the canonical t-structure of the finite-dimensional derived category $D_{fd}(\Gamma)$ of the Ginzburg algebra Γ [50] of (Q,W). To prove our main theorems we develop a new method in chapter 8 that we call the *(categorical) mutation method.* Unlike the algorithm of Cecotti et al. in [46] our method is based on the category associated to quiver with potential.

As application we have triangulated categories associated with quivers with potential in mind but our method works for a general category with similar properties.

One advantage of the mutation method is that it allows to read of the refined Donaldson-Thomas invariant. Classically Donaldson-Thomas (DT) invariants were introduced as counting invariants for the moduli space of stable sheaves on a Calabi-Yau threefold. [55] Kontsevich and Soibelman develop a theory for generalized Donaldson-Thomas invariants for 3-Calabi-Yau categories in [60]. Given a quiver with potential (Q,W) we have a associated Calabi-Yau category: the derived category of the Ginzburg algebra of (Q,W). A quiver with underlying graph a Dynkin diagram has a trivial potential, i.e. $W = 0$. Reineke introduced for this case a refined Donaldson-Thomas invariant in [58]. It is defined as a product of quantum dilogarithms in (the completion of) quantum affine space ordered by the decreasing phase of stable objects in the category of quiver representations. The refined DT invariant for a quiver with potential is defined in a similar way following [57]. Instead of the "(much) better definition [of the refined Donaldson-Thomas invariant]"[1] of Kontsevich and Soibelman we choose to follow this approach since the work of [60] seems to bear on conjectures, see [57].

For a generic potential we can indefinitely mutate (Q,W). Inspired by the mutation algorithm in [46] we study mutations/tilting via the space of stability conditions. In sections 8.1 and 8.2 we prove the following

Theorem 2 Let (Q,W) be a 2-acyclic quiver Q with generic potential W such that we have a discrete central charge on the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ with finitely many stable objects. Then the sequence of stable objects of $\mathscr{A}$ in the order of decreasing phase defines a sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$. Moreover, (Q,W) is Jacobi-finite.

This result together with Theorem 3 is a natural complement of the work [57]. It explains the relationship between stable objects and sequences of simple tilts and bridges the gap between the physics and the mathematics literature. Since Gaiotto, Moore and Neitzke are using a version of the mutation method implicitly in their work on triangulations of Riemann surfaces by trajectories of quadratic differentials [56] Theorem 2 should have applications to the work of Bridgeland and Smith [45] deeply inspired by

[1] B. Keller: On cluster theory and quantum dilogarithm identities, Representation Theory of Algebras and Related Topics (Tokyo) (A. Skowronski, K. Yamagata eds.), European Mathematical Society, 2011, 85-116, p.95.

[56].

Note that in Theorem 2 we use a weaker definition of a discrete central charge than the one given in [57], see section 8.1 of this work. Let $\mathscr{H}_Q := mod - kQ$ be the category of finite-dimensional representations of an acyclic quiver Q inside the bounded derived category of $\mathscr{H}_Q$. In this case we show that the stable objects of $\mathscr{H}_Q$ induce a sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$ in $D_{fd}(\Gamma)$ (Corollary 8.2.9). In section 8.3 we study the relationship to maximal green sequences. These were introduced by Keller in [57] as a combinatorial counter part to a sequence of simple tilts as above: Mutating strictly at green (frozen) vertices in the principal extension of a quiver corresponds to tilting at objects in $\mathscr{A}$ and tilting at red vertices corresponds to tilting at objects in $\mathscr{A}[-1]$. A sequence of mutations at green vertices is maximal if all frozen vertices are red in the final quiver. As an application we reproduce some well known results on maximal green sequences for Dynkin and more general acyclic quivers.

As an important consequence of Theorem 2 a result of B. Keller [51, 57] implies the following

Theorem 3 Let (Q,W) be a quiver with generic potential as in Theorem 2. Then the refined Donaldson-Thomas invariant associated to (Q,W) does not depend on the chosen discrete central charge $Z : K(\mathscr{A}) \to \mathbb{C}$ with finitely many stable objects.

This is an important new case since we do not assume that the potential is polynomial. It is a non-trivial test of the general case that remains unproven. Conjecture 3.2 in [57] claims that the refined DT invariant is independent of a chosen discrete central charge (with possibly infinitely many stable objects) for a polynomial potential. Beside general potentials one requires a general result for not necessary discrete central charges with possibly infinitely many stable objects. To prove such a statement one needs methods different to the approach in this thesis. The methods of Kontsevich and Soibelman [60] suggest that the refined DT invariant should be independent in general.

Chapter 2

Lattices

In this section we review lattice theory as presented in the monograph [68] or in [14]. We closely follow the outline in [65]. The integer cohomology of a complex torus or a K3 surface is a lattice. The main result in this section is Theorem 2.0.2 which will be crucial in chapter 7 for the embedding of the cohomology lattice of a complex torus into the cohomology lattice of the associated Kummer surface given in [3, 5].

A *lattice* M is a free $\mathbb{Z}$-module of finite rank together with a non-degenerate symmetric bilinear pairing $\langle\ ,\ \rangle_M : M \times M \to \mathbb{Z}$. A lattice M is *even* if $\langle x,x\rangle_M \in 2\mathbb{Z}$ for all $x \in M$ (and *odd* otherwise). We call this the *parity* of the lattice. A lattice M is called *positive-definite* if $\langle x,x\rangle_M > 0$ for all $x \in M \setminus \{0\}$ and *negative-definite* if $\langle x,x\rangle_M < 0$ for all $x \in M \setminus \{0\}$. It is *indefinite*, if it is neither positive-definite nor negative-definite.

A *homomorphism* of lattices is a homomorphism of $\mathbb{Z}$-modules that preserves the bilinear pairing. An *embedding* is an injective homomorphism. The dual $\mathbb{Z}$-module of a lattice $(M, \langle\ ,\ \rangle_M)$ is

$$\begin{aligned} M^* &= Hom(M,\mathbb{Z}) \\ &\cong \{x \in M \otimes \mathbb{Q} |\, \langle x,y\rangle_M \in \mathbb{Z}, \forall y \in M\} \end{aligned}$$

with its natural $\mathbb{Q}$-valued pairing. We have a canonical injective homomorphism $M \hookrightarrow M^*$ given by the map $x \mapsto \langle x,\ \rangle_M$. A lattice M is *unimodular* if this injective homomorphism is an isometry.

Here are two examples of lattices important for the study of K3 surfaces: The hyperbolic lattice U is $\mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. If e_1, e_2 is the standard basis of $\mathbb{Z} \oplus \mathbb{Z}$ then the matrix $(\langle e_i, e_j\rangle)$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

U is even and unimodular. Another important example is the even and unimodular E_8-lattice of rank 8 with matrix $(\langle e_i, e_j \rangle)$ given by the Cartan matrix of the root system E_8

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Changing all signs defines the lattice $-E_8$.

Any quadratic form can be diagonalised over the real numbers. The number s_+ of positive and the number s_- of negative entries in every diagonalisation are the same. We call $s_+ - s_-$ the *index* of the quadratic form.

Theorem 2.0.1. *Any indefinite unimodular lattice is determined by its rank, index and its parity (up to isometry).*

The following results lead to Theorem 2.0.2. Given a lattice M the finite quotient $A_M = M^*/M$ is the *discriminant group* of M. The $\mathbb{Q}$-valued pairing on M^* induces a nondegenerate symmetric $\mathbb{Q}/\mathbb{Z}$-valued bilinear pairing

$$b_M : A_M \times A_M \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

If M is even this yields a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form, the discriminant form,

$$q_M : A_M \longrightarrow \mathbb{Q}/2\mathbb{Z}.$$

Let M be an even lattice. A subgroup $G \subset A_M$ is *isotropic* if $q_M|_G = 0$. An even lattice L together with an embedding $M \hookrightarrow L$ such that the quotient L/M is finite is called an *overlattice*.

Proposition 2.0.1. *[14] Let M be an even lattice. There is a bijection between isotropic subgroups G_L of A_M and overlattices L of M.*

Proof. We follow the proof in [14]. Given an isotropic subgroup G, define the lattice L as the $\mathbb{Z}$-module $L := \{x \in M^* | [x] \in G\}$ together with the nondegenerate symmetric bilinear pairing induced by b_M. This defines an embedding $M \hookrightarrow L$ and the quotient L/M is the group G itself. Conversely, given an overlattice L the natural embedding $M \hookrightarrow M^*$ factors through the embeddings

$$M \hookrightarrow L \hookrightarrow L^* \hookrightarrow M^* \tag{2.0.1}$$

and thus the group $G_L = L/M \subset M^*/M$ is isotropic. □

By the three embeddings 2.0.1 we have the inclusions $G_L \subset L^*/M \subset A_M$. L^*/M is the orthogonal complement of G_L in A_M with respect to the pairing b_M. Thus we have $G_L^{\perp}/G_L = A_L$ and we can write the discriminant form q_L in terms of the form q_M as

$$q_L = (q_M|_{G_L^{\perp}})/G_L. \tag{2.0.2}$$

An embedding $M \hookrightarrow L$ is *primitive* if the quotient L/M is a free $\mathbb{Z}$-module and we call M a *primitive sublattice* of L in this case. The orthogonal complement $M^{\perp} = \{x \in L | \langle x, m \rangle = 0, \forall m \in M\}$ of a sublattice M of L is a primitive sublattice. A sublattice M is primitive if and only if $M = M^{\perp\perp}$.

We consider a primitive embedding $M \hookrightarrow L$ of even lattices M and L. Let the lattice K be isomorphic to the orthogonal complement of M in L. Then L is an overlattice of $M \oplus K$: $M \oplus K \hookrightarrow L$. By Proposition 2.0.1 this overlattice is uniquely determined by the isotropic group $G_L = L/M \oplus K \subset A_{M\oplus K} = A_M \oplus A_K$. We have the following composition of homomorphisms

$$G_L = L/(M \oplus K) \hookrightarrow L^*/(M \oplus K) \to (M \oplus K)^*/(M \oplus K). \tag{2.0.3}$$

The homomorphism given by the homomorphism 2.0.3 together with the projection $A_M \oplus A_K \to A_M$ gives a homomorphism $\phi_M : G_L \to A_M$. ϕ_M is injective. This can be seen for example by an argument given in [66]: Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \oplus K & \longrightarrow & L & \longrightarrow & L/(M \oplus K) & \longrightarrow & 0\ . \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi_M} & & \\ 0 & \longrightarrow & M & \longrightarrow & M^* & \longrightarrow & A_M & \longrightarrow & 0 \end{array}$$

The kernel of the map $L \to M^*$ given by $l \mapsto \langle l, \ \rangle_M$ is $K^{\perp\perp}$. By the snake lemma this yields the short exact sequence of the kernels

$$0 \longrightarrow K \longrightarrow K^{\perp\perp} \longrightarrow ker(\phi_M) \longrightarrow 0.$$

Since K is a primitive sublattice we have $K = K^{\perp\perp}$.

Similary, we obtain an injective homomorphism $\phi_K : G_L \to A_K$. If L is unimodular $L \cong L^*$ the maps ϕ_M, ϕ_K are isomorphisms. The isomorphism $h := \phi_K \circ \phi_M^{-1} : A_M \to A_K$ satisfies $q_K \circ h = -q_M$.

Theorem 2.0.2. *[14, 13] Let L be an even unimodular lattice and $M \hookrightarrow L$ a primitive sublattice. Let K be a sublattice isomorphic to the orthogonal complement of M in L. Then the embedding $M \hookrightarrow L$ is uniquely determined by an isomorphism $h : M^*/M \to K^*/K$ where the discriminant forms are related by $q_K \circ h = -q_M$. Moreover,*

$$L \cong \{(m,k) \in M^* \oplus K^* | h([m]) = [k]\} .$$

Chapter 3

K3 Surfaces

We want to study stability conditions on the derived category of coherent sheaves on projective Kummer surfaces. One of the fascinating aspects of this is the close interaction of homological algebra and classical algebraic geometry. In this section we review the theory of K3 surfaces [15, 69, 12].

Definition 3.0.1. A *K3 surface* is a compact complex surface X with trivial canonical bundle and $H^1(X, \mathscr{O}_X) = 0$.

Here are two examples:

1. The Fermat surface is the smooth quartic hypersurface $X \subset \mathbb{P}^3$ defined by $x_0^4 + x_1^4 + x_2^4 + x_3^4$.
2. Let T be a complex torus and $i : T \to T$ the involution $t \mapsto -t$ induced from $\mathbb{C}^2$. The minimal resolution $X \to T/i$ of the quotient surface T/i is a K3 surface, called the *Kummer surface* of T. The induced rational map $\pi : T \to X$ of degree 2 defined outside the fixed points of $i : T \to T$ induces a map on cohomology. The nowhere-vanishing 2-form on T is i-invariant so this map preserves the Hodge structures. Let $NS(T) = H^{1,1}(T) \cap H^2(T, \mathbb{Z})$ be the Néron-Severi lattice of the complex Kähler manifold T. If $NS(T)$ contains an element of positive square-length this induces an element of positive square-length in the Néron-Severi lattice $NS(X)$ of the Kummer surface X. Conversely, any such element in $NS(X)$ implies the existence of an element of positive square-length in $NS(T)$. By a famous theorem of Kodaira [70] the complex torus is thus an Abelian surface if and only if the associated Kummer surface is projective.

The intersection product

$$(\, , \,) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

makes the cohomology group $H^2(X,\mathbb{Z})$ into an even, unimodular lattice of rank 22. By Theorem 2.0.1 we can identify with the lattices introduced in chapter 2

$$H^2(X,\mathbb{Z}) \cong 2(-E_8) \oplus 3U. \tag{3.0.1}$$

The choice of an isomorphism in 3.0.1 is called a *marking*.

We have the following deep result:

Theorem 3.0.3. *[16] All K3 surfaces are Kähler manifolds.*

There is a weight-two Hodge structure on $H^2(X,\mathbb{Z})$ given by

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

By the Hodge signature theorem the signature of the intersection product on $H^{1,1}(X,\mathbb{R}) = H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ is $(1, h^{1,1}-1)$. Therefore the set $\left\{x \in H^{1,1}(X,\mathbb{R}) | (x,x) > 0\right\}$ has two connected components. The *positive cone* is the component containing the *Kähler cone*, i.e. the set of all classes in $H^{1,1}(X,\mathbb{R})$ that can be represented by a closed positive (1,1)-form. For a K3 surface we have the following description of its Kähler classes:

Theorem 3.0.4. *[72] The Kähler cone of a K3 surface consists if all elements $x \in H^{1,1}(X,\mathbb{R})$ of the positive cone such that $(x,C) > 0$ for all smooth rational curves $C \subset X$. A class $x \in H^{1,1}(X,\mathbb{R})$ is contained in the closure of the Kähler cone if and only if x is contained in the closure of the positive cone and $(x,C) \geq 0$ for all smooth rational curves $C \subset X$.*

Any ample line bundle L on X defines a class $c_1(L) \in NS(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ in the Kähler cone. Therefore the *ample cone* in $NS(X) \otimes \mathbb{R}$ spanned by all ample classes is contained in the Kähler cone and is described analogously to the above theorem.

The closure of the ample or Kähler cone becomes important when we consider resolutions of quotient surfaces. A compact complex surface X is called an *orbifold K3 surface* if X has at most simple singular points and its minimal resolution $Y \to X$ is a K3 surface. An example is the Kummer surface.

Definition 3.0.2. [17] Let Y be the minimal resolution of a orbifold K3 surface X. An *orbifold Kähler (ample) class* x is a class in the closure of the Kähler (ample) cone of Y such that $(x,C) = 0$ for a smooth rational curve $C \subset Y$ precisely if C is a (-2)-curve contracted by the minimal resolution $Y \to X$.

Theorem 3.0.5. *[71] All K3 surfaces are diffeomorphic as differentiable manifolds.*

The idea is now instead of viewing K3 surfaces as different complex surfaces to view them as different complex structures on a specific K3 surface as a real differentiable manifold. Let us consider a K3 surface viewed as a real differentiable manifold with a complex structure. Since the canonical bundle of a K3 surface X is trivial there is a nowhere-vanishing holomorphic 2-form Ω on X. This 2-form satisfies:

$$i)\ d\Omega = 0,\ ii)\ \Omega \wedge \Omega = 0 \text{ and } iii)\ \Omega \wedge \overline{\Omega} > 0.$$

Decomposing $\Omega = \Omega_1 + i\Omega_2$ into its real and imaginary part this conditions define a positive definite, oriented 2-plane in $H^2(X,\mathbb{R})$ with orthogonal basis given by $\Omega_1, \Omega_2 \in H^2(X,\mathbb{R})$:

$$\langle \Omega_1, \Omega_2 \rangle = 0,\ \langle \Omega_1, \Omega_1 \rangle = \langle \Omega_2, \Omega_2 \rangle > 0.$$

The choice of a complex structure defines a positive definite, oriented 2-plane with respect to the lattice of integral cohomolgy $H^2(X,\mathbb{Z})$. The converse statement is true due to the global Torelli theorem in the following form

Theorem 3.0.6. *[8, 9, 73, 17] The complex structures of a K3 surface (including orbifolds) are in bijection to positive definite, oriented two-planes in $H^2(X,\mathbb{Z}) \otimes \mathbb{R}$.*

A similar statement holds for complex structures on complex tori.

Chapter 4

Representations of Quivers

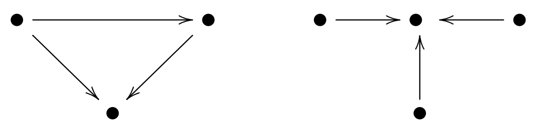
In this chapter we review the theory of representations of (finite) quivers following [63, 64]. If not stated otherwise proofs are taken from [64]. For example the category of representations of an acyclic quiver with n vertices is an Abelian category such that every object is of finite length and there are exactly n simple objects in this category associated to the simple representations for the n vertices. The definition of Bridgeland stability conditions on associated triangulated categories is therefore straightforward and we will apply these in chapter 8 as a tool to study the category itself.

4.1 Quivers

Definition 4.1.1. A *quiver* Q is a directed graph given by a set of vertices Q_0, a set of arrows Q_1 between them and head and tail maps

$$h,t : Q_1 \longrightarrow Q_0.$$

Here are some examples:



A vertex of a quiver is called a *sink*, if there are only arrows in the quiver directing to this vertex and no arrow starting at this vertex. A vertex is called a *source* if there are only arrows starting at this vertex and no ingoing arrows. In each of the two examples we can identify a source and a sink.

Let k be a field. A finite-dimensional *representation of a (finite) quiver* Q assigns to every vertex $i \in Q_0$ a (finite-dimensional) k-vector space V_i and to every arrow $a \in Q_1$ a linear map

$$\phi_a : V_{t(a)} \longrightarrow V_{h(a)}.$$

A *morphism* ϕ of representations V and W of Q is a collection of linear maps $f_i : V_i \to W_i$ such that the diagrams

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{\phi_a} & V_{h(a)} \\ \downarrow f_{t(a)} & & \downarrow f_{h(a)} \\ W_{t(a)} & \xrightarrow{\phi'_a} & W_{h(a)} \end{array}$$

commute for all $a \in Q_1$. A morphism of two representations of a quiver is injective (surjective) if all these linear maps are injective (surjective). If $\phi : V \to W$ is a morphism such that the maps ϕ_i for all $i \in Q_0$ are inclusion maps, we call V a *subrepresentation of* W. We define the *direct sum* $V \oplus W$ of two representations of Q by taking direct sums of the vector spaces V_i and W_i together with direct sums of the associated maps. The category $\mathscr{H}_Q$ of finite-dimensional representations of Q is an Abelian category. This can be checked directly or follows from Proposition 4.1.1.

A representation of Q is called *indecomposable* if we can not decompose it in a (non-trivial) direct sum of representations of Q.

Let us consider the A_2 quiver

$$1 \longrightarrow 2 \,.$$

Its indecomposable representations (up to isomorphism) are

$$0 \longrightarrow k, \qquad k \xrightarrow{id} k, \qquad k \longrightarrow 0 \,.$$

The first and the last representation are called *simple* representations. They are simple objects of $\mathscr{H}_Q$. This means they do not have proper subobjects in $\mathscr{H}_Q$ or equivalently, they are not the middle part of any non-trivial short exact sequence in $\mathscr{H}_Q$. In general, given a quiver there are simple representations S_i given by assigning a one-dimensional vector space to the vertex i and the zero-space to all vertices $j \neq i$.

Note that we have a short exact sequence

$$0 \longrightarrow (0 \to k) \longrightarrow (k \to k) \longrightarrow (k \to 0) \longrightarrow 0. \qquad (4.1.1)$$

Here is an indecomposable representation for our example of the D_4 quiver :

$$\begin{array}{ccccc} 0 \oplus k & \longrightarrow & k \oplus k & \longleftarrow & k \oplus 0 \,. \\ & & \uparrow & & \\ & & \Delta & & \end{array}$$

Here $\Delta = \{(x,x)|x \in k\}$ is the diagonal in $k \oplus k$ and all maps are the obvious inclusion maps. A theorem of Gabriel says that a quiver Q has finitely many isomorphism classes of indecomposable representations if and only if its underlying graph is of ADE-type [67].

A non-trivial *path* in a quiver Q is a sequence of arrows $a_n \cdots a_0$ with $h(a_{i-1}) = t(a_i)$ for $i = 1, \ldots, n$:

$$\bullet \xrightarrow{a_0} \bullet \xrightarrow{a_1} \bullet \longrightarrow \cdots \xrightarrow{a_{n-1}} \bullet \xrightarrow{a_n} \bullet.$$

We denote by e_i the trivial path at vertex $i \in Q_0$.

Definition 4.1.2. The *path algebra kQ* of a quiver Q is the $k-$vector space with basis given by all paths in Q with product given by composition of paths: The product of two paths p and q is the composition pq if $t(p) = h(q)$ and zero else.

