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On a class of anisotropic problems

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Contents

Abstract

This thesis contains two parts. Both of them have the same thema— anisotropy.

In the first part, I study various problems of partial differential equations with a so-called anisotropic Laplace (or Finsler Laplace) operator, which has profound background both in the theory of anisotropic and nonhomogenous media and in Finsler or Minkowski geometry. We start with an overdetermined boundary value problem. It is proved that the only domain such that the PDE admits a solution must be a Wulff shape. We also study the properties of the first eigenvalue of the anisotropic Laplacian. Our result is two folds. One is a Brunn-Minkowski type inequality. The other is Payne-Weinberger type optimal anisotropic Poincaré inequality. The last chapter in this part will be devoted to a complete blow-up analysis of anisotropic Liouville equations in two dimensions. We establish a Moser-Trudinger type inequality. Then the Brezis-Merle type compactness-concentration phenomena is studied. Finally, we get some partial existence result.

The second part of this thesis concerns some interesting geometric problems incorporating with the anisotropy. We first establish the fundamentals in Minkowski geometry and then focus on the anisotropic Minkowski problem. It is a problem of prescribing the anisotropic Gauss-Kronecker curvature for a closed strongly convex hypersurface in Minkowski space as a function on its anisotropic normals. In conclusion, we completely solve the anisotropic Minkowski problem.

Introduction

In a long historical work [Wu1901], Wulff considered a variational problem of an anisotropic geometric functional in the physical model of crystal growth. He stated without proof that among closed convex hypersurfaces with constant enclosed volume, the so-called Wulff shape minimizes the anisotropic surface energy. His work initiated lots of works on the theory of phase transitions, in particular in the case of anisotropic and nonhomogenous media.

On the other hand, such anisotropy appears naturally in a wide class of geometry—Finsler geometry. Finsler geometry appeared originally in the habilitation thesis by Riemann in 1854 and had been developed by Finsler in his thesis in 1918 [Fin1918]. A typical and important example of Finsler geometry is Minkowski geometry, which was named after Minkowski. He initialed the study of Minkowski geometry since the fundamental work [Mi1897].

In the last couple of decades, many mathematicians contribute on such topics of anisotropy, aiming to extend the phenomenon in isotropic theory and Riemannian geometry to that in anisotropic theory and Finsler or Minkowski geometry. Such extensions sometimes require new development on the techniques since the structures in anisotropic theory and Finsler or Minkowski geometry are more complicated. The aim of this thesis is to investigate many interesting phenomenon of the anisotropy, including analytic and geometric aspects.

The first part of this thesis is devoted to study various problems of elliptic PDEs associating with a special Laplace operator, which is referred to as anisotropic Laplacian or Finsler-Laplacian. It becomes one of the most natural and important operators in both anisotropic theory and Finsler geometry. We will use the terminology "anisotropic Laplacian" in the whole thesis.

For simplicity, we focus on PDEs on a normed space $(\mathbb{R}^n, F)$, which is the simplest example of Finsler manifolds, so-called Minkowski space. The interest of anisotropic Laplacian lies on its nonlinearity, which turns out to be the major difference from the standard Laplacian Δ on $\mathbb{R}^n$. Nevertheless, as we will see, many results with respect to Δ can be extended to the anisotropic Laplacian.

Given a norm F on $\mathbb{R}^n$, the anisotropic Laplacian is defined by

$$Qu := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (F(\nabla u)F_{\xi_i}(\nabla u)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \xi_i} \left(\frac{1}{2} F^2 \right) (\nabla u) \right),$$

When $F(\xi) = |\xi| = (\sum_{i=1}^n |\xi_i|^2)^{1/2}$, the anisotropic Laplacian $Q = \Delta$, the usual Laplacian. In general, the anisotropic Laplacian is a quasilinear elliptic operator of divergent type. It appears in the Euler-Lagrange equations which involve functionals containing the expressions $\int_\Omega F(\nabla u(x))^2 dx$, which is in fact the Dirichlet energy of u in Minkowski space.

Anisotropic Laplacian is closely related to a convex hypersurface in $\mathbb{R}^n$, which is called the Wulff shape (or equilibrium crystal shape) of F. As we said at the first beginning, the study of the Wulff shape was initiated in Wulff's work [Wu1901] on crystal shapes The Wulff shape $\mathcal{W}_F$ is the unique minimum (up to translations) of the surface energy $\int_{\partial\Omega} F(\nu) d\mathcal{H}^{n-1}$ among regular domains Ω with constant enclosed volume. (see e.g. [Ta78, BM91, FM91]).

In Chapter 1, we will state some fundamental properties of the anisotropic Laplacian and the Wulff shape. Besides that, the anisotropic mean curvature and the convex symmetrization will be introduced, which are the fundamental concepts and tools to investigate the anisotropic Laplacian.

The second part of this thesis is devoted to some study of geometry of convex hypersurfaces in Minkowski spaces. In Minkowski geometry, we are always given a Minkowski norm F. The Wulff shape plays the role in relative geometry as the standard sphere in classical Euclidean geometry. A Riemannian metric G is well defined as $\mathrm{Hess}(\frac{1}{2}F^2)$ in $\mathbb{R}^n$, which varies from point to point. In turn, the most natural Riemannian metric for a hypersurface in the Minkowski space is the restriction of G. The anisotropic normal of a hypersurface M is defined as the map from M to the Wulff shape. Also we are able to define the anisotropic second fundamental form for a hypersurface in Minkowski spaces. A detailed description of this will be contained in Chapter 6. With the well defined anisotropic geometric quantities, we can study many geometric problems. However, due to the geometric complications, such problems are more difficulty to settle. We will focus on an anisotropic Minkowski problem as an example.

In the following, we shall briefly introduce the major contents in this thesis.

Overdetermined PDE with anisotropic Laplacian (Work in [Xia1])

In a seminal paper by Serrin [Se71], the author considered for a connected bounded domain $\Omega \in \mathbb{R}^n$, the following boundary value problem:

$$\begin{cases} \Delta u &= -1 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \\ |\nabla u| &= c & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

for a positive constant c. He proved that if the problem (0.1) admits a C^2 solution, then Ω is necessarily a ball and the solution is radial symmetric. Since then, lots of results appeared to give other characterizations of the ball by PDEs with different operators. All these operators have in common that they are constituted by the Euclidean norm of ∇u. We wonder if there is an anisotropic counterpart of such results. In other words, we want to characterize the Wulff shape by PDEs with operators constituted by the norm associated with that Wulff shape. This is the case. In Chapter 2, we shall study the overdetermined boundary value problem with anisotropic Laplacian:

$$\begin{cases} Qu &= -1 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \\ F(\nabla u) &= c & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

for a positive constant c.

We will prove the following

Theorem 0.1. *Let $F : \mathbb{R}^n \to [0, +\infty)$ be a norm of class $C^3(\mathbb{R}^n \backslash \{0\})$ and F^2 is strongly convex in $\mathbb{R}^n \backslash \{0\}$. If the overdetermined boundary value problem (0.2) has a weak solution in a connected bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, then up to translation and scaling, $\partial\Omega$ is of Wulff shape.*

As mentioned above, when $F(\xi) = |\xi|$, the Wulff shape is just the unit sphere and $Q = \Delta$. In this case Theorem 0.1 is just the classical result of Serrin [Se71].

Serrin [Se71] employed the moving plane method, initialed by Alexandrov, to prove the classical result for F the Euclidean norm. This method has subsequently been used in many further symmetry results for elliptic equations. See e.g. [GNN79, CL91a]. A short proof was presented by Weinberger [We71]. This is the first successful attempt to use an associated P -function. By using some integral identities and the maximum principle, he shows that the Hessian matrix of u is a multiple of the identity, which leads to the conclusion. There have been many generalizations of Serrin's work to more general equations. For instance, the overdetermined problem with a possibly degenerate elliptic operator was studied in [FGK06, FaK08, GL89, Ph88]. See also [BNST08] for an overdetermined problem for fully nonlinear operators. In these works they all gave a characterization of ball or sphere by an overdetermined problem of certain elliptic equation. The approaches mostly rely on some modifications of that of Serrin or Weinberger. Particularly, in [FGK06, FaK08], besides the use of P-function and a Pohozaev identity as Weinberger, the authors looked at a geometric quantity of every level set of the solution, the mean curvature. They proved that the mean curvature for all level sets is constant. By a classification result due to Alexandrov, they were able to claim the boundary, which itself is a level set, must be a sphere.

Since the anisotropic Laplacian and Wulff shape are both closely related to the anisotropic mean curvature (See Chapter 1), we are able to utilize the geometric approach to prove Theorem 0.1. By using the Pohozaev identity, the maximum principle on a suitable P-function and an interpretation of the anisotropic mean curvature of level sets by the operator Q, we show that the anisotropic mean curvature of any level set of u is constant. A recent result of He et al. in [HLMG09], which generalizes the classical Alexandrov Theorem, implies that every level set has Wulff shape.

We should mention that, Theorem 0.1 was also obtained by Cianchi and Salani in [CiSa09]. However, the technique of their proof was different.

Green's function of anisotropic Laplacian (Work in [Xia3])

Green's function for Laplace operator plays a quite important role in analysis of PDE. It is well known that every Green's function can be decomposed into a singular part and a regular part. The singular part can be explicitly represented, which is just the fundamental solution of Laplace equation. Moreover, Green's function offers a formula, which is called Green's representation formula, to represent the solutions of Poisson equations.

In Chapter 3, we will study the Green's function for anisotropic Laplacian. Such study is motivated by a recent work by Ferone and Kawohl [FeK09], where they constructed the fundamental solutions for the anisotropic Laplacian,

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2\kappa_2}\log(F^0(x)), & \text{for } n = 2, \\ \dfrac{1}{n(2-n)\kappa_n} F^0(x)^{2-n}, & \text{for } n > 2. \end{cases}$$

In [FeK09], the authors asked whether there exists Green's function for the anisotropic Laplacian. We give an affirmative answer to this.

Theorem 0.2. *Let $F : \mathbb{R}^n \to [0, +\infty)$ be a norm of class $C^2(\mathbb{R}^n\backslash\{0\})$ and F^2 is strongly convex in $\mathbb{R}^n\backslash\{0\}$. Then there exists a unique function $G(\cdot, 0) \in C^{1,\alpha}(\Omega\backslash\{0\})$ with $|\nabla G| \in L^1(\Omega)$ and $G/\Gamma \in L^\infty(\Omega)$, satisfying*

$$\begin{cases} -QG(\cdot,0) = \delta_0 & \text{in } \Omega \\ G(\cdot,0) = \phi & \text{on } \partial\Omega, \end{cases}$$

where $\phi \in L^\infty \bigcap W^{1,2}(\Omega)$. Moreover, $G = \Gamma + g$ with $g \in C^0(\Omega)$ satisfying

$$\lim_{x\to 0}(F^0(x))^{n-1}\nabla g(x) = 0$$

Theorem 0.2 will offer an efficient tool when we do blow-up analysis of PDE with the anisotropic Laplacian. However, the Green's representation formula cannot hold for the anisotropic Laplacian, due to its nonlinearity.

First eigenvalue of anisotropic Laplacian (Work in [Xia2, Xia4])

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and ν be the outward normal of its boundary $\partial\Omega$. The first eigenvalue λ_1 of the anisotropic Laplacian Q is defined by the smallest positive constant such that there exists a nonconstant function u satisfying

$$-Qu = \lambda_1 u \quad \text{in } \Omega \tag{0.3}$$

with the Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega \tag{0.4}$$

or the Neumann boundary condition

$$\langle F_\xi(\nabla u), \nu\rangle = 0 \text{ on } \partial\Omega. \tag{0.5}$$

We call λ_1 the *first Dirichlet (Neumann resp.) eigenvalue* and u the *first Dirichlet (Neumann resp.) eigenfunction* . Denote them by λ_1^D (λ_1^N resp.) and u^D (u^N resp.). Here $\langle F_\xi(\nabla u), \nu\rangle = \sum_{i=1}^n F_{\xi_i}(\nabla u)\nu^i$ and $\nu = (\nu^1, \cdots, \nu^n)$. (0.5) is a natural Neumann boundary condition for the anisotropic Laplacian. When $F(\xi) = |\xi|$, $\langle F_\xi(\nabla u), \nu\rangle = \frac{\partial u}{\partial \nu}$.

The first Dirichlet (Neumann, resp.) eigenvalue can be formulated as a variational problem by

$$\lambda_1^D(\Omega) = \inf\left\{ \frac{\int_\Omega F^2(\nabla u)dx}{\int_\Omega u^2 dx} \,\middle|\, 0 \neq u \in W_0^{1,2}(\Omega) \right\}. \tag{0.6}$$

$$\lambda_1^N(\Omega) = \inf\left\{ \frac{\int_\Omega F^2(\nabla u)dx}{\int_\Omega u^2 dx} \,\middle|\, 0 \neq u \in W^{1,2}(\Omega), \int_\Omega u dx = 0 \right\}. \tag{0.7}$$

From the theory of the direct method in the calculus of variations, we easily see that $\lambda_1^D(\Omega)$ ($\lambda_1^N(\Omega)$ resp.) can be attained by some u^D (u^Nresp.), which is the weak solution of (0.3) with the Dirichlet (Neumann resp.) boundary condition. Moreover, we can say more about the first Dirichlet eigenfunction u^D. Since replacing u^D by $|u^D|$ does not change λ_1^D, we can assume that u^D is nonnegative. Furthermore, in [BFK03], the authors showed that the first Dirichlet eigenfunction u^D is unique, positive and log-concave.

Our investigation includes two folds. One is to establish a Brunn-Minkowski type inequality for the first Dirichlet eigenvalue. The other is to give estimates for the first eigenvalue.

The Brunn-Minkowski inequality play a very improtant role in the stduy of convex bodies and convex functions. See for instance [Gar02, Sch93]. We first explain what do we mean by a Brunn-Minkowski type inequality.

Let $\mathcal{K}^n$ be the family of n-dimentional convex bodies. Assume that W is a functional defined in $\mathcal{K}^n$

$$W : \mathcal{K}^n \to (0, +\infty),$$

which is homogeneous of order $\alpha \neq 0$.

Definition 0.3. *We say that W satisfies a Brunn-Minkowski type inequality if the following inequality*

$$W\big((1-t)K_0 + tK_1\big)^{1/\alpha} \geq (1-t)W(K_0)^{1/\alpha} + tW(K_1)^{1/\alpha} \tag{0.8}$$

holds for all $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0, 1]$.

The original Brunn-Minkowski inequality is established for the volume of convex bodies, which is an n-homogeneous functional. It was extended to inequalities for various geometric quantities of convex bodies, especially for functionals arising from the calculus of variations and related to partial differential equations. The first result of this kind of functionals is due to Brascamp and Lieb [BL76]: the first eigenvalue of Laplacian Δ, which is defined by $\inf\{\int_K |\nabla u|^2 dx, u \in W_0^{1,2}(int(K)), \int_K |u|^2 dx = 1\}$, satisfies a Brunn-Minkowski type inequality (0.8) with $\alpha = -2$. Subsequently Borell ([Bo83, Bo85]) proved the same kind of results with appropriate α for the Newton capacity and the torsional rigidity. These results have been recently generalized in [Co05, CCS06, CoSa03] for the first eigenvalue of the p-Laplace operator, p-capacity and p-torsional rigidity. See also [Sa05, LMX10] for Brunn-Minkowski inequality for fully nonlinear operator.

These extension of Brunn-Minkowski inequalities have some common features. The functional $W(K)$ can be rewritten as the energy integral of a function u, which is the solution of one corresponding second-order elliptic partial differential equation, that is,

$$W(K) = \int_K |\nabla u|^2 dx.$$

For instance, when u is the solution of the following boundary-value problem:

$$\begin{cases} \Delta u + \lambda u = 0, u > 0 & \text{in } int(K) \\ u = 0, & \text{on } \partial K, \end{cases}$$

the corresponding functional $W(K)$ is just the eigenvalue of the Laplacian operator.

In Chapter 4 we establish a Brunn-Minkowski type inequality for the first eigenvalue of the anisotropic Laplacian Q with respect to a given convex function F. We have the following main theorem:

Theorem 0.4. *Let $F : \mathbb{R}^n \to [0, +\infty)$ be a norm of class $C^1(\mathbb{R}^n \backslash \{0\})$. Let K_i be a convex body in $\mathbb{R}^n$, $i = 0, 1$. For $t \in [0, 1]$, we set $K_t = (1-t)K_0 + tK_1$. $\lambda(K)$ is the first Dirichlet eigenvalue of the anisotropic Laplacian for K. Then $\lambda(K)$ is homogeneous of degree -2 and we have the following Brunn-Minkowski inequality:*

$$\lambda^{-\frac{1}{2}}(K_t) \geq (1-t)\lambda^{-\frac{1}{2}}(K_0) + t\lambda^{-\frac{1}{2}}(K_1).$$

The other topic on the first eigenvalue is to give upper and lower bounds. Finding a lower bound for the first eigenvalue is always an interesting problem. In [BFK03, GS01], the authors proved the Faber-Krahn type inequality for the first Dirichlet eigenvalue of the anisotropic Laplacian. A Cheeger type estimate for the first eigenvalue of the anisotropic Laplacian involving isoperimetric constant was also obtained there. In Chapter 4, we shall prove the Payne-Weinberger type sharp estimate [PaWe60] of the first eigenvalue in terms of some geometric quantity, such as the diameter with respect to F.

Theorem 0.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$. Let λ_1^N be the first Neumann eigenvalue of theanisotropic Laplacian. Assume that $\partial\Omega$ is weakly convex. Then λ_1^N satisfies*

$$\lambda_1^N \geq \frac{\pi^2}{d_F^2}, \tag{0.9}$$

where d_F is the diameter of Ω with respect to F. Moreover, equality in (0.9) holds if and only if Ω is a segment in $\mathbb{R}$.

Estimate (0.9) for the Neumann boundary problem is optimal. This is in fact a generalization of the classical result of Payne-Weinberger in [PaWe60] on an optimal estimate of the first Neumann eigenvalue of the ordinary Laplacian.See also [Be03]. There are many interesting generalizations. Here we just mention its generalization to Riemannian manifolds, since we will use the methods developed there. It should be also interesting to ask if the methods of [PaWe60] and [Be03] work to reprove our result, since there are lots of motivations in computational mathematics.

For a smooth compact n-dimensional Riemannian manifold (M, g) with nonnegative Ricci curvature and diameter d, possibly with boundary, the first Neumann eigenvalue λ_1 of Laplace operator Δ is defined to be the smallest positive constant such that there is a nonconstant function u satisfying

$$-\Delta u = \lambda_1 u \text{ in } M,$$

with

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M,$$

if ∂M is not empty, where ν denotes the outward normal of ∂M. A fundamental work of Li [LiP79], Li-Yau [LiYa80], Zhong-Yang [ZY84] gives us the following optimal estimate

$$\lambda_1 \geq \frac{\pi^2}{d^2}, \tag{0.10}$$

where d is the diameter of M with respect to g. Li-Yau [LiYa80] derived a gradient estimate for the eigenfunction u and proved that $\lambda_1 \geq \frac{\pi^2}{4d^2}$ and Li [LiP79] used another auxiliary function to obtain a better estimate $\lambda_1 \geq \frac{\pi^2}{2d^2}$. Finally, Zhong-Yang [ZY84] was able to use a more precise auxiliary function to get the sharp estimate $\lambda_1 \geq \frac{\pi^2}{d^2}$, which is optimal in the sense that the lower bound is achieved by a circle or a segment. Recently Hang-Wang [HaWa07] proved that equality in (0.10) holds if and only if M is a circle or a segment. For the related work see also [Kr92, CW97, BQ00]. Very recently these results were generalized to the p-Laplacian in [Va11] and to the Laplacian on Alexandrov spaces in [QZZ11].

For the Dirichlet problem we have

Theorem 0.6. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$. Assume that λ_1^D are the first Dirichlet eigenvalue of the anisotropic Laplacian. Assume further that $\partial\Omega$ is F-mean convex. Then λ_1^N satisfies*

$$\lambda_1^D \geq \frac{\pi^2}{4i_F^2}, \tag{0.11}$$

where i_F is the inscribed radius of Ω with respect to F.

Estimate (0.11) is by no mean optimal.

Our idea to prove the result on the Dirichlet eigenvalue is based on the gradient estimate technique for eigenfunctions of Li-Yau [LiP79, LiYa80]. This idea also works for the first Neumann eigenvalue to get a rough estimate, say $\lambda_1^N \geq \frac{\pi^2}{2d_F^2}$. However, for getting the sharp estimate of the first Neumann eigenvalue (0.9), the method of Zhong-Yang seems hard to apply. Instead, we adopt the technique based on gradient comparison with a one dimensional model function, which was developed by Kröger [Kr92] and improved by Chen-Wang [CW97] and Bakry-Qian [BQ00]. Surprisingly, we find that the one dimensional model coincides with that for the Laplacian case. In fact, this must be the case because when we consider F in $\mathbb{R}$, it can only be $F(x) = c|x|$ with $c > 0$, a multiple of the standard Euclidean norm. In order to get the gradient comparison theorem, we need a Bochner type formula (2.7), A Kato type inequality (4.25) and a refined inequality (4.26), which was referred to as the “extended Curvature-Dimension inequality” in the context of Bakry-Qian [BQ00]. Interestingly, the proof of these inequalities sounds more “naturally” than the proof of their counterpart for the usual Laplace operator. These inequalities may have their own interest. Another difficulty we encounter is to handle the boundary maximum due to the different representation of the Neumann boundary condition (0.5). We find a suitable vector field V to avoid this difficulty. With the gradient comparison theorem, we are able to follow step by step the work of Bakry-Qian [BQ00] to get the sharp estimate.

Anisotropic Liouville equations in two dimensions (Work in[Xia3])

Let Ω be a bounded domain in $\mathbb{R}^2$. The Moser-Trudinger inequality says that the functional

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - 8\pi \log \int_\Omega e^u \tag{0.12}$$

is bounded below for any $u \in W_0^{1,2}(\Omega)$. The corresponding Euler-Lagrange equation for $J(u)$ is the so-called Liouville equation

$$-\Delta u = \lambda \frac{e^u}{\int_\Omega e^u} \tag{0.13}$$

for some constant $\lambda > 0$, which was first studied by Liouville in 1853 in [Li1853]. The functional (0.12) and Equation (0.13) have been intensively studied by many mathematicians, for there are many applications in geometric and physical problems, for example, in the problem of prescribing Gaussian curvature [ChY87, CD87, CGY93], in the theory of the mean field equation [DJLW97, DJLW99, CLi02, CLi03, Dj08, Ma08] and in the Chern-Simons theory [SpYa92, Tar96, DJLW98, StTa98, NoTa99]. See also survey articles [Lin07] and [Tar10].

In the celebrated paper by Brezis and Merle [BM91], they initiated the study of the blow-up analysis for the Liouville equation

$$-\Delta u = V(x)e^u \tag{0.14}$$

with $V(x) \in L^p(\Omega)$ and $e^u \in L^{p'}$ for $1 < p \leq \infty$ and $p' = \frac{p}{p-1}$. They first showed that any solution of (0.14) belongs to L^∞, and further they analyzed the convergence of a sequence of solutions of (0.14) and obtained a compactness-concentration type result. Their results initiate many works on the asymptotic behavior of blow-up solutions and also the existence of solutions of Liouville equation (0.13).

We will generalize the blow-up analysis for equation (0.13) to a Liouville type equation with the anisotropic Laplacian. In other words, we consider the following quasilinear equations,

$$-Qu = V(x)e^u, \tag{0.15}$$

and

$$-Qu = \lambda \frac{e^u}{\int_\Omega e^u}. \tag{0.16}$$

As in the isotropic case, Equation (0.15) has a corresponding functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega F(\nabla u)^2 - \lambda \log \int_\Omega e^u,$$

for $u \in W_0^{1,2}(\Omega)$. By using a convex symmetrization approach proposed in [AFTL97] and an argument of Moser [Mo71], we first prove a Moser-Trudinger type inequality.

Theorem 0.7. *Let $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$ and Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $u \in W_0^{1,n}(\Omega)$ and $\int_\Omega F(\nabla u)^n dx \leq 1$. Then there exists a constant $C(n)$, such that*

$$\int_\Omega \exp[\lambda u^{\frac{n}{n-1}}]dx \leq C(n)|\Omega|,$$

where $\lambda \leq \lambda_n = n^{\frac{n}{n-1}}\kappa_n^{\frac{1}{n-1}}$ and κ_n is the Lebesgue measure of the Wulff ball of radius 1. λ_n is optimal in the sense that if $\lambda > \lambda_n$ we can find a sequence (u_k) such that $\int_\Omega \exp[\lambda u_k^{\frac{n}{n-1}}]dx$ diverges.

As a direct consequence, we have that $J_\lambda(u)$ is bounded below if and only if $\lambda \leq 8\kappa$ ($\kappa = \kappa_2$). In the isotropic case, $\kappa = \pi$.

To study the asymptotic behavior of convergence and the existence of solutions, we first prove the following Brezis-Merle type compactness-concentration result.

Theorem 0.8. *Let $F : \mathbb{R}^2 \to [0, +\infty)$ be a norm of class $C^2(\mathbb{R}^2\backslash\{0\})$ and F^2 is strongly convex in $\mathbb{R}^2\backslash\{0\}$. Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of weak solutions of*

$$-Qu_n = V_n(x)e^{u_n} \text{ in } \Omega, \tag{0.17}$$

with

$$V_n \geq 0, \quad \|V_n\|_{L^p} \leq C_1 \text{ for some } 1 < p \leq \infty,$$
$$\|e^{u_n}\|_{L^{p'}} \leq C_2.$$

Define the blow-up set as follows:

$$S = \{x \in \Omega : \exists x_n \in \Omega \text{ such that } x_n \to x \text{ and } u_n(x_n) \to +\infty\}.$$

Then, one of the following possibilities happens (after taking subsequences):

(i) u_n is bounded in $L^\infty_{loc}(\Omega)$;

(ii) $u_n \to -\infty$ uniformly on any compact subsets of Ω;

(iii) $S = \{p_1, \cdots, p_m\}$ *is a finite, nonempty set, and* $u_n \to -\infty$ *uniformly on any compact subset of* $\Omega \setminus S$. *In addition,* $V_n e^{u_n} \rightharpoonup \Sigma_{i=1}^m \alpha_i \delta_{p_i}$ *in the sense of measures on* Ω *with* $\alpha_i \geq \frac{4\kappa}{p'}$ *for any* i.

In order to prove Theorem 0.8, we first need a Brezis-Merle type inequality. as we have seen, due to the nonlinearity of Q, we have no Green representation formula. Owing to this, we can not use the argument given in [BM91]. Here we use a level set method in [ReWe95], together with the convex symmetrization in [AFTL97], to prove the Brezis-Merle type inequality, which yields Theorem 0.8.

From Theorem 0.8 it is natural to ask if α_i is multiple of 8κ. We give an affirmative answer, under an extra boundary condition.

Theorem 0.9. *Let* F *as in Theorem 0.8. Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *and* (u_n) *be a sequence of weak solutions of (0.17) with*

$$\int_\Omega e^{u_n} \leq C.$$

(V_n) is a sequence of Lipschitz continuous functions satisfying

$$V_n \geq 0, \quad V_n \to V \text{ uniformly in } C^0(\overline{\Omega}), \|\nabla V_n\|_{L^\infty(\Omega)} \leq C,$$

In addition, we assume that

$$\max_{\partial\Omega} u_n - \min_{\partial\Omega} u_n \leq C.$$

Then if blow-up happens only at one point ((iii) in Theorem 0.8), the blow-up value $\alpha = 8\kappa$.

This is a generalization of the result of Li [LiY99] in the isotropic case. See also [LiSh94]. The approach in [LiY99] is based on a Harnack type inequality, which relies strongly on the method of moving plane. However, we have no idea here how to derive a method of moving plane for the anisotropic case. Fortunately, we can get this result by only analyzing the local Pohozaev identity. The expansion of the Green function, which will be proved in Chapter 3, is also crucial for the proof. This approach was proposed by Bartolucci and Tarantello in [BT02], where they worked on singular Liouville equations.

The second main goal of chapter 5 is to prove existence results for (0.16) with vanishing Dirichlet boundary value. By using the direct method in calculus of variations, it can be seen easily from the Moser-Trudinger inequality that for $\lambda < 8\kappa$, (0.16) with vanishing Dirichlet boundary value admits a solution. For general λ we need the following compactness result.

Theorem 0.10. *Let F as in Theorem 0.8. Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of solutions to*

$$\begin{cases} -Qu &= \lambda \dfrac{Ve^u}{\int_\Omega Ve^u dx} \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{cases} \tag{0.18}$$

with

$$\min_\Omega V > 0, \quad \max_\Omega V + \|\nabla V\|_{L^\infty(\Omega)} < \infty.$$

Then for any compact interval $\Lambda \subset (8\kappa(m-1), 8\kappa m)$ and $\lambda \in \Lambda$, $m \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$u(x) \le C \text{ for } x \in \Omega.$$

Theorem 0.10 is a direct consequence of the following

Theorem 0.11. *Let F as in Theorem 0.8. Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of solutions to*

$$\begin{cases} -Qu_n &= \lambda_n \dfrac{V_n e^{u_n}}{\int_\Omega V_n e^{u_n} dx} \quad \text{in } \Omega \\ u_n &= 0 \quad \text{in } \partial\Omega, \end{cases}$$

with

$$\lim_{n\to\infty} \min_\Omega V_n > 0, \quad \lim_{n\to\infty} (\max_\Omega V_n + \|\nabla V_n\|_{L^\infty(\Omega)}) < \infty.$$

Suppose, in addition, that

$$0 < \lambda_n \le C, \quad \max_\Omega u_n \to +\infty.$$

Then there exists a finite set $S = \{p_1, \cdots, p_m\} \subset \Omega$ such that

$$u_n(x) \to \sum_{i=1}^m 8\kappa G(x, p_i) \text{ in } C^{1,\beta}(\overline{\Omega} \setminus S),$$

$$\lambda_n \frac{V_n e^{u_n}}{\int_\Omega V_n e^{u_n} dx} \to \sum_{i=1}^m 8\kappa \delta_{p_i}$$

in the sense of measures in $\overline{\Omega}$, for some $0 < \beta < 1$. Here $G(x, p_i)$ and δ_{p_i} are the Green function of Q and the Dirac function with singularity p_i respectively. In particular, We have for some $m \in \mathbb{N}$,

$$\lambda_n \to 8\kappa m.$$

Like Theorem 0.8, Theorem 0.11 is also proved through blow-up analysis. We first show that the set of blow-up points is finite. Then by using Pohozaev identity, we are able to exclude the boundary blow-up. Finally, by applying Theorem 0.8 and Theorem 0.9, we obtain the result.

With the help of Theorem 0.10, we can prove the following existence result.

Theorem 0.12. *Let F as in Theorem 0.8. Let Ω be a smooth bounded domain whose complement contains at least one bounded region and V be as in Theorem 0.10. Then (0.18) admits a solution for all $\lambda \in (8\kappa, 16\kappa)$.*

Anisotropic Minkowski problem (Work in [Xia5])

The Minkowski problem is a well known problem in the classical differential geometry: given a positive function K on $\mathbb{S}^n$, can one find a closed strongly convex hypersurface whose Gauss-Kronecker curvature is given by K as a function on its normals? This problem has been solved by the works of Minkowski [Mi1897], Alexandrov [Al37], Lewy [Le38], Nirenberg [Ni53], Pogorelov [Po53] and eventually Cheng-Yau [CY76]. As is well known, the solvability of the Minkowski problem is equivalent to that of a Monge-Ampére equation. The analytic method of Nirenberg, Pogorelov and Cheng-Yau to the Minkowski problem led to significant development of the theory of the Monge-Ampére equation. Many generalized problems around convex hypersurfaces with other prescribed curvature functions were considered intensively in recent years, see e.g. [GG02] and [GM03]. Most of them can be formulated as fully nonlinear elliptic equations. We refer to the lecture note of Guan [Gu04] for a complete description of the fully nonlinear elliptic equations arising from geometry, particularly the Minkowski problem.

We will investigate in Part II an analogous Minkowski type problem which incorporates the anisotropy. We will call it the anisotropic Minkowski problem. It arises from a wide geometry, first studied by Minkowski [Mi1897, Mi1903], the relative or Minkowski differential geometry, where the role of sphere can be assumed by some other smooth convex hypersurfaces, in contrast with Euclidean geometry. As we said before, Minkowski geometry is a special example of Finsler geometry. The questions arising from relative or Minkowski geometry were intensively investigated by a number of mathematicians, see e.g. [BF34, Bu49, Re76, Gag93, Th96, An01] and so on.

In Minkowski or relative geometry, we are always given a Minkowski norm.

Definition 0.13. *A function $F : \mathbb{R}^{n+1} \to [0, +\infty)$ is called a Minkowski norm if*

(i) F is a norm of $\mathbb{R}^{n+1}$, i.e., F is a convex, 1-homogeneous function satisfying $F(x) > 0$ when $x \neq 0$;

(ii) $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$*;*

(iii) F *satisfies a uniformly elliptic condition: Hess*$(\frac{1}{2}F^2)$ *is positive definite in* $\mathbb{R}^{n+1} \setminus \{0\}$.

For an n -dimensional oriented hypersurface M in $\mathbb{R}^{n+1}$, the area in relative geometry should be computed as $\int_M d\mu_F = \int_M F^0(\bar{\nu}) d\mathcal{H}^n$, with $\bar{\nu}$ the standard normal and F^0 the dual norm of F. The anisotropic Gauss map (anisotropic normal) of M is a map from M to the Wulff shape $\mathcal{W}_F$. Using such anisotropic Gauss map, the anisotropic curvatures can be well defined. The major difference between relative geometry and Euclidean geometry lies on the fact that the metric we consider in $\mathbb{R}^{n+1}$ is not Euclidean metric any more, but a new one G instead, depending on the second derivative of F (see (6.1) below), which varies from point to point. As well, the metric on M is chosen as g, the restriction of G on M, but not that of the Euclidean metric. This arises serious complications and difficulties for the geometric problems. We will review the definitions and foundations of relative geometry in Chapter 6. The setting here is largely motivated by the work of Andrews [An01], though the notations appear differently in Part II.

As in the classical differential geometry, when M is a closed strongly convex hypersurface in $\mathbb{R}^{n+1}$, the anisotropic Gauss map defines a diffeomorphism between M and $\mathcal{W}_F$. Therefore, M can be reparametrized by the inverse anisotropic Gauss map. In turn, the anisotropic curvatures can be viewed as functions on $\mathcal{W}_F$. In particular, the anisotropic Gauss-Kronecker curvature $K(z)$ for $z \in \mathcal{W}_F$ must satisfy (see (7.1) below)

$$\int_{\mathcal{W}_F} G(z)(z, E^\alpha) \frac{1}{K(z)} d\mu_F = 0, \forall \alpha = 1, \cdots, n+1, \tag{0.19}$$

where E^α is the standard coordinate vectors in $\mathbb{R}^{n+1}$.

The anisotropic Minkowski problem is the converse of the previous statement, namely, given a positive function K on $\mathcal{W}_F$, can one find a closed strongly convex hypersurface whose anisotropic Gauss-Kronecker curvature is given by K as a function on its anisotropic normals?

In Chapter 7, we solve the anisotropic Minkowski problem. The main result is the following

Theorem 0.14. *Let* F *be a Minkowski norm in* $\mathbb{R}^{n+1}$*. Let* K *be a positive function in* $C^k(\mathcal{W})$ *with* $k \geq 2$ *and satisfy the condition* (0.19)*. Then there is a* $C^{k+1,\alpha}(\forall 0 < \alpha < 1)$ *closed strongly convex hypersurface* M *in* $\mathbb{R}^{n+1}$ *whose anisotropic Gauss-Kronecker curvature is* K *as a function on its anisotropic normals. Moreover,* M *is unique up to translations.*

Remark 0.15. *It can be seen from the proof that the smoothness of* F*,(ii) in Definition 6.1, can be assumed only in* $C^{k+3}(\mathbb{R}^{n+1} \setminus \{0\})$*.*

As in the classical Minkowski problem, we can reduce Theorem 0.14 to the solvability of a Monge-Ampére type equation on the anisotropic support function S,

$$\det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}) = \frac{1}{K} \quad \text{on } \mathcal{W}. \tag{0.20}$$

Here we give two remarks for (0.20). First, the covariant derivatives of S are all corresponding to the Riemannian metric g, which is the restriction of G on $\mathcal{W}_F$, but not restriction of the Euclidean metric on $\mathcal{W}_F$. Second, Q_{ijk} is a 3-tensor on $\mathcal{W}_F$, which corresponds to the third derivative of F. Hence in general, it does not vanish. In fact, it vanishes if and only if F is quadratic, in which case $\mathcal{W}_F$ is an ellipsoid. This causes major difficulty when we prove a priori estimates for the Monge-Ampére equation.

As usual, we will apply the method of continuity to solve (0.20). The first issue is the a priori estimates for solutions of (0.20). By modifying Cheng-Yau's proof in [CY76], we are able to give a uniformly upper bound of the anisotropic outer radius of M, which leads to the C^0 estimate. To proceed to higher order estimates, it seems necessary to derive a uniformly positive lower bound of the anisotropic inner radius of M. Cheng-Yau's proof is highly nontrivial and seems not applicable. We apply instead a new idea, which combines an inequality of Andrews [An01] and a uniformly positive lower bound of the anisotropic outer radius, to give an explicit uniformly positive lower bound of the anisotropic inner radius of M. The difficulty arises when we deal with the C^2 estimate. In the classical one, there is no gradient term in the equation. Also the simple representation of Gauss equation on the sphere makes the C^2 estimate possible without deriving C^1 estimate. Our situation is much more complicated due to both the gradient term in (0.20) and more complicated Gauss equation (see Lemma 6.7). It seems indispensable to derive the C^1 estimate first. Fortunately, since we already have the positive lower and upper bound of S, we can choose an auxiliary function as the sum of gradient part and some lower order part, explicitly, we choose $W = \log|\nabla S|^2 + e^{\alpha(m_2 - S)}$, where m_2 is the upper bound of S, α is some large constant. With this choice, we are able to use the maximum principle to obtain bounds for W and then bounds for $|\nabla S|$. The C^2 estimate cannot be proved as usual either. Here we adopt some idea of Yau's proof in [Ya78] for Calabi conjecture and Guan-Li's proof [GL10] for more general complex Monge-Ampére equation. We choose an auxiliary function $\Phi = \log(a + \Delta S) + e^{\beta(m_2 - S)}$, where a, β are some constant. Then it is possible to derive bound for Φ and then bound for $|\nabla^2 S|$.

Besides the a priori estimates for solutions of (0.20), we also need to prove the openness of sets of solutions. Thus it is necessary to study the linearized operator L_S of $S \mapsto \det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij})$. In the classical proof, the divergence free property of Newton transformation $\frac{\partial \det(u_{ij})}{\partial u_{ij}}$ is quite important to prove the self-adjointness of L_S. Here such property fails. However, by using the explicit Gauss equation, we still be able to prove the self-adjointness of L_S with respect to the anisotropic measure $d\mu_F$ (see Lemma 7.11). The kernel of L_S is explicitly derived as well (see Lemma 7.13). With these at hand, the openness can be proved in a standard way. The uniqueness part in Theorem 0.14 follows easily from Lemma 7.11 and 7.13.

Part I

Analytic aspects of anisotropic (Finsler) Laplacian

Chapter 1

Introduction to anisotropic Laplacian

1.1 Norm in $\mathbb{R}^n$

Definition 1.1. *A function $F : \mathbb{R}^n \to \mathbb{R}^+$ is called a **norm** in $\mathbb{R}^n$, when it satisfies the following three properties:*

(i) F is nonnegative and $F(\xi) = 0$ if and only if $\xi = 0$;

(ii) F is even, positively homogeneous of degree 1, i.e.,

$$F(t\xi) = |t|F(\xi) \text{ for any } t \in \mathbb{R}, \quad \xi \in \mathbb{R}^n;$$

(iii) F is convex, i.e., for any $0 \le t \le 1, \quad \xi_1, \xi_2 \in \mathbb{R}^n$,

$$F((1-t)\xi_1 + t\xi_2) \le (1-t)F(\xi_1) + tF(\xi_2).$$

Property (iii) can be replaced by triangle inequality, that is,

(iii)' $F(\xi_1 + \xi_2) \le F(\xi_1) + F(\xi_2)$ for any $\xi_1, \xi_2 \in \mathbb{R}^n$.

We say F is a weak norm if condition (ii) is weakened to be

$$F(t\xi) = tF(\xi) \text{ for any } t > 0, \quad \xi \in \mathbb{R}^n.$$

In other words, weak norm needs not to be even (In most case, the even assumption is not significant. For simplicity, we always work on an even norm.) It is easy to see that for every norm F, there exist two positive numbers $a, b > 0$ such that

$$a|\xi| \le F(\xi) \le b|\xi| \text{ for any } \xi \in \mathbb{R}^n, \tag{1.1}$$

where $|\cdot|$ denotes standard Euclidean norm. Consequently, a norm must be Lipschitz continuous. However, a norm can never be differentiable at the origin.

The following properties are easy consequences of 1-homogeneity and convexity of F.

Proposition 1.2. *Let F be a norm in $\mathbb{R}^n$, then the following holds:*

(i) if $F \in C^1(\mathbb{R}^n\backslash\{0\})$, then for $\xi \in \mathbb{R}^n\backslash\{0\}, t \neq 0$,

$$F_{\xi_i}(\xi)\xi_i = F(\xi), \quad F_{\xi_i}(t\xi) = sign(t)F_{\xi_i}(\xi);$$

(ii) if $F \in C^2(\mathbb{R}^n\backslash\{0\})$, then for $\xi \in \mathbb{R}^n\backslash\{0\}, t \neq 0$,

$$\sum_{j=1}^{n} F_{\xi_i\xi_j}(\xi)\xi_j = 0 \text{ for any } i = 1, 2, \ldots, n, \quad F_{\xi_i\xi_j}(t\xi) = \frac{1}{|t|}F_{\xi_i\xi_j}(\xi).$$

(iii) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$;
$|F_\xi(x)| \leq C$ for any $x \neq 0$ if $F \in C^1(\mathbb{R}^n\backslash\{0\})$.

For later use, some additional assumption may be posed on a norm F.

We say $F^2(\xi) := F(\xi)^2$ is **strictly convex** in $\mathbb{R}^n$ if for any $0 < t < 1$, $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$,

$$F^2((1-t)\xi_1 + t\xi_2) < (1-t)F^2(\xi_1) + tF^2(\xi_2).$$

We say F^2 is **strongly convex** in $\mathbb{R}^n\backslash\{0\}$ if $F \in C^2(\mathbb{R}^n\backslash\{0\})$ and Hess(F^2) is positive definite in $\mathbb{R}^n\backslash\{0\}$, i.e., for any $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n\backslash\{0\}$, there exists a positive constant γ such that

$$F_{\xi_i\xi_j}(\xi)\eta_i\eta_j \geq \gamma|\eta|^2.$$

It is clear that strong convexity of F^2 in $\mathbb{R}^n\backslash\{0\}$ implies strict convexity of F^2 in $\mathbb{R}^n$.

We now introduce a related **dual norm** F^0 on $\mathbb{R}^n$.

$F^0 : \mathbb{R}^n \to \mathbb{R}$ is defined to be the support function of $K := \{x \in \mathbb{R}^n | F(x) \leq 1\}$, namely,

$$F^0(x) := \sup_{\xi \in K}\langle x, \xi\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. It is easy to verify that F^0 is also a convex, even, 1-positively homogeneous function. Actually F^0 is dual to F (see for instance [Sch93]) in the sense that

$$F^0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)} \quad \text{and} \quad F(\xi) = \sup_{x \neq 0} \frac{\langle x, \xi \rangle}{F^0(x)}.$$

Hence the Cauchy-Schwarz inequality holds in the sense that

$$\langle \xi, x \rangle \leq F(\xi) F^0(x). \tag{1.2}$$

Differentiability of F in $\mathbb{R}^n \backslash \{0\}$ depends on the convexity of $K^0 := \{x \in \mathbb{R}^n | F^0(x) \leq 1\}$. In fact, F is differentiable in $\mathbb{R}^n \backslash \{0\}$ if and only if K^0 is strictly convex, namely, the tangent space of K^0 intersect with K at only one point. The same holds for F^0 and K (See [Sch93],Cor. 1.7.3).

The following properties between F and F^0 is fundamental but very useful.

Proposition 1.3. *Let $F \in C^1(\mathbb{R}^n \backslash \{0\})$ be a norm in $\mathbb{R}^n$ such that F^0 is also in $C^1(\mathbb{R}^n \backslash \{0\})$, then*

(i) $F(\nabla F^0(x)) = 1, \quad F^0(\nabla F(\xi)) = 1$ *for* $x, \xi \neq 0$;

(ii) $F^0(x) \nabla F(\nabla F^0(x)) = x, \quad F(\xi) \nabla F^0(\nabla F(\xi)) = \xi$ *for* $x, \xi \neq 0$.

Here $\nabla F = (F_{\xi_1}, \cdots, F_{\xi_n})$ *and* $\nabla F^0 = (F^0_{x_1}, \cdots, F^0_{x_n})$.

Proof. For any $x \neq 0$, there exists $\xi_x \neq 0$ such that

$$F^0(x) = \frac{\langle x, \xi_x \rangle}{F(\xi_x)}. \tag{1.3}$$

Meanwhile,

$$F(\xi_x) = \frac{\langle x, \xi_x \rangle}{F^0(x)} = \max_{y \neq 0} \frac{\langle y, \xi_x \rangle}{F^0(y)}. \tag{1.4}$$

Hence the function $g(y) := F^0(y) F(\xi_x) - \langle y, \xi_x \rangle$ attains its minimum at x, which implies

$$\nabla g(x) = \nabla F^0(x) F(\xi_x) - \xi_x = 0. \tag{1.5}$$

Acting F on both sides and using 1-homogeneity, we get

$$F(\xi_x) F(\nabla F^0(x)) = F(\xi_x), \tag{1.6}$$

which leads to

$$F(\nabla F^0(x)) = 1$$

for any $x \neq 0$. We observe that for $\xi = \nabla F^0(x)$, we have

$$F^0(x) = \frac{\langle x, \xi \rangle}{F(\xi)}$$

by using $F(\nabla F^0(x)) = 1$ and Proposition 1.2 (i). This implies $\xi = \nabla F^0(x)$ minimizes $h(\eta) := F^0(x)F(\eta) - \langle x, \eta \rangle$. Hence

$$\nabla h(\nabla F^0(x)) = F^0(x)\nabla F(\nabla F^0(x)) - x = 0, \tag{1.7}$$

which is just the first equality in (ii). For the other two equalities, we just interchange the role of x and ξ and use the same argument as above. □

In an equivalent but more geometric way, F can be defined as a nonnegative, even and convex function on a $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$. This kind of definition often appears in some more geometric context, such as geometry of submanifolds. Since we will use some results in that context, we need to clarify the equivalence of the assumption on F in these different definitions. (In general, when defining F on $\mathbb{S}^{n-1}$, the condition that F is even does not appear, namely, the norm can be weakened to be not even.)

To be precise, restrict F on $\mathbb{S}^{n-1}$, denote also by $F := F|_{\mathbb{S}^{n-1}} \in C^1(\mathbb{S}^{n-1})$.

Proposition 1.4. *The following three statements about F are equivalent:*

(i) F^2 is strongly convex in $\mathbb{R}^n \backslash \{0\}$;

(ii) The restriction of

$$F_{\xi\xi}(\xi) = (F_{\xi_i\xi_j}(\xi))_{i,j=1}^n$$

on $T_\xi\mathbb{S}^{n-1}$ is a positive definite endomorphism $T_\xi\mathbb{S}^{n-1} \to T_\xi\mathbb{S}^{n-1}$ for all $\xi \in \mathbb{S}^{n-1}$, i.e., there exists a positive constant λ, such that for any $\xi \in \mathbb{S}^{n-1}$, $V \in T_\xi\mathbb{S}^{n-1}$, we have

$$F_{\xi_i\xi_j}(\xi)V_iV_j \geq \lambda|V|^2.$$

(iii) $Hess_{\mathbb{S}^{n-1}}F + FI|_\xi$ is positive definite for any $\xi \in \mathbb{S}^{n-1}$. Here $Hess_{\mathbb{S}^{n-1}}F$ denotes the Hessian of F on $\mathbb{S}^{n-1}$ and I the identity map on $T_\xi\mathbb{S}^{n-1}$.

Proof. For simplicity of notation we use $F_i = F_{\xi_i}, F_{ij} = F_{\xi_i\xi_j}$.