For example the path algebra of the quiver with one loop

$$\bullet \circlearrowleft$$

is isomorphic to the ring $k[x]$ of polynomials in one variable and the path algebra of quiver

$$\bullet \rightarrow \bullet$$

is isomorphic to the algebra of 2×2 lower triangular matrices

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} |a,b,c \in k \right\}.$$

Proposition 4.1.1. *The category $\mathscr{H}_Q$ of finite-dimensional representations of a quiver Q is equivalent to the category of finitely generated left kQ-modules.*

Proof. Given a finite-dimensional representation V of Q we define a left kQ-module as a vector space as

$$V = \bigoplus_{i \in Q_0} V_i.$$

For every arrow $a \in Q_1$ we have an associated linear map $\phi_a : V_i \rightarrow V_j$. For the trivial path this is just the identity. We define an action of the path algebra kQ on the vector space V by linear extension of

$$e_i v = \begin{cases} v & v \in V_i \\ 0 & v \in V_j, j \neq i \end{cases}$$

for $i \in Q_0$ and

$$av = \begin{cases} \phi_a(v) & v \in V_{t(a)}, \\ 0 & v \in V_j, j \neq t(a) \end{cases}$$

for $a \in Q_1$. A morphism between two representations V and W in $\mathscr{H}_Q$ induces a morphism between $\bigoplus_{i \in Q_0} V_i$ and $\bigoplus_{i \in Q_0} W_i$ that gives a kQ-linear map. Conversely, given a finitely-generated left kQ-module V take the vector space $V_i = e_i V$ for all $i \in Q_0$ and introduce a linear map

$$\begin{aligned} \phi_a : V_{t(a)} &\longrightarrow V_{h(a)} \\ v &\longmapsto a(v). \end{aligned}$$

A morphism

$$\phi : V \longrightarrow W$$

of kQ-modules V and W sends $e_i V$ to $e_i W$ since $\phi(e_i v) = e_i \cdot \phi(v) \in e_i W$. Given a map of kQ-modules this observation allows to define a morphism between the associated representations of Q. The two constructions described above are inverse to the each other. □

In chapter 8 we study Abelian categories of finite length, i.e. every object in the category has finite length, and finitely many simple objects. A first basic observation is the following:

Proposition 4.1.2. *The category $\mathscr{H}_Q$ of finite-dimensional representations of a quiver Q is of finite length.*

Proof. This follows from the fact that proper subspaces of finite-dimensional vector spaces have strictly smaller dimension: By definition a representation E of a quiver is simple or we have a non-trivial short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0.$$

If B is not simple we can go on in the same way. This process has to terminate since we are dealing with finite-dimensional vector spaces and we get a simple representation S and a surjective map $f : E \to S$. Taking the kernel E^1 of f the quotient E/E^1 is isomorphic to S. If we repeat this process we get a finite filtration

$$\cdots \subset E^3 \subset E^2 \subset E^1 \subset E$$

with each quotient E^{i-1}/E^i a simple object. By the same argument as above after finitely many, say n steps we get $E^n = 0$. Renumbering $E_n := E, E_{n-1} := E^1, \ldots$ we get a Jordan-Hölder filtration for E. □

We call a quiver Q *acyclic* if it does not contain oriented cycles. In this case the category of representations of Q is of finite length with finitely many simple objects given by the simple representations S_i with a one-dimensional vector space at the vertex $i \in Q_0$ and the zero space at every vertex different to vertex i. The category of finite-dimensional representations of an acyclic quiver is equivalent to the category of finite-dimensional modules over the finite-dimensional path algebra. Indeed, the path algebra of an acyclic quiver is finite-dimensional and the finitely-generated modules over a finite-dimensional algebra are exactly the finite-dimensional modules over this algebra.

Let us consider the quiver with a cycle of length two:

$$\bullet \leftrightarrows \bullet.$$

Representations of dimension vector $(1,1)$ where both maps are isomorphisms are simple and parametrized by $\mathbb{C}^*$.

4.2 Stable representations

We are interested in the category $\mathscr{H}_Q$ of representations of an acyclic quiver Q. We denote its n simple representations associated to the n vertices of Q by S_i for $i \in Q_0$. Let $K(\mathscr{H}_Q)$ be the Grothendieck group of $\mathscr{H}_Q$, i.e. the free Abelian group generated by objects in $\mathscr{H}_Q$ modulo the subgroup generated by all elements of the form $B - A - C$ for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

The following notions will be generalized in chapters 6 and 8:

Definition 4.2.1. A *central charge* (or *stability function*) is a group homomorphism

$$Z : K(\mathscr{H}_Q) \longrightarrow \mathbb{C}$$

such that $Z(E) \in \overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}_{<0}$ for every object $E \in \mathscr{H}_Q$.

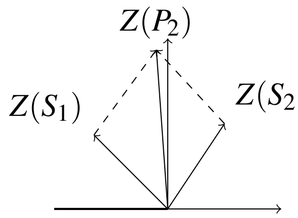
By Proposition 4.1.2 we can write the class of an object of $\mathscr{H}_Q$ as positive linear combination of the classes of the simple objects $S_1, \ldots, S_n$. Therefore a central charge is given by the numbers $Z(S_1), \ldots, Z(S_n) \in \overline{\mathbb{H}}$. A central charge of an object E comes with a phase $\phi \in (0,1]$: $Z(E) = rexp(i\pi\phi), r \in \mathbb{R}_{>0}$.

Definition 4.2.2. An object $E \in \mathscr{H}_Q$ is *(semi)stable* if for every proper subobject $F \subset E$ the phase of F is strictly smaller than the phase of E (the phase of F is smaller than or equal to the phase of E).

Here is an example: Let us consider the quiver

$$1 \longrightarrow 2 .$$

The stable representations are indecomposable. [74] We want the simple objects S_1, S_2 to have different phases. Then we have two choices: 1. $\phi(S_1) > \phi(S_2)$ and 2. $\phi(S_2) > \phi(S_1)$. Let us denote the representation $k \to k$ by P_2. The short exact sequence 4.1.1 tells us $Z(P_2) = Z(S_1) + Z(S_2)$. Therefore in the first case the stable objects are S_1, S_2, P_2 and in the second case S_1, S_2.



For a second example consider the Kronecker quiver

$$1 \rightrightarrows 2 .$$

If the phase of S_1 is strictly greater than the phase of S_2, the simple representations are the only stable objects. If the phase of S_2 is strictly greater than the phase of S_1 the stable objects of dimension vector $(1,1)$ are precisely the representations in the $\mathbb{P}^1$-family with dimension vector $(1,1)$. In this case infinitely many stable objects lie on a ray in the upper half plane.

Chapter 5

Triangulated Categories

We review derived categories and t-structures on triangulated categories [75, 77, 80]. We collect results and useful facts that will be used frequently in the following chapters. If not stated otherwise all proofs through the proof of Lemma 5.3.1 are taken from [75]. t-structures are a tool to see Abelian subcategories in a triangulated category. More precisely, the heart of a t-structure of a triangulated category is an Abelian subcategory. This construction is crucial for the definition of stability conditions. In fact, to give a Bridgeland stability condition is equivalent to give a heart of a bounded t-structure together with a central charge on its heart fulfilling some condition. The standard example of a t-structure on a derived category of an Abelian category is the one given by the Abelian category embedded in the derived category.

We fix some notations: For a full subcategory $\mathscr{F} \subset \mathscr{D}$ of an additive category $\mathscr{D}$ we write $\mathscr{F}^{\perp}$ for the full subcategory

$$\{E \in \mathscr{D} | Hom(F,E) = 0, \forall F \in \mathscr{F}\}.$$

Given full subcategories $\mathscr{A}, \mathscr{B}$ of a triangulated category $\mathscr{D}$ (see section 5.2) then $\langle \mathscr{A}, \mathscr{B} \rangle$ is the extension-closed subcategory generated by objects in $\mathscr{A}$ and $\mathscr{B}$.

5.1 Derived categories and localisation

The category $Kom(\mathscr{A})$ of complexes of an Abelian category $\mathscr{A}$ has as objects complexes M in $\mathscr{A}$

$$\cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$$

with $d^i \circ d^{i-1} = 0$ for all i and as morphisms a sequence of morphisms $f^i : M^i \to N^i$ such that $f^{i+1} d^i_M = d^i_N f^i$ for all i. Given a complex M, the

$i-th$ *cohomology object* is the quotient

$$H^i(M) = ker(d^i)/im(d^{i-1}).$$

A morphism of complexes $f : M \to N$ is a *quasi-isomorphism* if the induced morphisms $H^i(f) : H^i(M) \to H^i(N)$ are isomorphisms for all i.

Given $Kom(\mathscr{A})$ we can formally invert the quasi-isomorphism to get the derived category of $\mathscr{A}$: We localize $Kom(\mathscr{A})$ at the class of quasi-isomorphisms.

Proposition 5.1.1. *Given $Kom(\mathscr{A})$ for an Abelian category $\mathscr{A}$, there is a category $\mathscr{D}(\mathscr{A})$ together with a functor*

$$Q : Kom(\mathscr{A}) \longrightarrow \mathscr{D}(\mathscr{A})$$

such that:

1. *if $s : a \to b$ is a quasi-isomorphism in $Kom(\mathscr{A})$ $Q(s) : Q(a) \to Q(b)$ is an isomorphism in $\mathscr{D}(\mathscr{A})$,*

2. *Q is universal: if $Q' : Kom(\mathscr{A}) \longrightarrow \mathscr{D}'$ is another functor which inverts quasi-isomorphisms, i.e. a functor Q' which fulfills the 1. point, then there is a functor $F : \mathscr{D}(\mathscr{A}) \to \mathscr{D}'$ such that $Q' \cong F \circ Q$.*

Proof. Given $Kom(\mathscr{A})$, we define the graph Γ with vertices the objects of $Kom(\mathscr{A})$ and directed edges defined by the morphisms in $Kom(\mathscr{A})$. For every quasi-isomorphism $s : a \to b$ we insert a new edge $s^{-1} : b \to a$. We denote the new graph by Γ^* with paths of the form $f_1 \cdot f_2 \cdot \dots \cdot f_r$ with f_i is a morphism in $Kom(\mathscr{A})$ or of the form s^{-1} for a quasi-isomorphism s. Next we introduce an equivalence relation $\sim$ on the set of finite paths in Γ^*:

1. For each quasi-isomorphism $s : a \to b$ we set $s \cdot s^{-1} \sim id_b$ and $s^{-1} \cdot s \sim id_a$,

2. $g \cdot f \sim g \circ f$ for composable morphisms $f : a \to b$ and $g : b \to c$.

$\mathscr{D}(\mathscr{A})$ is the category with objects the vertices in Γ^* and morphisms the equivalence classes of finite paths. □

The derived category can be constructed as the localisation of the homotopy category at the quasi-isomorphisms. Let

$$f, g : M \longrightarrow N$$

be morphisms in $Kom(\mathscr{A})$. f and g are *homotopic* if there are morphisms

$$h^i : M^i \longrightarrow N^{i-1}$$

such that

$$g^i - f^i = d_N^{i-1} \circ h^i + h^{i+1} \circ d_M^i$$

for all i. The homotopy category $K(\mathscr{A})$ is the category with the same objects as $Kom(\mathscr{A})$ and morphisms given by homotopy equivalence classes of morphisms of complexes.

Lemma 5.1.1. *Let $Q : Kom(\mathscr{A}) \longrightarrow \mathscr{D}(\mathscr{A})$ be the functor in Proposition 5.1.1. If $f, g : M \longrightarrow N$ are homotopic morphisms in $Kom(\mathscr{A})$, then $Q(f) = Q(g)$.*

Therefore the functor Q factors via the homotopy category. The key observation of Verdier [78] was to prove that the set of quasi-isomorphisms satisfy the Ore conditions in $K(\mathscr{A})$:

Proposition 5.1.2. *[78] Let $\mathscr{A}$ be an Abelian category and $K(\mathscr{A})$ the homotopy category of $\mathscr{A}$.*

1. *If $f : M \to N$ and $s : N' \to N$ are morphisms in $K(\mathscr{A})$ where s is a quasi-morphism, then there is a complex M' and morphisms of complexes $g : M' \to N'$ and $t : M' \to M$ where t is a quasi-isomorphism, such that the diagram*

$$\begin{array}{ccc} M' & \xrightarrow{g} & N' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow{f} & N \end{array}$$

   commutes.

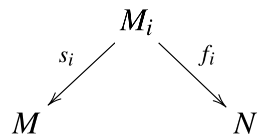
2. *If $f : M \to N$ is a morphism in $K(\mathscr{A})$, then there is a quasi-isomorphism $s : M' \to M$ with $f \circ s = 0$ in $K(\mathscr{A})$ precisely if there is a quasi-isomorphism $t : N \to N'$ with $t \circ f = 0$ in $K(\mathscr{A})$.*

Here is an analogy from ring theory: Clearly, you can localize the integers at the subset of all non-zero elements of $\mathbb{Z}$ and define a new ring $\mathbb{Q}$. More generally you can localize a commutative ring at its subset of all non-zero elements. You can still localize a noncommutative ring at some subset if the subset fulfills some conditions, the Ore conditions. [79]

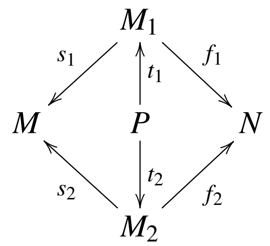
The consequence is that the localisation functor factors via the homotopy category $K(\mathscr{A})$ and the induced functor $\tilde{Q} : K(\mathscr{A}) \to \mathscr{D}(\mathscr{A})$ can be described as follows: A morphism $f : M \to N$ in $\mathscr{D}(\mathscr{A})$ can be represented by a roof in $K(\mathscr{A})$

$$\begin{array}{ccccc} & & M' & & \\ & {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} & \\ M & & & & N \end{array}$$

with $s : M' \to M$ a quasi-isomorphism. Two roofs with $i = 1,2$



represent the same morphism in $\mathscr{D}(\mathscr{A})$ if there is a commutative diagram in $K(\mathscr{A})$ of the form



with quasi-isomorphisms t_1, t_2.

The derived category $\mathscr{D}(\mathscr{A})$ of an Abelian category $\mathscr{A}$ is additive. The role of short exact sequences is played by distinguished triangles. To define these we first introduce shifts of complexes. For this fix an integer n. For a complex M in $Kom(\mathscr{A})$ define a new complex $M[n]$ by setting $M[n]^i = M^{i+n}$ and $d^i_{M[n]} := (-1)^n d^{i+n}_M$ and for a morphism $f : M \to N$ of complexes define a morphism $f[n] : M[n] \to N[n]$ by $f[n]^i := f^{i+n}$. The *mapping cone* of $f : M \to N$ is the complex $C(f)$ with

$$C(f)^i = M^{i+1} \oplus N^i$$

and with differential

$$d^i_{C(f)}(m,n) = (-d^{i+1}_M(m), f^{i+1}(m) + d^i_N(n)).$$

We can find morphisms of complexes fitting into the sequence

$$M \xrightarrow{f} N \longrightarrow C(f) \longrightarrow M[1].$$

The shift functor $[n]$ descends to a functor in the derived category $\mathscr{D}(\mathscr{A})$. A distinguished triangle in $\mathscr{D}(\mathscr{A})$ is a triple of objects (A,B,C) and morphisms

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

which is isomorphic to a triple given by a mapping cone. We write sometimes for a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow$$

without the object $A[1]$ on the right.

5.2 Triangulated categories

Triangulated categories axiomatize the properties of the shift functor and distinguished triangles in derived categories of Abelian categories.

Let $\mathscr{D}$ be an additive category with an auto-equivalence $[1]:\mathscr{D}\to\mathscr{D}$. A *triangle* in $\mathscr{D}$ is a sequence of morphisms of the form

$$A \to B \to C \to A[1].$$

Definition 5.2.1. A *morphism of triangles* is a commutative diagram of the form

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1] \end{array} .$$

Definition 5.2.2. A *triangulated category* $\mathscr{D}$ is an additive category with an auto-equivalence $[1]:\mathscr{D}\to\mathscr{D}$ and a set of distinguished triangles (d.t.) satisfying the following axioms:

1. A triangle isomorphic to a d.t. is a d.t.

2. The triangle $A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$ is a d.t.

3. For all morphisms

$$f: A \to B$$

there exists a d.t.

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

.

4. A triangle is

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a d.t. if and only if

$$B \xrightarrow{-g} C \xrightarrow{-h} A[1] \xrightarrow{-f[1]} B[1]$$

is a d.t.

5. Given two d.t.'s

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

and

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} A'[1]$$

and morphisms $\alpha : A \to A'$ and $\beta : B \to B'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma : C \to C'$ giving rise to a morphism of d.t.'s:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\alpha[1]}\downarrow \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & A'[1]
\end{array} .$$

6. Given three d.t.'s

$$A \xrightarrow{f} B \xrightarrow{h} C' \longrightarrow A[1] ,$$

$$B \xrightarrow{g} C \xrightarrow{k} A' \longrightarrow B[1] ,$$

$$A \xrightarrow{g\circ f} C \xrightarrow{l} B' \longrightarrow A[1] ,$$

there exists a d.t. $C' \xrightarrow{u} B' \xrightarrow{v} A' \xrightarrow{w} C'[1]$ making the following diagram (the octahedron diagram) commute:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{h} & C' & \longrightarrow & A[1] \\
{\scriptstyle id}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle id}\downarrow \\
A & \xrightarrow{g\circ f} & C & \xrightarrow{l} & B' & \longrightarrow & A[1] \\
{\scriptstyle f}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle f[1]}\downarrow \\
B & \xrightarrow{g} & C & \xrightarrow{k} & A' & \longrightarrow & B[1] \\
{\scriptstyle h}\downarrow & & {\scriptstyle l}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle h[1]}\downarrow \\
C' & \xrightarrow{u} & B' & \xrightarrow{v} & A' & \xrightarrow{w} & C'[1]
\end{array} .$$

Proposition 5.2.1. *The derived category of an Abelian category is a triangulated category with auto-equivalence* $[1]$ *given by the shift functor and with set of distinguished triangles given by the distinguished triangles defined in section 5.1.*

Here are some observations:

Lemma 5.2.1. *Let $\mathscr{D}$ be a triangulated category and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

a distinguished triangle. Then we have $g \circ f = 0$.

Proof. By axiom 1 and 4 of Definition 5.2.2 we have the following morphism of distinguished triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{id} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ {\scriptstyle id}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle id[1]} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & A[1] \end{array} .$$

In particular, $g \circ f = 0$. □

We prove the following important Proposition since we will frequently use it.

Proposition 5.2.2. *Let $\mathscr{D}$ be a triangulated category and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

a distinguished triangle. Then for any $W \in \mathscr{D}$ the sequence

$$Hom(W,A) \xrightarrow{f\circ} Hom(W,B) \xrightarrow{g\circ} Hom(W,C)$$

is exact.

Proof. Let $\varphi : W \to B$ be a morphism with $g \circ \varphi = 0$. We consider the d.t. $A \xrightarrow{id} A \longrightarrow 0 \longrightarrow A[1]$. By axioms 4 and 5 of Definition 5.2.2 there is a morphism $\psi : W \to A$ giving rise to a morphism of d.t.'s:

$$\begin{array}{ccccccc} W & \longrightarrow & 0 & \longrightarrow & W[1] & \xrightarrow{-id[1]} & W[1] \\ {\scriptstyle \varphi}\downarrow & & \downarrow & & \downarrow{\scriptstyle \psi[1]} & & \downarrow{\scriptstyle \varphi[1]} \\ B & \xrightarrow{-g} & C & \xrightarrow{-h} & A[1] & \xrightarrow{-f[1]} & B[1] \end{array} .$$

Thus there is a morphism $\psi : W \to A$ such that $\varphi = f \circ \psi$. Using Lemma 5.2.1 we see that the image of $f\circ$ lies in the kernel of the morphism $g\circ$. This finishes the proof. □

5.3 t-Structures and tilting

An Abelian category $\mathscr{A}$ is embedded in its derived category $\mathscr{D}(\mathscr{A})$ as the subcategory of complexes whose cohomology is concentrated in degree zero. t-structures allow to see different Abelian subcategories inside a triangulated category.

Definition 5.3.1. [29] A *t-structure on a triangulated category* $\mathscr{D}$ is a full subcategory $\mathscr{F} \subset \mathscr{D}$ such that

1. $\mathscr{F}[1] \subset \mathscr{F}$

2. for every object $E \in \mathscr{D}$ there is a distinguished triangle in $\mathscr{D}$

$$F \longrightarrow E \longrightarrow G \longrightarrow$$

   with $F \in \mathscr{F}$ and $G \in \mathscr{F}^{\perp}$

with

$$\mathscr{F}^{\perp} = \{E \in \mathscr{D} | Hom(F,E) = 0, \forall F \in \mathscr{F}\}.$$

The *heart of a t-structure* $\mathscr{F} \subset \mathscr{D}$ is the full subcategory $\mathscr{A} = \mathscr{F} \cap \mathscr{F}^{\perp}[1]$.

Proposition 5.3.1. *[29] The heart of a t-structure of a triangulated category is an Abelian category.*

The short exact sequences in the heart of a t-structure of a triangulated category are precisely the exact triangles in the triangulated category all of whose vertices are objects of the heart. We will use this fact frequently.

A t-structure $\mathscr{F} \subset \mathscr{D}$ is *bounded* if

$$\mathscr{D} = \bigcup_{i,j \in \mathbb{Z}} \mathscr{F}[i] \cap \mathscr{F}^{\perp}[j].$$

Given the heart $\mathscr{A}$ of a bounded t-structure $\mathscr{F} \subset \mathscr{D}$ the t-structure is the extension-closed subcategory

$$\mathscr{F} = \langle \mathscr{A}, \mathscr{A}[1], \mathscr{A}[2], \ldots \rangle.$$

If $\mathscr{A}_1, \mathscr{A}_2 \subset \mathscr{D}$ are hearts of bounded t-structures on the triangulated category $\mathscr{D}$ such that $\mathscr{A}_1 \subset \mathscr{A}_2$ then $\mathscr{A}_1 = \mathscr{A}_2$.