(i) ⇒(ii). The assumption means

$$(F(\xi)F_{ij}(\xi) + F_i(\xi)F_j(\xi))\zeta_i\zeta_j > 0 \tag{1.8}$$

for any vector $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n \backslash \{0\}$. If $F_\xi(\xi)$ is parallel to ξ, then statement (ii) is clearly true from (1.8), because in this case any $V \in T_\xi \mathbb{S}^{n-1}$ satisfies $\langle V, F_\xi(\xi)\rangle = 0$. If $F_\xi(\xi)$ is not parallel to ξ, then $F_\xi(\xi)^\perp := \{V \in \mathbb{R}^n \,|\, \langle V, F_\xi(\xi)\rangle = 0\}$ and ξ span the whole space $\mathbb{R}^n$. Hence for any $V \in T_\xi \mathbb{S}^{n-1}$, we have $V = \zeta + \lambda\xi$ for some $\zeta \in F_\xi(\xi)^\perp$ and some $\lambda \in \mathbb{R}$. Putting $\zeta = V - \lambda\xi$ into (1.8) we have

$$\begin{aligned} 0 &< (F(\xi)F_{ij}(\xi) + F_i(\xi)F_j(\xi))\zeta_i\zeta_j = F(\xi)F_{ij}(\xi)\zeta_i\zeta_j \\ &= F(\xi)F_{ij}(\xi)(V_i - \lambda\xi_i)(V_j - \lambda\xi_j) = F(\xi)\lambda^2 F_{ij}(\xi)V_iV_j, \end{aligned}$$

where we have used Proposition 1.2(ii).

(ii) $\Rightarrow$(i). Any $V \in \mathbb{R}^n \backslash \{0\}$ is decomposed into $V = \eta + \lambda\xi$ for some $\lambda \in \mathbb{R}$ and $\eta \in T_\xi \mathbb{S}^{n-1}$. From (ii) we have

$$\begin{aligned} (F(\xi)F_{ij}(\xi) + F_i(\xi)F_j(\xi))V_iV_j &= F(\xi)F_{ij}(\xi)\eta_i\eta_j + F_i(\xi)F_j(\xi)V_iV_j \\ &> F_i(\xi)F_j(\xi)V_iV_j \geq 0. \end{aligned}$$

Here we have used again Proposition 1.2(ii).

(ii)$\Leftrightarrow$(iii). Assume $\{e_\alpha\}_{\alpha=1}^{n-1}$ is an orthonormal basis of $T_\xi \mathbb{S}^{n-1}$ and $\{\varepsilon_i\}_{i=1^n}$ is the standard coordinate basis of $\mathbb{R}^n$. Let $e_\alpha = e_\alpha^i \varepsilon_i$. Denote by $\overline{\nabla}$ and D the covariant derivative on $\mathbb{S}^{n-1}$and $\mathbb{R}^n$ respectively. Then we have

$$\begin{aligned} (Hess_{\mathbb{S}^{n-1}} F + FI)|_\xi(e_\alpha, e_\beta) &= e_\alpha e_\beta F - (\overline{\nabla}_{e_\alpha} e_\beta)F + F\delta_{\alpha\beta} \\ &= D_{e_\alpha} D_{e_\beta} F + (D_{e_\alpha} e_\beta - \overline{\nabla}_{e_\alpha} e_\beta)F + F\delta_{\alpha\beta} \\ &= D_{e_\alpha} D_{e_\beta} F + h_{\alpha\beta}\langle -\xi, F_\xi(\xi)\rangle + F\delta_{\alpha\beta} \\ &= D_{e_\alpha} D_{e_\beta} F = e_\alpha^i e_\beta^j F_{ij}, \end{aligned}$$

where we have used the second fundamental form $h_{\alpha\beta} = \delta_{\alpha\beta}$ on $\mathbb{S}^{n-1}$ and Proposition 1.2(i). Thus we conclude that (ii) is equivalent to (iii). □

To end this section, we give some typical norms in $\mathbb{R}^n$.

Examples 1.5. *(i) The Euclidean norm $F(\xi) = (\sum_{i=1}^n |\xi_i|^2)^{\frac{1}{2}}$ in $\mathbb{R}^n$ is a norm of class $C^\infty(\mathbb{R}^n \backslash \{0\})$ with F^2 strongly convex in $\mathbb{R}^n \backslash \{0\}$.*

(ii) For a symmetric positive definite $n \times n$ matrix A , $F(\xi) = \langle A\xi, \xi\rangle$ in $\mathbb{R}^n$ is also a $C^\infty(\mathbb{R}^n \backslash \{0\})$ norm with F^2 strongly convex in $\mathbb{R}^n \backslash \{0\}$ (Riemannian metric).

(iii) The norm $F(\xi) = (\sum_{i=1}^n |\xi_i|^p)^{\frac{1}{p}}$ in $\mathbb{R}^n$ for $p \geq 1$ is called p-norm. Its dual norm is $F^0(\xi) = (\sum_{i=1}^n |\xi_i|^q)^{\frac{1}{q}}$ with $q = \frac{p}{p-1}$. The norm $F(\xi) = \max\{|\xi_1|, \cdots, |\xi_n|\}$ is called ∞-norm. Such p-norm has some disadvantage on its smoothness and convexity. For $1 \leq p < 2$, $F \notin C^2(\mathbb{R}^n \backslash \{0\})$. For $p > 2$, F is not strongly convex. Nevertheless, we can approximate it by a sequence of strongly convex norm of class $C^\infty(\mathbb{R}^n \backslash \{0\})$.

(iv) *For numerical explorations, a particular strongly convex norm of class* $C^\infty(\mathbb{R}^2\backslash\{0\})$ *in* $\mathbb{R}^2$ *is* $F(\xi_1,\xi_2)=\sqrt{\sqrt{\xi_1^4+\xi_2^4}+[\xi_1^2+\xi_2^2]}$.

1.2 Anisotropic Laplacian, Wulff shape

1.2.1 Anisotropic Laplacian

Assume that $F\in C^1(\mathbb{R}^n\backslash\{0\})$. The **anisotropic Laplacian** on $\mathbb{R}^n$ is defined by

$$Qu:=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(F(\nabla u)F_{\xi_i}(\nabla u)\right)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)\right). \tag{1.9}$$

When $F(\xi)=|\xi|=(\sum_{i=1}^n|\xi_i|^2)^{1/2}$, the anisotropic Laplacian $Q=\Delta$, the usual Laplacian. In general, anisotropic Laplacian is a nonlinear (or quasilinear) elliptic operator of divergent type.

This operator comes also from the theory of Calculus of Variations. When we derive the Euler-Lagrange equation which involves the parametric functionals containing the expression

$$\int F^2(\nabla u(x))dx, \tag{1.10}$$

anisotropic Laplacian becomes the second order differential operator in the equation.

In the following chapters, we will investigate the equation

$$-Qu(x)=f(x,u,\nabla u)\text{ in }\Omega\subset\mathbb{R}^n \tag{1.11}$$

with various $f(x,u,\nabla u)$. Since $F\in C^1(\mathbb{R}^n\backslash\{0\})$, (1.11) should be understood in the weak sense that

$$\int_\Omega\sum_{i=1}^{n}\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)\varphi_i dx=\int_\Omega f(x,u,\nabla u)\varphi dx\text{ for any }\varphi\in W_0^{1,2}(\Omega). \tag{1.12}$$

1.2.2 Wulff shape

Another concept we shall introduce in this section is **Wulff shape** (sometimes also **equilibrium crystal shape** or **indicatrix**). Consider the map

$$\Phi:\mathbb{S}^{n-1}\to\mathbb{R}^n,\qquad \Phi(\xi)=F_\xi(\xi).$$

Its image $\Phi(\mathbb{S}^{n-1})$ is a C^1 (symmetric) convex, compact hypersurface in $\mathbb{R}^n$, which is called the **Wulff shape** of F. When $F(\xi) = |\xi|$, the Wulff shape is nothing but the (n-1)-dimensional sphere $\mathbb{S}^{n-1}$. On the other hand, Wulff shape can be interpreted by the dual norm F^0.

Proposition 1.6. $\Phi(\mathbb{S}^{n-1}) = \{x \in \mathbb{R}^n | F^0(x) = 1\}$.

Proof. If $x = F_\xi(\xi)$ for some $\xi \in S^{n-1}$, by Proposition 1.3 (i), $F^0(x) = F^0(F_\xi(\xi)) = 1$. So we need only to prove $\{x \in \mathbb{R}^n | F^0(x) = 1\} \subset \Phi(\mathbb{S}^{n-1})$. Suppose $F^0(x) = 1$, by definition, $\langle x, \xi\rangle = F(\xi)$ for some $\xi \neq 0$, $\langle x, \tilde{\xi}\rangle \leq F(\eta)$ for any $\eta \neq 0$. Hence $g(\eta) := \langle x - F_\xi(\eta), \eta\rangle = \langle x, \eta\rangle - F(\eta)$ attains its maximum 0 at ξ. By taking derivative of g and evaluating at ξ, we know $x - F_\xi(\xi) = 0$. Therefore, $x = F_\xi(\xi)$, which completes the proof. □

In Part I, we denote $\mathcal{W}_F := \{x \in \mathbb{R}^n \mid F^0(x) \leq 1\}$ and $\kappa_n := |\mathcal{W}_F|$, the Lebesgue measure of $\mathcal{W}_F$. We also use the notation $\mathcal{W}_r(x_0) := \{x \in \mathbb{R}^n \mid F^0(x - x_0) \leq r\}$. We call $\mathcal{W}_r(x_0)$ a **Wulff ball** of radius r with center at x_0 . By Proposition 1.6, we see that $\partial\mathcal{W}_F$ is the Wulff shape.

Remark 1.7. *We have already seen that for every norm F, there exists a (symmetric) convex, compact hypersurface in $\mathbb{R}^n$ corresponding to it. Conversely, given a (symmetric) convex,compact hypersurface M in $\mathbb{R}^n$, denoting K as the convex body enclosed by M, the function $F^0(\xi) = \inf\{\alpha > 0 | \xi \in \alpha K\}$ is a norm, whose dual norm F has M as its Wulff shape.*

1.3 F-mean curvature

1.3.1 F-mean curvature

In this subsection we assume that $F \in C^2(\mathbb{R}^n\backslash\{0\})$ and F^2 is strongly convex in $\mathbb{R}^n\backslash\{0\}$.

We briefly recall geometry of submanifolds in this section and introduce F-mean curvature (or anisotropic mean curvature).

Let (M^{n-1}, g) be an $(n-1)$-dimensional, oriented, compact Riemannian manifold without boundary and $X : M^{n-1} \to \mathbb{R}^n$ be a a smooth immersion of M^{n-1} into $\mathbb{R}^n$. We denote by dX and $\nu : M \to \mathbb{S}^{n-1}$ the differential map of X and the corresponding Gauss map (outward normal) respectively. Let $S = dX^{-1} \circ d\nu$ and $h(\cdot, \cdot) := g(S(\cdot), \cdot)$ be the classical Weingarten operator and the second fundamental form respectively. Set

$$A_F = dX^{-1} \circ F_{\xi\xi}(\nu) \circ dX, \quad S_F = A_F \circ S, \quad h_F(\cdot, \cdot) := g(S_F(\cdot), \cdot).$$

It follows from Proposition 1.4 that A_F is a symmetric positive definite $(1,1)$-tensor. S_F and h_F are called F**-Weingarten operator** and F**-second fundamental form** respectively. F**-mean curvature** of the immersion X is defined as

$$H_F = \mathrm{tr}_g(h_F),$$

where tr_g denotes the trace of tensor with respect to g. $\overrightarrow{H_F} = -H_F \nu$ are called F**-mean curvature vector**.

The F-mean curvature comes from a variational problem related to elliptic parametric functionals of the type

$$\mathcal{F}(X) = \int_M F(\nu) d\mathcal{H}^{n-1}. \tag{1.13}$$

Critical points of $\mathcal{F}$ can be characterized as hypersurfaces with vanishing F-mean curvature. Precisely, consider a variation $X_t = X + t\varphi$ with $\varphi \in C_0^\infty(M, \mathbb{R}^n)$, the first variation of the $\mathcal{F}$ reads as

$$\delta_\varphi \mathcal{F}(X) = -\int_{\partial\Omega} \langle \overrightarrow{H_F}, \varphi \rangle d\mathcal{H}^{n-1}.$$

There are lots of works concerning the F-minimal surfaces, constant F-mean curvature surfaces and anisotropic mean curvature flow. See for example [Bel04, Cl04, Gi06, HLMG09, KoPa10].

For simplicity, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, we derive the local representation formula of H_F for the boundary $\partial\Omega$. Let $\{e_\alpha\}_{\alpha=1}^{n-1}$ be a basis of the tangent space $T_p(\partial\Omega)$ and $g_{\alpha\beta}$ and $h_{\alpha\beta}$ be the first and second fundamental form of $\partial\Omega$ respectively. Moreover let $(g^{\alpha\beta})$ be the inverse matrix of $(g_{\alpha\beta})$ and $\overline{\nabla}$ the covariant derivative in $\mathbb{R}^n$. Then

$$(h_F)_{\alpha\beta} = \langle F_{\xi\xi} \circ \overline{\nabla}_{e_\alpha} \nu, e_\beta \rangle, \tag{1.14}$$

$$H_F = \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta} (h_F)_{\alpha\beta}. \tag{1.15}$$

$\partial\Omega$ is called **weakly F-convex (F-mean convex, resp.)** if $(h_F)_{\alpha\beta}$ is nonnegative definite ($H_F \geq 0$ resp.). It is easy to see from the convexity of F that $(h_F)_{\alpha\beta}$ being nonnegative definite is equivalent that the ordinary second fundamental form $h_{\alpha\beta}$ being nonnegative definite, in other words, there is no difference between weakly F-convex and weakly convex. However, F-mean convex is different from mean convex.

1.3.2 Relation with anisotropic Laplacian and Wulff shape

The relationship of Wulff shape and F-mean curvature are very close. It is well known that an *embedded* compact hypersurface without boundary in $\mathbb{R}^n$ with constant mean curvature must be a standard sphere. This is the famous Alexandrov Theorem. Another results due to Hopf says that a *topological sphere* immersed in $\mathbb{R}^3$ with constant mean curvature must be a standard sphere.

Similarly, the Wulff shape can be characterized as an compact connected hypersurface with constant F-mean curvature. We list these Theorems, which are due to He-Li-Ma-Ge and Koiso-Palmer respectively.

Theorem 1.8 ([HLMG09],Th. 1.3)**.** *Let $X : M \to \mathbb{R}^n$ be an* embedded *compact connected hypersurface without boundary in the Euclidean space. If $H_F(M)$ is constant, then up to translations and rescaling, M is the Wulff shape.*

Theorem 1.9 ([KoPa10],Th. 1.1)**.** *let $X : \Sigma \to \mathbb{R}^n$ be a smooth immersion of a compact genus zero surface without boundary with constant F-mean curvature. Then up to translations and rescaling, Simga is the 2-dimensional Wulff shape.*

Remark 1.10. *In [HLMG09] and [KoPa10], their assumption on F is purely on $\mathbb{S}^{n-1}$, for example, $Hess_{\mathbb{S}^{n-1}}F + FI$ is positive definite on $\mathbb{S}^{n-1}$. However, by Proposition 1.4, these assumptions are equivalent to ours.*

Anisotropic Laplacian and F-mean curvature are related by the following theorem, which also gives a formula for F-mean curvature of a level set of some function.

Theorem 1.11. *Let u be a C^2 function with a regular level set $S_t := \{x \in \overline{\Omega} | u = t\}$. Let $H_F(S_t)$ be the F-mean curvature of the level set S_t. We then have*

$$Qu(x) = H_F(S_t)\frac{\partial u}{\partial \nu_F} + \frac{\partial^2 u}{\partial \nu_F^2}$$

for $x \in S_t$ with $\nabla u(x) \neq 0$, where $\nu_F := F_\xi(\nu)$.

Proof. We first derive a representation for graphic hypersurface. Assume that a surface $M \subset \mathbb{R}^n$ is given by the graph of a function f in a domain of $\mathbb{R}^{n-1}$:

$$x_n = f(x'), \qquad x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

In the proof, the Greek indices $1 \leq \alpha, \beta, \gamma \leq n-1$, the Roman indices $1 \leq i, j, k \leq n$,

A basis of the tangent space of M is given locally by

$$e_\alpha = (0, \dots, 1, \dots, f_\alpha),$$

where 1 is on the α-th coordinate. The first fundamental form for the graph of f is given by

$$g_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} + f_\alpha f_\beta.$$

Here $\delta_{\alpha\beta}$ denotes the Kronecker symbol. One can compute that

$$g^{\alpha\beta} = \delta_{\alpha\beta} - \frac{f_\alpha f_\beta}{1 + |\nabla f|^2},$$

where $\nabla f = (f_1, f_2, \ldots, f_{n-1})$. The unit normal vector field is given by

$$\nu = \frac{1}{\sqrt{1 + |\nabla f|^2}}(-\nabla f, 1).$$

We calculate the first derivative of ν and obtain

$$\nu_\alpha = (\nu_\alpha^1, \nu_\alpha^2, \ldots, \nu_\alpha^n),$$

where

$$\nu_\alpha^\beta = -\frac{f_{\beta\alpha}}{\sqrt{1 + |\nabla f|^2}} + \frac{f_\beta f_\gamma f_{\gamma\alpha}}{\sqrt{1 + |\nabla f|^2}^3}, \tag{1.16}$$

$$\nu_\alpha^n = -\frac{f_\gamma f_{\gamma\alpha}}{\sqrt{1 + |\nabla f|^2}^3}. \tag{1.17}$$

Now we derive the F-mean curvature by the local representation (1.14) and (1.15).

$$\begin{aligned} H_F(M) &= g^{\alpha\beta} F_{ij}(\nu) \nu_\alpha^i e_\beta^j \\ &= g^{\alpha\beta}(F_{i\beta}(\nu) + F_{in}(\nu) f_\beta) \nu_\alpha^i \\ &= \left(F_{i\alpha}(\nu) + \frac{(F_{in}(\nu) - F_{i\beta}(\nu) f_\beta) f_\alpha}{1 + |\nabla f|^2} \right) \nu_\alpha^i. \end{aligned}$$

From Proposition 1.2 (ii), we have

$$F_{ij}(\nu)\nu^j = 0$$

for any i, that is

$$F_{in}(\nu) - F_{i\beta}(\nu) f_\beta = 0.$$

Therefore the F-mean curvature of M is

$$H_F(M) = F_{i\alpha}(\nu) \nu_\alpha^i, \tag{1.18}$$

if M is a graph of f.

We return to compute the F-mean curvature of the level set S_t and show that

$$H_F(S_t) = sign\left(\frac{\partial u}{\partial \nu_F}\right) F_{ij}(\nabla u)u_{ij}. \tag{1.19}$$

We shall locally work around a point x_0 with $u(x_0) = t$ and $|\nabla u(x_0)| \neq 0$. Without loss of generality we assume that $u_n(x_0) \neq 0$. By the implicit function theorem, S_t can be locally represented as a graph of a function f, i.e.

$$S_t = (x', f(x')), \qquad x' = (x_1, x_2, \ldots, x_{n-1}).$$

Then

$$u(x', f(x')) = t. \tag{1.20}$$

Taking the first and second derivative of (1.20), we obtain that

$$f_\alpha = -\frac{u_\alpha}{u_n}, \tag{1.21}$$

$$f_{\alpha\beta} = -\frac{u_{\alpha\beta} + u_{\alpha n} f_\beta}{u_n} + \frac{u_\alpha(u_{n\beta} + u_{nn} f_\beta)}{u_n^2}. \tag{1.22}$$

It follows from (1.16), (1.17), (1.21) and (1.22) that

$$\nu_\alpha^\beta = \frac{|u_n|}{u_n}\left(\frac{1}{|\nabla u|}(u_{\beta\alpha} + u_{\beta n} f_\alpha) - \frac{u_\beta u_i}{|\nabla u|^3}(u_{i\alpha} + u_{in} f_\alpha)\right), \tag{1.23}$$

$$\nu_\alpha^n = \frac{|u_n|}{u_n}\left(\frac{1}{|\nabla u|}(u_{n\alpha} + u_{nn} f_\alpha) - \frac{u_n u_i}{|\nabla u|^3}(u_{i\alpha} + u_{in} f_\alpha)\right). \tag{1.24}$$

By (1.18), (1.23) and (1.24), we have

$$\begin{aligned} H_F(S_t) &= F_{\beta\alpha}(\nu)\nu_\alpha^\beta + F_{n\alpha}(\nu)\nu_\alpha^n \\ &= \frac{|u_n|}{u_n}\left(F_{i\alpha}(\nu)\frac{1}{|\nabla u|}(u_{i\alpha} + u_{in} f_\alpha) - F_{jn}(\nu)\frac{u_j}{|\nabla u|^3}u_i(u_{i\alpha} + u_{in} f_\alpha)\right) \end{aligned} \tag{1.25}$$

Note that $\nu = \frac{|u_n|\nabla u}{u_n|\nabla u|}$. Again from Proposition 1.2 (ii) we have

$$F_{jn}(\nu)u_j = 0,$$

$$F_{i\alpha}(\nu)u_\alpha + F_{in}(\nu)u_n = 0.$$

Applying these two equalities and (1.21) to (1.25) and noting that $F_{ij}(\nu) = F_{ij}(\frac{|u_n|\nabla u}{u_n|\nabla u|}) = |\nabla u| F_{ij}(\nabla u)$, we obtain

$$\begin{aligned} H_F(S_t) &= \frac{|u_n|}{u_n} F_{i\alpha}(\nabla u)(u_{i\alpha} + u_{in} f_\alpha) \\ &= \frac{|u_n|}{u_n} \left(F_{i\alpha}(\nabla u) u_{i\alpha} - F_{i\alpha}(\nabla u) u_{in} \frac{u_\alpha}{u_n} \right) \\ &= \frac{|u_n|}{u_n} \left(F_{i\alpha}(\nabla u) u_{i\alpha} + F_{in}(\nabla u) u_{in} \right) \\ &= \frac{|u_n|}{u_n} F_{ij}(\nabla u) u_{ij}. \end{aligned}$$

On the other hand, by Proposition 1.2(i), we have

$$\frac{\partial u}{\partial \nu_F} = \langle F_\xi(\nu), \nabla u \rangle = \frac{|u_n|}{u_n} F(\nabla u).$$

Now we have the relationship:

$$Qu(x) = (F F_{ij}(\nabla u) + F_i(\nabla u) F_j(\nabla u)) u_{ij} = H_F(S_t) \frac{\partial u}{\partial \nu_F} + \frac{\partial^2 u}{\partial \nu_F^2},$$

for $x \in S_t$ with $\nabla u(x) \neq 0$. □

Remark 1.12. *Theorem 1.11 is an extension for the classical mean curvature formula:*

$$\Delta u = H u_\nu + u_{\nu\nu}.$$

1.4 Convex symmetrization

This section is devoted to an important tool for the investigation of anisotropic Laplacian: convex symmetrization, which was introduced in [AFTL97] as a generalization of Schwarz symmetrization. We will use the results frequently in future chapters.

1.4.1 Polyá-Szegö principle

Consider a measurable function u on a smooth bounded domain $\Omega \subset \mathbb{R}^n$. The one dimensional **decreasing rearrangement** of u is

$$u^*(t) = \sup \{ s \geq 0 : |\{ x \in \Omega : |u(x)| > s \}| > t \}, \text{ for } t \in \mathbb{R}.$$

The classical **Schwarz symmetrization** of u is defined as

$$u^\sharp(x) = u^*\big(\omega_n |x|^n\big), \text{ for } x \in \Omega^\sharp,$$

where ω_n is the Lebesgue measure of the unit sphere in $\mathbb{R}^n$ and $\Omega^\sharp$ is the ball centered at the origin having the same measure as Ω. Similarly, we define **convex symmetrization** of u with respect to F as

$$u^\star(x) = u^*\Big(\kappa_n F^0(x)^n\Big), \text{ for } x \in \Omega^\star.$$

Here $\kappa_n F^0(x)^n$ is just the Lebesgue measure of a homothetic Wulff ball with radius $F^0(x)$ and $\Omega^\star$ is the homothetic Wulff ball centered at the origin having the same measure as Ω.

The motivation is to find such a convex symmetrization to minimize the parametric functional (1.10). Such kind of property was named as Polyá-Szegö principle. (See [Tal76], [PuSe86])

In [AFTL97], the authors proved Polya-Szegö principle for the parametric functional (1.10).

Theorem 1.13 ([AFTL97], Th. 3.1)**.** *(i) Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $u \in W^{1,p}(\Omega)$ for $p \geq 1$. Then $u^\star \in W^{1,p}(\Omega^\star)$ and*

$$\int_\Omega F^p(\nabla u)dx \geq \int_{\Omega^\star} F^p(\nabla u^\star)dx. \tag{1.26}$$

(ii) Let $u \in W_0^{1,p}(\mathbb{R}^n)$ for $p \geq 1$. Then $u^\star \in W_0^{1,p}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} F^p(\nabla u)dx \geq \int_{\mathbb{R}^n} F^p(\nabla u^\star)dx. \tag{1.27}$$

The proof combines coarea formula and isoperimetric inequality. For a smooth bounded domain $\Omega \subset \mathbb{R}^n$, and a function $u \in W^{1,p}(\Omega)$ for some $p \geq 1$, the *anisotropic perimeter* of Ω is defined by

$$P_F(\Omega) := \int_{\partial\Omega} F(\nu) d\mathcal{H}^{n-1}$$

which coincides with parametric functional (1.13).