Let us fix some notation: For two hearts $\mathscr{A}_1, \mathscr{A}_2$ with associated bounded t-structures $\mathscr{F}_1, \mathscr{F}_2 \subset \mathscr{D}$ we write $\mathscr{A}_1 \leq \mathscr{A}_2$ if and only if $\mathscr{F}_2 \subset \mathscr{F}_1$.

The following useful fact is well-known. A proof can be found in [76].

Proposition 5.3.2. *Let $\mathscr{A}$ be the heart of a bounded t-structure of the triangulated category $\mathscr{D}$. Then $Ext^1_{\mathscr{A}}(X,Y)$ with $X,Y \in \mathscr{A}$ can be identified with the Abelian group $Hom_{\mathscr{D}}(X,Y[1])$.*

Lemma 5.3.1. *The Abelian category $\mathscr{A}$ of the derived category $\mathscr{D}(\mathscr{A})$ is the heart of the t-structure of $\mathscr{D}(\mathscr{A})$ given by*

$$\mathscr{F} = \left\{E \in \mathscr{D}(\mathscr{A}) | H^i(E) = 0, \forall i > 0\right\},$$

This is the standard (or canonical) t-structure of a derived category.

Proof. First we note that $\mathscr{F}[1] \subset \mathscr{F}$. Next we want to show that $\mathscr{F}^{\perp} = \mathscr{D}^{\geq 0}[-1]$ with

$$\mathscr{D}^{\geq 0} = \left\{E \in \mathscr{D}(\mathscr{A}) | H^i(E) = 0, \forall i < 0\right\}.$$

We represent a morphism $f : M \to N$ in $\mathscr{D}(\mathscr{A})$ with $M \in \mathscr{F}$ and $N \in \mathscr{D}^{\geq 0}[-1]$ by a roof

$$M \xleftarrow{\ s\ } K \xrightarrow{\ f\ } N$$

in $K(\mathscr{A})$ with quasi-isomorphism s. Since $N \in \mathscr{D}^{\geq 0}[-1]$ the complex $N/\tau_{\leq 0}N$ is quasi-isomorphic to N. Here the truncation functor $\tau_{\leq 0}$ is defined by

$$\tau_{\leq 0}N := \begin{cases} N^i & i < 0, \\ ker(d^0) & i = 0 \\ 0 & i > 0 \end{cases}$$

Thus we can assume that $N^i = 0$ for $i < 0$ and that $d^0_N : N^0 \to N^1$ is injective. The objects M and K are quasi-isomorphic and thus we have $K \in \mathscr{F}$ since $M \in \mathscr{F}$. Therefore the canonical morphism $r : \tau_{\leq 0}K \to K$ is a quasi-isomorphism and we can represent the morphism $f : M \to N$ by the roof

$$M \xleftarrow{\ sr\ } \tau_{\leq 0}K \xrightarrow{\ fr\ } N$$

Next we want to prove that $fr = 0$. Since $N^i = 0$ for $i < 0$ for $i \neq 0$, we have that $N^i = 0$ or $(\tau_{\leq 0}K)^i = 0$, i.e. $(fr)^i = 0$ for $i \neq 0$. For $i = 0$ we have $d^0_N(fr)^0 = (fr)^1 d^0_{\tau_{\leq 0}K} = 0$. Since d^0_N is injective $(fr)^0 = 0$ and so we conclude $fr = 0$.

The 2. assumption in Def. 5.3.1 follows from the short exact sequence of complexes

$$0 \longrightarrow \tau_{\leq 0}M \longrightarrow M \longrightarrow M/\tau_{\leq 0}M \longrightarrow 0$$

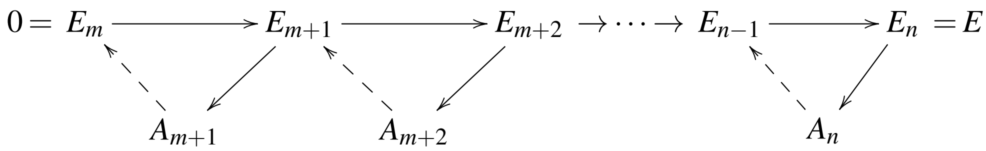
that induce a distinguished triangle

$$\tau_{\leq 0}M \longrightarrow M \longrightarrow M/\tau_{\leq 0}M \longrightarrow$$

in $\mathscr{D}(\mathscr{A})$. □

Lemma 5.3.2. *[26] A bounded t-structure is determined by its heart. Moreover, if $\mathscr{A} \subset \mathscr{D}$ is a full additive subcategory of a triangulated category $\mathscr{D}$, then $\mathscr{A}$ is the heart of a bounded t-structure on $\mathscr{D}$ if and only if the following conditions hold:*

1. *if A and B are objects of $\mathscr{A}$, then $Hom_{\mathscr{D}}(A, B[k]) = 0$ for $k < 0$,*
2. *for every non-zero object $E \in \mathscr{D}$ there are integers $m < n$ and a collection of triangles*



with $A_i[i] \in \mathscr{A}$ for all i.

The objects $A_i[i] \in \mathscr{A}$ are called cohomology objects of E with respect to the given t-structure in analogy to the standard t-structure of the derived category of an Abelian category and are denoted $H^i(E)$.

Definition 5.3.2. [27] A *torsion pair* in an Abelian category $\mathscr{A}$ is a pair of full subcategories $(\mathscr{T}, \mathscr{F})$ satisfying

1. $Hom_{\mathscr{A}}(T, F) = 0$ for all $T \in \mathscr{T}$ and $F \in \mathscr{F}$;
2. every object $E \in \mathscr{A}$ fits into a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0 \qquad (5.3.1)$$

for some pair of objects $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

The objects of $\mathscr{T}$ are called *torsion* and the objects of $\mathscr{F}$ are called *torsion-free*, $\mathscr{T}$ is called *torsion class* and $\mathscr{F}$ *torsion-free class*.

Lemma 5.3.3. *[27] Let $(\mathscr{T},\mathscr{F})$ be a torsion pair in an Abelian category $\mathscr{A}$. The subcategories $\mathscr{T}$ and $\mathscr{F}$ are closed under extensions. Moreover, $\mathscr{T}$ is closed under quotients and $\mathscr{F}$ is closed under subobjects.*

Proof. First note that $\mathscr{F}=\mathscr{T}^{\perp}$. Indeed, for $E\in\mathscr{A}$ with $Hom(T,E)=$ for all $T\in\mathscr{T}$ the morphism from the torsion object to the object E in the short exact sequence 5.3.1 is zero. Thus $E\in\mathscr{F}$. In the same way follows $\mathscr{T}={}^{\perp}\mathscr{F}$. Now take an extension E of two torsion objects $T_1,T_2\in\mathscr{T}$

$$0\longrightarrow T_1\longrightarrow E\longrightarrow T_2\longrightarrow 0.$$

Applying the functor $Hom(\ ,F)$ for torsion-free objects $F\in\mathscr{F}$ we see that objects in $\mathscr{T}$ are closed under extensions. In the same way we see that torsion objects are closed under quotients. Similarly, the second statement follows. □

In fact, a full subcategory $\mathscr{T}$ of an Abelian category $\mathscr{A}$ is a torsion class if and only if it is closed under quotients, direct sums and extensions and a full subcategory $\mathscr{F}$ is a torsion free class if and only if it is closed under subobjects, direct products and extensions. [27]

Proposition 5.3.3. *[34] Let $\mathscr{A}$ be the heart of a bounded t-structure on a triangulated category $\mathscr{D}$. Denote by $H^i(E)\in\mathscr{A}$ the i-th cohomology object of E with respect to this t-structure. Let $(\mathscr{T},\mathscr{F})$ be a torsion pair in $\mathscr{A}$. Then the full subcategory*

$$\mathscr{A}^*=\left\{E\in\mathscr{D}|H^i(E)=0\,for\,i\notin\{-1,0\},H^{-1}(E)\in\mathscr{F},H^0(E)\in\mathscr{T}\right\}$$

is the heart of a bounded t-structure on $\mathscr{D}$.

We say $\mathscr{A}^*$ is obtained from $\mathscr{A}$ by *tilting* with respect to the torsion pair $(\mathscr{T},\mathscr{F})$. The pair $(\mathscr{F}[1],\mathscr{T})$ is a torsion pair in $\mathscr{A}^*$.

Suppose $\mathscr{A}\subset\mathscr{D}$ is the heart of a bounded t-structure on $\mathscr{D}$ such that every object of $\mathscr{A}$ is of finite length. Given a simple object $S\in\mathscr{A}$ we denote by $\langle S\rangle$ the full subcategory of objects $E\in\mathscr{A}$ whose simple factors in the Jordan-Hölder filtration are isomorphic to S. By the remark after Lemma 5.3.3 we can either view $\langle S\rangle$ as the torsion class of a torsion pair on $\mathscr{A}$ with torsion-free class

$$\mathscr{F}=\{E\in\mathscr{A}|Hom_{\mathscr{A}}(S,E)=0\}$$

or as the torsion-free class with torsion class

$$\mathscr{T}=\{E\in\mathscr{A}|Hom_{\mathscr{A}}(E,S)=0\}.$$

Applying the functors $Hom(S, \ )$ and $Hom(\ ,S)$ we see that there is no non-trivial morphism between these torsion and torsion free classes. E.g., by its very definition the subcategory $\mathscr{F} = \{E \in \mathscr{A} | Hom_{\mathscr{A}}(S,E) = 0\}$ is closed under extensions and subobjects.

The new hearts after tilting are

$$L_S(\mathscr{A}) = \left\{E \in \mathscr{D} | H^i(E) = 0 \text{ for } i \notin \{0,1\}, H^0(E) \in \mathscr{F}, H^1(E) \in \langle S \rangle \right\},$$
$$R_S(\mathscr{A}) = \left\{E \in \mathscr{D} | H^i(E) = 0 \text{ for } i \notin \{-1,0\}, H^{-1}(E) \in \langle S \rangle, H^0(E) \in \mathscr{T} \right\}.$$

$L_S(\mathscr{A})$ (respectively $R_S(\mathscr{A})$) is called *the left* (respectively *the right*) *tilt of* $\mathscr{A}$ *at the simple* S. $S[-1]$ is a simple object in $L_S(\mathscr{A})$ and if this heart is again of finite length we have $R_{S[-1]}L_S(\mathscr{A}) = \mathscr{A}$. Similarly, if $R_S(\mathscr{A})$ has finite length, we have $L_{S[1]}R_S(\mathscr{A}) = \mathscr{A}$.

We conclude this subsection with two useful observations that will become important in chapter 8. The following lemma of Bridgeland gives a composition of left-tilts.

Lemma 5.3.4. *[62] Let* $(\mathscr{T}, \mathscr{F})$ *be a torsion pair in* $\mathscr{A}$ *and* $(\mathscr{T}', \mathscr{F}')$ *a torsion pair in* $\mathscr{A}^* = \langle \mathscr{F}, \mathscr{T}[-1] \rangle$ *If* $\mathscr{T}' \subset \mathscr{F}$*, then the left-tilt* $\mathscr{A}^{**} = \langle \mathscr{F}', \mathscr{T}'[-1] \rangle$ *of* $\mathscr{A}^*$ *equals a left-tilt of* $\mathscr{A}$*.*

The following observation is immediate but important for chapter 8:

Lemma 5.3.5. *If* $\mathscr{A}^* = \langle \mathscr{F}, \mathscr{T}[-1] \rangle$ *is the left-tilt of* $\mathscr{A}$ *with respect to a torsion pair* $(\mathscr{T}, \mathscr{F})$*, then the simple objects of* $\mathscr{A}^*$ *lie in* $\mathscr{A}$ *or in* $\mathscr{A}[-1]$.

Proof. We have a short exact sequence in $\mathscr{A}^*$ for every object $S \in \mathscr{A}^*$

$$0 \longrightarrow E \longrightarrow S \longrightarrow F \longrightarrow 0$$

with $E \in \mathscr{F} \subset \mathscr{A}$ and $F \in \mathscr{T}[-1] \subset \mathscr{A}[-1]$. If S is simple we have $S \cong E$ or $S \cong F$. □

5.4 Derived categories of dg algebras

Let A be a differential graded algebra. A *(right) differential graded (dg) module M over A* is a graded A-module equipped with a differential d such that

$$d(ma) = d(m)a + (-1)^{|m|} m d(a)$$

where $m \in M$ is homogeneous of degree $|m|$ and $a \in A$. A morphism of dg A-modules is a quasi-isomorphism if it induces a quasi-isomorphism in the underlying complexes. The derived category $\mathscr{D}(A)$ of A is now the localization of the category of dg A-modules at the class of quasi-isomorphisms.

Let A be a dg algebra with $H^i(A) = 0$ for $i > 0$. Let $\mathscr{D}_{\leq 0}$ be the full subcategory of $\mathscr{D}(A)$ consisting of dg modules M such that the homology $H^i(M) = 0$ for $i > 0$. Then the subcategory $\mathscr{D}_{\leq 0}$ defines a t-structure on $\mathscr{D}(A)$. The functor $M \mapsto H^0(M)$ induces an equivalence of the heart of this canonical t-structure and the category of all (right) $H^0(A)$-modules. [53]

Chapter 6

Bridgeland Stability Conditions on Triangulated Categories

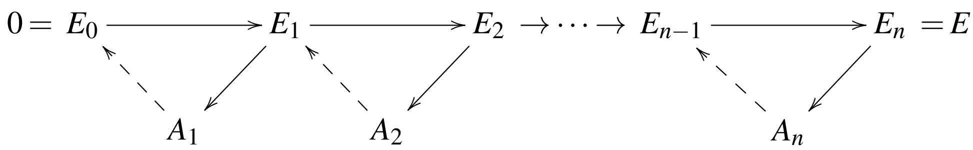
We review stability conditions on a triangulated category $\mathscr{D}$ introduced by Bridgeland in [26]. We denote by $K(\mathscr{D})$ the corresponding Grothendieck group of $\mathscr{D}$, i.e. the free Abelian group generated by objects in $\mathscr{D}$ modulo the subgroup generated by all elements of the form $B-A-C$ for all exact triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow .$$

An well-known fact is that the Grothendieck group of the heart of a bounded t-structure on a triangulated category and the Grothendieck group of the triangulated category can be identified. For a proof see [76].

Definition 6.0.1. [26] A *(Bridgeland) stability condition on a triangulated category* $\mathscr{D}$ consists of a group homomorphism $Z : K(\mathscr{D}) \to \mathbb{C}$ called the *central charge* and of full additive subcategories $\mathscr{P}(\phi) \subset \mathscr{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

1. if $0 \neq E \in \mathscr{P}(\phi)$, then $Z(E) = m(E)exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$;
2. $\forall \phi \in \mathbb{R}, \mathscr{P}(\phi+1) = \mathscr{P}(\phi)\,[1]$;
3. if $\phi_1 > \phi_2$ and $A_j \in \mathscr{P}(\phi_j)$, then $Hom_{\mathscr{D}}(A_1, A_2) = 0$;
4. for $0 \neq E \in \mathscr{D}$, there is a finite sequence of real numbers $\phi_1 > \cdots > \phi_n$ and a collection of triangles



with $A_j \in \mathscr{P}(\phi_j)$ for all j.

We will construct examples of stability conditions at the end of this chapter and in Proposition 7.4.3.

The filtration given in axiom 4 is unique up to isomorphism due to axiom 3. We recall some results of [26]. The subcategory $\mathscr{P}(\phi)$ is Abelian and its non-zero objects are said to be semistable of phase ϕ for a stability condition $\sigma = (Z, \mathscr{P})$. We call its simple objects *stable*. The objects A_i in Definition 6.0.1 are called *semistable factors of E* with respect to σ. We write for an non-zero object $\phi_\sigma^+ := \phi_1$ and $\phi_\sigma^- := \phi_n$. An object E fulfills $\phi_\sigma^+ = \phi_\sigma^-$ precisely if $E \in \mathscr{P}$ for some $\phi \in \mathbb{R}$. The *mass* of E is defined to be $m_\sigma(E) = \sum_i |Z(A_i)| \in \mathbb{R}$. For an interval $I \subset \mathbb{R}$ we define $\mathscr{P}(I)$ to be the extension-closed subcategory of $\mathscr{D}$ generated by the subcategories $\mathscr{P}(\phi)$ for all $\phi \in I$.

Here are some useful facts:

1. All morphisms $E_i \to E$ and the morphism $E \to A_n$ in the filtration in axiom 4 of Definition 6.0.1 are non-trivial.
2. If $E_1 \in \mathscr{P}(\phi), E_2 \in \mathscr{P}(I)$ and $\phi > t$ for all $t \in I$, then $Hom(E_1, E_2) = 0$.

The second observation follows from the fact that the object E_2 is generated by semistable objects with phases in the interval I.

Let $\mathscr{P}(> \phi)$ be the extension-closed subcategory generated by the subcategories $\mathscr{P}(\psi)$ for all $\psi > \phi$. The subcategory $\mathscr{F} = \mathscr{P}(> \phi)$ is closed under shifts and there is no non-trivial morphism from $\mathscr{F}$ to $\mathscr{F}^\perp = \mathscr{P}(\leq \phi)$. Indeed, $\mathscr{P}(> \phi)$ is a t-structure on the triangulated category $\mathscr{D}$ with heart

$$\mathscr{A} = \mathscr{P}(> \phi) \cap \mathscr{P}(\leq \phi)[1] = \mathscr{P}((\phi, \phi + 1]).$$

A stability condition is *locally-finite* if there exists some $\varepsilon > 0$ such that for all $\phi \in \mathbb{R}$ each quasi-Abelian subcategory[1] $\mathscr{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length. In this case $\mathscr{P}(\phi)$ is of finite length and every semistable object has a finite Jordan-Hölder filtration into stable objects of the same phase. We denote by $Stab(\mathscr{D})$ the set of locally finite stability conditions on a triangulated category $\mathscr{D}$.

[1] For quasi-Abelian subcategories see chapter 4 in [26].

$Stab(\mathscr{D})$ has a topology induced by the generalised metric[2]:

$$d(\sigma,\tau) = sup_{0\neq E\in\mathscr{D}}\left\{\left|\phi^-_\tau(E)-\phi^-_\sigma(E)\right|,\left|\phi^+_\tau(E)-\phi^+_\sigma(E)\right|,\left|log\frac{m_\tau(E)}{m_\sigma(E)}\right|\right\}.$$

with $\sigma = (Z,\mathscr{P})$ and $\tau = (Z',\mathscr{Q})$. If $d(\sigma,\tau) = 0$ this implies for an object $E \neq 0$ of $\mathscr{P}(\phi)$: $\phi^+_\sigma(E) = \phi^-_\sigma(E) = \phi^+_\tau(E) = \phi^-_\tau(E)$ and therefore E is also an element of $\mathscr{Q}(\phi)$, i.e. $\mathscr{P} = \mathscr{Q}$. If E is semistable with respect to σ, then $m_\sigma(E) = |Z(E)|$. Therefore the central charges of σ and τ agree on the semistable objects. Since every object in $\mathscr{D}$ has a filtration like the one in Definition 6.0.1 the central charges agree on all objects of $\mathscr{D}$. Thus $d(\sigma,\tau) = 0$ implies $\sigma = \tau$ and d is indeed a metric.

There is an action of the group of auto-equivalences $Aut(\mathscr{D})$ of the derived category $\mathscr{D}$ on $Stab(\mathscr{D})$. For $\sigma = (Z,\mathscr{P}) \in Stab(\mathscr{D})$ and $\Phi \in Aut(\mathscr{D})$ define the new stability condition $\Phi(\sigma) = (Z\circ\Phi_*^{-1},\mathscr{P}')$ with $\mathscr{P}'(\phi) = \Phi(\mathscr{P}(\phi))$. Here Φ_* is the induced automorphism of $K(\mathscr{D})$ of Φ. Note that auto-equivalences preserve the generalised metric.

The universal covering $\widetilde{GL^+(2,\mathbb{R})}$ of $GL^+(2,\mathbb{R})$ acts on the metric space $Stab(\mathscr{D})$ on the right in the following way: We think of an element of $\widetilde{GL^+(2,\mathbb{R})}$ as a pair (G,f) with $G\in GL^+(2,\mathbb{R})$ and an increasing function $f:\mathbb{R}\to\mathbb{R}$ with $f(\phi+1) = f(\phi)+1$ such that $Gexp(i\pi\phi)/|exp(i\pi\phi)| = exp(2i\pi f(\phi))$ for all $\phi\in\mathbb{R}$. Here we identify $\mathbb{C} = \mathbb{R}^2$. A pair $(G,f) \in \widetilde{GL^+(2,\mathbb{R})}$ maps $\sigma = (Z,\mathscr{P}) \in Stab(\mathscr{D})$ to $(Z',P') = (G^{-1}\circ Z,\mathscr{P}\circ f)$.

Example: The subgroup $\mathbb{C}\hookrightarrow\widetilde{GL^+(2,\mathbb{R})}$ acts freely on $Stab(\mathscr{D})$ for a triangulated category $\mathscr{D}$ by sending a complex number λ and a stability condition $(Z,\mathscr{P})$ to a stability condition $(Z',\mathscr{P}')$ where $Z'(E) = exp(-i\pi\lambda)Z(E)$ and $\mathscr{P}'(\phi) = \mathscr{P}(\phi+Re(\lambda))$. Note that this is for $\lambda = n\in\mathbb{Z}$ just the action of the shift functor $[n]$.

Definition 6.0.2. Let $\mathscr{A}$ be an Abelian category with Grothendieck group $K(\mathscr{A})$. A *central charge* (or *stability function*) on $\mathscr{A}$ is a group homomorphism $Z: K(\mathscr{A})\to\mathbb{C}$ such that for any non-zero $E\in\mathscr{A}$, $Z(E)$ lies in the upper half plane

$$\overline{\mathbb{H}} := \{rexp(i\pi\phi)|r>0, 0<\phi\leq 1)\}\subset\mathbb{C}. \qquad (6.0.1)$$

We saw an example of a central charge on the Abelian category of finite-dimensional representations of an acyclic quiver in section 4.2.

[2]This generalised metric has the usual properties of a metric but can take the value ∞.

We say a non-zero object $E \in \mathscr{A}$ is *semistable* (resp. *stable*) with respect to the central charge Z if every proper subobject $0 \neq A \subset E$ satisfies $\phi(A) \leq \phi(E)$ ($\phi(A) < \phi(E)$). The central charge Z has the Harder-Narasimhan (HN) property if every non-zero object $E \in \mathscr{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E$$

where the semistable factors $F_j = E_j / E_{j-1}$ fulfill

$$\phi(F_1) > \phi(F_2) > \ldots > \phi(F_n).$$

The following useful proposition implies the existence of a HN filtration for a central charge on an Abelian category of finite length:

Proposition 6.0.1. *[26] Given an Abelian category $\mathscr{A}$ with central charge such that*

1. *there are no infinite sequences of subobjects in $\mathscr{A}$*

$$\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_2 \subset E_1$$

   with $\phi(E_{i+1}) > \phi(E_i)$ for all i,

2. *there are no infinite sequences of quotients in $\mathscr{A}$*

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots$$

   with $\phi(E_i) > \phi(E_{i+1})$. Then $\mathscr{A}$ has the HN property.

A Harder-Narasimhan filtration is unique. The basic reason is the following standard

Lemma 6.0.1. *Let be given a central charge on an Abelian category $\mathscr{A}$. If E_1 is semistable of phase ϕ_1 and E_2 is semistable of phase ϕ_2 with respect to this central charge, then $Hom_{\mathscr{A}}(E_1, E_2) = 0$, if $\phi_1 > \phi_2$.*

Proof. For every non-zero map $f : E_1 \to E_2$ we have the following two short exact sequences in $\mathscr{A}$:

$$0 \longrightarrow ker\ f \longrightarrow E_1 \longrightarrow im\ f \longrightarrow 0,$$
$$0 \longrightarrow im\ f \longrightarrow E_2 \longrightarrow coker\ f \longrightarrow 0.$$

Note that a non-zero object E fulfills $\phi(A) \leq \phi(E)$ for every non-zero subobject A if and only if every non-zero quotient $E \twoheadrightarrow Q$ fulfills $\phi(E) \leq \phi(Q)$, since the central charge is additive on short exact sequences. Thus the first sequence implies $\phi(E_1) \leq \phi(im\ f)$ and the second $\phi(im\ f) \leq \phi(E_2)$. This is a contradiction. □

The following proposition is useful for the construction of stability conditions. We give a full proof since this construction is crucial for the following results and especially for Proposition 6.0.4 underlying the mutation method of chapter 8.

Proposition 6.0.2. *[26] To give a stability condition on a triangulated category $\mathscr{D}$ is equivalent to giving a bounded t-structure on $\mathscr{D}$ and a central charge on its heart which has the Harder-Narasimhan property.*

Proof. We follow the proof in [26] and demonstrate a statement in detail that was not proven there. Given a heart $\mathscr{A}$ of a bounded t-structure on $\mathscr{D}$ and a central charge with HN property we define the subcategories $\mathscr{P}(\phi)$ to be the full additive categories given by semistable objects of $\mathscr{A}$ of phase $\phi \in (0,1]$ together with the zero objects of $\mathscr{D}$. We continue by the rule $\mathscr{P}(\phi+1) = \mathscr{P}(\phi)[1]$. Property 3 of Definition 6.0.1 follows from Lemma 5.3.2 and Lemma 6.0.1. The filtration of any non-zero object of $\mathscr{D}$ as in axiom 4 of Definition 6.0.1 can be obtained by combining the filtration of Lemma 5.3.2 and the HN filtration in $\mathscr{A}$. Conversely, given a stability condition $\sigma = (Z, \mathscr{P})$ on a triangulated category $\mathscr{D}$ the full subcategory $\mathscr{A} = \mathscr{P}((0,1])$ is the heart of a bounded t-structure on $\mathscr{D}$. Identifying the Grothendieck groups $K(\mathscr{A})$ and $K(\mathscr{D})$ the central charge $Z: K(\mathscr{D}) \to \mathbb{C}$ defines a central charge on $\mathscr{A}$.

The semistable objects of the categories $\mathscr{P}(\phi)$ are the semistable objects of $\mathscr{A}$ with respect to this central charge on $\mathscr{A}$: Note first that the semistable factors of an object $0 \neq E \in \mathscr{A} = \mathscr{P}((0,1])$ have to lie in $\mathscr{P}((0,1])$: Let us assume $\phi(A_1) > 1$ for the phase of the semistable factor A_1 of an object $E \in \mathscr{P}((0,1])$ in the filtration of Definition 6.0.1. But this gives a contradiction since this would imply $Hom(A_1, E) = 0$. E is generated by objects in $\mathscr{P}(\phi)$ with $\phi \in (0,1]$. If $\phi(A_n) \leq 0$ then $Hom(E, A_n) = 0$ but $E \to A_n$ is non-trivial.

Take an object $0 \neq E \in \mathscr{P}(\phi)$ with $\phi \in (0,1]$ and assume there is a subobject $F \subset E$ in $\mathscr{A} = \mathscr{P}((0,1])$ with $\phi(F) > \phi(E)$ where $\phi(E)$ and $\phi(F)$ are the phases with respect to the central charge on $\mathscr{A}$. Since all semistable factors of F with respect to σ are objects in $\mathscr{P}(\phi)$ for some $\phi \in (0,1]$ we have for the semistable factor A_1 of F that $\phi(A_1) \geq \phi(F)$ in $\mathscr{A}$ by the additivity of the central charge on short exact sequences. But then $\phi(A_1) \geq \phi(F) > \phi(E)$ in $\mathscr{A}$. Since E and A_1 are semistable in σ $Hom(A_1, E) = 0$ but we have a non-trivial morphism $A_1 \to F \to E$. This implies that the HN filtration in axiom 4 of Definition 6.0.1of an object in $\mathscr{A}$ is a HN filtration in $\mathscr{A}$. Let now $0 \neq E \in \mathscr{A}$ be semistable with respect to the central charge on $\mathscr{A}$. If E would have a non-trivial filtration like in axiom 4 of Definition 6.0.1 then this filtration would be a non-trivial HN filtration in $\mathscr{A}$. By uniqueness of HN filtrations E must be semistable with respect to the stability condition σ. This finishes the proof. □

The following theorem allows to deform stability conditions by deforming the associated central charge:

Theorem 6.0.2. *[26] Let $\mathscr{D}$ be a triangulated category. For each connected component $\Sigma \subset Stab(\mathscr{D})$ there is a linear subspace $V(\Sigma) \subset Hom_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$ with a well-defined linear topology and a local homeomorphism $\Sigma \to V(\Sigma)$ which maps a stability condition $(Z,\mathscr{P})$ to its central charge Z.*

Therefore each component of $Stab(\mathscr{D})$ is a complex manifold locally homeomorphic to the complex vector space $V(\Sigma)$.

In the following we follow a setup in [33]. We will study examples in chapter 8.

Proposition 6.0.3. *[33] Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure on a triangulated category $\mathscr{D}$. Let $\mathscr{A}$ be of finite length with finitely many simple objects $S_1,\dots,S_n$. Then the subset $U(\mathscr{A})$ of $Stab(\mathscr{D})$ consisting of locally-finite stability conditions with heart $\mathscr{A} = \mathscr{P}((0,1])$ is isomorphic to $\overline{\mathbb{H}}^n$.*

Proof. This is Lemma 5.2 stated in [33] without proof. The classes of the simple objects $S_1,\dots,S_n$ build a basis of the Grothendieck groups of $\mathscr{A}$ and $\mathscr{D}$. We define a central charge on $K(\mathscr{A})$ by assigning a number in $\overline{\mathbb{H}}$ to the class of every simple object S_i. A central charge on $\mathscr{A}$ has automatically the HN property by Proposition 6.0.1 since $\mathscr{A}$ is of finite length. By Proposition 6.0.2 this defines a stability condition $\sigma = (Z,\mathscr{P})$ on $\mathscr{D}$. By Lemma 4.3 in [26] the strict short exact sequences in the quasi-Abelian category $\mathscr{P}((0,1))$ are precisely the triangles in $\mathscr{D}$ all of whose vertices are objects of $\mathscr{P}((0,1))$. Since $\mathscr{A} = \mathscr{P}((0,1])$ is of finite length there are no infinite chains

$$\cdots \subset E_3 \subset E_2 \subset E_1$$

of strict monomorphisms in $\mathscr{P}((0,1))$. Analogously, every chain of strict epimorphisms in $\mathscr{P}((0,1))$ has to stabilize. Therefore $\mathscr{P}((0,1))$ is of finite length and this implies the stability condition defined in this proof is locally-finite.

Conversely, every locally-finite stability condition with heart $\mathscr{A}$ defines a unique central charge on $\mathscr{A}$. □

We can deform the central charge by rotating it counter-clockwise by a small angle. A natural question is what happens if exactly one simple object S of $\mathscr{A}$ rotates out of the upper half-plane $\overline{\mathbb{H}}$ on the left? Since the space of stability conditions is a complex manifold there is a stability condition with this deformed central charge, but it has a heart different to $\mathscr{A}$. The following Proposition gives an answer to the question:

Proposition 6.0.4. *([33], Lemma 5.5) Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure on $\mathscr{D}$ and suppose $\mathscr{A}$ has finite length with finitely many simple objects. Then the codimension one subset of $U(\mathscr{A})$ where the simple S has phase 1 and all other simples have phases in $(0,1)$ is the intersection $U(\mathscr{A}) \cap \overline{U(\mathscr{B})}$ precisely if $\mathscr{B} = L_S(\mathscr{A})$ where $L_S(\mathscr{A})$ is the left-tilt of $\mathscr{A}$ at the simple object S.*

To clarify Proposition 6.0.4 let us consider a stability condition σ in $U(\mathscr{A})$ such that the simple object S lies on the negative real axis and all other simple objects lie above the real axis in $\overline{\mathbb{H}}$. In a small neighborhood U of σ the simple objects of $\mathscr{A}$ will remain stable (see e.g. chapter 7 of [45]). Note that $d(\sigma,\tau) \leq \varepsilon$ for two stability conditions $\sigma = (Z,\mathscr{P})$ and $\tau = (W,\mathscr{Q})$ implies for any $0 \neq E \in \mathscr{Q}(\phi)$ that $\phi^+_\sigma \leq \phi + \varepsilon$ and $\phi^-_\sigma \geq \phi - \varepsilon$ and thus $\mathscr{Q}(\phi) \subset \mathscr{P}([\phi-\varepsilon,\phi+\varepsilon])$. Thus by shrinking the neighborhood U we can assume that $Re\ Z(S) < 0$ and all other simple objects still lie above the real axis in $\overline{\mathbb{H}}$. Now we split U into two pieces: U_1 with $Im\ Z(S) \geq 0$ and U_2 with $Im\ Z(S) < 0$. The simple objects of $\mathscr{A}$ lie in the heart $\mathscr{P}((0,1])$ for every stability condition in U_1 and thus we have $\mathscr{A} = \mathscr{P}((0,1])$ for every stability condition in U_1. The object S is stable in U_2 with phase in the interval $(1,3/2)$ und thus $S[-1]$ is contained in the heart $\mathscr{P}((0,1])$ of any stability condition in U_2. Shrinking U, one can further show that all simple objects obtained by tilting the heart $\mathscr{A}$ at the simple S are likewise in the heart $\mathscr{P}((0,1])$.

The tilted subcategories $L_S(\mathscr{A})$ and $R_S(\mathscr{A})$ need not to have finite length again. The next idea is to consider hearts of finite length with finitely many simple objects such that the tilted heart is again of finite length such that we can tilt again at a simple object of the tilted heart and so on. Proposition 6.0.4 tells us that in this case we can glue together regions in $Stab(\mathscr{D})$ each isomorphic to $\overline{\mathbb{H}}^n$ along boundaries corresponding to hearts related by simple tilts. We will take up this idea in chapter 8.

Chapter 7

Stability Conditions on Kummer Surfaces

7.1 Moduli spaces of superconformal field theories

In this section we discuss the moduli space of N=(4,4) SCFTs with central charge $c = 6$. We follow in this section the version of [3, 5]. For a pedagogical introduction see [7]. Let X be a two-dimensional Calabi-Yau manifold, i.e. a complex tori or a K3 surface. We have a pairing induced by the intersection product on the even cohomology $H^{even}(X,\mathbb{R}) \cong \mathbb{R}^{4,4+\delta}$. We choose a marking, that is an isometry $H^{even}(X,\mathbb{Z}) \cong L$ where L is the unique even unimodular lattice $\mathbb{Z}^{4,4+\delta}$ with $\delta = 0$ for a complex torus and $\delta = 16$ for a K3 surface as explained in chapter 3. In the latter case this is of course just the K3 lattice $4U \oplus 2(-E_8)$. The moduli space of SCFTs associated to complex tori or K3 surfaces are given by the following

Theorem 7.1.1. *[10] Every connected component of the moduli space of SCFTs associated to Calabi-Yau 2-folds is either of the form $\mathscr{M}_{tori} = \mathscr{M}^0$ or $\mathscr{M}_{K3} = \mathscr{M}^{16}$ where:*

$$\mathscr{M}^{\delta} \cong O^+(4,4+\delta;\mathbb{Z})\backslash O^+(4,4+\delta;\mathbb{R})/SO(4)\times O(4+\delta).$$

Points $x \in \tilde{\mathscr{M}}^{\delta}$ in the Grassmannian

$$\tilde{\mathscr{M}}^{\delta} = O^+(4,4+\delta;\mathbb{R})/SO(4)\times O(4+\delta)$$

correspond to positive definite oriented four-planes in $\mathbb{R}^{4,4+\delta}$ whose position is given by its relative position to the reference lattice L.

Let us choose a marking $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{3,3+\delta}$. The Torelli theorem 3.0.6 tells us that complex structures on two-dimensional complex tori or K3 surfaces X are in 1:1 correspondence with positive definite oriented two-planes

$\Omega \subset H^2(X,\mathbb{Z}) \otimes R \cong \mathbb{R}^{3,3+\delta}$ that are specified by its relative position to $\mathbb{Z}^{3,3+\delta}$.

Definition 7.1.1. Let $x \subset H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$ be a positive oriented four-plane specifying a SCFT on X. A *geometric interpretation of this SCFT* is a choice of null vectors $\upsilon^0, \upsilon \in H^{even}(X,\mathbb{Z})$ along with a decomposition of x into two perpendicular oriented two-planes $x = \Omega \perp \mho$ such that $\langle \upsilon^0, \upsilon^0 \rangle = \langle \upsilon, \upsilon \rangle = 0$, $\langle \upsilon^0, \upsilon \rangle = 1$, and $\Omega \perp \upsilon^0, \upsilon$.

Lemma 7.1.2. *[10] Let $x \subset H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$ be a positive definite oriented four-plane with geometric interpretation $\upsilon^0, \upsilon \in H^{even}(X,\mathbb{Z})$, where υ^0, υ are interpreted as generators of $H^0(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$, respectively, and a decomposition $x = \Omega \perp \mho$. Then one finds $\omega \in H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$ and $B \in H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$ with*

$$\mho = \mathbb{R} \left\langle \omega - \langle B, \omega \rangle \upsilon, \xi_4 = \upsilon^0 + B + \left(V - \frac{1}{2} \langle B, B \rangle \right) \upsilon \right\rangle \tag{7.1.1}$$

with $\omega, B \in H^2(X,\mathbb{R}) := H^{even}(X,\mathbb{R}) \cap \upsilon^\perp \cap (\upsilon^0)^\perp$,$V \in \mathbb{R}_+$ and $\omega^2 \in \mathbb{R}_+$. B and V are determined uniquely and ω is unique (up to scaling).

The picture is that a SCFT associated to a Calabi-Yau 2-fold can be realized by a non-linear σ model. It is important to note that the mentioned moduli space of SCFTs associated to K3 surfaces also contains ill-defined conformal field theories. Namely, a positive definite oriented four-plane $x \in \widetilde{\mathscr{M}}^{16}$ corresponds to such a theory if and only if there is a class $\delta \in H^{even}(X,\mathbb{Z})$ with $\delta \perp x$ and $\langle \delta, \delta \rangle = -2$. String theory tells us that the field theory gets extra massless particles at these points in the moduli space and breaks down. For physical details see [11]. For complex tori there are no such ill-defined SCFTs.

7.2 Orbifold conformal field theories on K3

We are interested in SCFTs with geometric interpretations on Kummer surfaces coming from orbifolding of SCFTs on complex tori since later we want to induce stability conditions on projective Kummer surfaces from the associated Abelian surfaces.

We consider a complex torus $T = \mathbb{C}^2/\Lambda$ where Λ is a lattice of rank 4 identified with $H^1(T,\mathbb{Z})$. Let $\mu_1, \ldots, \mu_4$ denote generators of $H^1(T,\mathbb{Z})$. Recall from chapter 3 that we have a $G = \mathbb{Z}_2$ action on T. We have a minimal resolution of its sixteen singularities:

$$X := \widetilde{T/G} \longrightarrow T/G$$

where X is the Kummer surface associated to T. This resolution introduces 16 rational two-cycles which we label by elements of $\mathbb{F}_2^4 \cong \frac{1}{2}\Lambda/\Lambda$ and we denote their Poincaré duals by E_i with $i \in \mathbb{F}_2^4$. The Kummer lattice Π is the smallest primitive sublattice of the Picard lattice Pic(X)=NS(X) containing $\left\{E_i | i \in \mathbb{F}_2^4\right\}$. It is spanned by $\left\{E_i | i \in \mathbb{F}_2^4\right\}$ and $\left\{1/2\sum_{i\in H} E_i | H \subset \mathbb{F}_2^4 \text{ a hyperplane}\right\}$ [24]. (For a review see e.g. [12]). We want to find an injective map from the moduli space of SCFTs on a two-dimensional complex torus T to the moduli space of SCFTs on the corresponding Kummer surface X. This was done by Nahm and Wendland [3, 5] generalizing results of Nikulin [24]:

Let $\pi : T \to X$ be the induced rational map of degree 2 defined outside the fixed points of the $\mathbb{Z}_2$ action. The induced map on the cohomology gives a primitive embedding $\pi_* : H^2(T,\mathbb{Z})(2) \hookrightarrow H^2(X,\mathbb{Z})$ [24, 23].[1] We define $K := \pi_* H^2(T,\mathbb{Z})$. The lattice K obeys $K \oplus \Pi \subset H^2(X,\mathbb{Z}) \subset K^* \oplus \Pi^*$ where $K \oplus \Pi \subset H^2(X,\mathbb{Z})$ is a primitive sublattice with the same rank as $H^2(X,\mathbb{Z})$. $H^2(X,\mathbb{Z})$ is even and unimodular. This embedding defines the isomorphism

$$\begin{aligned} \gamma : K^*/K &\longrightarrow \Pi^*/\Pi \\ \frac{1}{2}\pi_*(\mu_j \wedge \mu_k) &\longmapsto \frac{1}{2}\sum_{i\in P_{jk}} E_i \end{aligned} \tag{7.2.1}$$

where $P_{jk} = \left\{a = (a_1,a_2,a_3,a_4) \in \mathbb{F}_2^4 | a_l = 0, \forall l \neq j,k\right\}$ with $j,k \in \{1,2,3,4\}$. Conversely, with this isomorphism we can describe the lattice $H^2(X,\mathbb{Z})$ using Theorem 2.0.2. We find in our case

$$H^2(X,\mathbb{Z}) \cong \{(\kappa,\pi) \in K^* \oplus \Pi^* | \gamma(\bar{\kappa}) = \bar{\pi}\}.$$

Hence $H^2(X,\mathbb{Z})$ is generated by

1. $\pi_* H^2(T,\mathbb{Z}) \cong H^2(T,\mathbb{Z})(2)$,
2. the elements of the Kummer lattice Π,
3. and forms of the form $\frac{1}{2}\pi_*(\mu_j \wedge \mu_k) + \frac{1}{2}\sum_{i\in P_{jk}} E_i$.

Let υ^0 respectively υ be generators of $H^0(T,\mathbb{Z})$ respectively $H^4(T,\mathbb{Z})$. We introduce:

$$\begin{aligned} \hat{\upsilon} &:= \pi_* \upsilon, \\ \hat{\upsilon}^0 &:= \frac{1}{2}\pi_* \upsilon^0 - \frac{1}{4}\sum_{i\in\mathbb{F}_2^4} E_i + \pi_* \upsilon \end{aligned} \tag{7.2.2}$$

[1] Here and in the following L(2) means a lattice L with quadratic form scaled by 2.

We define $\hat{E}_i := -\frac{1}{2}\hat{\upsilon} + E_i$. Then we have the following

Proposition 7.2.1. *[3, 5] The lattice generated by $\hat{\upsilon}$, $\hat{\upsilon}^0$ and*

$$\left\{\frac{1}{2}\pi_*(\mu_j \wedge \mu_k) + \frac{1}{2}\sum_{i\in P_{jk}} \hat{E}_{i+l}; l \in \mathbb{F}_2^4\right\} and \left\{\hat{E}_i, i \in \mathbb{F}_2^4\right\} \tag{7.2.3}$$

is isomorphic to $\mathbb{Z}^{4,20}$.

In [3, 5, 6] it is argued that this gives the unique embedding which is compatible with all symmetries of the respective SCFTs. Using the generators given in Proposition 7.2.1 we can regard a positive definite, oriented four-plane $x \subset H^{even}(T,\mathbb{Z})\otimes\mathbb{R}$ as a four-plane in $H^{even}(X,\mathbb{Z})\otimes\mathbb{R}$.