Set $\Omega_t := \{x \in \Omega | u(x) > t\}$. It is well known that the *co-area formula*

$$-\frac{d}{dt}\int_{\Omega_t} f(x)dx = \int_{\partial\Omega_t} \frac{f(x)}{|\nabla u|} d\mathcal{H}^{n-1} \tag{1.28}$$

and the *isoperimetric inequality*

$$P_F(\Omega) \geq P_F(\Omega^\star) = n\kappa_n^{\frac{1}{n}} |\Omega|^{1-\frac{1}{n}} \tag{1.29}$$

hold. Moreover, equality in (1.29) holds if and only if Ω is a Wulff ball. When $F(\xi) = |\xi|$, the standard norm in $\mathbb{R}^n$, (1.29) reduces to the classical isoperimetric inequality. For the proof of (1.28) and (1.29), we refer to [Bu49, FR60, FM91, AFTL97].

We now prove Theorem 1.13.

Proof of Theorem 1.13. We only prove (i), since (ii) is very similar.

By using coarea formula (1.28) and Hölder inequality, we have

$$\begin{aligned} \int_\Omega F^p(\nabla u)dx &= \int_{\inf u}^{\sup u} -\frac{d}{dt}\left(\int_{\Omega_t} F^p(\nabla u)\right)dt \\ &= \int_{\inf u}^{\sup u}\int_{\partial\Omega_t} \frac{F^p(\nabla u)}{|\nabla u|} d\mathcal{H}^{n-1}dt \\ &\geq \int_{\inf u}^{\sup u}\left(\int_{\partial\Omega_t}\frac{F(\nabla u)}{|\nabla u|}d\mathcal{H}^{n-1}\right)^p\left(\int_{\partial\Omega_t}\frac{1}{|\nabla u|}d\mathcal{H}^{n-1}\right)^{1-p} dt. \end{aligned} \tag{1.30}$$

On the other hand, by using isoperimetric inequality (1.29), we have

$$\int_{\partial\Omega_t}\frac{F(\nabla u)}{|\nabla u|}d\mathcal{H}^{n-1} = P_F(\Omega_t) \geq P_F(\Omega_t^\star) = \int_{\partial\Omega_t^\star}\frac{F(\nabla u^\star)}{|\nabla u^\star|}d\mathcal{H}^{n-1}. \tag{1.31}$$

It follows again from coarea formula (1.28) that

$$\int_{\partial\Omega_t}\frac{1}{|\nabla u|}d\mathcal{H}^{n-1} = -\frac{d}{dt}|\Omega_t| = -\frac{d}{dt}|\Omega_t^\star| = \int_{\partial\Omega_t^\star}\frac{1}{|\nabla u^\star|}d\mathcal{H}^{n-1}, \tag{1.32}$$

where we have also used the fact $|\Omega_t| = |\Omega_t^\star|$ for every t. Since $\sup u^\star = \sup u$ and $\inf u^\star = \inf u$, we see easily from (1.30), (1.31) and (1.32) that

$$\int_\Omega F^p(\nabla u)dx \geq \int_{\Omega^\star} F^p(\nabla u^\star)dx.$$

□

A direct corollary of Theorem 1.13 is a sharp Sobolev type inequality.

Corollary 1.14 ([AFTL97], Cor. 3.2)**.** *Let $u \in W_0^{1,p}(\mathbb{R}^n)$ for $p \geq 1$. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \frac{\omega_n^{1/n}}{\kappa_n^{1/n}} c_{n,p}\left(\int_{\mathbb{R}^n} F^p(\nabla u)dx\right)^{\frac{1}{p}}. \tag{1.33}$$

Here $p^* = \frac{p}{p-1}$, $c_{n,p}$ is the best constant in classical Sobolev inequality in [Tal76].

Proof. We first claim that

$$\int_{\mathbb{R}^n} F^p(\nabla u^\star)dx = \frac{\kappa_n^{\frac{p}{n}}}{\omega_n^{\frac{p}{n}}}\int_{\mathbb{R}^n} |\nabla u^\sharp|^p dx. \tag{1.34}$$

Proof of (1.34). It's clear that

$$\nabla u^\star(x) = (u^*)'(\kappa_n F^0(x)^n) n\kappa_n F^0(x)^{n-1}\nabla F^0$$

Noting that u^* decreases and using Proposition 1.3(i), we have

$$F(\nabla u^\star(x)) = -(u^*)'(\kappa_n F^0(x)^n) n\kappa_n F^0(x)^{n-1}.$$

Set $\mu(t) = |\{u^\star > t\}|$ and $r(t) = \left(\frac{1}{\kappa_n}\mu(t)\right)^{\frac{1}{n}}$. It is easy to see that $\{u^\star = t\} = \{x \in \mathbb{R}^n : F^0(x) = r(t)\}$. Hence

$$\begin{aligned}
&\int_{\{u^\star=t\}} \frac{F^p(\nabla u^\star)}{|\nabla u^\star|} d\mathcal{H}^{n-1} \\
= \;& \left(-(u^*)'(\mu(t)) n\kappa_n r(t)^{n-1}\right)^p \int_{\{u^\star=t\}} \frac{1}{|\nabla u^\star|} d\mathcal{H}^{n-1} \\
= \;& \left(-(u^*)'(\mu(t)) n\kappa_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}}\right)^p \mu(t)' \\
= \;& \frac{\kappa_n^{\frac{p}{n}}}{\omega_n^{\frac{p}{n}}} \left(-(u^*)'(\mu(t)) n\omega_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}}\right)^p \mu(t)'.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} F^p(\nabla u^\star) dx &= \int_{\inf u}^{\sup u} \int_{\{u^\star=t\}} \frac{F^p(\nabla u^\star)}{|\nabla u^\star|} d\mathcal{H}^{n-1} dt \\
&= \frac{\kappa_n^{\frac{p}{n}}}{\omega_n^{\frac{p}{n}}} \int_{\inf u^\star}^{\sup u^\star} \left(-(u^*)'(\mu(t)) n\omega_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}}\right)^p \mu(t)' dt. \quad (1.35)
\end{aligned}$$

On the other hand, thanks to the facts that $\mu(t) = |\{u^\sharp > t\}|$, $\inf u^\star = \inf u^\sharp$ and $\sup u^\star = \sup u^\sharp$, a similar calculation shows that

$$\int_{\mathbb{R}^n} |\nabla u^\sharp|^p dx = \int_{\inf u^\star}^{\sup u^\star} \left(-(u^*)'(\mu(t)) n\omega_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}}\right)^p \mu(t)' dt. \quad (1.36)$$

(1.35) and (1.36) lead to the claim (1.34).

Corollary 1.14 follows from (1.34), Theorem 1.13 and the classical Sobolev inequality easily. Indeed,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} F^p(\nabla u) dx\right)^{\frac{1}{p}} &\geq \left(\int_{\mathbb{R}^n} F^p(\nabla u^\star) dx\right)^{\frac{1}{p}} = \frac{\kappa_n^{\frac{1}{n}}}{\omega_n^{\frac{1}{n}}} \left(\int_{\mathbb{R}^n} |\nabla u^\sharp|^p dx\right)^{\frac{1}{p}} \\
&\geq \frac{\kappa_n^{\frac{1}{n}}}{\omega_n^{\frac{1}{n}}} \frac{1}{c_{n,p}} \|u^\sharp\|_{L^{p^*}(\mathbb{R}^n)} = \frac{\kappa_n^{\frac{1}{n}}}{\omega_n^{\frac{1}{n}}} \frac{1}{c_{n,p}} \|u\|_{L^{p^*}(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark 1.15. *Theorem 1.14 is sharp since the equality in the Sobolev type inequality can hold. In fact, take*

$$u_0(x) = \frac{1}{(C + (F^0(x))^q)^{\frac{n-p}{p}}}, \tag{1.37}$$

with C *determined by* $\|u\|_{L^p} = 1$ *and* $q = \frac{p}{p-1}$*. By simple computation, equality holds. In [CENV04], the authors find a mass-transportation approach to prove the Sobolev type inequality. Moreover, they showed that equality holds if and only if* $u = u_0$*.*

1.4.2 Comparison theorem involving anisotropic Laplacian

Another important property about convex symmetrization which was proved in [AFTL97] is a comparison theorem involving the anisotropic Laplacian.

Theorem 1.16 ([AFTL97], pp. 289)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a smooth bounded domain and* $u, v \in W_0^{1,2}(\Omega)$ *satisfy*

$$-\text{div}(a(x, u, \nabla u)) = f(x), \quad -Qv = f^\star(x), \tag{1.38}$$

where $f \in L^{\frac{2n}{n+2}}(\Omega)$ *if* $n \geq 3$*,* $f \in L^p(\Omega)$*,* $p > 1$ *if* $n = 2$*, and* $a(x, \eta, \xi)$ *are vector-valued Carathéodory function satisfying*

$$\langle a(x, \eta, \xi), \xi \rangle \geq F^2(\xi) \ a.e. \ x \in \Omega, \quad \eta \in \mathbb{R}, \quad \xi \in \mathbb{R}^n. \tag{1.39}$$

Then we have

$$u^\star \leq v \ \text{in} \ \Omega^\star.$$

Proof. It is clear that $v(x) = v(F^0(x))$ is symmetric with respect to F, and satisfies the following ODE:

$$\begin{cases} \frac{1}{r^{n-1}} (-r^{n-1} v'(r))' = f^\star(x) \text{ in } [0, R], \\ v(R) = 0, \qquad v'(0) = 0, \end{cases}$$

where $r = F^0(x)$, $R > 0$ is the constant such that $|\Omega| = \kappa_n R^n$. Therefore

$$-v'(r) = \frac{1}{r^{n-1}} \int_0^r t^{n-1} f^\star(\kappa_n t^n) dt,$$

By changing variables $\tilde{t} = \kappa_n t^n$ and then $\tilde{s} = \kappa_n s^n$, we obtain

$$\begin{aligned} v(r) &= -\int_r^R v'(s) ds = \int_r^R \frac{1}{s^{n-1}} \int_0^s t^{n-1} f^\star(\kappa_n t^n) dt ds \\ &= \int_r^R \frac{1}{s^{n-1}} \int_0^{\kappa_n s^n} \frac{1}{n\kappa_n} f^\star(\tilde{t}) d\tilde{t} ds \\ &= \int_{\kappa_n r^n}^{\kappa_n R^n} \frac{1}{n^2 \kappa_n^{\frac{2}{n}}} \frac{1}{\tilde{s}^{2-\frac{2}{n}}} \int_0^{\tilde{s}} f^\star(\tilde{t}) d\tilde{t} d\tilde{s}. \end{aligned} \tag{1.40}$$

On the other hand, the first equation in (1.38) holds in weak sense as (1.12). For $h > 0$, $t > 0$, choose a test function

$$\varphi = \begin{cases} h & \text{if } |u| > t+h, \\ (|u|-t)sign(u), & \text{if } t < |u| \leq t+h, \\ 0, & \text{if } |u| \leq t, \end{cases}$$

in (1.12), we get

$$-\frac{d}{dt}\int_{\{|u|>t\}} \langle a(x,u,\nabla u), \nabla u\rangle dx = \int_{\{|u|>t\}} f dx.$$

Using the assumption (1.39), we obtain

$$-\frac{d}{dt}\int_{\{|u|>t\}} F^2(\nabla u)dx \leq -\frac{d}{dt}\int_{\{|u|>t\}} \langle a(x,u,\nabla u), \nabla u\rangle dx = \int_0^{\mu(t)} f^*(s)ds.$$

As in the proof of Theorem 1.13, it follows from isoperimetric inequality, coarea formula and Hölder inequality that

$$-\frac{d}{dt}\int_{\{|u|>t\}} F^2(\nabla u)dx \geq \frac{1}{-\mu'(t)}P_F(\{|u|>t\})^2 = \frac{1}{-\mu'(t)}n^2\kappa_n^{\frac{2}{n}}\mu(t)^{2-\frac{2}{n}}.$$

Therefore,

$$1 \leq \frac{-\mu'(t)}{n^2\kappa_n^{\frac{2}{n}}\mu(t)^{2-\frac{2}{n}}}\int_0^{\mu(t)} f^*(s)ds.$$

An integration among $[0, u^*(s)]$ gives

$$\begin{aligned} u^*(s) &\leq \int_0^{u^*(s)} \frac{-\mu'(t)}{n^2\kappa_n^{\frac{2}{n}}\mu(t)^{2-\frac{2}{n}}}\int_0^{\mu(t)} f^*(\tilde{s})d\tilde{s} \\ &= \int_s^{|\Omega|} \frac{1}{n^2\kappa_n^{\frac{2}{n}}}\frac{1}{\tilde{s}^{2-\frac{2}{n}}}\int_0^{\tilde{s}} f^\star(\tilde{t})d\tilde{t}d\tilde{s}. \end{aligned}$$

Comparing with (1.40), we find that

$$u^*(\kappa_n r^n) \leq v(r),$$

which leads to $u^*(x) \leq v(F^0(x)) = v(x)$. □

Chapter 2

Overdetermined problem for anisotropic Laplacian

This chapter is devoted to study the anisotropic overdetermined boundary value problem.

Thoughout this chapter, we assume that the norm $F \in C^3(\mathbb{R}^n \backslash \{0\})$ and F^2 is strongly convex in $\mathbb{R}^n \backslash \{0\}$. For a connected bounded domain $\Omega \in \mathbb{R}^n$ we consider the following boundary value problem

$$\left\{ \begin{array}{rcll} -Qu & = & 1 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \\ F(\nabla u) & = & c & \text{on } \partial\Omega \end{array} \right. \tag{2.1}$$

for a positive constant c.

Theorem 2.1. *Let F be a norm of class $C^3(\mathbb{R}^n \backslash \{0\})$ and F^2 is strongly convex in $\mathbb{R}^n \backslash \{0\}$. If the overdetermined boundary value problem (2.1) has a weak solution in a connected bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, then up to translation and scaling, $\partial\Omega$ is a Wulff shape.*

By a weak solution we mean that the solution $u \in W_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega F(\nabla u) F_\xi(\nabla u) \cdot \nabla v dx = \int_\Omega v dx \text{ for any } v \in W_0^{1,2}(\Omega), \tag{2.2}$$

together with the condition $F(\nabla u) = c$ on $\partial\Omega$. It was observed in [BFK03] that any weak solution of (2.2), $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$ (see also [To84]). Hence the conditon $F(\nabla u) = c$ on $\partial\Omega$ is well-defined. Note that u may not be in $C^2(\overline{\Omega})$.

As mentioned in the introduction, when $F(\xi) = |\xi|$, the Wulff shape is just the unit sphere and $Q = \Delta$. In this case Theorem 2.1 is just the classical result of Serrin [Se71]. Serrin's result was first proved by the method of moving planes, which is based on the maximum principle. It is clear that one could not use directly the method of moving planes to prove Theorem 2.1.

If $\partial\Omega$ has Wulff shape with respect to F, then there is an explicit function u satisfying (2.1).

Lemma 2.2. *Let $\Omega = \mathcal{W}_{nc}(x_0)$ and $u(x) = \frac{1}{2n}(n^2c^2 - (F^0(x-x_0))^2)$. Then u is a weak solution to (2.1).*

Proof. A direct calculation yields $\nabla u = -\frac{1}{n}F^0(x-x_0)\nabla F^0(x-x_0)$. Using Proposition 1.3, we have

$$F(\nabla u) = \frac{1}{n}F^0(x-x_0), F_\xi(\nabla u) = \frac{x-x_0}{F^0(x-x_0)} \text{ for } x \neq x_0. \tag{2.3}$$

Hence $F(\nabla u)F_\xi(\nabla u) = \frac{1}{n}(x-x_0)$ for all x (note that $\nabla u(x_0) = 0$) and $-Qu = 1$. In view of $F^0(x-x_0) = nc$ on $\partial\Omega$, we also have $u = 0$ and $F(\nabla u) = c$ on $\partial\Omega$. The proof is completed. □

Theorem 2.1 in fact gives a characterization of the Wulff shape by an overdetermined problem (2.1).

The proof of Theorem 2.1 goes along the line of Farina and Kawohl [FaK08]. There they proved that any level set of u has constant mean curvature. Then by Alexandrov's classical Theorem (See [Al58], [Ro72]) that the only compact connected hypersurface with constant mean curvature embedded in the Euclidean space is sphere, they conclude that the level sets must be spheres.

Our proof will involve the anisotropic mean curvature, which was defined in Chapter 1. By using a Pohozaev identity, a maximum principle on a so-called P-function and an interpretation of the anisotropic mean curvature of level sets by the operator Q, we show that the anisotropic mean curvature of any level set of u is constant. The generalized Alexandrov Theorem (Theorem 1.8) implies that every level set has Wulff shape.

2.1 P-function

Proposition 2.3. *Let u be a weak solution to the overdetermined boundary value problem (2.1). Then the P-function, which is defined as*

$$P(x) := \frac{1}{2}F^2(\nabla u(x)) + \frac{1}{n}u(x), \tag{2.4}$$

attains its maximum on $\partial\Omega$. Moreover, if P is not constant in Ω, then any maximum point for P in Ω is necessary to satisfy $\nabla u = 0$.

For simplicity, from now on we will follow the summation convention and frequently use the notations $F = F(\nabla u)$, $F_i = F_{\xi_i}(\nabla u)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and so on. Denote

$$
\begin{aligned}
a_{ij}(\nabla u)(x) &:= \frac{\partial^2}{\partial \xi_i \partial \xi_j}\left(\frac{1}{2}F^2\right)(\nabla u(x)) = (F_iF_j + FF_{ij})(\nabla u(x)), \\
a_{ijk}(\nabla u)(x) &:= \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u(x)).
\end{aligned}
\tag{2.5}
$$

In the following we shall write it simply by a_{ij} and a_{ijk} if no confusion appears. With these notations, we can rewrite the anisotropic Laplacian (1.9) as

$$
Qu = a_{ij}u_{ij} = \frac{\partial}{\partial x_i}(a_{ij}u_j). \tag{2.6}
$$

For the function $\frac{1}{2}F^2(\nabla u)$ we have a Bochner type formula.

Lemma 2.4 (Bochner Formula)**.** *At a point where $\nabla u \neq 0$, we have*

$$
\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\left(\frac{1}{2}F^2(\nabla u)\right)\right) = a_{ij}a_{kl}u_{ik}u_{jl} + \frac{\partial}{\partial x_k}(Qu)\frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u). \tag{2.7}
$$

Proof. The formula is derived from a direct computation.

$$
\begin{aligned}
a_{ij}(\nabla u)\left(\frac{1}{2}F^2(\nabla u)\right)_{ij} &= a_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u)u_{ik}\right) \\
&= a_{ij}\frac{\partial^2}{\partial \xi_k \partial \xi_l}\left(\frac{1}{2}F^2\right)(\nabla u)u_{ik}u_{jl} + a_{ij}\frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u)u_{ijk} \\
&= a_{ij}a_{kl}u_{ik}u_{jl} + \frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u)\left(\frac{\partial}{\partial x_k}(a_{ij}u_{ij}) - (\frac{\partial}{\partial x_k}a_{ij})u_{ij}\right).
\end{aligned}
$$

Taking into account of (2.6) and

$$
\frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)\frac{\partial}{\partial x_k}a_{ij} = a_{ijl}\frac{\partial}{\partial x_l}\left(\frac{1}{2}F^2(\nabla u)\right),
$$

we get (2.7).

□

When $F(\xi) = |\xi|$, (2.7) is just the usual Bochner formula

$$
\frac{1}{2}\Delta(|\nabla u|^2) = |D^2u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle.
$$

Now we prove Proposition 2.3.

Proof of Proposition 2.3: Set $\mathcal{C} = \{x \in \Omega | \nabla u(x) = 0\}$, we know from classical elliptic regularity theory that $u \in C^{2,\alpha}(\Omega \setminus \mathcal{C})$ and hence $P \in C^1(\Omega \setminus \mathcal{C})$. However, we can see below that $\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}P\right) \in C(\Omega \setminus \mathcal{C})$. The following calculations are all taken in $\Omega \setminus \mathcal{C}$.

Taking the first derivative of P , we have

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2(\nabla u)\right) = P_i - \frac{1}{n}u_i.$$

Hence it follows from the Bochner formula (2.7) that

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}P\right) &= a_{ij}a_{kl}u_{ik}u_{jl} + \frac{\partial}{\partial x_j}(Qu)\frac{\partial}{\partial \xi_k}\left(\frac{1}{2}F^2\right)(\nabla u) + \frac{1}{n}\frac{\partial}{\partial x_i}(a_{ij}u_j) \\ &= a_{ij}a_{kl}u_{ik}u_{jl} - \frac{1}{n}Qu. \end{aligned} \tag{2.8}$$

Since $u \in C^{2,\alpha}(\Omega \setminus \mathcal{C})$, the right hand side of (2.8) defines a continuous function in $\Omega \setminus \mathcal{C}$.

We estimate the term $a_{ij}a_{kl}u_{ik}u_{jl}$ by the following lemma.

Lemma 2.5.

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{1}{n}(Qu)^2.$$

Proof. Since the matrix $A := (a_{ij})_{i,j}$ is positive definite, we can write $A = O^T\Lambda O$ for some orthogonal matrix O and diagonal matrix $\Lambda = diag(\mu_1, \mu_2, \cdots, \mu_n)$ with $\mu_i \geq 0$ for any $i = 1, 2, \cdots, n$. Set $U = (u_{ij})_{i,j}$ and $\widetilde{U} = OUO^T = (\tilde{u}_{ij})_{i,j}$. Then we have

$$\begin{aligned} a_{ij}a_{kl}u_{lj}u_{ki} &= \mathrm{tr}(O^T\Lambda OUO^T\Lambda OU) = tr(\Lambda OUO^T\Lambda OUO^T) \\ &= \mathrm{tr}(\Lambda\widetilde{U}\Lambda\widetilde{U}) = \mu_i\mu_j\tilde{u}_{ij}^2 \\ &\geq \sum_{i=1}^n \mu_i^2\tilde{u}_{ii}^2 \geq \frac{1}{n}(\sum_{i=1}^n \mu_i\tilde{u}_{ii})^2. \end{aligned}$$

On the other hand,

$$Qu = a_{ij}u_{ij} = \mathrm{tr}(O^T\Lambda OU) = \mathrm{tr}(\Lambda\widetilde{U}) = \sum_{i=1}^n \mu_i\tilde{u}_{ii}.$$

Hence we get the desired inequality. □

Using Lemma 2.5 in (2.8) and $Qu = -1$, we have

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}P\right) + b_lP_l \geq 0 \text{ in } \Omega \setminus \mathcal{C}, \tag{2.9}$$

where $b_l := a_{ijl}(\nabla u)u_{ij}$.

Because of the uniform ellipticity of a_{ij} in $\Omega \setminus \mathcal{C}$, we can apply the maximum principle to (2.9) and conclude that P attains its maximum on $\partial\Omega$ or $\mathcal{C}$.

In order to exclude the possibility of maximality on $\mathcal{C}$, we proceed through a pertubation argument. Set $V(\xi) := \frac{1}{2}F^2(\xi) \in C^1(\mathbb{R}^n) \bigcap C^3(\mathbb{R}^n \setminus \{0\})$. Since V is 2-homogeneous, there exists $\lambda, \Lambda > 0$ such that

$$\lambda|\zeta|^2 \leq V_{\xi_i\xi_j}(\xi)\zeta_i\zeta_j \leq \Lambda|\zeta|^2 \text{ for any } \xi \neq 0, \zeta \in \mathbb{R}^n.$$

By a standard convolution argument, we can find a family of functions $\{V^\varepsilon\}$ in $C^\infty(\mathbb{R}^n)$ such that

$$V^\varepsilon \to V \text{ uniformly in any compact sets in } \mathbb{R}^n, V^\varepsilon \to V \text{ in } C^3_{loc}(\mathbb{R}^n \setminus \{0\}) \text{ as } \varepsilon \to 0,$$
$$\frac{\lambda}{2}|\zeta|^2 \leq V^\varepsilon_{\xi_i\xi_j}(\xi)\zeta_i\zeta_j \leq 2\Lambda|\zeta|^2 \text{ for any } \xi, \zeta \in \mathbb{R}^n. \tag{2.10}$$

From direct method in calculus of variations, there exists a unique minimizer $u^\varepsilon \in W^{1,2}_0(\Omega)$ of the functional $\int_\Omega V^\varepsilon(\nabla w) - w dx$, which is the weak solution to

$$a^\varepsilon_{ij}(\nabla w)w_{ij} = -1 \text{ in } \Omega, w|_{\partial\Omega} = 0,$$

where $a^\varepsilon_{ij}(\xi) := \frac{\partial^2 V^\varepsilon}{\partial\xi_i\xi_j}(\xi)$. The elliptic regularity theory tells us that $u^\varepsilon \in C^\infty(\overline{\Omega})$. Moreover, since the elliptic constants of a^ε_{ij} is independent of ε, we have u^ε is unfiormly bounded in $C^\infty(\overline{\Omega})$. It follows from uniqueness of u and the convergence (2.10) that

$$u^\varepsilon \to u \text{ in } C^1(\overline{\Omega}). \tag{2.11}$$

Define $g(x) = a_{ij}(\nabla u)P_{ij} + b_l P_l$ in $\Omega \setminus \mathcal{C}$ and $g(x) = 0$ in $\mathcal{C}$. Hence $g \in L^p(\Omega)$ for any $p \geq 1$ and $g \geq 0$ in Ω. Choose a sequence of continunous vector-valued functions $\{b^\varepsilon\}$ in $C^0(\overline{\Omega}, \mathbb{R}^n)$ such that

$$b^\varepsilon \to b \text{ uniformly in any compact sets in } \Omega \setminus \mathcal{C}. \tag{2.12}$$

Consider now the solution P^ε to

$$\begin{cases} a^\varepsilon_{ij}(\nabla u^\varepsilon)P^\varepsilon_{ij} + b^\varepsilon_i P^\varepsilon_i &= g(x) \geq 0 \quad \text{in} \quad \Omega \\ P^\varepsilon &= \frac{1}{2}c^2 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.13}$$

Thanks to the ellipticity of (2.13), we know from maximum principle that P^ε attains its maximum on $\partial\Omega$, that is,

$$\max_{\overline{\Omega}} P^\varepsilon(x) = \max_{\partial\Omega} P^\varepsilon(x) = \max_{\overline{\Omega\setminus U}} P^\varepsilon(x) \tag{2.14}$$

for any neighborhood U of $\mathcal{C}$. On the other hand, the convergences in (2.10–2.12) and L^p regularity lead to

$$P^\varepsilon \to P \text{ in } C^2_{loc}(\overline{\Omega} \setminus \mathcal{C}).$$

Therefore, by taking $\varepsilon \to 0$ in (2.14), we obtain

$$\max_{\Omega \setminus \mathcal{C}} P(x) = \max_{\partial\Omega} P(x).$$

Suppose there exists some point $x_0 \in \Omega$ such that $P(x_0) > \max_{\partial\Omega} P(x)$, then x_0 must belong to the interior of $\mathcal{C}$. However, the interior of $\mathcal{C}$ is empty. This follows directly from equation (2.1). In fact, an intergration on a ball $B \subset \mathcal{C}$ would give a contradiction via the divergence theorem. Hence P attains its maximum over $\overline{\Omega}$ on $\partial\Omega$. Moreover, if P is not constant in Ω, then any maximum point for P in Ω belongs necessarily to $\mathcal{C}$. The proof is completed. □

2.2 Pohozaev identity

For convenience of later use, we prove here a general Pohozaev identity.

Theorem 2.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let F be a norm of class $C^2(\mathbb{R}^n \backslash \{0\})$ and F^2 is stongly convex in $\mathbb{R}^n \backslash \{0\}$. Let f be a continuous function on Ω. Assume that $u \in C^1(\overline{\Omega})$ is a weak solution to*

$$\begin{cases} -Qu = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Then the following identity

$$\frac{1}{2}\int_{\partial\Omega} F^2(\nabla u)\langle x, \nu\rangle d\mathcal{H}^{n-1} + \frac{n-2}{2}\int_\Omega F^2(\nabla u)dx = \int_\Omega \langle x, \nu\rangle f(x)dx \tag{2.16}$$

holds

Proof. The original proof required that $u \in C^2(\overline{\Omega})$, which is not available here. We use an approximation argument.

Denote $V(\xi) := \frac{1}{2}F^2(\xi)$. There exists a family of smooth convex functions $\{V^\varepsilon\}$ such that $V^\varepsilon \to V$ in $C^1_{loc}(\mathbb{R}^n)$. Also we can find $f^\varepsilon \in C^\infty(\Omega)$ such that $f^\varepsilon \to f$ uniformly. Let u^ε be the unique minimizer of $\inf_\Omega V^\varepsilon(\nabla u) - f^\varepsilon u dx$ in $W^{1,2}_0(\Omega)$. It then solves

$$\begin{cases} -\text{div}(\nabla_\xi V^\varepsilon(\nabla u^\varepsilon)) = f^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

It follows from standard elliptic regularity theory that $u^\varepsilon \in C^\infty(\overline{\Omega})$. Moreover, since the elliptic constants can be chosen to be is independent of ε, we have u^ε is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$. It follows from uniqueness of u and the convergence (2.10) that $u^\varepsilon \to u$ in $C^1(\overline{\Omega})$. Multiplying $\langle x, \nabla u^\varepsilon\rangle$ to the both sides of (2.17) and intergrating by parts twice, we have

$$\begin{aligned}
&\int_\Omega \langle x, \nabla u\rangle f^\varepsilon(x) = \int_\Omega -\langle x, \nabla u^\varepsilon\rangle \mathrm{div}(\nabla_\xi V^\varepsilon(\nabla u^\varepsilon)) \\
= & \int_{\partial\Omega} \langle x, \nabla u\rangle \langle \nabla_\xi V^\varepsilon(\nabla u^\varepsilon), \nu\rangle - \int_\Omega \langle \nabla_\xi V^\varepsilon(\nabla u^\varepsilon), \nabla u^\varepsilon\rangle - \int_\Omega \nabla_\xi V^\varepsilon(\nabla u^\varepsilon) u^\varepsilon_{ij} x_j\rangle \\
= & \int_{\partial\Omega} \langle x, \nabla u\rangle \langle \nabla_\xi V^\varepsilon(\nabla u^\varepsilon), \nu\rangle - \int_\Omega \langle \nabla_\xi V^\varepsilon(\nabla u^\varepsilon), \nabla u^\varepsilon\rangle \\
& - \int_{\partial\Omega} V^\varepsilon(\nabla u^\varepsilon)\langle x, \nu\rangle + \int_\Omega n V^\varepsilon(\nabla u^\varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\begin{aligned}
\int_\Omega \langle x, \nabla u\rangle f(x) \;=\; & \int_{\partial\Omega} \langle x, \nabla u\rangle \langle \nabla_\xi V(\nabla u^\varepsilon), \nu\rangle - \int_\Omega \langle \nabla_\xi V(\nabla u), \nabla u\rangle \\
& - \int_{\partial\Omega} V(\nabla u)\langle x, \nu\rangle + \int_\Omega n V(\nabla u).
\end{aligned}$$

Since $u = 0$ on $\partial\Omega$, we have $\nabla u = u_\nu \nu$. Taking into account of $\langle \nabla_\xi V(\nabla u^\varepsilon), \nu\rangle = F^2(\nabla u)$, we conclude

$$\int_\Omega \langle x, \nabla u\rangle f(x) = \frac{1}{2}\int_{\partial\Omega} F^2(\nabla u)\langle x, \nu\rangle + \frac{n-2}{2}\int_\Omega F^2(\nabla u).$$

□

From Theorem 2.6, in our case we obtain

Proposition 2.7. *Let u be a weak solution to problem (2.1). $P(x)$ is defined as (2.4). Then the following identity*

$$\int_\Omega P(x)dx = \frac{1}{2}c^2|\Omega| \tag{2.18}$$

holds, where $|\Omega|$ is the n-dimensional volume of Ω.

Proof. We see from the general Pohozaev identity (2.15) that

$$\frac{1}{2}\int_{\partial\Omega} F^2(\nabla u)\langle x, \nu\rangle d\mathcal{H}^{n-1} + \frac{n-2}{2}\int_\Omega F^2(\nabla u)dx = n\int_\Omega u dx. \tag{2.19}$$

Since $F^2(\nabla u)|_{\partial\Omega} = c$, using integration by parts, we have

$$\int_{\partial\Omega} F^2(\nabla u)\langle x, \nu\rangle d\mathcal{H}^{n-1} = c^2\int_{\partial\Omega}\langle x, \nu\rangle d\mathcal{H}^{n-1} = nc^2|\Omega|. \tag{2.20}$$

Another integration by parts yields that

$$\int_\Omega F^2(\nabla u)dx = \int_\Omega u dx.$$

Therefore,

$$\int_\Omega P(x)dx = \int_\Omega \frac{1}{2}F^2(\nabla u) + \frac{1}{n}udx = \int_\Omega u - \frac{n-2}{2n}F^2(\nabla u)dx. \tag{2.21}$$

Combining (2.19–2.21), we get the disired equality. □

2.3 Proof of Theorem 2.1

From Proposition 2.3 and 2.7, we immediately obtain the following

Corollary 2.8. *Let u be a weak solution to the overdetermined boundary value problem (2.1). Then*

$$F^2(\nabla u) + \frac{2}{n}u \equiv c^2 \quad in \quad \Omega. \tag{2.22}$$

Now we are ready to prove our main theorem, Theorem 2.1.

Proof of Theorem 2.1. By Corollary 2.8, we claim that

∇u *vanishes only at points where u attains its maximum in Ω and the maximum in this case must be $\frac{n}{2}c^2$.*

Indeed, if $\nabla u(x_0) = 0$, then $F(\nabla u(x_0)) = 0$, by (2.22), $u(x_0) = \frac{n}{2}c^2$. On the other hand, $u(x) = \frac{n}{2}(c^2 - F^2(\nabla u(x)) \leq \frac{n}{2}c^2$ in Ω, so $u(x_0) = \max_\Omega u$. From this claim we know that u is positive in Ω. Otherwise there is a $x_0 \in \Omega$ with $u(x_0) = \inf_{x\in\overline{\Omega}} u(x) \leq 0 < \frac{n}{2}c^2$. Hence from this claim $\nabla u(x_0)$ does not vanish, a contradiction. Again from this claim, we easily see that $\nu = \frac{\nabla u}{|\nabla u|}$ is well defined on the open set $U := \{x \in \Omega | 0 < u(x) < \max_\Omega u\}$. We define on U

$$\nu_F = F_\xi(\nu) = F_\xi(\nabla u).$$

Note that

$$\frac{\partial u}{\partial \nu_F} = F(\nabla u) = \sqrt{c^2 - \frac{2}{n}u} := g(u), \tag{2.23}$$

$$\frac{\partial^2 u}{\partial \nu_F^2} = F_i(\nabla u)\nu_F^i = F_i(\nabla u)F_j(\nabla u)u_{ij}. \tag{2.24}$$

From (2.23), we also have

$$2\frac{\partial u}{\partial \nu_F}\frac{\partial^2 u}{\partial \nu_F^2} = \frac{\partial}{\partial \nu_F}(\frac{\partial u}{\partial \nu_F})^2 = \frac{\partial}{\partial \nu_F}(g(u)^2) = 2g(u)g'(u)\frac{\partial u}{\partial \nu_F},$$

which leads to

$$g(u)g'(u) = \frac{\partial^2 u}{\partial \nu_F^2} = F_i(\nabla u)F_j(\nabla u)u_{ij},$$

for $\frac{\partial u}{\partial \nu_F} = F(\nabla u) \neq 0$ on U.

Using the formula in Theorem 1.11 and the first equation of (2.1), we obtain the F-mean curvature of the level set S_t $(0 < t < \max_\Omega u)$:

$$\begin{aligned} H_F(S_t) &= \frac{1}{F(\nabla u)}(Qu(x) - \frac{\partial^2 u}{\partial \nu_F^2}) \\ &= \frac{1}{g(u)}(-1 - g(u)g'(u)). \end{aligned}$$

The above equality just means that every level set of u at height t between 0 and $\max u$ is a (perhaps not connected) hypersurface of constant F-mean curvature. By Theorem 1.8, each connected component of it must be of Wulff shape, up to translation and rescaling. Namely each connected component is a translation of $\mathcal{W}_r$ with the same r determined by t.

We claim that Ω is simply connected. Otherwise, $\partial\Omega = S_0$ contains two connected components. Thus for small $\delta > 0$, S_δ contains two connected components. However, each connected component of S_δ is a translation of $\mathcal{W}_r$, with the same r. This is impossible. Therefore Ω is simply connected and $\partial\Omega = S_0$ is also a Wulff shape, up to translation and rescaling. This finishes the proof of Theorem 2.1. □

Chapter 3

Anisotropic harmonic functions

In this chapter, we focus on the anisotropic harmonic functions, which are solutions of

$$-Qu = 0. \tag{3.1}$$

Throughout this chapter we shall assume that $F \in C^2(\mathbb{R}^n\backslash\{0\})$ and F^2 is strongly convex.

Many aspects of anisotropic harmonic functions are similar to harmonic functions in $\mathbb{R}^n$. At the same time, it lacks some beautiful properties of harmonic functions due to the nonlinearty and degeneracy of Q. We give some for example.

(1) As a special case of general degenerate elliptic equations of divengence type (see [HKM93]), the Liouville theorem holds, which states that a positive anisotropic harmonic function must be a constant. We also have the Harnack inequality, which states that the supremum of an anisotropic harmonic function can be estimated by its infimum on any compact sets. On the other hand, an anisotropic harmonic function can only be a $C^{1,\alpha}$ function but do not neccessarily belongs to C^2, and a harmonic function is analytic. Also the mean value property can only hold for anisotropic harmonic functions under a very restrictive assumption (See [FeK09]).

(2) Because of the special structure of anisotropic Laplacian, a fundamental solution can be constructed similarly as Laplacian, by using the dual norm F^0. Moreover, we shall prove that the Green's function of the anisotropic Laplacian exists and appears a significant decomposition. However, we have no Green's representation formula due to the nonlinearty. In spite of this disadvantage, the Green's function is powerful when we do blow-up analysis in the following chapter.

The major subject in this chapter is to study the Green's function of the anisotropic Laplacian.

3.1 Green's function of anisotropic Laplacian

The fundamental solution $\Gamma(x)$ for the operator Q is defined as follows:

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2\kappa_2}\log(F^0(x)), & \text{for } n = 2, \\ \dfrac{1}{n(2-n)\kappa_n} F^0(x)^{2-n}, & \text{for } n > 2. \end{cases}$$

Recall that κ_n is the n-dimensional Lebesgue measure of the Wulff shape $\mathcal{W}_F$.

Theorem 3.1 ([FeK09]). *The function Γ satisfies*

$$-Q\Gamma = \delta_0,$$

in the sense of measures, where δ_0 denotes the Dirac measure at the origin.

Proof. For simplicity, we prove only the case $n = 2$. A similar computation as Lemma 2.2 leads to $Q\Gamma(x) = 0$ for $x \neq 0$. Since

$$\int_{\mathcal{W}_\varepsilon(0)} \langle FF_\xi(\nabla\Gamma), \nabla\varphi(x)\rangle dx = \int_{\mathcal{W}_\varepsilon(0)} \langle \frac{x}{(F^0(x))^2}, \nabla\varphi(x)\rangle dx = O(\varepsilon),$$

we see that

$$\begin{aligned} \int_{\mathbb{R}^n} -Q\Gamma\varphi dx &= \int_{\mathbb{R}^n} \langle FF_\xi(\nabla\Gamma), \nabla\varphi\rangle dx \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus\mathcal{W}_\varepsilon(0)} \langle FF_\xi(\nabla\Gamma), \nabla\varphi\rangle dx \\ &= -\lim_{\varepsilon\to 0} \int_{\partial\mathcal{W}_\varepsilon(0)} \langle F(\nabla\Gamma)F_\xi(\nabla\Gamma), \nu\rangle \varphi d\mathcal{H}^1. \end{aligned}$$

In the last equality we used $Qu(x) = 0$ for $x \neq 0$. Hence to prove $-Q\Gamma = \delta_0$, it is sufficient to show that

$$-\lim_{\varepsilon\to 0} \int_{\partial\mathcal{W}_\varepsilon} \langle F(\nabla\Gamma)F_\xi(\nabla\Gamma), \nu\rangle \varphi d\mathcal{H}^1 = \varphi(0) \text{ for any } \varphi \in C_0^\infty(\mathbb{R}^n).$$

In fact,

$$\begin{aligned} & -\int_{\partial\mathcal{W}_\varepsilon} \langle F(\nabla\Gamma)F_\xi(\nabla\Gamma), \nu\rangle \varphi d\mathcal{H}^1 \\ = & -\int_{\partial\mathcal{W}_\varepsilon} \langle \frac{x}{(F^0(x))^2}, \frac{\nabla F^0}{|\nabla F^0|}\rangle \varphi d\mathcal{H}^1 \\ = & \int_{\partial\mathcal{W}_\varepsilon} \frac{1}{\varepsilon}\frac{1}{|\nabla F^0|}\varphi d\mathcal{H}^1 = -\int_{\partial\mathcal{W}_1} \frac{1}{2\kappa_2}\frac{1}{|\nabla F^0|(x)}\varphi(\epsilon x) dx. \end{aligned}$$

Letting $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} - \int_{\partial \mathcal{W}_\epsilon} F(\nabla u)\langle F_\xi(\nabla u), \nu\rangle \varphi = \frac{1}{2\kappa_2}\varphi(0)\int_{\partial \mathcal{W}_1} \frac{1}{|\nabla F^0|(x)} dx = \varphi(0).$$

Here we used $\int_{\partial \mathcal{W}_1} \frac{1}{|\nabla F^0|(x)} dx = 2\kappa_2$ by integration by parts.

□

Theorem 3.1 inspires us to find the Green's function for Q.

Assume that Ω is an open set in $\mathbb{R}^n$, containing 0, $\Omega^* = \Omega \setminus \{0\}$. By a result of Serrin (See [Se71]), if u satisfies $-Qu = 0$ and is bounded below in Ω^*, then either the singularity at 0 is removable, or u/Γ is bounded in some neighborhood of 0. (In fact, Serrin proved that $u/\log|x|$ is bounded in some neighborhood of 0. However, in view of (1.1), it is equivalent to say u/Γ is bounded in some neighborhood of 0 for our Γ.)

Our first purpose is to describe the behavior of u near the origin when it is not removable. We shall write $\Gamma(r) = \Gamma(x)$ whenever $F^0(x) = r$.

Theorem 3.2. *Assume u satisfies 3.1 in Ω^* such that $u(x)/\Gamma(x)$ remains bounded in some neighborhood of 0. Then there exists a real number γ and $g \in C^0(\Omega)$ such that*

$$u = \gamma\Gamma + g. \tag{3.2}$$

Moreover, when $\gamma \neq 0$, the following relation holds

$$\lim_{x\to 0}(F^0(x))^{n-1}\nabla g(x) = 0 \tag{3.3}$$

and u satisfies

$$-Qu = \gamma\delta_0 \tag{3.4}$$

in the sense of measures in Ω.

Before we prove this theorem, we state a strong comparison theorem for anisotropic Laplacian (The original statement is for more general degenerate elliptic equations, our anisotropic Laplacian is a special case).

Theorem 3.3 ([To83], Prop. 3.3.2, [Da98], Th. 1.4)**.** *Let $u_1, u_2 \in C^1(\Omega)$ satisfy $-Qu = 0$ in Ω and $u_1 \geq u_2$. If u_1 is not equal to u_2, then $u_1 > u_2$ in Ω.*

Proof of Theorem 3.2. For simplicity, we prove only the case $n = 2$.

Without loss of generality, we may assume $\overline{\mathcal{W}_1} \subset \Omega$. Due to translating invariance of the equation, we may assume $\max_{\partial \mathcal{W}_{\frac{1}{2}}} u = 0$. Let

$$\gamma = \limsup_{x\to 0} u(x)/\Gamma(x), \quad \tilde{\gamma}(r) = \max_{r \leq F^0(x) \leq \frac{1}{2}} u(x)/\Gamma(x).$$

We consider the case $\gamma > 0$. (otherwise we use $\liminf_{x\to 0} u(x)/\Gamma(x)$ instead) From the strong comparison principle, Theorem 3.3, $\tilde{\gamma}(r)$ is nonincreasing, and there exists x_r with $F^0(x_r) = r$ such that

$$\tilde{\gamma}(r) = \max_{\partial \mathcal{W}_r} u(x)/\Gamma(x) = u(x_r)/\Gamma(x_r).$$

It's also clear that $\tilde{\gamma}(\frac{1}{2}) = 0$ and $\lim_{r\to 0} \tilde{\gamma}(r) = \gamma$. We introduce for $0 < r \leq \frac{1}{2}$ a function in $\mathcal{W}_{1/(2r)} \setminus \{0\}$

$$v_r(x) = u(rx)/\Gamma(r).$$

It's clear that v_r satisfies (3.1) in $\mathcal{W}_{1/(2r)} \setminus \{0\}$. The boundedness of u/Γ in a neighborhood of 0 gives

$$|v_r(x)| \leq C\left(1 + \frac{|\log F^0(x)|}{\log(1/r)}\right) \tag{3.5}$$

for $x \in \mathcal{W}_{1/(2r)} \setminus \{0\}$. Moreover, from the scale invariance of (3.1) and $C^{1,\alpha}$ estimates for quasilinear equations, we have the following a priori estimates: for any $R > 0$ and $0 < |x| < |y| < R$,

$$|\nabla v_r(x)| \leq C|x|^{-1}|v_r|_{L^\infty(B_{2R}\setminus B_{|x|/2})},$$

$$\frac{|\nabla v_r(x) - \nabla v_r(y)|}{|x-y|^\alpha} \leq C|x|^{-1-\alpha}|v_r|_{L^\infty(B_{2R}\setminus B_{|x|/2})}.$$

Hence for any compact set $K \subset\subset \mathbb{R}^2 \setminus \{0\}$ and some C_K independent of r, we have

$$\|v_r\|_{C^{1,\alpha}}(K) \leq C_K.$$

By Ascoli-Arzela's Theorem, we can find a sequence $r_j \to 0$ such that $v_{r_j} \to v$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$, where $v \in C^1(\mathbb{R}^2 \setminus \{0\})$ also satisfies (3.1). In view of (3.5), v is bounded. From Serrin's result (See [Se71]), 0 is a removable singularity and v can be extended to $\tilde{v} \in C^1(\mathbb{R}^2)$. Consequently, from Liouville Theorem (See [HKM93]), v must be a constant. For the sequence $\xi_j = x_{r_j}/r_j, F^0(\xi_j) = 1$, we know from the definition of γ that

$$v_{r_j}(\xi_j) \to \gamma.$$

This means the constant function $v = \gamma$. Therefore,

$$\lim_{r\to 0} v_r(x) = \gamma \text{ and hence } \lim_{x\to 0} u(x)/\Gamma(x) = \gamma.$$

We now consider two sequence of functions

$$V_\epsilon^+(x) = (\gamma+\epsilon)\Gamma(x) - (\gamma+\epsilon)\Gamma(\frac{1}{2}) + \max_{\partial\mathcal{W}_{\frac{1}{2}}} u,$$

$$V_\epsilon^-(x) = (\gamma - \epsilon)\Gamma(x) - (\gamma - \epsilon)\Gamma(\frac{1}{2}) + \min_{\partial \mathcal{W}_{\frac{1}{2}}} u.$$

They both satisfies (3.1) in $\mathcal{W}_{1/2} \setminus \{0\}$ and from the comparison principle we obtain $V_\epsilon^- \leq u \leq V_\epsilon^+$, which implies the boundedness of $u - \gamma\Gamma$ when $\epsilon \to 0$.

Next we prove the continuity of $u - \gamma\Gamma$ at 0 and (3.3). We look at the points where the bounded function $u - \gamma\Gamma$ achieves its supremum in $\overline{\mathcal{W}_{1/2}}$. Set $\lambda = \sup_{\overline{\mathcal{W}_{1/2}}}(u - \gamma\Gamma)$.

Case (i). λ achieves at some point in $\mathcal{W}_{1/2} \setminus \{0\}$. It follows from comparison principle that $u - \gamma\Gamma$ is a constant, hence we are done.