Theorem 7.2.1. *[3, 5] For a geometric interpretation of a SCFT $x_T = \Omega\perp\mho$ on a complex torus T with ω, V_T, B_T as in Lemma 7.1.2 the corresponding orbifold conformal field theory $x = \pi_*\Omega\perp\pi_*\mho$ has a geometric interpretation $\hat{\upsilon}$, $\hat{\upsilon}^0$ with $\pi_*\omega, V = \frac{V_T}{2}, B$ where*

$$\begin{aligned} B &= \frac{1}{2}\pi_* B_T + \frac{1}{2}B_{\mathbb{Z}}, \\ B_{\mathbb{Z}} &= \frac{1}{2}\sum_{i\in\mathbb{F}_2^4}\hat{E}_i. \end{aligned} \tag{7.2.4}$$

Proof. Using the embedding $H^{even}(T,\mathbb{Z})\otimes\mathbb{R} \hookrightarrow H^{even}(X,\mathbb{Z})\otimes\mathbb{R}$ given in Proposition 7.2.1 we calculate [3, 5]

$$\begin{aligned} \pi_*\left(\omega - \langle B_T,\omega\rangle\,\upsilon\right) &= \pi_*\omega - \langle \pi_* B,\omega\rangle\,\hat{\upsilon}, \\ \frac{1}{2}\pi_*\left(\upsilon^0 + B_T + \left(V_T - \frac{1}{2}\|B_T\|^2\right)\upsilon\right) &= \hat{\upsilon}^0 + \frac{1}{2}\pi_* B_T + \frac{1}{2}B_{\mathbb{Z}} \\ &\quad + \left(\frac{V_T}{2} - \frac{1}{2}\left\|\frac{1}{2}\pi_* B_T + \frac{1}{2}B_{\mathbb{Z}}\right\|^2\right)\hat{\upsilon}. \end{aligned}$$

This proves the theorem. □

For Proposition 7.4.1 the following observation is crucial:

Corollary 7.2.2. *[3, 5] Let $x = \pi_*\Omega\perp\pi_*\mho \subset H^{even}(X,\mathbb{Z})\otimes\mathbb{R}$ be the four-plane induced from a positive-definite, oriented four-plane $x_T = \Omega\perp\mho \subset H^{even}(T,\mathbb{Z})\otimes\mathbb{R}$ as in Theorem 7.2.1. Then $x^\perp \cap H^{even}(X,\mathbb{Z})$ does not contain (-2) classes.*

Proof. Let Ω be the positive-definite, oriented two-plane defined by the complex structure for the torus T. We choose a basis of the orthogonal complement $x^\perp \subset H^{even}(X,\mathbb{Z})\otimes\mathbb{R}$. For example:

1. $\hat{E}_i + \frac{1}{2}\hat{\upsilon}, i \in \mathbb{F}_2^4$,
2. $\pi_* \eta_i - \langle \pi_* \eta_i, B \rangle \, \hat{\upsilon}, i = 1, \ldots, 3$,
3. $\hat{\upsilon}^0 + B - \left(V + \frac{1}{2} \|B\|^2 \right) \hat{\upsilon}$.

The $\eta_i, i = 1, \ldots, 3$ are an orthogonal basis of the orthogonal complement of $\text{span}_{\mathbb{R}} \langle \omega, \Omega \rangle$ in $H^2(T, \mathbb{Z}) \otimes \mathbb{R}$. Then the $\pi_* \eta_i, i = 1, \ldots, 3$ build together with the sixteen $E_i, i \in \mathbb{F}_2^4$ an orthogonal basis of the orthogonal complement of $\text{span}_{\mathbb{R}} \langle \pi_* \omega, \pi_* \Omega \rangle$ in $H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ with ω as in Lemma 7.1.2. B is as in Theorem 7.2.1. Note that $\langle E_i, E_i \rangle = -2$ but E_i is not an element of our lattice. If we then try to build a (-2) class in $x^\perp$ from our ansatz we run into contradictions. □

7.3 Generalized Calabi-Yau Structures

In this section we introduce generalized Calabi-Yau structures of Hitchin [18] following [19, 36]. This is also relevant for stability conditions on twisted surfaces as we will see in section 6.

The Mukai pairing on the even integral cohomology $H^{even}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ is defined by

$$\langle (a_0, a_2, a_4), (b_0, b_2, b_4) \rangle := -a_0 \wedge b_4 + a_2 \wedge b_2 - a_4 \wedge b_0.$$

For an Abelian or K3 surface X the Mukai lattice is $H^{even}(X, \mathbb{Z})$ equipped with the Mukai pairing that differs from the intersection pairing in signs. Note that the hyperbolic lattice U with basis υ, υ^0 is isomorphic to $-U$ via

$$\begin{aligned} \upsilon &\longmapsto -\upsilon, \\ \upsilon^0 &\longmapsto \upsilon^0. \end{aligned}$$

We make the following choice: From now on we will work in the Mukai lattice.

Definition 7.3.1. Let Ω be a holomorphic 2-two form on an Abelian or K3 surface X defining a complex structure. For a rational B-field $B \in H^2(X, \mathbb{Q})$ a *generalized Calabi-Yau structure on X* is defined by

$$\varphi := exp(B)\Omega = \Omega + B \wedge \Omega \in H^2(X) \oplus H^4(X).$$

We define a Hodge structure of weight two on the Mukai lattice by

$$\widetilde{H}^{2,0}(X) := \mathbb{C}[\varphi]$$

We write $\widetilde{H}(X,B,\mathbb{Z})$ for the lattice equipped with this Hodge structure and the Mukai pairing.

Definition 7.3.2. Let $\varphi = exp(B)\Omega$ be a generalized Calabi-Yau structure. The *generalized transcendental lattice* $T(X,B)$ is the minimal primitive sublattice of $H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$, such that $\varphi \in T(X,B) \otimes \mathbb{C}$.

$T(X,0) = T(X) = NS(X)^{\perp}$ is the transcendental lattice and $NS(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z})$ is the Néron-Severi lattice.

Definition 7.3.3. Let X be a smooth complex projective variety. The *(cohomological) Brauer group* is the torsion part of $H^2(X,\mathscr{O}_X^*)$ in the analytic topology: $Br(X) = H^2(X,\mathscr{O}_X^*)_{tor}$.[2]

For an introduction to Brauer classes see [20] or [21]. Eventually we introduce twisted surfaces:

Definition 7.3.4. A *twisted Abelian or K3 surface (X,α)* consists of an Abelian or K3 surface X together with a class $\alpha \in Br(X)$. Two twisted surfaces $(X,\alpha),(Y,\alpha')$ are isomorphic if there is an isomorphism $f : X \cong Y$ with $f^*\alpha' = \alpha$.

The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^* \longrightarrow 1$$

gives the long exact sequence

$$\longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathscr{O}_X) \longrightarrow H^2(X,\mathscr{O}_X^*) \longrightarrow H^3(X,\mathbb{Z}) \longrightarrow .$$

For an Abelian or K3 surface $H_1(X,\mathbb{Z})$ and therefore $H^3(X,\mathbb{Z})$ is torsion free. So an n-torsion element of $H^2(X,\mathscr{O}_X^*)$ is always in the image of the exponential map for a $B^{0,2} \in H^2(X,\mathscr{O}_X)$ such that $nB^{0,2} \in H^2(X,\mathbb{Z})$ for a positive integer n. For a rational B-field $B \in H^2(X,\mathbb{Q})$ we use the induced homomorphism

$$\begin{aligned} B : T(X) &\longrightarrow \mathbb{Q} \\ \gamma &\longmapsto \int_X \gamma \wedge B \end{aligned}$$

(modulo $\mathbb{Z}$) to introduce

$$T(X,\alpha_B) := ker\{B : T(X) \to \mathbb{Q}/\mathbb{Z}\}. \tag{7.3.1}$$

The details can be found in [19, 22].

[2] Equivalently, we could define the Brauer group as the torsion part of $H^2_{et}(X,\mathscr{O}_X^*)$ in the Étale topology.

7.4 Stability conditions on K3 surfaces

Theorem 7.2.1 gives an embedding of the moduli space of SCFTs associated to a complex torus into the moduli space of SCFTs associated to the corresponding Kummer surface. We are interested in the question if this embedding has a lift to Bridgeland stability conditions. In the following we show that this is indeed the case.

The abstract lattice $\mathbb{Z}^{4,20}$ is isometric to the even cohomology lattice $H^{even}(X,\mathbb{Z})$ equipped with the Mukai (or intersection) pairing such that the generators υ^0 respectively υ of the hyperbolic lattice U are identified with $1 \in H^0(X,\mathbb{Z})$ respectively $[pt] \in H^4(X,\mathbb{Z})$ (using Poincaré duality). The lattice $\mathbb{Z}^{4,20}$ is also isometric to the lattice defined in Proposition 7.2.1. We will identify these lattices in this section frequently.

Moduli spaces of N=(2,2) SCFTs can be seen as moduli spaces of generalized Calabi-Yau structures [19]. Since we have an embedding of orbifold conformal field theories it is natural to ask if there is a relation between the structures we introduced in section 4 for an Abelian surface A and the associated Kummer surface $X = Km$ A.

Lemma 7.4.1. *Let (A,α_{B_A}) be a twisted Abelian surface and (X,α_B) the associated twisted Kummer surface with B-field lift $B_A \in H^2(A,\mathbb{Q})$ as described above and B as in Theorem* 7.2.1. *Then we have a Hodge isometry $T(A,B_A)(2) \cong T(X,B)$.*

Proof. For a rational B-field B we have a Hodge isometry

$$T(X,\alpha_B) \cong T(X,B)$$

This was proven for K3 surfaces in [19] and also works for Abelian surfaces. The isomorphism in Theorem 2.0.2 defined by the map (7.2.1) sends $\pi_* H^2(T,\mathbb{Z})$ to $\pi_* H^2(T,\mathbb{Z})$. We know that the ordinary transcendental lattices of an Abelian surface A and its Kummer surface X are Hodge isometric (up to a factor of 2) [23, 24]

$$T(A)(2) \cong T(X). \tag{7.4.1}$$

The Hodge isometry (7.4.1) can be enhanced by (7.3.1) to a Hodge isometry $T(A,\alpha_{B_A})(2) \cong T(X,\alpha_B)$. □

So we have natural isometries of the above transcendental lattices for B-fields associated with orbifold CFTs. An isometry between these two transcendental lattices was also noted in [25].

Let us first consider untwisted surfaces with B-field $B \in NS(X) \otimes \mathbb{R}$. We consider a algebraic K3 surface X following [30] and use the Mukai pairing on the integral cohomology lattice. We denote the bounded derived categories of coherent sheaves on X by $D^b(X) := D^b(\mathrm{Coh}\, X)$. Let $NS(X)$ be the Néron-Severi lattice. The Mukai lattice is $\mathscr{N}(X) = H^0(X,\mathbb{Z}) \oplus NS(X) \oplus H^4(X,\mathbb{Z})$. Recall that the Mukai vector $v(E)$ of an object $E \in D^b(X)$ is defined by

$$v(E) = (r(E), c_1(E), s(E)) = ch(E)\sqrt{td(X)} \in \mathscr{N}(X)$$

where $ch(E)$ is the Chern character and $s(E) = ch_2(E) + r(E)$. We define an open subset

$$\mathscr{P}(X) \subset \mathscr{N}(X) \otimes \mathbb{C}$$

consisting of vectors whose real and imaginary part span positive definite two-planes in $\mathscr{N}(X) \otimes \mathbb{R}$. $\mathscr{P}(X)$ consists of two connected components that are exchanged by complex conjugation. We have a free action of $GL^+(2,\mathbb{R})$ by the identification $\mathscr{N}(X) \otimes \mathbb{C} \cong \mathscr{N}(X) \otimes \mathbb{R}^2$. A section of this action is provided by the submanifold

$$\mathscr{Q}(X) = \left\{\mho \in \mathscr{P}(X) | \langle \mho, \mho \rangle = 0, \left\langle \mho, \bar{\mho} \right\rangle > 0, r(\mho) = 1\right\} \subset \mathscr{N}(X) \otimes \mathbb{C}.$$

$r(\mho)$ projects $\mho \in \mathscr{N}(X) \otimes \mathbb{C}$ into $H^0(X,\mathbb{C})$. We can identify $\mathscr{Q}(X)$ with the tube domain

$$\left\{B + i\omega \in NS(X) \otimes \mathbb{C} | \omega^2 > 0\right\}$$

by

$$\mho = exp(B + i\omega) = \upsilon^0 + B + i\omega + \frac{1}{2}(B^2 - \omega^2)\upsilon + i\langle B, \omega \rangle \upsilon$$

with $\upsilon^0 = 1 \in H^0(X,\mathbb{Z})$ and $\upsilon = [pt] \in H^4(X,\mathbb{Z})$. We denote by $\mathscr{P}^+(X) \subset \mathscr{P}(X)$ the connected component containing vectors of the form $exp(B + i\omega)$ for an ample $\mathbb{R}$-divisor class $\omega \in NS(X) \otimes \mathbb{R}$. Let $\Delta(X) = \{\delta \in \mathscr{N}(X) | \langle \delta, \delta \rangle = -2\}$ be the root system. For each $\delta \in \Delta(X)$ we have a complex hyperplane

$$\delta^\perp = \{\mho \in \mathscr{N}(X) \otimes \mathbb{C} | \langle \mho, \delta \rangle = 0\} \subset \mathscr{N}(X) \otimes \mathbb{C}.$$

We denote by

$$\mathscr{P}_0^+(X) = \mathscr{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset \mathscr{N}(X) \otimes \mathbb{C}.$$

Note that there are no spherical objects in $D^b(A)$ on an Abelian surface A [26]. The following Proposition is underlying the main result of this chapter Theorem 7.5.3.

Proposition 7.4.1. *Let A be an Abelian surface and $X = Km\,A$ the corresponding Kummer surface. Then we have an embedding $\mathscr{P}^+(A) \hookrightarrow \mathscr{P}_0^+(X)$.*

Proof. An element of $\mathscr{P}^+(A)$ is of the form $exp(B+i\omega)\circ g$ for $g \in GL^+(2,\mathbb{R})$, $B \in NS(A)\otimes\mathbb{R}$ and $\omega \in NS(A)\otimes\mathbb{R}$ with $\omega^2 > 0$ [26]. Here $\circ$ is the free action of an element $g \in GL^+(2,\mathbb{R})$ on $NS(A)\otimes\mathbb{R}$. Let π_* be the map induced by the rational map $\pi : A \to X$. The action of $GL^+(2,\mathbb{R})$ and the map π_* commute. By Proposition 7.2.1 we have an injective map

$$i : H^{even}(A,\mathbb{Z})\otimes\mathbb{R} \hookrightarrow H^{even}(X,\mathbb{Z})\otimes\mathbb{R}. \tag{7.4.2}$$

The 2-plane Ω given by the complex structure of the Abelian surface A defines the complex structure on X by the 2-plane $\pi_*\Omega$. Therefore $\mathscr{N}(A)$ is mapped to $\mathscr{N}(X)$ and we get an induced map form $\mathscr{P}(A)$ to $\mathscr{P}(X)$. The proof of Theorem 7.2.1 shows that vectors of the form $1/2\pi_*(exp(B_T + i\omega))$ for $B_T, \omega \in NS(A)\otimes\mathbb{R}$ are sent to vectors

$$\hat{v}^0 + B + \frac{1}{2}\left(B^2 - \left(\frac{1}{2}\pi_*\omega\right)^2\right)\hat{v} + i\left(\frac{1}{2}\pi_*\omega + \left\langle B, \frac{1}{2}\pi_*\omega\right\rangle \hat{v}\right) \tag{7.4.3}$$

in $\mathscr{N}(X)\otimes\mathbb{C}$ with B as in Lemma 7.4.1. The elements of $\mathscr{N}(X)$ are contained in the orthogonal complement of $H^{2,0}(X) = \mathbb{C}[\pi_*\Omega]$ where $\pi_*\Omega = \pi_*\Omega_1 + i\pi_*\Omega_2$.[3] By corollary 7.2.2 we know that there are no roots of $H^{even}(X,\mathbb{Z})$ in the orthogonal complement of the 4-plane spanned by $\pi_*\Omega_1, \pi_*\Omega_2$ and the real and imaginary part of a vector of the form (7.4.3) in $H^{even}(X,\mathbb{Z})\otimes\mathbb{R}$. This implies that there are no roots in the orthogonal complement of an induced element in $\mathscr{P}(X)$. Since $\pi_*\omega$ is an orbifold ample class in the closure of the ample cone, $\mathscr{P}^+(A)$ is mapped to $\mathscr{P}^+(X)$. □

The results of [30] can be generalized for twisted surfaces [32]. We want to formulate a similar result to Proposition 7.4.1 in the twisted case as well. From a physics point of view it is natural to consider more general B-fields. This suggests the twisted approach.

Any class $\alpha \in Br(X) = H^2(X, \mathscr{O}_X^*)_{tor}$ can be represented by a *Čech* 2-cocycle $\left\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathscr{O}_X^*)\right\}$ on an analytic open cover $\{U_i\}$ of X.

Definition 7.4.1. An (α_{ijk})*-twisted coherent sheaf* E consists of pairs $(\{E_i\}, \{\varphi_{ij}\})$ such that E_i is a coherent sheaf on U_i and $\varphi_{ij} : E_j|_{U_i\cap U_j} \to E_i|_{U_i\cap U_j}$ are isomorphisms satisfying the following conditions:

[3] By abuse of notation we denote the holomorphic two-form defining the complex structure and the 2-plane defined by it with the same symbol.

1. $\varphi_{ii} = id$
2. $\varphi_{ji} = \varphi_{ij}^{-1}$
3. $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot id$.

We denote the equivalence class of such Abelian categories of twisted coherent sheaves by $Coh(X, \alpha)$ and the bounded derived category by $D^b(X, \alpha)$. For details consult [20]. For a realization of the following notions one has to fix a B-field lift B of the Brauer class α such that $\alpha = \alpha_B = exp(B^{0,2})$. The twisted Chern character

$$ch^B : D^b(X, \alpha_B) \longrightarrow \widetilde{H}(X, B, \mathbb{Z})$$

introduced in [36] identifies the numerical Grothendieck group with the twisted Néron-Severi group $NS(X, \alpha_B) := \widetilde{H}^{1,1}(X, B, \mathbb{Z})$. Here $\widetilde{H}^{1,1}(X, B, \mathbb{Z})$ is the $(1,1)$-part of the Hodge structure associated to the B-field B as defined after Definition 7.3.1. As in the untwisted case we denote by

$$\mathscr{P}(X, \alpha_B) \subset NS(X, \alpha_B) \otimes \mathbb{C}$$

the open subset of vectors whose real and imaginary part span a positive plane in $NS(X, \alpha_B) \otimes \mathbb{R}$. Let $\mathscr{P}^+(X, \alpha_B) \subset \mathscr{P}(X, \alpha_B)$ be the component containing vectors of the form $\exp(B + i\omega)$, where $B \in H^2(X, \mathbb{Q})$ is a B-field lift of α_B and ω a real ample class.

Proposition 7.4.2. *Let (A, α_{B_A}) be a twisted Abelian surface and (X, α_B) the twisted Kummer surface with X the Kummer surface of A and Brauer class $\alpha_B := exp(B^{0,2})$ with B-field B as in Lemma* 7.4.1. *Then we have an embedding $\mathscr{P}^+(A, \alpha_{B_A}) \hookrightarrow \mathscr{P}_0^+(X, \alpha_B)$.*

Proof. Let A be an Abelian surface with Brauer class α_{B_A} with B-field lift $B_A \in H^2(A, \mathbb{Q})$. Then the induced B-field B is an element of $H^2(X, \mathbb{Q})$ and thus induces a Brauer class $\alpha_B := exp(B^{0,2})$ by the remark after Definition 7.3.4. $NS(A, \alpha_{B_A})$ is embedded into $NS(X, \alpha_B)$, since we have $\pi_*\Omega + \langle B, \pi_*\Omega \rangle \hat{\upsilon} = \pi_*(\Omega + \langle B_A, \Omega \rangle \upsilon)$. Using similar arguments as for Proposition 7.4.1 finishes the proof. □

We consider the bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective variety X over the complex numbers. In this case we say a stability condition is numerical if the central charge $Z : K(X) \to \mathbb{C}$ factors through the quotient group and we have the identification $\mathscr{N}(X) = K(X)/K(X)^{\perp}$. Let us write $Stab(X)$ for the set of all locally finite numerical stability conditions on $\mathscr{D}^b(X)$. The Euler form χ is non-degenerate on $\mathscr{N}(X) \otimes \mathbb{C}$, so the central charge takes the form

$$Z(E) = -\chi(p(\sigma), v(E))$$

for some vector $p(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$, defining a map $p : Stab(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C}$. In this case Theorem 6.0.2 takes the following form:

Theorem 7.4.2. *[26] For each connected component $Stab^*(X) \subset Stab(X)$, there is a linear subspace $V \subset \mathcal{N}(X) \otimes \mathbb{C}$ such that*

$$p : Stab^*(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C}$$

is a local homeomorphism onto an open subset of the subspace V. In particular, $Stab^*(X)$ is a finite-dimensional complex manifold.

Recall from chapter 6 that auto-equivalences act continuously on the space of locally-finite stability conditions. We have the following description of the stability manifold for algebraic K3 surfaces X:

Theorem 7.4.3. *[30] There is a distinguished connected component $Stab^\dagger(X) \subset Stab(X)$ which is mapped by p onto the open subset $\mathscr{P}_0^+(X)$. The induced map*

$$p : Stab^\dagger(X) \to \mathscr{P}_0^+(X)$$

is a covering map. We denote by $Aut_0^\dagger(D^b(X))$ the subgroup of cohomological trivial auto-equivalences of $D^b(X)$ which preserve the connected component $Stab^\dagger(X)$. $Aut_0^\dagger(D^b(X))$ acts freely on $Stab^\dagger(X)$ and is the group of deck transformations of this covering.