Case (ii). λ achieves at 0. Define

$$\lambda(r) = \max_{r \leq F^0(x) \leq 1/2}(u - \gamma\Gamma) = \max_{\partial \mathcal{W}_r}(u - \gamma\Gamma).$$

Then $\lambda(r) \uparrow \lambda$ as $r \downarrow 0$, and there exists x_r with $|x_r| = r$ such that $\lambda(r) = u(x_r) - \gamma\Gamma(x_r)$. We introduce for $0 < r \leq \frac{1}{2}$ the function

$$w_r(x) = u(rx) - \gamma\Gamma(r)$$

in $\mathcal{W}_{1/(2r)} \setminus \{0\}$. The function w_r satisfies (3.1). We also have $|w_r - \gamma\Gamma| \leq C_0$ for $C_0 = \sup_{\mathcal{W}_{1/2}\setminus\{0\}} |u - \gamma\Gamma|$. This implies that w_r is bounded on any compact subset of $\mathcal{W}_{1/(2r)} \setminus \{0\}$. Similarly as v_r, we have for any compact set $K \subset\subset \mathbb{R}^2 \setminus \{0\}$ and some C_K independent of r,

$$\|w_r\|_{C^{1,\alpha}}(K) \leq C_K.$$

Consequently, there exists a sequence $r_j \to 0$ such that $w_{r_j} \to w$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$, where $w \in C^1(\mathbb{R}^2 \setminus \{0\})$ also satisfies (3.1). For the sequence $\xi_j = x_{r_j}/r_j$, $F^0(\xi_j) = 1$, which may be assumed to converge to $\xi^0 \in \partial\mathcal{W}_1$, we have

$$w_{r_j}(\xi_j) - \gamma\Gamma(\xi_j) = u(x_{r_j}) - \gamma\Gamma(x_{r_j}) \to \lambda.$$

Hence

$$w(x) \leq \gamma\Gamma(x) + \lambda \text{ and } w(\xi^0) = \gamma\Gamma(\xi^0) + \lambda.$$

By comparison principle, $w(x) = \gamma\Gamma(x) + \lambda$ and hence $w_r \to \gamma\Gamma + \lambda$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$. This implies

$$\lim_{x \to 0}(u - \gamma\Gamma) = \lambda, \quad \lim_{x \to 0} F^0(x)\nabla u(x) = \gamma\nabla\Gamma(\frac{x}{F^0(x)}). \tag{3.6}$$

The above equalities lead to the continuity of $u - \gamma\Gamma$ and (3.3).

Case (iii). λ achieves on $\partial\mathcal{W}_{1/2}$. We define w_r as in case 2, $w_r \to w$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$ and $|w - \gamma\Gamma| \le C_0$. We now look at the points where $w - \gamma\Gamma$ achieves its supremum in $\mathbb{R}^2$. Set $\tilde{\lambda} = \sup_{\mathbb{R}^2}(u - \gamma\Gamma)$.

If $\tilde{\lambda}$ is achieved at some point in $\mathbb{R}^2 \setminus \{0\}$, then $w - \gamma\Gamma$ equals to some constant by strong maximum principle, which implies $u(rx) - \gamma\Gamma(rx) \to \tilde{\lambda}$ in $C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$ as $r \to 0$. For any fixed $\epsilon > 0$, there exists n_0 such that for $n \ge n_0$ and $x \in \partial\mathcal{W}_1$, we have

$$\gamma\Gamma(r_n x) + \tilde{\lambda} - \epsilon \le u(r_n x) \le \gamma\Gamma(r_{n_0} x) + \tilde{\lambda} + \epsilon.$$

Applying maximum principle in $\mathcal{W}_{r_{n_0}} \setminus \mathcal{W}_{r_n}$ we obtain

$$\gamma\Gamma(x) + \tilde{\lambda} - \epsilon \le u(r_n x) \le \gamma\Gamma(x) + \tilde{\lambda} + \epsilon,$$

which leads to (3.6) with λ replaced by $\tilde{\lambda}$.

If $\tilde{\lambda}$ is achieved at 0, we simply argue as case 2 with w instead of u to deduce

$$\lim_{x\to 0}(w - \gamma\Gamma) = \tilde{\lambda} \text{ and hence } \lim_{x\to 0}\lim_{r_n\to 0}(u(r_n x) - \gamma\Gamma(r_n x)) = \tilde{\lambda}. \tag{3.7}$$

If $\tilde{\lambda}$ is achieved at ∞, the same idea in case 2 can be applied when we define $\lambda(R) = \max_{1/2 \le F^0(x) \le R}(w - \gamma\Gamma) = \max_{\partial\mathcal{W}_R}(w - \gamma\Gamma)$ and let R tend to ∞. We obtain

$$\lim_{x\to\infty}(w - \gamma\Gamma) = \tilde{\lambda}, \lim_{x\to\infty}\lim_{r_n\to 0}(u(r_n x) - \gamma\Gamma(r_n x)) = \tilde{\lambda}. \tag{3.8}$$

As long as we have (3.7) or (3.8), we can use maximum principle again to conclude (3.6) as before.

Now it remains to prove (3.4). In view of (3.1), it is sufficient to show that

$$\lim_{\epsilon\to 0} -\int_{\partial\mathcal{W}_\epsilon} F(\nabla u)\langle F_\xi(\nabla u), \nu\rangle\phi = \gamma\phi(0)$$

for any $\phi \in C^1_0(\Omega)$. Here $\nu = \frac{\nabla F^0}{|\nabla F^0|}$ is the unit outward normal. Using (3.2), (3.3) and Proposition 1.2, We have on $\partial\mathcal{W}_\epsilon$,

$$F(\nabla u) = F(\gamma\nabla\Gamma + \nabla g) = F(-\frac{\gamma}{2\kappa}\frac{\nabla F^0}{F^0} + o(\frac{1}{F^0})) = \frac{\gamma}{2\kappa\epsilon} + o(\frac{1}{\epsilon}),$$

$$\begin{aligned}
\langle F_\xi(\nabla u), \nu\rangle &= \langle F_\xi(\nabla u), \frac{\nabla F^0}{|\nabla F^0|}\rangle \\
&= \left\langle F_\xi(\nabla u), (-\frac{2\kappa}{\gamma}F^0)\frac{\nabla u - o(\frac{1}{F^0})}{|\nabla F^0|}\right\rangle \\
&= -\frac{2\kappa\epsilon}{\gamma}\left(\frac{F(\nabla u)}{|\nabla F^0|} - \frac{o(\frac{1}{\epsilon})}{|\nabla F^0|}\right) = -(1 + o(1))\frac{1}{|\nabla F^0|},
\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\partial\mathcal{W}_\epsilon} F(\nabla u)\langle F_\xi(\nabla u),\nu\rangle\phi &= -\int_{\partial\mathcal{W}_\epsilon}(\frac{\gamma}{2\kappa\epsilon}+o(\frac{1}{\epsilon}))\,(1+o(1))\,\frac{1}{|\nabla F^0|}\phi\\ &= -\int_{\partial\mathcal{W}_1}(\frac{\gamma}{2\kappa}+o(1))\frac{1}{|\nabla F^0|(x)}\phi(\epsilon x)dx.\end{aligned}$$

Letting $\epsilon\to 0$, we have

$$\lim_{\epsilon\to 0}-\int_{\partial\mathcal{W}_\epsilon} F(\nabla u)\langle F_\xi(\nabla u),\nu\rangle\phi=\frac{\gamma}{2\kappa}\phi(0)\int_{\partial\mathcal{W}_1}\frac{1}{|\nabla F^0|(x)}dx=\gamma\phi(0).$$

Here we used $\int_{\partial\mathcal{W}_1}\frac{1}{|\nabla F^0|(x)}dx=2\kappa$ by integration by parts. We complete the proof of Theorem 3.2. □

As a consequence of Theorem 3.2, the singular Dirichlet problem can be uniquely solved.

Theorem 3.4. *There exists a unique function $G(\cdot,0)\in C^{1,\alpha}(\Omega^*)$ with $|\nabla G|\in L^1(\Omega)$ and $G/\Gamma\in L^\infty(\Omega)$, satisfying*

$$\begin{cases} -QG(\cdot,0) &= \delta_0 \quad \text{in } \Omega\\ \quad G(\cdot,0) &= \phi \quad \text{on } \partial\Omega,\end{cases}\tag{3.9}$$

where $\phi\in L^\infty\bigcap W^{1,2}(\Omega)$. Moreover, $G=\Gamma+g$ with $g\in C^0(\Omega)$ satisfying (3.3).

Proof. First, we prove the uniqueness. Suppose $u_i, i=1,2$ are two solutions of (3.9). By virtue of (3.2) and (3.3), we know

$$u_1-u_2\in L^\infty(\Omega),\lim_{x\to 0}F^0(x)\nabla(u_1-u_2)=0.\tag{3.10}$$

By integration by parts, we have for r small,

$$\begin{aligned}&\int_{\Omega\setminus\mathcal{W}_r}(F(\nabla u_1)F_\xi(\nabla u_1)-F(\nabla u_2)F_\xi(\nabla u_2))\nabla(u_1-u_2)\\ =&-\int_{\partial\mathcal{W}_r}\langle F(\nabla u_1)F_\xi(\nabla u_1)-F(\nabla u_2)F_\xi(\nabla u_2),\nu\rangle(u_1-u_2).\end{aligned}$$

Using (3.10), we deduce that the RHS tends to 0 as $r\to 0$. On the other hand, it follows from Lemma 5.6 in Chapter 5 that the LHS is larger than $C\int_{\Omega\setminus\mathcal{W}_r}F^2(\nabla u_1-\nabla u_2)dx$. Hence $\nabla(u_1-u_2)=0$. Combining with the boundary condition, we conclude $u_1=u_2$.

For the existence, we consider the solutions u_ϵ to the following problem:

$$\begin{cases} -Qu_\epsilon &= 0 \quad \text{in } \Omega\setminus\mathcal{W}_\epsilon\\ u_\epsilon &= \Gamma(\epsilon) \quad \text{on } \partial\mathcal{W}_\epsilon\\ u_\epsilon &= \phi \quad \text{on } \partial\Omega.\end{cases}$$

By a weak comparison principle, we obtain $|u_\epsilon - \Gamma| \leq C_1$, where $C_1 = \sup_{\partial\Omega} \Gamma$. Using the $C^{1,\alpha}$ estimates and Ascoli-Arzela's Theorem, we can extract a subsequence u_{ϵ_n}, which converges to a $u \in C^1(\Omega^*)$ as $\epsilon_n \to 0$ in C^1_{loc} topology. Clearly, u/Γ is bounded in a neighborhood of 0. Therefore, from Theorem 3.2, we conclude that u satisfies (3.9).

□

Chapter 4

First eigenvalue of anisotropic Laplacian

In this chapter, we investigate the eigenvalue problem of the anisotropic Laplacian. We emphysis that throughout this chapter we only assume that F is a norm of class $C^1(\mathbb{R}^n\backslash\{0\})$.

Though our assumption on F is only convex and $C^1(\mathbb{R}^n\backslash\{0\})$, we may carry out the proof in this chapter under more regularity assumption that $F \in C^3(\mathbb{R}^n \setminus \{0\})$ and F is a strongly convex norm on $\mathbb{R}^n$ without of loss generality. In fact, for any norm $F \in C^1(\mathbb{R}^n \setminus \{0\})$, there exists a sequence $F_\varepsilon \in C^3(\mathbb{R}^n \setminus \{0\})$ such that the strongly convex norm $\widetilde{F}_\varepsilon := \sqrt{F_\varepsilon^2 + \varepsilon|x|^2}$ converges to F uniformly in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$, then the corresponding first eigenvalue $(\lambda_1)_\varepsilon$ of anisotropic Laplacian with respect to $\widetilde{F}_\varepsilon$, converges to λ_1 as well. Here $|\cdot|$ denotes the Euclidean norm.

Therefore, in the following sections, we assume that $F \in C^3(\mathbb{R}^n \setminus \{0\})$ and F is a strongly convex norm on $\mathbb{R}^n$. Thus (0.3) is degenerate elliptic among Ω and uniformly elliptic in $\Omega \setminus \mathcal{C}$, where $\mathcal{C} := \{x \in \Omega | \nabla u(x) = 0\}$ denotes the set of degenerate points. The standard regularity theory for degenerate elliptic equation (see e.g. [BFK03, To84]) implies that $u \in C^{1,\alpha}(\Omega) \bigcap C^{2,\alpha}(\Omega \setminus \mathcal{C})$.

4.1 Brunn-Minkowski inequality

In this section we establish a Brunn-Minkowski type inequality for the first Dirichlet eigenvalue of the anisotropic Laplacian. We have the following main theorem:

Theorem 4.1. *Let K_i be two convex bounded open sets (convex body) in $\mathbb{R}^n$, $i = 0, 1$. For $t \in [0, 1]$, we set $K_t = (1-t)K_0 + tK_1$. Let $\lambda(K_t)$ be the first Dirichlet eigenvalue of the anisotropic Laplacian on K_t. Then $\lambda(K_t)$ is homogeneous of degree -2 and the following Brunn-Minkowski inequality holds:*

$$\lambda^{-\frac{1}{2}}(K_t) \geq (1-t)\lambda^{-\frac{1}{2}}(K_0) + t\lambda^{-\frac{1}{2}}(K_1). \tag{4.1}$$

It's easy to check that $\lambda_1(K)$ is homogeneous of degree -2. Indeed, if $\lambda_1(K)$ attains its infimum at $u(x)$, i.e. u satisfies $\int_K |u|^2 dx = 1$ and $\lambda_1(K) = \int_K F(\nabla u)^2 dx$. For $y = tx \in tK$, set $v(x) = t^{-n/2}u(x)$, then $\int_{tK} |v(y)|^2 dy = 1$ and $\lambda_1(tK) \leq \int_{tK} F(\nabla v(y))^2 dy = t^{-2} \int_K F(\nabla u)^2 dx = t^{-2}\lambda_1(K)$; on the other hand $\lambda_1(K) \leq t^2\lambda_1(tK)$ can be obtained similarly.

In order to prove the Brunn-Minkowski inequality, we first recall some elementary concept in convex analysis, for details we refer to [Ro72].

Definition 4.2. *For any convex set K and any convex function f defined on K, the function f^*, which is defined by*

$$f^*(x^*) = \sup_{x \in K}\{\langle x, x^*\rangle - f(x)\}$$

for $x^* \in \mathbb{R}^n$, is called the conjugate function of f.

To be not confused with the notation of the decreasing rearrangement in Chapter 1, we remark that in this section f^* always denote the conjugate function.

Definition 4.3. *A convex function f is called essentially smooth on K, if f is differentiable in K and*

$$\lim_{i\to\infty} |\nabla f(x_i)| = +\infty,$$

when x_i tends to some point on ∂K.

Definition 4.4. *For two convex functions f_0 and f_1, whose definition domains are K_0 and K_1 respectively, we called the function $\tilde{f}$, which is defined by*

$$\tilde{f}(z) = \inf\{(1-t)f_0(x) + tf_1(y) : x \in K_0, y \in K_1, z = (1-t)x + ty\}$$

for $z \in K_t$, the infimal convolution of f_0 and f_1.

We recall some properties of the conjugate function and the infimal convolution.

Proposition 4.5 ([Ro72])**.** *Assume f, f_0, f_1 are convex functions defined on K, K_0, K_1 respectively. f^* and $\tilde{f}$ denote the conjugate function and the infimal convolution of f respectively. Then we have the following:*

(i) $(f^*)^* = f$.

(ii) f is essentially smooth if and only if f^* is strictly convex.

(iii) $\tilde{f}^* = (1-t)f_0^* + tf_1^*$.

(iv) If f is essentially smooth on K, then ∇f^* is the inverse of ∇f, i.e.

$$\nabla f^* = (\nabla f)^{-1}, \quad \nabla f^*(\nabla f(x)) = x.$$

(v) If f_i is essentially smooth and strictly convex on K, then $\tilde{f}$ is also essentially smooth and strictly convex.

(vi) If f_i^ε converges uniformly to f_i as $\varepsilon \to 0$ for $i = 0, 1$, then $\tilde{f}^\varepsilon$ converges uniformly to $\tilde{f}$.

(vii) If f^ε is differentiable on K and converges uniformly to f as $\varepsilon \to 0$, then ∇f^ε converges uniformly to ∇f.

Now we prove a property for convex function, which play a crucial role in the proof of our main theorem.

Lemma 4.6 ([CCS06]). *For $i = 0, 1$, let K_i be convex open sets. Let $f_i \in C^1(K_i)$ be a strictly convex function such that $\lim_{x\to\partial K} f_i(x) = +\infty$. Let $\tilde{f}$ denote the infimal convolution of f_0 and f_1. For $t \in [0, 1]$, Set $K_t = (1-t)K_0 + tK_1$. Then for every $z \in K_t$, there exist $x \in K_0$ and $y \in K_1$ such that*

$$z = (1-t)x + ty, \tag{4.2}$$

$$\tilde{f}(z) = (1-t)f_0(x) + tf_1(y), \tag{4.3}$$

$$\nabla\tilde{f}(z) = \nabla f_0(x) = \nabla f_1(y). \tag{4.4}$$

Moreover, if f_0 and f_1 are twice differentiable at x and y respectively, and $D^2 f_0(x) > 0$, $D^2 f_1(y) > 0$, then $\tilde{f}$ is twice differentiable at z and

$$D^2\tilde{f}(z) = \left[(1-t)\left(D^2 f_0(x)\right)^{-1} + t\left(D^2 f_1(y)\right)^{-1}\right]^{-1}. \tag{4.5}$$

Proof. The proof was given in [CCS06], we sketch it here.

By Definition 4.3, f_i is essentially smooth and strictly convex in K_i and by Proposition 4.5 (i) , (ii) and (v), f_i^* and $\tilde{f}$ are also essentially smooth and strictly convex in $\mathbb{R}^n$.

Fix $z \in K_t$, by Definition 4.4 and $\lim_{x\to\partial K} f_i(x) = +\infty$, there exist $x \in K_0$ and $y \in K_1$ such that (4.2) and (4.3) hold. That is, for x, y satisfying $z = (1-t)x+ty$, the function $T(x, y) := (1-t)f_0(x)+tf_1(y)$ attains its infimum at (x, y). By Lagrangian Multiples Theorem, there exist a constant $\lambda \in \mathbb{R}$, such that

$$(1-t)\nabla f_0(x) - \lambda(1-t) = 0,$$

$$t\nabla f_1(y) - \lambda t = 0.$$

Hence,

$$\nabla f_0(x) = \nabla f_1(y) = \lambda.$$

Using Proposition 4.5 (iii) and (iv), we obtain that

$$\begin{aligned} \nabla\tilde{f}^*(\lambda) &= (1-t)\nabla f_0^*(\lambda) + t\nabla f_1^*(\lambda) \\ &= (1-t)(\nabla f_0)^{-1}(\lambda) + t(\nabla f_1)^{-1}(\lambda) \\ &= (1-t)x + ty = z = \nabla\tilde{f}^*(\nabla\tilde{f}(z)). \end{aligned} \tag{4.6}$$

Since $\tilde{f}$ is strictly convex, we know that $\nabla \tilde{f}^* = (\nabla \tilde{f})^{-1}$ is injective. It follows from (4.6) that

$$\lambda = \nabla \tilde{f}(z),$$

which leads to (4.4). if f_0 and f_1 are twice differentiable at x and y respectively, and $D^2 f_0(x), D^2 f_1(y) > 0$, then f_i^* are twice differentiable at λ and

$$D^2 f_i^*(\lambda) = D(\nabla f_i^*)(\lambda) = D((\nabla f_i)^{-1}(\lambda)) = (D^2 f_0(x))^{-1} = (D^2 f_1(y))^{-1}. \quad (4.7)$$

Therefore, $\tilde{f}^*$ are twice differentiable at λ and $D^2 \tilde{f}^*(\lambda) > 0$. Consequently, $\tilde{f}$ are twice differentiable at z and the same computation as (4.7) yields

$$D^2 \tilde{f}(z) = (D^2 \tilde{f}^*(\lambda))^{-1} = [(1-t) D^2 f_0^*(\lambda) + t D^2 f_1^*(\lambda)]^{-1},$$

where we also used Proposition 4.5 (iii). Combining with (4.7), we obtain the equality (4.5). □

Now we are ready to prove Theorem 4.1.

Assume u_i be the first Dirichlet eigenfunctions with respect to the sets K_i for $i = 0, 1$. Consider the function

$$v_i(x) = -\log u_i(x), \quad x \in K_i.$$

They satisfy corresponding equations:

$$\begin{cases} Qv_i = \lambda(K_i) + F(\nabla v_i)^2 & \text{in} \quad K_i \\ \lim_{x \to \partial K_i} v_i = +\infty & \text{on} \quad \partial K_i \end{cases} \quad (4.8)$$

In [KaNo08], the author observed that $v_i \in C^{1,\alpha}(K_i)$ and is convex in K_i. Our proof will be based on this property. Denote

$$\mathcal{C}_i = \{x \in K_i | \nabla v_i(x) = 0\}.$$

Since the operator Qv_i is uniformly elliptic in any compact sets of $K_i \backslash \mathcal{C}_i$, we know by the standard regularity theory for elliptic equations that $v_i \in C^2(K_i \backslash \mathcal{C}_i)$.

For $\varepsilon > 0$, we define

$$v_i^\varepsilon(x) = v_i(x) + \varepsilon (F^0(x))^2,$$

By the strictly convexity of $(F^0(x))^2$, the function $v_i^\varepsilon \in C^2(K_i \backslash \mathcal{C}_i) \bigcap C^{1,\alpha}(K_i)$ and is strictly convex in K_i. Obviously v_i^ε converges uniformly to v_i in K_i. By proposition 4.5 (vii), ∇v_i^ε also converges uniformly to ∇v_i in K_i.

We denote by $\tilde{v}$ the infimal convolution of v_0 and v_1, $\tilde{v}^\varepsilon$ the infimal convolution of v_0^ε and v_1^ε, again by proposition 4.5 (vi) and (vii), we know that $\tilde{v}^\varepsilon \in C^1(K_t)$ and $\tilde{v}^\varepsilon, \nabla \tilde{v}^\varepsilon$ converges uniformly to $\tilde{v}, \nabla \tilde{v}$ in K_t respectively. Let $\mathcal{C}_t = \{x \in K_t | \nabla \tilde{v}(x) = 0\}$, from Lemma 4.6 it is not difficult to see that

$$\mathcal{C}_t = (1-t)\mathcal{C}_0 + t\mathcal{C}_1.$$

We have the following

Lemma 4.7. *For $\tilde{v}^\varepsilon$ defined above, the following inequality holds:*

$$Q\tilde{v}^\varepsilon(z) \le (1-t)\lambda(K_0) + t\lambda(K_1) + F^2(\nabla\tilde{v}^\varepsilon(z)) + R^\varepsilon(z), \tag{4.9}$$

where R^ε converges uniformly to zero in any compact subset of $K_t\backslash\mathcal{C}_t$ as $\varepsilon \to 0$.

Proof. By Lemma 4.6, for a fixed $z \in K_t\backslash\mathcal{C}_t$ there exists a unique $(x_0^\varepsilon, x_1^\varepsilon) \in K_0 \times K_1$, such that

$$\begin{aligned} z &= (1-t)x_0^\varepsilon + tx_1^\varepsilon, && (4.10)\\ \tilde{v}^\varepsilon(z) &= (1-t)v_0^\varepsilon(x_0^\varepsilon) + tv_1^\varepsilon(x_1^\varepsilon), && (4.11)\\ \nabla\tilde{v}^\varepsilon(z) &= \nabla v_0^\varepsilon(x_0^\varepsilon) = \nabla v_1^\varepsilon(x_1^\varepsilon). && (4.12) \end{aligned}$$

Since $\nabla\tilde{v}(z) \neq 0$, it follows that for sufficiently small ε,

$$\nabla v_i^\varepsilon(x_i^\varepsilon) \neq 0, \quad i = 0, 1. \tag{4.13}$$

Particularly,

$$x_i^\varepsilon \in K_i\backslash\mathcal{C}_i, \quad i = 0, 1. \tag{4.14}$$

Recall that $v_i^\varepsilon \in C^2(K_i\backslash\mathcal{C}_i)$, by Lemma 4.6 again, we know that $\tilde{v}^\varepsilon \in C^2(K_t\backslash\mathcal{C}_t)$ and the following equality holds:

$$D^2\tilde{v}^\varepsilon(z) = \left[(1-t)\left(D^2v_0^\varepsilon(x^\varepsilon)\right)^{-1} + t\left(D^2v_1^\varepsilon(y^\varepsilon)\right)^{-1}\right]^{-1} \tag{4.15}$$

for $z \in K_t\backslash\mathcal{C}_t$.

We denote $a_{ij}(\xi) = F(\xi)F_{\xi_i\xi_j}(\xi) + F_{\xi_i}(\xi)F_{\xi_j}(\xi)$ for $\xi \in \mathbb{R}^n\backslash\{0\}$. From (4.12), we can set

$$a_{ij} := a_{ij}(\nabla\tilde{v}^\varepsilon(z)) = a_{ij}(\nabla v_0^\varepsilon(x_0^\varepsilon)) = a_{ij}(\nabla v_1^\varepsilon(x_1^\varepsilon)).$$

Let A denote the matrix $(a_{ij})_{n\times n}$. It is positive definite and symmetric.

Now we need the following proposition.

Proposition 4.8. *Assume A, B, C are symmetric and positive definite $n \times n$ matrices. Then we have the following inequality:*

$$\operatorname{tr}\big(A((1-t)B + tC)^{-1}\big) \le (1-t)\operatorname{tr}(AB^{-1}) + t\operatorname{tr}(AC^{-1}), \tag{4.16}$$

for any $t \in [0, 1]$.

Proof. Since A is symmetric and positive definite, there exists an orthogonal matrix O, such that

$$A = O\Lambda O^T,$$

where $\Lambda = \text{diag}\{\mu_1, \mu_2, \cdots, \mu_n\}$ and $\{\mu_i\}_{i=1}^n$ are the positive eigenvalues of A. Set $\tilde{B} = O^T BO, \tilde{C} = O^T CO$. It is clear that $\tilde{B}, \tilde{C}$ are also symmetric and positive definite.

For $i = 1, \cdots, n$, let $v_i = \sqrt{\lambda_i} e_i$ be n row vectors, where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. By using Lemma 1 in [CoSa03], for each i we have

$$v_i((1-t)\tilde{B} + t\tilde{C})^{-1} v_i^T \leq (1-t) v_i \tilde{B}^{-1} v_i^T + t v_i \tilde{C}^{-1} v_i^T. \tag{4.17}$$

On the other hand,

$$\begin{aligned} \text{tr}(AB^{-1}) &= \text{tr}(O\Lambda O^T B^{-1}) = \text{tr}(\Lambda \tilde{B}^{-1}) = \text{tr}(\Lambda^{\frac{1}{2}} \tilde{B}^{-1} \Lambda^{\frac{1}{2}}) \\ &= \sum_{i=1}^n v_i \tilde{B}^{-1} v_i^T. \end{aligned}$$

The same computation gives

$$\text{tr}(AC^{-1}) = \sum_{i=1}^n v_i \tilde{C}^{-1} v_i^T,$$

$$\text{tr}\big(A((1-t)B + tC)^{-1}\big) = \sum_{i=1}^n v_i((1-t)\tilde{B} + t\tilde{C})^{-1} v_i^T.$$

Thus we conclude (4.16) from (4.17). □

We return to the proof of Lemma 4.7. Because of the strict convexity of v_0^ε and v_1^ε, we can apply (4.16) to the matrix A defined above, $B = (D^2 v_0^\varepsilon(x_0^\varepsilon))^{-1}$ and $C = (D^2 v_1^\varepsilon(x_1^\varepsilon))^{-1}$ to obtain that

$$\begin{aligned} \text{tr}(A \cdot D^2 \tilde{v}^\varepsilon(z)) &= \text{tr}\left(A \cdot \big((1-t)(D^2 v_0^\varepsilon(x_0^\varepsilon))^{-1} + t(D^2 v_1^\varepsilon(x_1^\varepsilon))^{-1}\big)^{-1}\right) \\ &\leq (1-t)\text{tr}(A \cdot D^2 v_0^\varepsilon(x_0^\varepsilon)) + t\text{tr}(A \cdot D^2 v_1^\varepsilon(x_1^\varepsilon)). \end{aligned}$$

Here we also used (4.15). By the fact that $Qv = \text{tr}(A \cdot D^2 v)$, we have

$$Q\tilde{v}^\varepsilon(z) \leq (1-t) Q v_0^\varepsilon(x_0^\varepsilon) + t Q v_1^\varepsilon(x_1^\varepsilon) \tag{4.18}$$

for $z \in K_t \backslash \mathcal{C}_t$.

To arrive (4.9), we compute

$$
\begin{aligned}
Qv_i^\varepsilon(x_i^\varepsilon) &= \sum_{j,k=1}^n a_{jk}(\nabla v_i^\varepsilon(x_i^\varepsilon))\frac{\partial^2 v_i^\varepsilon}{\partial x_k \partial x_j}(x_i^\varepsilon) \\
&= \sum_{j,k=1}^n a_{jk}(\nabla v_i + \varepsilon \nabla (F^0)^2)\big(\frac{\partial^2 v_i}{\partial x_k \partial x_j} + \frac{\partial^2 (F^0)^2}{\partial x_k \partial x_j}\big) \\
&= Qv_i + \sum_{j,k=1}^n \big(a_{jk}(\nabla v_i + \varepsilon \nabla (F^0)^2) - a_{jk}(\nabla v_i)\big)\frac{\partial^2 v_i}{\partial x_k \partial x_j} \\
&\quad + \sum_{j,k=1}^n \varepsilon a_{jk}\big(\nabla v_i + \varepsilon \nabla (F^0)^2\big)\frac{\partial^2 (F^0)^2}{\partial x_k \partial x_j} \\
&= Qv_i(x_i^\varepsilon) + R_i^\varepsilon(x_i^\varepsilon),
\end{aligned}
\tag{4.19}
$$

where

$$
\begin{aligned}
R_i^\varepsilon(x_i^\varepsilon) &= \sum_{j,k=1}^n \big(a_{jk}(\nabla v_i + \varepsilon \nabla (F^0)^2) - a_{jk}(\nabla v_i)\big)\frac{\partial^2 v_i}{\partial x_k \partial x_j} \\
&\quad + \sum_{j,k=1}^n \varepsilon a_{jk}\big(\nabla v_i + \varepsilon \nabla (F^0)^2\big)\frac{\partial^2 (F^0)^2}{\partial x_k \partial x_j}.
\end{aligned}
$$

From (4.18), (4.19), (4.8) and (4.12), it follows that

$$
\begin{aligned}
Q\tilde{v}^\varepsilon(z) &\leq (1-t)Qv_0(x_0^\varepsilon) + tQv_1(x_1^\varepsilon) + (1-t)R_0^\varepsilon(x_0^\varepsilon) + tR_1^\varepsilon(x_1^\varepsilon) \\
&= (1-t)\lambda_1(K_0) + t\lambda_1(K_1) + F^2(\nabla \tilde{v}^\varepsilon(z)) \\
&\quad + (1-t)[R_0^\varepsilon(x_0^\varepsilon) + F^2(\nabla v_0(x_0^\varepsilon)) - F^2(\nabla v_0^\varepsilon(x_0^\varepsilon))] \\
&\quad + t[R_1^\varepsilon(x_1^\varepsilon) + F^2(\nabla v_1(x_1^\varepsilon)) - F^2(\nabla v_1^\varepsilon(x_1^\varepsilon))] \\
&= (1-t)\lambda_1(K_0) + t\lambda_1(K_1) + F^2(\nabla \tilde{v}^\varepsilon(z)) + R^\varepsilon(z),
\end{aligned}
$$

where

$$
\begin{aligned}
R^\varepsilon(z) &= (1-t)[R_0^\varepsilon(x_0^\varepsilon) + F^2(\nabla v_0(x_0^\varepsilon)) - F^2(\nabla v_0^\varepsilon(x_0^\varepsilon))] \\
&\quad + t[R_1^\varepsilon(x_1^\varepsilon) + F^2(\nabla v_1(x_1^\varepsilon)) - F^2(\nabla v_1^\varepsilon(x_1^\varepsilon))].
\end{aligned}
$$

Finally we show that $R^\varepsilon(z)$ converges uniformly to zero in any compact subset of $K_t \backslash \mathcal{C}_t$ as $\varepsilon \to 0$.

Indeed, it is clear that $R^\varepsilon(z) \to 0$ point-wise. If L is a compact subset of $K_t \backslash \mathcal{C}_t$, then for every $z \in L$, there exists ε_0 and two compact subsets L_i of $K_i \backslash \mathcal{C}_i$, such that, for $\varepsilon < \varepsilon_0$, $x_i^\varepsilon \in L_i$. We then obtain by the 0-homogeneity of $a_{jk}(\xi)$ that $a_{jk}\big(\nabla v_i + \varepsilon \nabla (F^0)^2\big), a_{jk}\big(\nabla v_i\big)$ is uniformly bounded in L_i. As v_i and $(F^0)^2$ belong to $C^2(L_i)$, we conclude that $R^\varepsilon(z)$ is uniformly bounded in L. □

Now set

$$\tilde{u}^\varepsilon(z) = e^{-\tilde{v}^\varepsilon(z)},$$
$$\tilde{u}(z) = e^{-\tilde{v}(z)}$$

for $z \in K_t$. Because of the uniform convergence of $\tilde{v}^\varepsilon$ and $\nabla\tilde{v}^\varepsilon$, we know that $\tilde{u}^\varepsilon, \nabla\tilde{u}^\varepsilon$ converge uniformly to $\tilde{u}, \nabla\tilde{u}$ respectively. Note that $\tilde{v}^\varepsilon \in C^1(K_t)$ and $\tilde{v}^\varepsilon \to +\infty$ on ∂K_t, we have $\tilde{u}^\varepsilon \in C^1(K_t)$ and $\tilde{u}^\varepsilon = 0$ on ∂K_t.

Lemma 4.9. *The following inequality holds for $\tilde{u}$:*

$$\int_{K_t} F(\nabla\tilde{u}(z))^2 dz \leq [(1-t)\lambda_1(K_0) + t\lambda_1(K_1)] \int_{\Omega_t} (\tilde{u}(z))^2 dz. \tag{4.20}$$

Proof. From Lemma 4.7, we deduce that $\tilde{u}^\varepsilon$ satisfies

$$Q\tilde{u}^\varepsilon(z) \geq -[(1-t)\lambda_1(K_0) + t\lambda_1(K_1)]\tilde{u}^\varepsilon(z) - R^\varepsilon(z)\tilde{u}^\varepsilon(z),\ z \in \Omega_t \backslash \mathcal{C}_t. \tag{4.21}$$

To prove Lemma 4.9 we use an approximation procedure. Assume that X_j and Y_j are two sequences of open sets such that $\mathcal{C}_t \subset Y_j \subset X_j \subset\subset K_t$, $\bar{Y}_j \subset X_j$ for every $j \in \mathbb{N}$ and X_j, Y_j converge to K_t, $\mathcal{C}_t$ in Hausdorff metric respectively as $j \to +\infty$.

From (4.21), we obtain for every $j \in \mathbb{N}$ that

$$\begin{aligned} \int_{X_j \backslash Y_j} \tilde{u}^\varepsilon(z) Q\tilde{u}^\varepsilon(z) dz \geq & - \int_{X_j \backslash Y_j} [(1-t)\lambda_1(K_0) + t\lambda_1(K_1)](\tilde{u}^\varepsilon(z))^2 dz \\ & - \int_{X_j \backslash Y_j} R^\varepsilon(z)(\tilde{u}^\varepsilon(z))^2 dz. \end{aligned}$$

It follows by integrating by parts that

$$\begin{aligned} & \int_{X_j \backslash Y_j} F(\nabla\tilde{u}^\varepsilon(z))^2 dz - \int_{\partial X_j \bigcup \partial Y_j} \tilde{u}^\varepsilon(z) F(\nabla\tilde{u}^\varepsilon(z)) F_{\xi_i}(\nabla\tilde{u}^\varepsilon(z)) \frac{\partial \tilde{u}^\varepsilon(z)}{\partial \nu} dz \\ & \leq \int_{X_j \backslash Y_j} [(1-t)\lambda_1(K_0) + t\lambda_1(K_1)](\tilde{u}^\varepsilon(z))^2 dz + \int_{X_j \backslash Y_j} R^\varepsilon(z)(\tilde{u}^\varepsilon(z))^2 dz. \end{aligned}$$

Letting $\varepsilon \to 0$ in the above inequality, by $F \in C^2(\mathbb{R}^n \backslash \{0\})$ and the uniform convergence of $\tilde{u}^\varepsilon$, $\nabla\tilde{u}^\varepsilon$ and R^ε in $X_j \backslash Y_j$, we have:

$$\begin{aligned} & \int_{X_j \backslash Y_j} F(\nabla\tilde{u}(z))^2 dz - \int_{\partial X_j \bigcup \partial Y_j} \tilde{u}(z) F(\nabla\tilde{u}(z)) F_{\xi_i}(\nabla\tilde{u}(z)) \frac{\partial \tilde{u}(z)}{\partial \nu} dz \\ & \qquad \leq \int_{X_j \backslash Y_j} [(1-t)\lambda_1(K_0) + t\lambda_1(K_1)](\tilde{u}(z))^2 dz. \end{aligned}$$

Finally, letting $j \to \infty$, since $\lim_{x\to\partial\mathcal{C}_t} \nabla\tilde{u}(x) = 0$, $\lim_{x\to\partial K_t} \tilde{u}(x) = 0$, $F(0) = 0$, $\nabla\tilde{u}$ is bounded in K_t, F_{ξ_i} is 0-homogeneous, we have:

$$\int_{K_t\setminus\mathcal{C}_t} F(\nabla\tilde{u}(z))^2 dz \le [(1-t)\lambda_1(K_0) + t\lambda_1(K_1)] \int_{K_t} (\tilde{u}(z))^2 dz.$$

As $F(\nabla\tilde{u}(z)) = 0$ in $\mathcal{C}_t$, we deduce (4.20). □

By Lemma 4.9, we obtain immediately

$$\lambda_1(K_t) \le \frac{\int_{K_t} F(\nabla\tilde{u}(z))^2 dz}{\int_{K_t} (\tilde{u}(z))^2 dz} \le (1-t)\lambda_1(K_0) + t\lambda_1(K_1) \tag{4.22}$$

In order to obtain the Brunn-Minkowski type inequality (4.1), we only need the following standard argument: For arbitrary K_0, K_1 and $t \in [0,1]$, let

$$K_0' = [\lambda_1(K_0)]^{\frac{1}{2}} K_0,$$

$$K_1' = [\lambda_1(K_1)]^{\frac{1}{2}} K_1,$$

$$t' = \frac{t[\lambda_1(K_1)]^{-\frac{1}{2}}}{(1-t)[\lambda(K_0)]^{-\frac{1}{2}} + t[\lambda_1(K_1)]^{-\frac{1}{2}}}$$

and apply (4.22) to K_0', K_1' and t', that is,

$$\lambda_1((1-t')K_0' + t'K_1') \le (1-t')\lambda_1(K_0') + t\lambda_1(K_1').$$

By using the -2 -homogeneity of $\lambda_1(K)$, we obtain (4.1). This complete the proof of Theorem 4.1.

4.2 Sharp lower bound for the first Neumann eigenvalue

We will prove the following theorem in this section.

Theorem 4.10. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$. Let λ_1^N be the first Neumann eigenvalue of the anisotropic Laplacian. Assume that $\partial\Omega$ is weakly convex. Then λ_1^N satisfies*

$$\lambda_1^N \ge \frac{\pi^2}{d_F^2}, \tag{4.23}$$

Equivalently, the optimal Poincaré inequality

$$\int_\Omega F^2(\nabla u) dx \ge \frac{\pi^2}{d_F^2} \int_\Omega u^2 dx \tag{4.24}$$

holds for any $u \in W^{1,2}(\Omega)$ with $\int_\Omega u dx = 0$. Moreover, equality in (4.23) holds if and only if Ω is a segment in $\mathbb{R}$.

In the theorem, d_F is the *diameter* of Ω with respect to the norm F on $\mathbb{R}^n$, which defines as follows. We say $\gamma : [0,1] \to \Omega$ a *minimal geodesic* from x_1 to x_2 if

$$d_F(x_1, x_2) := \int_0^1 F^0(\dot{\gamma}(t))dt = \inf \int_0^1 F^0(\dot{\tilde{\gamma}}(t))dt,$$

where the infimum takes on all C^1 curves $\tilde{\gamma}(t)$ in Ω from x_1 to x_2. In fact γ is a straight line and $d_F(x_1, x_2) = F^0(x_2 - x_1)$. We call $d_F(x_1, x_2)$ the F*-distance* between x_1 and x_2. d_F is defined by

$$d_F := \sup_{x_1, x_2 \in \overline{\Omega}} d_F(x_1, x_2).$$

Estimate (4.23) for the Neumann boundary problem is optimal, in the sense that equality can be attained for a one dimensional segment. This is in fact a generalization of the classical result of Payne-Weinberger in [PaWe60] on an optimal estimate of the first Neumann eigenvalue of the ordinary Laplacian.

First of all, we give an outline of our proof. The technique is based on a comparison theorem on the gradient of the first eigenfunction with that of a one dimensional (1-D) model function (Theorem 4.15), which was developed by Kröger [Kr92] and improved by Chen-Wang [CW97] and Bakry-Qian [BQ00]. By using a refined Hölder inequality, we find that the one dimensional model coincides with that in the Laplacian case, as presented in Theorem 4.15). It should be not so surprising, because when we consider F in $\mathbb{R}$, it can only be $F(x) = c|x|$ for some positive constant c. Since the 1-D model has been extensively studied in [BQ00], it also eases our situation, although we deal with a nonlinear operator. One difficulty we encounter is to handle the boundary maximum due to the different representation of the Neumann boundary condition (0.5). We find a suitable vector field V to avoid this difficulty. Another ingredient is a comparison theorem on the maxima of eigenfunction with that of the 1-D model function (Theorem 4.17). Everything in [BQ00] works except the boundedness of the Hessian of eigenfunctions around a critical point (since the eigenfunction is only $C^{1,\alpha}$ among Ω), which was used to prove (5.16). Here we avoid the use of the Hessian of eigenfunctions by using the comparison theorem on the gradient. With these comparison theorems at hand, we could follow step by step the work of Bakry-Qian [BQ00] to get Theorem 4.10. The proof for the rigidity part of Theorem 0.5 follows closely the work of Hang-Wang [HaWa07]. Here we need pay more attention on the points with vanishing $|\nabla u|$.

Before getting into the proof of Theorem 4.10, we prove two simple but important inequalities with respect to the anisotropic Laplacian. One is a Kato type inequality, the other is a refined Hölder inequality for the square of "anisotropic" norm of Hessian.

Lemma 4.11 (Kato inequality)**.** *At a point where* $\nabla u \neq 0$, *we have*

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq a_{ij}F_kF_lu_{ik}u_{jl}. \tag{4.25}$$

Proof. It is clear that

$$a_{ij}a_{kl}u_{ik}u_{jl} - a_{ij}F_kF_lu_{ik}u_{jl} = a_{ij}FF_{kl}u_{ik}u_{jl} = FF_iF_jF_{kl}u_{ik}u_{jl} + F^2F_{ij}F_{kl}u_{ik}u_{jl}.$$

Since (F_{ij}) is positive definite, we know the first term

$$FF_iF_jF_{kl}u_{ik}u_{jl} = FF_{kl}(F_iu_{ik})(F_ju_{jl}) \geq 0.$$

The second term $F_{ij}F_{kl}u_{ik}u_{jl}$ is nonnegative as well. Indeed, we can write the matrix $(F_{kl})_{k,l} = O^T\Lambda O$ for some orthogonal matrix O and diagonal matrix $\Lambda = diag(\mu_1, \mu_2, \cdots, \mu_n)$ with $\mu_i \geq 0$ for any $i = 1, 2, \cdots, n$. Set $U = (u_{ij})_{i,j}$ and $\widetilde{U} = OUO^T = (\tilde{u}_{ij})_{i,j}$. Then we have

$$\begin{aligned} F_{ij}F_{kl}u_{lj}u_{ki} &= tr(O^T\Lambda OUO^T\Lambda OU) = tr(\Lambda OUO^T\Lambda OUO^T) \\ &= tr(\Lambda\widetilde{U}\Lambda\widetilde{U}) = \mu_i\mu_j\tilde{u}_{ij}^2 \geq 0, \end{aligned}$$

and hence the proof of (4.25). □

Remark 4.12. *When* $F(\xi) = |\xi|$, *(4.25) is the usual Kato inequality*

$$|\nabla^2 u|^2 \geq |\nabla|\nabla u||^2.$$

Lemma 4.13 (refined Hölder inequality)**.** *At a point where* $\nabla u \neq 0$, *we have for* $N \geq n$,

$$a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{N} + \frac{N}{N-1}\left(\frac{a_{ij}u_{ij}}{N} - F_iF_ju_{ij}\right)^2. \tag{4.26}$$

Proof. We first handle the case $N = n$. Let

$$A = F_iF_ju_{ij} \quad \text{and } B = FF_{ij}u_{ij}.$$

The right hand side of (4.26) equals to

$$\frac{(A+B)^2}{n} + \frac{n}{n-1}\left(\frac{B}{n} - \frac{n-1}{n}A\right)^2 = A^2 + \frac{1}{n-1}B^2.$$

The left hand side of (4.26) is

$$A^2 + 2FF_iF_jF_{kl}u_{ik}u_{jl} + F^2F_{ij}F_{kl}u_{ik}u_{jl}.$$

Since (F_{ij}) is semi-positively definite, we know

$$FF_iF_jF_{kl}u_{ik}u_{jl} = FF_{kl}(F_iu_{ik})(F_ju_{jl}) \geq 0.$$

Using the same notations in the proof of Lemma 4.11, we have

$$F^2 F_{ij}F_{kl}u_{ik}u_{jl} = F^2\mu_i\mu_j\tilde{u}_{ij}^2 = F^2\mu_i^2\tilde{u}_{ii}^2 + F^2\sum_{i\neq k}\mu_i\mu_k\tilde{u}_{ik}^2 \geq F^2\mu_i^2\tilde{u}_{ii}^2,$$

$$B = FF_{ij}u_{ij} = tr(O^T\Lambda OU) = tr(\Lambda OUO^T) = \mu_i\tilde{u}_{ii}.$$

We claim that (F_{ij}) is a matrix of rank $n-1$, in other words, one of μ_i is zero. Firstly, $F_{ij}u_j = 0$. Secondly, for any nonzero $V \perp F_\xi(\nabla u)$, $F_{ij}V^iV^j = a_{ij}V^iV^j > 0$. The claim follows easily. Thus the Hölder inequality gives

$$F^2\mu_i^2\tilde{u}_{ii}^2 \geq \frac{1}{n-1}F^2(\mu_i\tilde{u}_{ii})^2 = \frac{1}{n-1}B^2.$$

Altogether we complete the proof of the case $N = n$.

For the case $N > n$, we only need to observe the following identity for any $a, b \in \mathbb{R}$:

$$\frac{a^2}{n} + \frac{n}{n-1}\left(\frac{a}{n} - b\right)^2 = \frac{a^2}{N} + \frac{N}{N-1}\left(\frac{a}{N} - b\right)^2 + \frac{N-n}{(N-1)(n-1)}(a-b)^2.$$

□

Remark 4.14. *When $F(\xi) = |\xi|$, then (4.26) is*

$$|\nabla^2 u|^2 \geq \frac{(\Delta u)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta u}{n} - \frac{u_iu_ju_{ij}}{|\nabla u|^2}\right)^2.$$

Let us start the proof of Theorem 4.10.

It is well-known that the existence of Neumann first eigenfunction can be obtained from the direct method in the calculus of variations. We note that the first Neumann eigenfunction must change sign, for its average vanishes.

We begin with the following gradient comparison theorem, which is the most crucial part for the proof of the sharp estimate. For simplicity, we write λ_1 instead of λ_1^N throughout this section.

Theorem 4.15. *Let Ω, u, λ_1 be as in Theorem 4.10. Let v be a solution of the one dimensional (1-D) problem on some interval (a, b):*

$$v'' - Tv' = -\lambda_1 v, \quad v'(a) = v'(b) = 0, \quad v' > 0, \tag{4.27}$$

with $T(t) = -\frac{n-1}{t}$ or 0. Assume that $[\min u, \max u] \subset [\min v, \max v]$, then

$$F(\nabla u)(x) \leq v'(v^{-1}(u(x))). \tag{4.28}$$

Proof. First, since $\int u = 0$, we know that $\min u < 0$ while $\max u > 0$. We may assume that $[\min u, \max u] \subset (\min v, \max v)$ by multiplying u by a constant $0 < c < 1$. If we prove the result for this u, then letting $c \to 1$ we have (4.28).

Under the condition $[\min u, \max u] \subset (\min v, \max v)$, v^{-1} is smooth on a neighborhood U of $[\min u, \max u]$.