The main difference in the case of Abelian surfaces is the absence of spherical objects giving rise to (-2)-classes. In fact there are no ill-behaved SCFTs on complex tori. For an Abelian surface A the Todd class is trivial thus the Mukai vector of an object $E \in D^b(A)$ is

$$v(E) = (r(E), c_1(E), ch_2(E)) \in \mathcal{N}(A) = H^0(A,\mathbb{Z}) \oplus NS(A) \oplus H^4(A,\mathbb{Z}).$$

We define $\mathscr{P}^+(A) \subset \mathcal{N}(A) \otimes \mathbb{C}$ to be the component of the set of vectors which span positive-definite two-planes containing vectors of the form $exp(B + i\omega)$ with $B, \omega \in NS(A) \otimes \mathbb{R}$ and ω ample.

Theorem 7.4.4. *[30] Let A be an Abelian surface. Then there is a connected component $Stab^\dagger(A) \subset Stab(A)$ which is mapped by p onto the open subset $\mathscr{P}^+(A) \subset \mathcal{N}(X) \otimes \mathbb{C}$, the induced map*

$$p : Stab^\dagger(A) \longrightarrow \mathscr{P}^+(A) \tag{7.4.4}$$

is the universal cover, and the group of deck transformations is generated by the double shift-functor.

$\mathscr{P}^+(A)$ is a $GL^+(2,\mathbb{R})$-bundle over the contractable space

$$\left\{B+i\omega \in NS(X)\otimes\mathbb{C}|\omega^2>0\right\}.$$

Thus the fundamental group $\pi_1(\mathscr{P}^+(A))\cong\mathbb{Z}$ is generated by the loop induced by the $\mathbb{C}^*$ action on $\mathscr{P}(A)$. This $\mathbb{C}^*$ action is given explicitly by the action of the rotation matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & cos\, t \end{pmatrix}$$

for $t\in[0,2\pi]$ on $\mathscr{P}(A)$ via the identification $\mathscr{N}\otimes\mathbb{C}\cong\mathscr{N}\otimes\mathbb{R}^2$.

We give an example of a stability condition on an algebraic K3 or an Abelian surface. For this we have to introduce a little more machinery. The standard t-structure of the derived category of coherent sheaves of a smooth projective variety has as its heart the Abelian category of coherent sheaves. For a K3 surface slope stability with this t-structure defines no stability condition since the stability function for any sheaf supported in dimension zero vanishes. The next simplest choice is the t-structure obtained by tilting [34]. For details see [30].

Let $\omega\in NS(X)\otimes\mathbb{R}$ be an element of the ample cone Amp(X) of an Abelian or an algebraic K3 surface X. We define the slope $\mu_\omega(E)$ of a torsion-free sheaf E on X to be

$$\mu_\omega(E)=\frac{c_1(E)\cdot\omega}{r(E)}.$$

Let $\mathscr{T}$ be the category consisting of sheaves whose torsion-free part have μ_ω-semistable Harder-Narasimhan factors with $\mu_\omega>B\cdot\omega$ and $\mathscr{F}$ the category consisting of torsion-free sheaves with μ_ω-semistable Harder-Narasimhan factors with $\mu_\omega\le B\cdot\omega$. $(\mathscr{T},\mathscr{F})$ defines a torsion pair in the Abelian category of coherent sheaves on X. Tilting with respect to this torsion pair gives a bounded t-structure on $D^b(X)$ with heart $\mathscr{A}(B,\omega)$ that depends on $B\cdot\omega$. As stability function on this heart we choose

$$Z_{(B,\omega)}(E)=(exp(B+i\omega),v(E)). \qquad (7.4.5)$$

Note that the central charge (7.4.5) is of the form guessed by physicists by mirror symmetry arguments. For a Calabi-Yau threefold we expect quantum corrections for this central charge [35].

Proposition 7.4.3. *[30] The pair $(Z_{(B,\omega)},\mathscr{A}(B,\omega))$ defines a stability condition if for all spherical sheaves E on X one has $Z(E)\notin\mathbb{R}_{\le 0}$. In particular, this holds whenever $\omega^2>2$.*

We denote the set of all stability conditions arising in this way by $V(X)$. We denote by $\Delta^+(X) \subset \Delta(X)$ elements $\delta \in \Delta(X)$ with $r(\delta) > 0$. We define the following subset of $\mathscr{Q}(X)$

$$\mathscr{L}(X) = \left\{ \Omega = exp(B + i\omega) \in \mathscr{Q}(X) | \omega \in Amp(X), \langle \Omega, \delta \rangle \notin \mathbb{R}_{\leq 0}, \forall \delta \in \Delta^+(X) \right\}.$$

The map p restricts to a homeomorphism [30]

$$p : V(X) \longrightarrow \mathscr{L}(X).$$

We use the free action of $\widetilde{GL^+(2,\mathbb{R})}$ on $V(X)$ to introduce $U(X) := V(X) \cdot \widetilde{GL^+(2,\mathbb{R})}$. The connected component $Stab^\dagger(X)$ is the unique one containing $U(X)$. $U(X)$ can be described as the stability conditions in $Stab^\dagger(X)$ for which all skyscraper sheaves $\mathscr{O}_p$ are stable of the same phase [30]. Since we have no spherical objects on an Abelian surface A in this case we have $Stab^\dagger(A) = U(A)$.

We say a set of objects $S \subset D^b(X)$ has bounded mass in a connected component $Stab^*(X) \subset Stab(X)$ if $sup\{m_\sigma(E) | E \in S\} < \infty$ for some point $\sigma \in Stab^*(X)$. This implies that the set of Mukai vectors $\{v(E) | E \in S\}$ is finite. We have a wall-and-chamber structure:

Proposition 7.4.4. *[30] Suppose that the subset $S \subset D^b(X)$ has bounded mass in $Stab^*(X)$ and fix a compact subset $B \subset Stab^*(X)$. Then there is a finite collection $\left\{W_\gamma | \gamma \in \Gamma\right\}$ of real codimension-one submanifolds of $Stab^*(X)$ such that any component*

$$C \subset B \backslash \bigcup_{\gamma \in \Gamma} W_\gamma$$

has the following property: if $E \in S$ is $\sigma-$semistable for $\sigma \in C$, then E is σ-semistable for all $\sigma \in C$. Moreover, if $E \in S$ has primitive Mukai vector, then E is σ-stable for all $\sigma \in C$.

Using this result Bridgeland proved the following theorem for the boundary $\partial U(X)$ of the open subset $U(X)$ that is contained in a locally finite union of codimension-one real submanifolds of Stab(X):

Theorem 7.4.5. *[30] Suppose that $\sigma \in \partial U(X)$ is a general point of the boundary of U(X), i.e. it lies on only one codimension-one submanifold of $Stab(X)$. Then exactly one of the following possibilities holds:*

1. *There is a rank r spherical vector bundle A such that the only σ-stable factors of the objects $\left\{\mathscr{O}_p | p \in X\right\}$ are A und $T_A(\mathscr{O}_p)$. Thus the Jordan-Holder filtration of each $\mathscr{O}_p$ is given by*

$$0 \longrightarrow A^{\oplus r} \longrightarrow \mathscr{O}_p \longrightarrow T_A(\mathscr{O}_p) \longrightarrow 0.$$

2. *There is a rank r spherical vector bundle A such that the only σ-stable factors of the objects $\{\mathscr{O}_p | p \in X\}$ are $A[2]$ and $T_A^{-1}(\mathscr{O}_p)$. Thus the Jordan-Holder filtration of each $\mathscr{O}_p$ is given by*

$$0 \longrightarrow T_A^{-1}(\mathscr{O}_p) \longrightarrow \mathscr{O}_p \longrightarrow A^{\oplus r}[2] \longrightarrow 0.$$

3. *There are a nonsingular rational curve $C \subset X$ and an integer k such that $\mathscr{O}_p$ is σ-stable for $p \notin C$ and such that the Jordan-Holder filtration of $\mathscr{O}_p$ for $p \in C$ is*

$$0 \longrightarrow \mathscr{O}_C(k+1) \longrightarrow \mathscr{O}_p \longrightarrow \mathscr{O}_C(k)[1] \longrightarrow 0.$$

Here $T_A(B)$ is the Seidel-Thomas twist of B with respect to the spherical object A [37].

7.5 Inducing stability conditions

Let $Stab^{\dagger}(X)$ be the distinguished connected component of the space of locally-finite stability conditions described in section 7.4. We have the following important observation:

Lemma 7.5.1. *Let A be an Abelian surface and $X = Km\,A$ the associated Kummer surface. Let $i : \mathscr{N}(A) \otimes \mathbb{C} \to \mathscr{N}(X) \otimes \mathbb{C}$ be the map induced from the injective map 7.4.2. Then for every element $z \in i(\mathscr{P}^+(A))$ there is a stability condition $\sigma \in Stab^{\dagger}(X)$ with $p(\sigma) = z$ for the map $p : Stab^*(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C}$.*

Proof. This is an immediate consequence of Proposition 7.4.1 and Theorem 7.4.3. □

We observed in Theorem 7.2.1 that a four-plane defining a SCFT on a two-dimensional complex torus T with B-field B_T and Kähler class ω is mapped to a four-plane defining a SCFT with B-field $B = \frac{1}{2}\pi_* B_T + \frac{1}{2}B_{\mathbb{Z}}$. $\pi_*\omega$ is an orbifold ample class orthogonal to the 16 classes $\{\hat{E}_i\}, i \in \mathbb{F}_2^4$. $\pi_*\omega$ is an element of the closure of the ample cone $\overline{Amp(X)} = Nef(X)$.

Let $U(X)$ be the subset of $Stab^{\dagger}(X)$ described in section 7.4.

Lemma 7.5.2. *Let $exp(B + i\pi_*\omega) \in i(\mathscr{P}^+(A))$ be as in Proposition 7.4.1 with $\omega^2 > 1$. Then there is a stability condition $\sigma \in \partial U(X)$ with $\pi(\sigma) = exp(B + i\pi_*\omega)$. This σ is an element of the codimension-one submanifolds associated to the 16 exceptional divisor classes.*

Proof. By the assumption $\omega^2 > 1$ we have $(\pi_*\omega)^2 > 2$ and thus we can apply Proposition 7.4.3. By the covering map property there is a stability condition σ on the boundary of $\partial U(X)$ with $p(\sigma) = exp(B + i\pi_*\omega)$. Every (-2) curve defines a boundary element of $U(X)$ as in the third case of Theorem 7.4.5 [38]. The $\pi_*\omega$ is an orbifold ample class orthogonal exactly on the 16 (-2) curves described in section 7.2. This finishes the proof. □

The covering $p : Stab^\dagger(X) \to \mathscr{P}_0^+(X))$ is normal [30].

Proposition 7.5.1. *There is an injective map from the group of deck transformations of $Stab^\dagger(A)$ to the group of deck transformations of $Stab^\dagger(X)$.*

Proof. Recall from the remark after Theorem 7.4.4 that the fundamental group $\pi_1(\mathscr{P}^+(A)) \cong \mathbb{Z}$ is generated by the loop induced by the $\mathbb{C}^*$ action on $\mathscr{P}(A)$. We choose base points l, l' and $\sigma \in Stab^\dagger(X)$ with $p(\sigma) = l'$. The induced map

$$\pi_1(\mathscr{P}^+(A), l) \longrightarrow \pi_1(\mathscr{P}_0^+(X), l')$$

is injective since the map π_* and the action of $GL^+(2,\mathbb{R})$ commute. The trivial element of $\pi_1(\mathscr{P}^+(A), l)$ is the only normal subgroup mapped to the normal subgroup $p_*(\pi_1(Stab^\dagger(X), \sigma)$ of $\pi_1(\mathscr{P}_0^+(X), l')$. □

From the discussion of the SCFT side of the story we expect that there is an embedding of the connected component $Stab^\dagger(A)$ into the distinguished connected component $Stab^\dagger(X)$.

Theorem 7.5.3. *Let $Stab^\dagger(A)$ be the (unique) maximal connected component of the space of stability conditions of an Abelian surface A and $Stab^\dagger(X)$ the distinguished connected component of Stab(X) of the Kummer surface $X = Km\,A$. Then every connected component of $p^{-1}(i(\mathscr{P}^+(A)))$ is homeomorphic to $Stab^\dagger(A)$.*

Proof. Let $i : \mathscr{N}(A) \otimes \mathbb{C} \to \mathscr{N}(X) \otimes \mathbb{C}$ be the linear map induced from the injective map 7.4.2. We have a homeomorphism $i(\mathscr{P}^+(A)) \cong \mathscr{P}^+(A)$ and thus the fundamental group $\pi_1(i(\mathscr{P}^+(A))) = \mathbb{Z}$ is also a free cyclic group generated by the loop induced from the $\mathbb{C}^*$ action on $\mathscr{P}^+(A)$. Note that $\mathscr{P}^+(A)$ is path connected. By Theorem 7.4.4 it is also locally path connected since $Stab^\dagger(A)$ is a manifold. A path component of the covering space $p^{-1}(i(\mathscr{P}^+(A)))$ is again a covering space. By the example before Definition 4.2.1 the generator of $\pi_1(i(\mathscr{P}^+(A)))$ lifts to a path corresponding to the double shift functor [2] that is no closed loop. The only closed loop in $\pi_1(i(\mathscr{P}^+(A)))$ lifting to a closed loop in the path component is the trivial loop, i.e. a path connected component of the covering space $p^{-1}(i(\mathscr{P}^+(A)))$ is simply connected. By uniqueness of the universal covering it is isomorphic to $Stab^\dagger(A)$. □

Lemma 7.5.4. *Deck transformations except double shifts exchange the path components of* $p^{-1}(i(\mathscr{P}^+(A)))$.

Proof. By the example before Definition 4.2.1 double shifts correspond to the lift of the continuous $\mathbb{C}^*$ action to the stability manifold and thus a double shift maps a path component to itself. Would any other deck transformation, i.e. any cohomological trivial auto-equivalence of $D^b(X)$ preserving the connected component $Stab^\dagger(X)$, map a path component of $p^{-1}(i(\mathscr{P}^+(A)))$ to itself this would be a deck transformation of this path component isomorphic to $Stab^\dagger(A)$ considered as a covering space. But such deck transformations are given exactly by the double shifts. □

Theorem 7.5.3 defines embeddings $Stab^\dagger(A) \hookrightarrow Stab^\dagger(X)$. In fact, we get one embedding up to deck transformations by the uniqueness of lifts, i.e. we can choose any of the path components of $p^{-1}(i(\mathscr{P}^+(A)))$ for an embedding. Any two such embeddings are then related by a deck transformation.

We have the following generalization of Theorem 7.5.3 in the case of twisted surfaces:

Theorem 7.5.5. *Let* $Stab^\dagger(A, \alpha_{B_A})$ *be the distinguished connected component of the space of stability conditions on* $D^b(A, \alpha_{B_A})$ *for a twisted Abelian surface* (A, α_{B_A}) *and* $Stab^\dagger(X, \alpha_B)$ *the distinguished connected component of the space of stability conditions on* $D^b(X, \alpha_B)$ *for the associated twisted Kummer surface* (X, α_B) *with B-field lift as in Proposition* 7.4.2. *Then we have an embedding* $Stab^\dagger(A, \alpha_{B_A}) \hookrightarrow Stab^\dagger(X, \alpha_B)$.

Proof. Using Proposition 7.4.2 this follows from exactly the same arguments as in the proof of Theorem 7.5.3. □

Chapter 8

Quivers with Potential

8.1 Mutation method

Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$. We assume $\mathscr{A}$ is of finite length, i.e. every objects in $\mathscr{A}$ is of finite length. Further we assume the Abelian category $\mathscr{A}$ has only finitely many simple objects (up to isomorphisms). In chapter 6 we described the subset $U(\mathscr{A}) \subset Stab(\mathscr{D})$ of stability conditions with heart $\mathscr{A} = \mathscr{P}((0,1])$. We are interested in the case where the tilt $L_S(\mathscr{A})$ of the heart $\mathscr{A}$ at any of its simple objects S is again of finite length. Then Proposition 6.0.4 allowed to glue the regions $U(\mathscr{A})$ and $U(L_S(\mathscr{A}))$ along their boundaries.

Definition 8.1.1. Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ and we assume $\mathscr{A}$ is of finite length with finitely many simple objects. We say we can *tilt $\mathscr{A}$ indefinitely* if the left-tilt $L_S(\mathscr{A})$ at any simple object of $\mathscr{A}$ is again of finite length. The tilt of $L_S(\mathscr{A})$ at any simple object of $L_S(\mathscr{A})$ is again of finite length and so on.

This is a strong condition on the heart of a t-structure of a triangulated category. We will study examples in section 8.2.

Here are two remarks:

1. The Grothendieck group $K(\mathscr{A})$ of an Abelian category $\mathscr{A}$ of finite length is a free Abelian group on the simple objects of $\mathscr{A}$. Since we can identify $K(\mathscr{A}) = K(\mathscr{D})$ any heart obtained by a finite sequence of simple tilts from a heart $\mathscr{A}$ that we can tilt indefinitely has the same (finite) number of simple objects.

2. If a heart of a bounded t-structure of a triangulated category is of finite length with finitely many simple objects and every simple object S fulfills $Ext^1(S,S) = 0$, then the tilt at any simple object is again a heart of finite length. [48]

Recall the definition of a central charge on an Abelian category in chapter 6.

Definition 8.1.2. Let $\mathscr{A}$ be an Abelian category. We call a central charge $Z: K(\mathscr{A}) \to \mathbb{C}$ *discrete* if different stable objects of $\mathscr{A}$ have different phases.

Note that this definition is weaker than the definition given in [57]. The full (Abelian) subcategory of $\mathscr{A}$ whose objects are the zero object and the semistable objects of a fixed phase has the stable objects as its simple objects. B. Keller defines a central charge to be discrete if these full subcategories are semi-simple with a unique stable object.

Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ of finite length with finitely many simple objects. Let us consider the n simple objects $S_1, \ldots, S_n$ of $\mathscr{A}$. For a discrete central charge $Z: K(\mathscr{A}) \to \mathbb{C}$ there must be a simple object S_i that is *left-most*, i.e. whose phase in $(0,1]$ is the bigger than the phases of the other simple objects. First we assume that the central charge of $S_1, \ldots, S_n$ lie in the upper half-plane above the real axis. Then we rotate the central charge $Z: K(\mathscr{A}) \to \mathbb{C}$ by rotating the complex numbers $Z(S_1), \ldots, Z(S_n)$ a bit counter-clockwise. This corresponds to deforming stability conditions in $U(\mathscr{A})$ within $U(\mathscr{A})$ until we reach a stability condition σ on the boundary of $U(\mathscr{A})$ corresponding to a central charge where the left-most simple object S_i of $\mathscr{A}$ lies on the negative real axis and all other simple objects still lie in the upper half-plane. Now we are in the situation of Proposition 6.0.4 and thus there is a neighborhood of σ that lies in $U(\mathscr{A}) \cup U(L_{S_i}(\mathscr{A}))$. If we rotate a bit further the corresponding stability conditions all lie in $U(L_{S_i}(\mathscr{A}))$. Now we can repeat the same process for the tilted heart $L_{S_i}(\mathscr{A})$ and proceed further with this procedure until (if possible) we accomplish a rotation by π. This algorithm describes a path through regions in $Stab(\mathscr{D})$ each isomorphic to $\overline{\mathbb{H}}^n$. This procedure is inspired by the mutation method in [46].

Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ of finite length with finitely many simple objects. We assume we can tilt $\mathscr{A}$ indefinitely and we have given a discrete central charge on $\mathscr{A}$. We summarize the steps of the *(categorical) mutation method*:

1. Start with a stability condition in $U(\mathscr{A})$ and deform it within $U(\mathscr{A})$ by rotating the central charge $Z: K(\mathscr{D}) \to \mathbb{C}$ counter-clockwise.

2. If the left-most simple object S leaves the upper half-plane tilt at this left-most simple S.

3. Deform within $U(L_S(\mathscr{A}))$ by rotating the central charge further till the

left-most object of $L_S(\mathscr{A})$ leaves the upper half-plane and tilt $L_S(\mathscr{A})$ at this simple object.

4. Repeat this procedure (if possible) until we accomplish a rotation by π.

A priori, two simple objects of a tilted heart could be both left-most. We exclude this possibility in Lemma 8.1.3 and we can therefore continue indefinitely with the mutation algorithm described above. First we describe the steps in the mutation method in the proof of Theorem 8.1.1 in detail. Bases on this proof we are then ready to prove Lemma 8.1.3. Now this is the key result:

Theorem 8.1.1. *Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ of finite length with finitely many simple objects $S_1, \dots, S_n$. We assume we can tilt $\mathscr{A}$ indefinitely and we have given a discrete central charge $Z : K(\mathscr{A}) \to \mathbb{C}$. Further we assume we terminate after finitely many steps in the mutation method, i.e. after finitely many tilts. Then the left-most simple objects of hearts appearing in the mutation method are the stable objects of $\mathscr{A}$. In the order of decreasing phase they give a sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$. In particular, we tilt at all initial simple objects $S_1, \dots, S_n$.*

Proof. In the mutation method we always tilt at objects in $\mathscr{A}$ as can be seen as follows: The first tilt is at a simple object in $\mathscr{A}$. Then the simple objects in the first tilted heart are in $\mathscr{A}$ or in $\mathscr{A}[-1]$ by Lemma 5.3.5. Since we tilt at the left-most simple of a heart we tilt next at an object in $\mathscr{A}$. This is because all objects in $\mathscr{A}[-1]$ will have a smaller phase than objects in $\mathscr{A}$ since we rotate counter-clockwise. It follows from Lemma 5.3.4 by induction that the simple objects of a tilted heart are in $\mathscr{A}$ or in $\mathscr{A}[-1]$: Let us assume that a heart $\mathscr{A}'$ appearing in the mutation method is the left-tilt of $\mathscr{A}$ with respect to some torsion pair $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}$. Then the simple objects of $\mathscr{A}'$ lie in $\mathscr{F} \subset \mathscr{A}$ or in $\mathscr{T}[-1] \subset \mathscr{A}[-1]$. We tilt next at an object S' in $\mathscr{A}$. Thus $S' \in \mathscr{F}$ and $\langle S' \rangle \subset \mathscr{F}$. By Lemma 5.3.4 the simple objects of the heart obtained by tilting $\mathscr{A}'$ at S' is the left-tilt of some torsion pair of $\mathscr{A}$.

As long as a simple object of a heart appearing in the mutation method lies in $\mathscr{A}$ and thus its central charge lies in $\overline{\mathbb{H}}$ we have not accomplished a rotation by π. The final heart $\mathscr{A}'$ obtained in the mutation method contains only simple objects in $\mathscr{A}[-1]$. We have therefore $\mathscr{A}' \subset \mathscr{A}[-1]$ and this implies $\mathscr{A}' = \mathscr{A}[-1]$. If all simple objects are in $\mathscr{A}[-1]$ we are in the final heart.

By the proof of Lemma 8.1.3 all left-most simple objects of a heart appearing in the mutation method are stable objects in $\mathscr{A}$. The phases of all other stable objects in $\mathscr{A}$ are smaller than the phase of the first left-most simple object S since we chose a discrete central charge. By the definition of the left-tilt at a simple object before Lemma 5.3.4 all stable objects except the left-most simple remain in the first tilt of $\mathscr{A}$ since there are no homomorphisms between S and the other stable objects by Lemma 6.0.1. In the first tilted heart the phases of the stable objects of $\mathscr{A}$ are equal or smaller than the new left-most simple object. If the phase of a stable object of $\mathscr{A}$ is equal to this left-most simple they are the same since we chose a discrete central charge. Otherwise the stable object remains in the next tilted heart and so on. Since we rotate the central charge further and further every stable object of $\mathscr{A}$ has to appear as a left-most simple of a heart. Therefore we tilt in the mutation method at all stable objects of $\mathscr{A}$. For every central charge, we tilt at all initial simple objects $S_1, \dots, S_n$ since these are stable for any central charge. □

Corollary 8.1.2. *For every heart $\mathscr{A}'$ appearing in the mutation method we have $\mathscr{A}[-1] \leq \mathscr{A}' \leq \mathscr{A}$.*

Proof. For two hearts $\mathscr{A}_1, \mathscr{A}_2$ with associated bounded t-structures $\mathscr{F}_1, \mathscr{F}_2 \subset \mathscr{D}$ we say $\mathscr{A}_1 \leq \mathscr{A}_2$ if and only if $\mathscr{F}_2 \subset \mathscr{F}_1$. Given the heart $\mathscr{A}$ of a bounded t-structure $\mathscr{F} \subset \mathscr{D}$ the t-structure is the extension-closed subcategory

$$\mathscr{F} = \langle \mathscr{A}, \mathscr{A}[1], \mathscr{A}[2], \dots \rangle .$$

By the proof of Theorem 8.1.1 every heart $\mathscr{A}'$ appearing in the mutation method is a left-tilt of the initial heart $\mathscr{A}$ at same torsion pair $(\mathscr{T}, \mathscr{F})$ in $\mathscr{A}$. Note that $\mathscr{A}' = \langle \mathscr{F}, \mathscr{T}[-1] \rangle$ thus we have $\mathscr{F} \subset \mathscr{A}'$ and $\mathscr{T} \subset \mathscr{A}'[1]$. Since the torsion pair $(\mathscr{T}, \mathscr{F})$ generates $\mathscr{A}$ we have $\mathscr{A}' \leq \mathscr{A}$. Further $\mathscr{F}$ and $\mathscr{T}[-1]$ lie in the t-structure

$$\langle \mathscr{A}[-1], \mathscr{A}, \mathscr{A}[1], \dots \rangle$$

associated to the heart $\mathscr{A}[-1]$. Thus we have $\mathscr{A}[-1] \leq \mathscr{A}'$. □

Lemma 8.1.3. *Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ with discrete central charge $Z : K(\mathscr{A}) \to \mathbb{C}$ as in Theorem 8.1.1. The phases of any simple objects of a heart in any step of the mutation method are distinct.*

Proof. The stable objects of a stability condition in $U(\mathscr{A})$ are precisely the stable objects in $\mathscr{A}$ with respect to the central charge $Z : K(\mathscr{A}) \to \mathbb{C}$ by the proof of Proposition 6.0.2. If we rotate this central charge as above all stable objects will remain stable since the phases of all objects in $\mathscr{A}$ change by the same phase. Rotating further we arrive at a stability condition $\sigma \in U(\mathscr{A})$ at the boundary of the region $U(\mathscr{A})$. The simple objects of $\mathscr{A}$ will remain stable in a neighborhood of σ (see e.g. chapter 7 of [45]). Instead of the upper half plane $\mathbb{H} \cup \mathbb{R}_{<0}$ we could have chosen the convention $\mathbb{H} \cup \mathbb{R}_{>0}$ in the definition of a central charge. We saw in chapter 5 that right-tilts and left-tilts are inverse to each other. Given any heart appearing in the mutation method we can perform the mutation method in clockwise direction with right-tilts instead of left-tilts. Thus the simple objects of a heart appearing in the mutation method are stable objects in the initial stability condition with heart $\mathscr{P}((0,1]) = \mathscr{A}$. Since we chose a discrete central charge they all have distinct phases. □

In Theorem 8.1.1 we assumed we rotate by finitely many steps. We can reformulate this assumption by the following

Lemma 8.1.4. *Let $\mathscr{A} \subset \mathscr{D}$ be the heart of a bounded t-structure of $\mathscr{D}$ with discrete central charge $Z : K(\mathscr{A}) \to \mathbb{C}$ as in Theorem 8.1.1. In the mutation method we rotate by finitely many steps if and only if there are only finitely many stable objects in $\mathscr{A}$.*

Proof. If we rotate by finitely many steps all stable objects of $\mathscr{A}$ will appear as a left-most simple object of a heart appearing in the mutation method. Thus we have finitely many stable objects. Conversely, let us assume we have finitely many stable objects in $\mathscr{A}$. All simple objects of hearts appearing in the mutation method are stable objects in $\mathscr{A}$ by the proof of Lemma 8.1.3. Thus there are only finitely many possibilities of such simple objects. If we would just go ahead with the mutation algorithm we would come back to a heart that already appeared in this process. But this is impossible since there are no oriented cycles in the exchange graph of directed simple tilts [48]. Therefore the mutation method must terminate after finitely many steps. □

8.2 Quivers with (super)potential

Let k be a field. In this section we consider examples of hearts of bounded t-structures of triangulated categories that we can tilt indefinitely. The first example is the category of finite-dimensional representations $\mathscr{H}_Q := mod - kQ$ of an acyclic quiver Q. $\mathscr{H}_Q$ is the heart of the standard t-structure on the bounded derived category $D^b(\mathscr{H}_Q)$ of $\mathscr{H}_Q$. By Theorem 5.7 in [48] every heart obtained from $\mathscr{H}_Q$ by a finite sequence of simple tilts in $D^b(\mathscr{H}_Q)$ is of finite length with finitely many simple objects. In the case of a Dynkin quiver Q Theorem 8.1.1 reads as follows:

Proposition 8.2.1. *[47] Let Q be a Dynkin quiver. Let us assume we have a discrete central charge on $\mathscr{H}_Q$. Then the stable representations of $\mathscr{H}_Q$ in the order of decreasing phase give a sequence of simple tilts from $\mathscr{H}_Q$ to $\mathscr{H}_Q[-1]$.*

Proof. By Theorem 5.7 in [48] we can tilt indefinitely. Since there are only finitely many indecomposable objects in $\mathscr{H}_Q$ there can only be finitely many stable objects. By Lemma 8.1.4 we terminate after finitely many steps in the mutation method. We finish the proof by applying Theorem 8.1.1. □

An example for a non-Dynkin quiver is the Kronecker quiver

$$1 \rightrightarrows 2 .$$

Let us denote by S_1 and S_2 the simple representations associated with the two vertices. If the phase of S_1 is strictly greater than the phase of S_2, the simples are the only stable objects and we tilt two times to get to the heart with simples $S_1[-1], S_2[-1]$. If the phase of S_2 is strictly greater than the phase of S_1 the stable objects are precisely the representations in the $\mathbb{P}^1$-family with dimension vector $(1,1)$ together with the postprojective and the preinjective representations. In this case infinitely many stable objects lie on a ray in the upper half plane.

Every acyclic quiver Q has a sink and a source. Therefore we can label the n vertices of Q in the following way: Take a sink of Q and label it by 1. Remove this vertex from the quiver. Take a sink in the remaining quiver and label it by 2. Going on in this way we label the vertices of Q from 1 to n. Then there are no arrows in Q from the vertex i to the vertex j if $1 \leq i < j \leq n$.

Lemma 8.2.1. *For any acyclic quiver Q there is a discrete central charge on $\mathscr{H}_Q$ such that the stable objects with respect to this central charge are precisely the simple objects of $\mathscr{H}_Q$.*

Proof. The labeling of the vertices of Q described in front of the Lemma implies the existence of an ordering of the simple objects $S_1, \dots, S_n$ of $\mathscr{H}_Q$ such that

$$Ext^1(S_i, S_j) = 0 \text{ for } 1 \leq i < j \leq n.$$

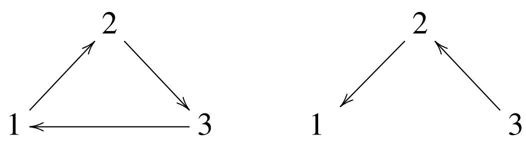
We define a central charge on $\mathscr{H}_Q$ by assigning complex numbers $Z(S_1), \dots, Z(S_n)$ in $\overline{\mathbb{H}}$ to the simple objects $S_1, \dots, S_n$ with decreasing phase from $Z(S_1)$ to $Z(S_n)$. Now we run the mutation method: Note that any simple object S_i is rigid, i.e. $Ext^1(S_i, S_i) = 0$. Thus we can calculate the new simple objects of the tilted heart with the help of Proposition 5.2 in [48] and the simple objects of any hearts appearing in the mutation method are again rigid. First we tilt at the simple object S_1. But since $Ext^1(S_1, S_i) = 0$ for $1 < i$ all other simple objects are simple objects in the new heart together with $S[-1]$. Then we tilt the new heart at S_2. Note that $Ext^1(S_2, S_1[-1]) = Hom(S_2, S_1) = 0$. The simple objects of the new heart are the objects $S_3, \dots, S_n$ together with the objects $S_1[-1]$ and $S_2[-1]$ and so on. Going on in this way we see that the mutation algorithm terminates after n steps. If there would be a stable object E except the simple objects $S_1, \ldots, S_n$ it must become left-most together with a simple object S_i in some heart $\mathscr{H}'$ appearing in the mutation method. But then the simple factors of the Jordan-Hölder filtration of E in $\mathscr{H}'$ must be isomorphic to S_i. Thus S_i is a subobject of E in $\mathscr{H}_Q$ of the same phase. Since E is stable we have $E \cong S_i$. □

The mutation method in [46] uses mutations of quivers with potential. An idea of Bridgeland was that mutation is modeled by tilting hearts [44]. This philosophy is behind Theorem 8.2.5. We now make contact with these original ideas.

Definition 8.2.1. Let Q be a finite, 2-acyclic[1] quiver and r be a vertex of Q. The *mutation* of Q at the vertex r is the new quiver $\mu_r(Q)$ obtained from Q by the rules:

1. for each $i \to r \to j$ add an arrow $i \to j$,
2. reverse all arrows with source or target r,
3. remove a maximal set of 2-cycles.

In the following example



the quivers are linked by a mutation at the vertex 2.

The category of representations of an acyclic quiver is a special case of the category of finite-dimensional modules over the Jacobi algebra of a quiver with potential [49]. Let $Q = (Q_0, Q_1)$ be a finite quiver with set of vertices Q_0 and set of arrows Q_1. We denote by kQ its path algebra, i.e. the algebra with basis given by all paths in Q and product given by composition of paths as in chapter 4. Let $\widehat{kQ}$ be the completion of kQ at the ideal generated by the arrows of Q. We consider the quotient of $\widehat{kQ}$ by the subspace $[\widehat{kQ}, \widehat{kQ}]$ of all commutators. It has a basis given by the cyclic paths of Q (up to cyclic permutation). For each arrow $a \in Q_1$ the

[1] We call a quiver $2-$*acyclic* if it does not contain loops ↻ and 2-cycles ⇆.

cyclic derivative is the linear map from the quotient to $\widehat{kQ}$ which takes an equivalence class of a path p to the sum

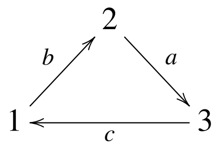
$$\sum_{p=uav} vu$$

taken over all decompositions $p = uav$. An element

$$W \in HH_0(\widehat{kQ}) = \frac{\widehat{\widehat{kQ}}}{[\widehat{kQ}, \widehat{kQ}]}$$

is called a *(super)potential* if it does not involve cycles of length ≤ 2.

For example the quiver



(8.2.1)

may have the potential $W = abc$ or $W = abcabc$.

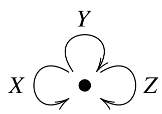
Definition 8.2.2. [49] Let (Q,W) be a quiver Q with potential W. The *Jacobi algebra* $\mathfrak{P}(Q,W)$ is the quotient of $\widehat{kQ}$ by the twosided ideal generated by the cyclic derivatives $\partial_a W$:

$$\mathfrak{P}(Q,W) := \widehat{kQ}/(\partial_a W, a \in Q_1).$$

We call a quiver with potential (Q,W) *Jacobi-finite* if its Jacobi algebra is finite-dimensional.

Here are two examples:

1. If Q is an acyclic quiver there is only one possible potential $W = 0$ and the Jacobi algebra is the path algebra kQ of Q.
2. Let us consider the quiver with one vertex and three loop arrows X, Y, Z:

   with potential $W = XYZ - XZY$. (123) and (132) are not cyclic permutations of each other and therefore W is non-zero. We calculate the cyclic derivatives:

$$\partial_X(W) = YZ - ZY$$
$$\partial_Y(W) = ZX - XZ$$
$$\partial_Z(W) = XY - YX.$$

Thus the Jacobi algebra is the power series ring $k[[X,Y,Z]]$.

We denote by $nil(\mathfrak{P}(Q,W))$ the category of finite-dimensional (right) modules over $\mathfrak{P}(Q,W)$. This is an Abelian category of finite length with simple objects the modules $S_i, i \in Q_1$. Given a quiver with potential we introduce next a triangulated category following [51]. This category has a canonical t-structure with heart equivalent to $nil(\mathfrak{P}(Q,W))$.

Definition 8.2.3. [50] Let (Q,W) be a quiver Q with potential W. The *Ginzburg algebra* $\Gamma(Q,W)$ of (Q,W) is the differential graded (dg) algebra constructed as follows: Let $\tilde{Q}$ be the graded quiver[2] with the same vertices as Q and whose arrows are

1. the arrows of Q (they all have degree 0),
2. an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
3. a loop $t_i : i \to i$ of degree -2 for each vertex $i \in Q_0$.

The underlying graded algebra of the Ginzburg algebra $\Gamma := \Gamma(Q,W)$ is the completion of the graded path algebra $k\tilde{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $\tilde{Q}$. The differential of $\Gamma(Q,W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

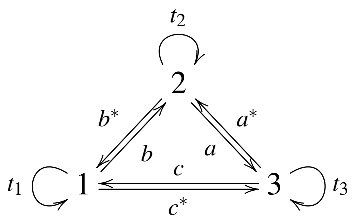
$$d(uv) = (du)v + (-1)^p u dv, \tag{8.2.2}$$

for all homogeneous u of degree p and all v defined by

1. $da = 0$ for each arrow a of Q,
2. $d(a^*) = \partial_a W$ for each arrow a of Q,
3. $d(t_i) = e_i(\sum_a [a,a^*])e_i$ for each vertex i of Q where e_i is the lazy path at i. The sum is over all arrows of Q.

[2]A graded quiver is a quiver where each arrow is equipped with an integer degree.

We call on the quiver (8.2.1) with potential $W = abc$ again. The graded quiver $\tilde{Q}$ is the following:



The differentials are as follows:

$$\begin{aligned} d(a^*) &= bc \\ d(b^*) &= ca \\ d(c^*) &= ab \\ d(t_1) &= cc^* - b^*b \\ d(t_2) &= bb^* - a^*a \\ d(t_3) &= aa^* - c^*c. \end{aligned}$$

Let us calculate the sum $\sum_u [u, \partial_u W]$ in this case as an illustrative example:

$$\begin{aligned} \sum_u [u, \partial_u W] &= [a, \partial_a W] + [b, \partial_b W] + [c, \partial_c W] \\ &= abc - bca + bca - cab + cab - abc \\ &= 0. \end{aligned}$$

This implies $d^2(t_i) = 0$.

For the convenience of the reader we prove the following two Lemmas stated in the original work [50].

Lemma 8.2.2. *[50] The differential d of the Ginzburg algebra fulfills:* $d^2 = 0$.

Proof. Using the rule 8.2.2 we only have to check $d^2(t_i) = 0$. We compute

$$d^2(t_i) = e_i(\sum_a [a, da^*])e_i = e_i(\sum_a [a, \partial_a W])e_i.$$

Then the statement follows from the identity $\sum_a [a, \partial_a W] = 0$. □

Lemma 8.2.3. *[50] Let* Γ *be the Ginzburg algebra of a quiver with potential* (Q,W)*. Then* $H^0(\Gamma) = \mathfrak{P}(Q,W)$.

Proof. This follows from the Definition 8.2.3: The Ginzburg algebra is concentrated in cohomological degrees ≤ 0. The degree zero part corresponds to $\widehat{kQ}$ and the degree -1 part corresponds to products of degree zero elements with an element of the form a^* for $a \in Q_1$. Using the rule 8.2.2 we see that the -1 part is mapped by the differential d to the twosided ideal generated by the cyclic derivatives $\partial_a W$ for $a \in Q_1$. □

Let $D(\Gamma)$ be the derived category of the Ginzburg algebra and $D_{fd}(\Gamma)$ be the full subcategory of $D(\Gamma)$ formed by dg modules whose homology is of finite total dimension ,i.e.

$$\sum_{i \in \mathbb{Z}} \dim H^i(M) < \infty.$$

The category $D_{fd}(\Gamma)$ is triangulated and 3-Calabi-Yau [52]. Let $\mathscr{F}$ be the full subcategory of $D(\Gamma)$ whose objects are the dg modules M such that $H^p(M) = 0$ for $p > 0$. The category $\mathscr{F}$ is a canonical t-structure of $D(\Gamma)$ whose truncation functors are given by the ordinary truncation functors in the category of complexes of vector spaces [53]. The heart $\mathscr{A}$ of the induced t-structure on $D_{fd}(\Gamma)$ is equivalent to $nil(\mathfrak{P}(Q,W))$. The simple $\mathfrak{P}(Q,W)$-modules S_i associated with the vertices of Q are made into Γ-modules via the morphism $\Gamma \to H^0(\Gamma)$. In $D_{fd}(\Gamma)$ they are 3-spherical objects, i.e. we have an isomorphism

$$Ext^*_\Gamma(S_i, S_i) \cong H^*(S^3, \mathbb{C}).$$

For spherical objects in triangulated categories see [54].

Definition 8.2.4. A *good extension* of mutations of quivers to quivers with potential is an extension such that we can mutate indefinitely and the mutation of the underlying quivers is given by the quiver mutation rule of Definition 8.2.1.

Derksen, Weyman and Zelevinsky proved the following important result in [49].

Theorem 8.2.4. *[49] The mutation of a 2-acyclic quiver $Q \mapsto \mu_r(Q)$ at a vertex r admits a good extension for potentials not belonging to a countable union of hypersurfaces $C \subset HH_0(\widehat{kQ})$.*

Definition 8.2.5. Let $C \subset HH_0(\widehat{kQ})$ be the union of hypersurfaces as in Theorem 8.2.4. We call the potentials not belonging to C *generic*.

If Q is an acyclic quiver there is only one generic potential: $W = 0$. For a precise definition of the extension of mutations to quivers with potential see [49]. The crucial fact for this work is that you can mutate a 2-acyclic quiver with a generic potential indefinitely. We denote the mutation of the quiver with potential (Q,W) in the sense of Theorem 8.2.4 at the vertex r by $\mu_r(Q,W)$.

The following Theorem of Keller and Yang realizes the idea of Bridgeland mentioned in the introduction that mutation should be modeled by tilting. The crucial consequence for us is that we can tilt the Abelian category associated with a quiver with generic potential indefinitely, see Lemma 8.2.6.

Theorem 8.2.5. *[53] Let (Q,W) be a 2-acyclic quiver with generic potential. Let Γ be the Ginzburg algebra of (Q,W) and Γ' be the Ginzburg algebra of $\mu_r(Q,W)$. Then there is a canonical equivalence of derived categories*

$$\Phi : D(\Gamma') \longrightarrow D(\Gamma)$$

inducing k-linear triangle equivalence of the subcategories

$$D_{fd}(\Gamma') \longrightarrow D_{fd}(\Gamma).$$

Let $\mathscr{A}'$ be the heart of the canonical t-structure on $D_{fd}(\Gamma')$. Then $\Phi(\mathscr{A}')$ is the heart of a new t-structure on $D_{fd}(\Gamma)$ given by the left-tilt at S_r of $\mathscr{A}$ in the sense of chapter 5, where $\mathscr{A}$ is the heart of the canonical t-structure on $D_{fd}(\Gamma)$.

Lemma 8.2.6. *Let (Q,W) be a 2-acyclic quiver Q with a generic potential W. Then we can tilt the heart $\mathscr{A}$ of the canonical t-structure on $D_{fd}(\Gamma)$ indefinitely.*

Proof. Let Γ be the Ginzburg algebra of (Q,W), Γ' the Ginzburg algebra of $\mu_r(Q,W)$ and Γ'' the Ginzburg algebra of $\mu_l(\mu_r(Q,W))$. Then we have a triangle equivalence $\Phi' : D_{fd}(\Gamma'') \longrightarrow D_{fd}(\Gamma')$. The heart $\mathscr{A}''$ of the canonical t-structure of $D_{fd}(\Gamma'')$ is sent by Φ' to the left-tilt of the canonical heart $\mathscr{A}'$ of $D_{fd}(\Gamma')$ tilted at the simple S'_l corresponding to the vertex l. Thus every object E in $L_{S'_l}(\mathscr{A}')$ fits into a triangle

$$F \longrightarrow E \longrightarrow T \longrightarrow$$

with $F \in \langle S'_l \rangle^{\perp} = \{E | Hom_{\mathscr{A}'}(S'_l, E) = 0\}$ and $T \in \langle S'_l \rangle [-1]$. Since $\Phi : D_{fd}(\Gamma') \to D_{fd}(\Gamma)$ is a triangle equivalence the object $\Phi(E)$ fits into the triangle

$$\Phi(F) \longrightarrow \Phi(E) \longrightarrow \Phi(T) \longrightarrow$$

with $\Phi(F) \in \langle \Phi(S'_l) \rangle^{\perp} = \{E | Hom_{\mathscr{A}}(\Phi(S'_l), E) = 0\}$ and $\Phi(T) \in \langle \Phi(S'_l) \rangle [-1]$. The object $\Phi(S'_l)$ is a simple object in the left-tilt

$L_{S_r}(\mathscr{A})$ of $\mathscr{A}$ at the simple object S_r corresponding to the vertex r and $(\langle\Phi(S'_l)\rangle^{\perp}, \langle\Phi(S'_l)\rangle[-1])$ is a torsion pair in $L_{S_r}(\mathscr{A})$. Thus the heart $\mathscr{A}''$ is mapped under the equivalence $\Phi \circ \Phi'$ to the heart $L_{\Phi(S'_l)}(L_{S_r}(\mathscr{A}))$. Both hearts are of finite length since $\mathscr{A}''$ is of finite length. We can mutate indefinitely and going on in this way we see that we can tilt $\mathscr{A}$ indefinitely. □

The important point for us is the following: The simple objects of $\mathscr{A}$ can be identified with the simple objects $S_1, \ldots, S_n$ of $nil(\mathfrak{P}(Q,W))$ for a quiver Q with n vertices. The heart $\mathscr{A}$ is of finite length with simple objects $S_1, \ldots, S_n$.

Theorem 8.2.7. *Let (Q,W) be a 2-acyclic quiver Q with generic potential W such that we have a discrete central charge on the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ with finitely many stable objects. Then the sequence of stable objects of $\mathscr{A}$ in the order of decreasing phase defines a sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$. Moreover, (Q,W) is Jacobi-finite.*

Proof. By Lemma 8.2.6 we can tilt the heart $\mathscr{A}$ indefinitely. We assume we have a discrete central charge so we can apply Theorem 8.1.1 since we terminate after finitely many steps in the mutation method by Lemma 8.1.4. To prove the last statement note that there is no sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$ if (Q,W) is not Jacobi-finite by the proof of Theorem 8.1 in [81]. □

In the proof of Lemma 8.2.6 we saw that a sequence of simple tilts of the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ for a quiver with potential (Q,W) defines a sequence of mutations of the initial quiver with potential (Q,W). Every tilt in a sequence of tilts of the initial heart $\mathscr{A}$ corresponds to a mutation.

Lemma 8.2.8. *Let Q be a 2-acyclic quiver with n vertices. The sequence of mutations of Q defined by the sequence of simple tilts in the mutation method linking the set $(S_1, \ldots, S_n)$ to the set $(S_1[-1], \ldots, S_n[-1])$ as in Theorem 8.2.7 gives back the original quiver Q (up to permutation of the vertices).*

Proof. Let $\mathscr{A}'$ be the canonical t-structure of $D_{fd}(\Gamma')$ with simple objects $S'_1, \ldots, S'_n$. Here Γ' is the Ginzburg algebra of the final quiver given by the sequence of mutations as in the statement of the lemma. By the proof of Lemma 8.2.6 the heart $\mathscr{A}'$ is equivalent to the initial heart shifted by [-1], $\mathscr{A}[-1] \subset D_{fd}(\Gamma)$, induced by a triangle equivalence $\Phi : D_{fd}(\Gamma') \to D_{fd}(\Gamma)$ where Γ is the Ginzburg algebra of the initial quiver with potential. Using Proposition 5.3.2 this gives the identifications $Ext^1_{\mathscr{A}'}(S'_i, S'_j) = Hom_{D(\Gamma')}(S'_i, S'_j[1]) = Hom_{D(\Gamma)}(\Phi(S'_i), \Phi(S'_j)[1]) =$

$Ext^1_{\mathscr{A}[-1]}(\Phi(S'_i),\Phi(S'_j))$. The objects $\Phi(S'_i)$ and $\Phi(S'_j)$ lie in $\mathscr{A}[-1]$ thus $Ext^1_{\mathscr{A}[-1]}(\Phi(S'_i),\Phi(S'_j))$ can be identified with $Ext^1_{\mathscr{A}}(S_k,S_l)$ for some simple objects S_k and S_l of $\mathscr{A}$. The dimensions of the Ext^1-groups between the simple objects $S_1,\dots,S_n$ associated to the vertices of a quiver Q are given by:

$$\dim Ext^1(S_i,S_j) = \#(\text{arrows } j \longrightarrow i \text{ in Q}).$$

This finishes the proof. □

Corollary 8.2.9. *Let $\mathscr{H}_Q := mod-kQ$ be the category of representations of an acyclic quiver Q. We assume we have finitely many stable objects with respect to a discrete central charge on $\mathscr{H}_Q$. Then the stable objects of $\mathscr{H}_Q$ in the order of decreasing phase induce a sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$, where $\mathscr{A}$ is the heart of the canonical t-structure of $D_{fd}(\Gamma)$ for the Ginzburg algebra $\Gamma = \Gamma(Q,W=0)$ of Q.*
Vice versa, given a discrete central charge on $\mathscr{A}$ with finitely many stable objects then the stable objects of $\mathscr{A}$ in the order of decreasing phase induce a sequence of simple tilts from $\mathscr{H}_Q$ to $\mathscr{H}_Q[-1]$.

Proof. By the proof of Proposition 8.2.1 the stable objects of $\mathscr{H}_Q$ in the order of decreasing phase are precisely the objects at that we tilt in a sequence of simple tilts from $\mathscr{H}_Q$ to $\mathscr{H}_Q[-1]$ in $D^b(\mathscr{H}_Q)$. $\mathscr{H}_Q$ is equivalent to the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ for the Ginzburg algebra $\Gamma = \Gamma(Q,W=0)$ of Q by Example 1 behind Definition 8.2.2. Since we can identify $\mathscr{H}_Q$ and $\mathscr{A}$ the stable objects of $\mathscr{H}_Q$ with respect to the given discrete central charge on $\mathscr{H}_Q$ can be identified with the stable objects of $\mathscr{A}$ with respect to the induced central charge. Now the Corollary follows from Theorem 8.2.7. □

An isomorphism of entire exchange graphs for the two derived categories associated to a acyclic quiver is constructed in [48].

8.3 Maximal green sequences

In this section we relate stable objects to maximal green mutation sequences as introduced by B. Keller in [57].

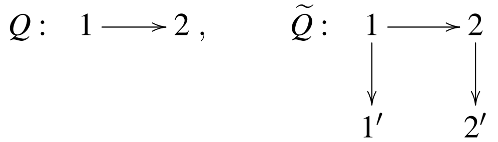
Let us consider a 2-acyclic quiver Q with n vertices. Associated with Q is a skew-symmetric matrix B whose coefficients b_{ij} are given by

$$\#(\text{arrows } i \longrightarrow j \text{ in Q}) - \#(\text{arrows } j \longrightarrow i \text{ in Q}) \qquad (8.3.1)$$

for all $1 \leq i, j \leq n$. The quiver Q is given by the matrix B up to an isomorphism of the vertices. If B is the matrix associated for the quiver Q, the skew-symmetric matrix B' associated with the mutation $\mu_k(Q)$ of Q at the vertex k is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + sgn(b_{ij}) max(0, b_{ik} b_{kj}) & \text{else.} \end{cases}$$

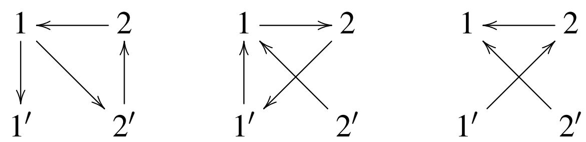
Let $\tilde{Q}$ be the *principal extension of* Q, i.e. the quiver obtained from Q by adding a new vertex $i' := i + n$ and a new arrow $i \to i'$ for each vertex $i \in Q_0$. The new vertices i' are called frozen and we will never mutate at them. Here is an example:



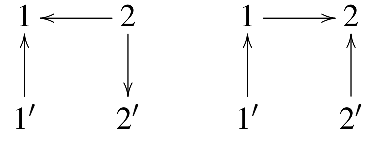
(8.3.2)

Definition 8.3.1. [57] A vertex j of a quiver in the mutation class of $\widetilde{Q}$ is called *green* if there are no arrows from a frozen vertex i' to j and *red* otherwise. A *green (mutation) sequence* on $\widetilde{Q}$ is a mutation sequence such that every mutation in the sequence is at a green vertex in the corresponding quiver. A green sequence is *maximal* if all vertices of the final quiver are red. The *length* of a green mutation sequence is the number of mutations in the sequence.

If in the example above we begin mutating at vertex 2 we find the maximal green sequence



Starting at vertex 1 we find



Let us consider a sequence of simple tilts as in Theorem 8.2.7. They define a sequence of nearby cluster collections (see section 7) and yield as combinatorial counterpart a sequence of green mutations [57, 51].

Proposition 8.3.1. *Let (Q,W) be a quiver with potential. Let $\mathscr{A}$ be the canonical heart of $D_{fd}(\Gamma)$ for $\Gamma = \Gamma(Q,W)$ with a discrete central charge as in Theorem 8.2.7. Then the stable objects of $\mathscr{A}$ define a maximal green mutation sequence with length given by the number of stable objects.*

Proof. Let $S_1,\dots,S_n$ be the simple objects of $\mathscr{A}$. Consider the $Ext-$quiver of $S_1,\dots,S_n$, i.e. the quiver with vertices labelled by the vertices of the simple objects $S_1,\dots,S_n$ and the numbers of arrows in the Ext-quiver between the vertices i and j is the dimension of $Ext^1(S_i,S_j)$. Take the principal extension of this Ext-quiver. By section 5.13 in [57] (or the proof of Theorem 7.9 in [51]) the sequence of hearts appearing in the mutation method as in Theorem 8.2.7 define a sequence of mutations of this quiver, since they define a sequence of simple tilts. The classes in the basis $[S_1],\dots,[S_n]$ of the simple objects of a heart $\mathscr{A}'$ appearing in the mutation method as in Theorem 8.2.7 encode the number of arrows from the non-frozen vertices to the frozen vertices. By the proof of Theorem 8.1.1 the simple objects of every heart $\mathscr{A}'$ appearing in the mutation method lie in $\mathscr{A}$ or $\mathscr{A}[-1]$. Since we tilt always at objects in $\mathscr{A}$ and finish with the heart $\mathscr{A}[-1]$ the sequence of mutations of quivers obtained in this way is a maximal green sequence. □

Let Q be an acyclic quiver. By Lemma 8.2.1 we can find a discrete central charge such that the stable objects are exactly the simple objects of $\mathscr{H}_Q$. Thus the set of maximal green mutations of Q is non-empty. If Q is a Dynkin quiver there is a discrete central charge with stable objects given by all indecomposable objects. Therefore we can find a maximal green sequence of length equal to the number of indecomposables.

8.4 Refined Donaldson-Thomas invariants

We can associate to a quiver with potential a refined Donaldson-Thomas invariant [58, 60]. In this section we choose $k = \mathbb{C}$ and we closely follow [57].

Let Q be a finite quiver with n vertices. The quantum affine space $\mathbb{A}_Q$ is the $\mathbb{Q}(q^{1/2})$-algebra generated by the variables $y^\alpha, \alpha \in \mathbb{N}^n$, subject to the relations

$$y^\alpha y^\beta = q^{1/2\lambda(\alpha,\beta)} y^{\alpha+\beta}$$

where $\lambda(\ ,\ )$ is the antisymmetrization of the Euler form of Q. Equivalently, $\mathbb{A}_Q$ is generated by the variables $y_i := y^{e_i}, 1 \leq i \leq n$ subject to the relations

$$y_i y_j = q^{\lambda(e_i,e_j)} y_j y_i.$$

We denote by $\hat{\mathbb{A}}_Q$ the completion of $\mathbb{A}_Q$ with respect to the ideal generated by the y_i.

Let (Q,W) be a quiver with potential and we assume we can find a discrete central charge Z on $nil(\mathfrak{P}(Q,W))$. The refined Donaldson-Thomas invariant is defined to be the product in $\hat{\mathbb{A}}_Q$

$$\mathbb{E}_{Q,W,Z} := \overrightarrow{\prod}_{M\ stable} \mathbb{E}(y^{\underline{dim}\ M}) \tag{8.4.1}$$

where the stable modules with respect to the discrete central charge appear in the order of decreasing phase. $\mathbb{E}(y)$ is the quantum dilogarithm [59], i.e. the element in the power series algebra $\mathbb{Q}(q^{1/2})[[y]]$ defined by

$$\mathbb{E}(y) = 1 + \frac{q^{1/2}}{q-1}y + \ldots + \frac{q^{n^2/2}}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}y^n + \ldots.$$

The invariant $\mathbb{E}_{Q,W,Z}$ is of course only well defined if it does not depend on the choice of a discrete central charge Z. (This is conjecture 3.2 in [57].) If it is well-defined we denote it by $\mathbb{E}_{Q,W}$.

The set of simple objects $(S_1,\ldots,S_n)$ of the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ is a cluster collection.

Definition 8.4.1. [60] A *cluster collection* S' is a sequence of objects $S'_1,\ldots,S'_n$ of $D_{fd}(\Gamma)$ such that

1. the S'_i are spherical,
2. $Ext^*(S'_i,S'_j)$ vanishes or is concentrated either in degree 1 or degree 2 for $i \neq j$,
3. the S'_i generate the triangulated category $D_{fd}(\Gamma)$.

In our case the cluster collection $S_1,\ldots,S_n$ is linked to the cluster collection $S_1[-1],\ldots,S_n[-1]$ by a sequence of simple tilts and permutations. The functor $[-1]$ is therefore a *reachable* functor for $D_{fd}(\Gamma)$ in the sense of [57, 61]. A functor $F : D_{fd}(\Gamma) \to D_{fd}(\Gamma)$ is reachable if there is a sequence of mutations and permutations from the initial cluster collection $(S_1,\ldots,S_n)$ to $(F(S_1),\ldots,F(S_n))$.

A quiver Q has a associated braid group $Braid(Q)$ which acts on $D_{fd}(\Gamma)$. Keller and Nicolás prove that there is a canonical bijection between the set of $Braid(Q)$-orbits of reachable cluster collections and reachable cluster-tilting sequences in the cluster category associated to $D_{fd}(Q)$

[57, 61]. A discrete central charge with finitely many stable objects induces reachable cluster collections. We can view the images of these cluster collections as 'stable' objects in the cluster category.

A cluster collection S' is *nearby* if the associated heart $\mathscr{A}'$ is the left-tilt of some torsion pair in $\mathscr{A}$. A sequence of simple tilts at objects of $\mathscr{A}$ starting at the initial cluster collection S gives a sequence of nearby cluster collections

$$S = S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^N.$$

For a sequence of reachable nearby cluster collections given by a sequence of simple tilts B. Keller introduced in [57] the invariant in $\mathbb{A}_Q$

$$\mathbb{E}(\varepsilon_1\beta_1)^{\varepsilon_1}\cdots\mathbb{E}(\varepsilon_N\beta_N)^{\varepsilon_N} \tag{8.4.2}$$

where $\beta_i, 1 \leq i \leq N$ is the class of the i-th simple object on that we tilt. If this object is an element of $\mathscr{A}$ we set $\varepsilon_i = +1$, if it is an element of $\mathscr{A}[-1]$ we set $\varepsilon_i = -1$.

Theorem 8.4.1. *[51, 57] Let be given sequences of reachable nearby cluster collections as described above with the same final nearby cluster collection. Then the invariant 8.4.2 does not depend on the choice of a sequence.*

Here is the main result of this chapter:

Theorem 8.4.2. *Let (Q,W) be a 2-acyclic quiver Q with generic potential W such that we have a discrete central charge on the heart $\mathscr{A}$ of the canonical t-structure of $D_{fd}(\Gamma)$ with finitely many stable objects. Then the refined Donaldson-Thomas invariant 8.4.1 associated to (Q,W) does not depend on the chosen discrete central charge $Z : K(\mathscr{A}) \to \mathbb{C}$ with finitely many stable objects.*

Proof. By the proof of Theorem 8.1.1 the simple objects of a heart appearing in the mutation method are a nearby cluster collection. The left-most simple objects of the hearts appearing in the mutation method all lie in $\mathscr{A}$. The refined DT invariant 8.4.1 is defined in terms of the stable objects of $\mathscr{A}$ in the order of decreasing phase. But these are precisely the left-most simple objects in the order they appear in the mutation method. Let $(S_1,\dots,S_n)$ be the simple objects of the initial heart $\mathscr{A}$. By Theorem 8.4.1 the refined DT invariant associated to (Q,W) in this case does not depend on the finite sequence of mutations from the cluster collection $(S_1,\dots,S_n)$ to $(S_1[-1],\dots,S_n[-1])$. Therefore the refined DT invariant is the same for every sequence of nearby cluster collections induced by a discrete central charge with finitely many stable objects. ☐

Note that the potential does not have to be polynomial. Thus we get an even stronger result than the conjecture 3.2 of B. Keller in [57] described in the introduction. This is an important new special case. In the case of a Dynkin quiver this proves the identities of Reineke [58].

Chapter 9

Conclusions and Outlook

In this thesis we considered Bridgeland stability conditions on triangulated categories and the closely related topic of BPS states. In our geometrical example of projective Kummer surfaces we constructed an embedding of the unique maximal component of the space of stability $Stab^{\dagger}(A)$ conditions of a Abelian surface into the distinguished component of the stability manifold $Stab^{\dagger}(X)$ of the associated Kummer surface. Therefore we have simply connected subspaces in $Stab^{\dagger}(X)$. The next goal would be to demonstrate that $Stab^{\dagger}(X)$ is simply connected. Unfortunately, it seems that other methods are needed to achieve this goal.

In the second project we did not investigate the space of stability conditions itself instead we used it as a tool to study refined Donaldson-Thomas invariants. We could confirm a conjecture of Bernhard Keller in a special case and actually formulate a stronger result (Theorem 8.4.2) in this case. To prove that the refined DT invariant is independent for a general central charge needs other methods like motivic Hall algebras and for this the theory of Kontsevich-Soibelman has to be further developed.

The proof of Theorem 8.2.7 is connected to (additive) categorification of cluster algebras [51] after Nagao. Categorification associates to every 2-acyclic quiver Q a generic potential W and thus the triangulated category $D_{fd}(\Gamma)$. For a quiver with n vertices we have a n-regular tree defined by iterated mutation at one of the n vertices. The statement of Theorem 8.2.7 is that stable objects of the canonical heart $\mathscr{A}$ of $D_{fd}(\Gamma)$ define a special path through this tree. The simple objects of all tilted hearts described by this path lie in $\mathscr{A}$ or in $\mathscr{A}[-1]$, what is the crucial fact in the categorification of cluster algebras associated with 2-acyclic quivers. Bridgeland stability conditions on triangulated categories are defined by a central charge on the heart of a t-structure with Harder-Narasimhan property. The space of stability conditions is a complex manifold and thus deforming the central charge deforms the stability condition. Therefore stability conditions are a natural tool to describe processes like tilting of hearts at simple objects as described above. Our methods should be useful for the categorification of cluster algebras. Here is an example: A quiver Q has a associated braid group $Braid(Q)$ which acts on $D_{fd}(\Gamma)$. Keller and Nicolás prove that there is a canonical bijection between the set of $Braid(Q)$-orbits of reachable cluster collections and reachable cluster-tilting sequences in the cluster category associated to $D_{fd}(Q)$ [57]. A discrete central charge with finitely many stable objects induces reachable cluster collections. We can view the images of these cluster collections as 'stable' objects in the cluster category.

A question that might be answered within the developed methods of this thesis is the following: Is any sequence of simple tilts from $\mathscr{A}$ to $\mathscr{A}[-1]$ induced by some central charge? In fact, Yu Qiu gave a counterexample for the case of an Dynkin quiver [47]. But one may still consider the unordered set of a sequence of simple tilts and ask if some permutation of such a sequence is induced by a central charge. This was conjectured by Qiu for Dynkin quivers in [47]. We concluded in section 8.2 that we can find a discrete central charge for any acyclic quiver such that the stable objects are precisely the simple objects. This statement is based on the fact that we can order the vertices of an acyclic quiver in such a way that there is no arrow from vertex i to vertex j if $i < j$. In fact, it is straightforward to prove the existence of a discrete central charge for an acyclic quiver without using the mutation method. For more general existence results our approach seems promising. This is left for future research.

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