Consider $P := \psi(u)(\frac{1}{2}F(\nabla u)^2 - \phi(u))$, where $\psi, \phi \in C^\infty(U)$ are two positive smooth functions to be determined later. We first assume that P attains its maximum at $x_0 \in \Omega$, and then we will consider the case that $x_0 \in \partial\Omega$. If $\nabla u(x_0) = 0$, $P \leq 0$ is obvious. Hence we assume $\nabla u(x_0) \neq 0$. From now on we compute at x_0. We use the notation (2.5) in Chapter 2. Since x_0 is the maximum of P, we have that

$$P_i(x_0) = 0, \tag{4.29}$$

$$a_{ij}(x_0)P_{ij}(x_0) \leq 0. \tag{4.30}$$

Equality in (4.29) gives

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2(\nabla u) - \phi(u)\right) = -\frac{\psi(u))_i}{\psi^2}P, \quad F_iF_ju_{ij} = \phi' - \frac{\psi'}{\psi^2}P. \tag{4.31}$$

Then we compute $a_{ij}P_{ij}$.

$$\begin{aligned} a_{ij}P_{ij} &= \frac{P}{\psi}a_{ij}(\psi(u))_{ij} + \psi a_{ij}\frac{\partial}{\partial x_i x_j}\left(\frac{1}{2}F^2(\nabla u) - (\phi(u))\right) \\ &\quad +2a_{ij}(\psi(u))_i\frac{\partial}{\partial x_j}\left(\frac{1}{2}F^2(\nabla u) - \phi(u)\right). \end{aligned}$$

It is easy to see from Proposition 1.2 that

$$\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2\right)(\nabla u)u_i = F^2(\nabla u), \quad a_{ij}u_iu_j = F^2(\nabla u), \quad a_{ijk}u_k = 0. \tag{4.32}$$

By using (4.31), (4.32), the Bochner formula (2.7) and eigenvalue equation (0.3), we get

$$\begin{aligned} a_{ij}P_{ij} &= (-\lambda_1 u\frac{\psi'}{\psi} + F^2\frac{\psi''}{\psi} - 2F^2\frac{\psi'^2}{\psi^2})P \\ &\quad +\psi(a_{ij}a_{kl}u_{ik}u_{jl} - \lambda_1 F^2) + \psi(\lambda_1 u\phi' - F^2\phi''). \end{aligned} \tag{4.33}$$

Applying Lemma 4.13 with $N = n$ to (4.33), replacing F^2 by $2\frac{P}{\psi} + \phi$ and using (4.31), (0.3), (4.30), we deduce

$$
\begin{aligned}
0 \geq a_{ij}P_{ij} \quad \geq \quad & (-\lambda_1 u\frac{\psi'}{\psi} + F^2\frac{\psi''}{\psi} - 2F^2\frac{\psi'^2}{\psi^2})P + \psi(\lambda_1 u\phi' - F^2\phi'') \\
& + \psi\left(\frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1}\left(\frac{a_{ij}u_{ij}}{n} - F_iF_ju_{ij}\right)^2 - \lambda_1 F^2\right) \\
= \quad & \frac{1}{\psi}\left[2\frac{\psi''}{\psi} - (4 - \frac{n}{n-1})\frac{\psi'^2}{\psi^2}\right]P^2 \\
& + \left[2\phi\left(\frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2}\right) - \frac{n+1}{n-1}\frac{\psi'}{\psi}\lambda_1 u - \frac{2n}{n-1}\frac{\psi'}{\psi}\phi' - 2\lambda_1 - 2\phi''\right]P \\
& + \psi\left[\frac{1}{n-1}\lambda_1^2u^2 + \frac{n+1}{n-1}\lambda_1 u\phi' + \frac{n}{n-1}\phi'^2 - 2\lambda_1\phi - 2\phi\phi''\right] \\
:= \quad & a_1P^2 + a_2P + a_3.
\end{aligned} \tag{4.34}
$$

We are lucky to observe that the coefficients a_i, $i = 1, 2, 3$, coincide with those appearing in the ordinary Laplacian case (see e.g. [BQ00], Lemma 1). The next step is to choose suitable positive functions ψ and ϕ such that $a_1, a_2 > 0$ everywhere and $a_3 = 0$, which had already be done in [BQ00]. For completeness, we sketch the main idea here.

Choose $\phi(u) = \frac{1}{2}v'(v^{-1}(u))^2$, where v is a solution of 1-D problem (4.27). One can compute that

$$\phi'(u) = v''(v^{-1}(u)), \phi''(u) = \frac{v'''}{v'}(v^{-1}(u)).$$

Setting $t = v^{-1}(u)$ and $u = v(t)$ we have

$$
\begin{aligned}
\frac{a_3(t)}{\psi} &= \frac{1}{n-1}\lambda_1^2v^2 + \frac{n+1}{n-1}\lambda_1 vv'' + \frac{n}{n-1}v''^2 - \lambda_1 v'^2 - v'v''' \\
&= -v'(v'' - Tv' + \lambda_1 v)' + \frac{1}{n-1}(v'' - Tv' + \lambda_1 v)(nv'' + Tv' + \lambda_1 v) = 0.
\end{aligned}
$$

Here we have used that T satisfies $T' = \frac{T^2}{n-1}$. For a_1, a_2, we introduce

$$X(t) = \lambda_1\frac{v(t)}{v'(t)}, \quad \psi(u) = \exp(\int h(v(t))), \quad f(t) = -h(v(t))v'(t).$$

With these notations, we have

$$f' = -h'v'^2 + f(T - X),$$

$$v'|^2_{v^{-1}}a_1\psi = 2f(T - X) - \frac{n-2}{n-1}f^2 - 2f' := 2(Q_1(f) - f'),$$

$$a_2 = f(\frac{3n-1}{n-1}T - 2X) - 2T(\frac{n}{n-1}T - X) - f^2 - f' := Q_2(f) - f'.$$

We may now use Corollary 3 in [BQ00], which says that there exists a bounded function f on $[\min u, \max u] \subset (\min v, \max v)$ such that $f' < \min\{Q_1(f), Q_2(f)\}$.

In view of (4.34), we know that by our choice of ψ and ϕ, $P(x_0) \leq 0$, and hence $P(x) \leq 0$ for every $x \in \Omega$, which leads to (4.28).

Now we consider the case $x_0 \in \partial\Omega$. Suppose that P attains its maximum at $x_0 \in \partial\Omega$. Consider a new vector field $V(x) = (V^i(x))_{i=1}^n$ defined on $\partial\Omega$ by

$$V^i(x) = \sum_{j=1}^n a_{ij}(\nabla u(x))\nu^j(x).$$

Thanks to the positivity of a_{ij}, $V(x)$ must point outward. Hence $\frac{\partial P}{\partial V}(x_0) \geq 0$.

On the other hand, we see from the Neumann boundary condition and homogeneity of F that

$$\frac{\partial u}{\partial V}(x_0) = u_i a_{ij}(\nabla u(x))\nu^j = FF_j\nu^j = 0.$$

Thus we have

$$0 \leq \frac{\partial P}{\partial V}(x_0) = \psi FF_i u_{ij} a_{jk}\nu^k. \tag{4.35}$$

Choose now local coordinate $\{e_i\}_{i=1,\cdots,n}$ around x_0 such that $e_n = \nu$ and $\{e_\alpha\}_{\alpha=1,\cdots,n-1}$ is the orthonormal basis of tangent space of $\partial\Omega$. Denote by $h_{\alpha\beta}$ the second fundamental form of $\partial\Omega$. By the assumption that $\partial\Omega$ is weakly convex, we know the matrix $(h_{\alpha\beta}) \geq 0$.

The Neumann boundary condition implies

$$F_i\nu^i(x_0) = F_n(x_0) = 0. \tag{4.36}$$

By taking tangential derivative of (4.36), we have

$$D_{e_\beta}(\sum_{i=1}^n F_i\nu^i)(x_0) = 0,$$

for any $\beta = 1, \cdots, n-1$. Computing $D_{e_\beta}(\sum_{i=1}^n F_i\nu^i)(x_0)$ explicitly, we have

$$\begin{aligned} 0 = D_{e_\beta}(\sum_{i=1}^n F_i\nu^i)(x_0) &= \sum_{i,j=1}^n F_{ij}u_{j\beta}\nu^i + \sum_{i=1}^n F_i\nu^i_\beta \\ &= \sum_{i,j=1}^n F_{ij}u_{j\beta}\nu^i + \sum_{i=1}^n\sum_{\gamma=1}^{n-1} F_i h_{\beta\gamma}e^i_\gamma \\ &= \sum_{j=1}^n F_{nj}u_{j\beta} + \sum_{\gamma=1}^{n-1} F_\gamma h_{\beta\gamma}. \end{aligned} \tag{4.37}$$

In the last equality we have used $\nu_n = 1$ and $\nu_\beta = 0$ for $\beta = 1, \cdots, n-1$ in the chosen coordinate.

Combining (4.35), (4.36) and (4.37), we obtain

$$\begin{aligned} 0 \le \frac{\partial P}{\partial V}(x_0) &= \sum_{i,j,k=1}^{n} \psi F F_i u_{ij} a_{jk} \nu^k = \psi F \sum_{\alpha=1}^{n-1} \sum_{j=1}^{n} F_\alpha u_{\alpha j} a_{jn} \\ &= \psi F \sum_{\alpha=1}^{n-1} \sum_{j=1}^{n} F_\alpha u_{\alpha j} F_{jn} = -\psi F \sum_{\alpha,\gamma=1}^{n-1} F_\alpha F_\gamma h_{\alpha\gamma} \le 0. \end{aligned}$$

Therefore we obtain that $\frac{\partial P}{\partial V}(x_0) = 0$. Since the tangent derivatives of P also vanishes, we have $\nabla P(x_0) = 0$. It's also the case that (4.30) holds. Thus the previous proof for an interior maximum also works in this case. This finishes the proof of Theorem 4.15. □

Remark 4.16. *Theorem 4.15 still holds if we replace $T(t) = \frac{n-1}{t}$ by $T(t) = \frac{N-1}{t}$ for $N > n$, but without change of the dimension of Ω. To see this, we need just to apply Lemma 4.13 with $N > n$.*

Another ingredient is a comparison theorem for the maxima of the eigenfunctions.

Theorem 4.17. *Let Ω, u, λ_1 be as in Theorem 4.10. Let v be a solution of the 1-D model problem on some interval $(0, \infty)$:*

$$v'' = -\frac{n-1}{t} v' - \lambda_1 v, \quad v(0) = -1, \quad v'(0) = 0.$$

Let b be the first number after 0 with $v'(b) = 0$ and denote $m = v(b)$. Then $\max u \ge m$.

Proof. We argue by contradiction. Suppose $\max(u) < m$. Then $[\min u, \max u] \subset [\min v, \max v]$. Note that $v' > 0$ in (a, b). Hence we could apply Theorem 4.15 for u and v. Denote by $d\mu_n = t^{n-1} dt$.

The same argument as Theorem 12 in [BQ00] implies that the ratio

$$R(c) = \frac{\int_{\{u \le c\}} u dx}{\int_{\{v \le c\}} v d\mu_n}$$

is increasing on $[\min(u), 0]$ and decreasing on $[0, \max(u)]$. Therefore, for $c \le -\frac{1}{2}$, we have that

$$\mathcal{L}^n(\{u \le c\}) \le 2 \int_{\{u \le c\}} |u| dx \le 2R(0) \int_{\{v \le c\}} |v| d\mu_n \le 2R(0) \mu_n(\{v \le c\}). \quad (4.38)$$

Let $c = -1 + \varepsilon$ for $\varepsilon > 0$ small. A simple calculation gives that $v''(0) = \frac{\lambda_1}{n}$. Hence for t close to a, $v''(t)$ has positive lower and upper bound. Together with $v'(0) = 0$, we see that $v(t) - v(0) \geq Ct^2$. Thus if $t \in \{v \leq -1 + \varepsilon\}$, then $t \in (0, C\varepsilon^{\frac{1}{2}})$. It follows that

$$\mu_n(\{v \leq -1 + \varepsilon\}) \leq \mu_n((0, C\varepsilon^{\frac{1}{2}})) \leq C\varepsilon^{n/2}. \tag{4.39}$$

On the other hand, we shall prove that

$$\mathcal{L}^n(\{u \leq -1 + \varepsilon\}) \geq \mathcal{L}^n(B(x_0, C\varepsilon^{\frac{1}{2}}) \bigcap \overline{\Omega}). \tag{4.40}$$

Let $x_0 \in \overline{\Omega}$ be such that $u(x_0) = -1$. For any $x \in B(x_0, \delta) \bigcap \overline{\Omega}$ with δ small, $u(x)$ is close to -1 and $s := v^{-1}(u(x))$ is close to 0. Thus we see again from the upper bound of v'' and $v'(0) = 0$ that $v'(s) \leq Cs$. Therefore, we have from Theorem 4.15 that $F(\nabla u(x)) \leq v'(v^{-1}(u(x))) \leq Cs$ and $F(\nabla v^{-1}(u(x))) = (v^{-1})'(u(x))F(x, \nabla u(x)) \leq 1$. In turn, we get

$$s = v^{-1}(u(x)) - v^{-1}(u(x_0)) \leq F(\nabla v^{-1}(u(\tilde{x})))\delta \leq \delta,$$

and

$$u(x) \leq u(x_0) + F(\nabla u(\tilde{\tilde{x}}))\delta \leq -1 + Cs\delta \leq -1 + C\delta^2,$$

for some $\tilde{x}, \tilde{\tilde{x}} \in B(x_0, \delta) \bigcap \overline{\Omega}$. Let $\varepsilon = C\delta^2$, we conclude $B(x_0, \delta) \bigcap \overline{\Omega} \subset \{u \leq -1 + \varepsilon\}$, which implies (4.40).

Combining (4.38), (4.39) and (4.40), we see that there exists some constant $C > 0$ such that

$$\mathcal{L}^n(B(x_0, r) \bigcap \overline{\Omega}) \leq Cr^n. \tag{4.41}$$

We shall get a contradiction as follows. Define v_N as a solution of the 1-D model problem on some interval $(0, \infty)$:

$$v'' = -\frac{N-1}{t}v' - \lambda_1 v, \quad v(0) = -1, \quad v'(0) = 0.$$

Let b_N be the first number after 0 with $v_N'(b) = 0$ and denote $m_N = v_N(b)$. Then m_N is continuous with respect to N. Since $\max(u) < m = m_n$, it follows from the continuity of m_N that $\max(u) < m_N$ for any $N > n$ close to N. From Remark 4.16, Theorem 4.15 still holds for $N > n$. Hence the above argument yields (4.41) with N instead of n for any $N > n$ close to N, i.e.,

$$m(B(x_0, r) \bigcap \overline{\Omega}) \leq Cr^N. \tag{4.42}$$

However, $B(x_0, r) \bigcap \overline{\Omega}$ is a set in $\mathbb{R}^n$, we have $m(B^{\pm}(x_0, r)) \geq Cr^n$ for $r > 0$ small. This causes a contradiction to (4.42). That means, our assumption $\max(u) < m$ can not hold. We get $\max(u) \geq m$. □

Following the idea of [BQ00], Besides the comparison theorem on the gradient and maxima, in order to prove Theorem 4.10, we need to study many properties of the 1-D models, such as the difference $\delta(a) = b(a) - a$ as a function of $a \in [0, +\infty]$, where $b(a)$ is the first number that $v'(b(a)) = 0$ (Note that $v' > 0$ in $(a, b(a))$). As we already saw in Theorem 4.15, the 1-D model (4.27) appears the same as that in the Laplacian case. Therefore, we can use directly the results of [BQ00] on the properties of the 1-D models. Here we use some simpler statement from [Va11].

We define $\delta(a)$ as a function of $a \in [0, +\infty]$ as follows. On one hand, we denote $\delta(\infty) = \frac{\pi}{\sqrt{\lambda_1}}$. This number comes from the 1-D model (4.27) with $T = 0$. In fact, it is easy to see that solutions of the 1-D model (4.27) with $T = 0$ can be explicitly written as

$$v(t) = \sin \sqrt{\lambda_1} t$$

up to dilations. Hence in this case, $b(a) - a = \frac{\pi}{\sqrt{\lambda_1}}$ for any $a \in \mathbb{R}$. On the other hand, we denote $\delta(a) = b(a) - a$ as a function of $a \in [0, +\infty)$ relative to the 1-D model (4.27) with $T = -\frac{n-1}{x}$.

The following property of $\delta(a)$ was proved in [BQ00, Va11].

Lemma 4.18 ([BQ00] or [Va11], Th. 5.3, Cor. 5.4)**.** *The function $\delta(a) : [0, \infty] \to \mathbb{R}^+$ is a continuous function such that*

$$\delta(a) > \frac{\pi}{\sqrt{\lambda_1}},$$

$$\delta(\infty) = \frac{\pi}{\sqrt{\lambda_1}}.$$

$m(a) := v(b(a)) < 1$, $\lim_{a\to\infty} m(a) = 1$ and $m(a) = 1$ if and only if $a = \infty$.

Now we are in a position to prove Theorem 4.10.

Proof of Theorem 4.10. Let u be an eigenfunction with eigenvalue λ_1. Since $\int u = 0$, we may assume $\min u = -1$ and $0 \le k = \max u \le 1$. Given a solution v to (4.27), denote $m(a) = v(b(a))$ with $b(a)$ the first number with $v'(b(a)) = 0$ after a.

Theorem 4.17 and Lemma 4.18 imply that for any eigenfunction u, there exists a solution v to (4.27) such that $\min v = \min u = -1$ and $\max v = \max u = k \le 1$.

We now get the expected estimate by using Theorem 4.15. Choosing $x_1, x_2 \in \overline{\Omega}$ with $u(x_1) = \min u = -1, u(x_2) = \max u = k$ and $\gamma(t) : [0, 1] \to \overline{\Omega}$ the minimal geodesic from x_1 to x_2. Consider the subset I of [0,1] such that $\frac{d}{dt} u(\gamma(t)) \ge 0$. By the gradient comparison estimate (4.28), Lemma 4.18 and Cauchy-Schwarz inequality

(1.2), we have

$$\begin{aligned} d_F &\geq \int_0^1 F^0(\dot{\gamma}(t))dt \geq \int_I F^0(\dot{\gamma}(t))dt \\ &\geq \int_0^1 \frac{1}{F(\nabla u)} \langle \nabla u, \dot{\gamma}(t) \rangle dt = \int_{-1}^k \frac{1}{F(\nabla u)} du \\ &\geq \int_{-1}^k \frac{1}{v'(v^{-1}(u))} du = \int_a^{b(a)} dt = \delta(a) \geq \frac{\pi}{\sqrt{\lambda_1}}, \end{aligned}$$

which leads to

$$\lambda_1 \geq \frac{\pi^2}{d_F^2}.$$

We are remained to prove the equality case. The idea of proof follows from [HaWa07]. Here we need to pay more attention on the points with vanishing ∇u.

Assume that $\lambda_1 = \frac{\pi^2}{d_F^2}$. It can be easily seen from the proof of Theorem 4.10 that $a = \infty$, which leads to $\max u = \max v = 1$ by Lemma 4.18. We will prove that Ω is in fact a segment in $\mathbb{R}$. We divide the proof into several steps.

Step 1: $S := \{x \in \overline{\Omega} | u(x) = \pm 1\} \subset \partial\Omega$.

Let $\mathcal{P} = F(\nabla u)^2 + \lambda_1 u^2$. After a simple calculation by using Bochner formula (2.7) and Kato inequality (4.25), we obtain

$$\begin{aligned} \frac{1}{2} a_{ij} \mathcal{P}_{ij} &= a_{ij} a_{kl} u_{ik} u_{jl} - \frac{1}{2} a_{ijl} u_{ij} \mathcal{P}_l - \lambda_1^2 u^2 \\ &\geq a_{ij} F_k F_l u_{ik} u_{jl} - \frac{1}{2} a_{ijl} u_{ij} \mathcal{P}_l - \lambda_1^2 u^2 \\ &= -\frac{1}{2} a_{ijl} u_{ij} \mathcal{P}_l + \frac{1}{4F^2} (a_{ij} \mathcal{P}_i \mathcal{P}_j - 4\lambda_1 u u_i \mathcal{P}_i) \text{ on } \Omega \setminus \mathcal{C}. \end{aligned}$$

Namely,

$$\frac{1}{2} a_{ij} \mathcal{P}_{ij} + b_i \mathcal{P}_i \geq 0 \text{ on } \Omega \setminus \mathcal{C} \tag{4.43}$$

for some $b_i \in C^0(\Omega)$. If $\mathcal{P}$ attains its maximum on $x_0 \in \partial\Omega$, then arguing as in Theorem 4.15, we have that $\nabla \mathcal{P}(x_0) = 0$. However, from the Hopf Theorem, $\nabla \mathcal{P}(x_0) \neq 0$, a contradiction. Hence $\mathcal{P}$ attains its maximum at $\mathcal{C}$, and therefore,

$$\mathcal{P} \leq \lambda_1. \tag{4.44}$$

Take any two points $x_1, x_2 \in S$ with $u(x_1) = -1, u(x_2) = 1$. Let

$$\gamma(t) = \left(1 - \frac{t}{F^0(x_2 - x_1)}\right) x_1 + \frac{t}{F^0(x_2 - x_1)} x_2 : [0, l] \to \overline{\Omega}$$

be the straight line from x_1 to x_2, where $l := F^0(x_2 - x_1)$ is the distance from x_1 to x_2 with respect to F. Denote $f(t) := u(\gamma(t))$. It is easy to see $F^0(\dot{\gamma}(t)) = 1$. It follows from (4.44) and Cauchy-Schwarz inequality (1.2) that

$$|f'(t)| = |\nabla u(\gamma(t)) \cdot \dot{\gamma}(t)| \leq F(\nabla u)(\gamma(t)) \leq \sqrt{\lambda_1(1 - f(t)^2)}. \tag{4.45}$$

Here we have used the Cauchy-Schwarz inequality (1.2) again. Hence

$$\begin{aligned} d_F \geq l &\geq \int_{\{0 \leq t \leq l,\, f'(t) > 0\}} dt \geq \int_0^l \frac{1}{\sqrt{\lambda_1}} \frac{f'(t)}{\sqrt{1 - f(t)^2}} dt \\ &= \frac{1}{\sqrt{\lambda_1}} \int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} dx = \frac{\pi}{\sqrt{\lambda_1}}. \end{aligned} \tag{4.46}$$

Since $d_F = \frac{\pi}{\sqrt{\lambda_1}}$, we must have $d_F = l$, which means $S \subset \partial\Omega$.

Step 2: $\mathcal{P} = F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ in $\overline{\Omega}$, hence $S \equiv \mathcal{C}$.

Indeed, from Step 1, we know that $\Omega^* := \overline{\Omega} \setminus S$ is connected. Let $E := \{x \in \Omega^* : \mathcal{P} = \lambda_1\}$. It is clear that E is closed. In view of (4.43), thanks to the strong maximum principle we know that E is also open. we now show that E is nonempty. Indeed, from the fact that all inequalities in (4.45) and (4.46) are equality, we obtain $f(t) = u(\gamma(t)) = -\cos\sqrt{\lambda_1}t$ for $t \in (0, l)$. Hence

$$\mathcal{P}(\gamma(t)) = f'(t)^2 + \lambda_1 f(t)^2 = \lambda_1.$$

Thus E is nonempty, open, closed in Ω^*. Therefore, we obtain $\mathcal{P} \equiv \lambda_1$ in $\overline{\Omega}$ (for $x \in S$, $\mathcal{P} = \lambda_1$ is obvious).

Step 3: Define $X = \frac{\nabla u}{F(\nabla u)}$ in Ω^* and X^* the cotangent vector given by $X^*(Y) = \langle X, Y \rangle$ for any tangent vector Y. Then in Ω^*, we claim that

$$D^2 u = -\lambda_1 u X^* \otimes X^*, \tag{4.47}$$

and moreover $X = \overrightarrow{c}$ for some constant vector $\overrightarrow{c}$.

First, taking derivative of $F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ gives

$$F_i F_j u_{ij} = -\lambda_1 u. \tag{4.48}$$

On the other hand, since $\mathcal{P} \equiv \lambda_1$, the proof of (4.43) leads to

$$a_{ij} a_{kl} u_{ik} u_{jl} = \lambda_1^2 u^2 = (F_i F_j u_{ij})^2. \tag{4.49}$$

(4.49) in fact gives that

$$F_{ij} F_{kl} u_{ik} u_{jl} = 0. \tag{4.50}$$

Set $X^\perp := \{V \in \mathbb{R}^n | V \perp X\}$. $X^\perp$ is an $(n-1)$-dim vector subspace. Note that (F_{ij}) is exactly matrix of rank $n-1$ (see the proof of Lemma 4.13) and $F_{ij}X^j = 0$. It follows from this fact and (4.50) that

$$u_{ij}V^iV^j = 0 \text{ for any } V \in X^\perp. \tag{4.51}$$

(4.48) and (4.51) imply (4.47), which in turn implies

$$u_{ij} = \frac{-\lambda_1 u u_i u_j}{F^2(\nabla u)}. \tag{4.52}$$

By differentiating X, we obtain from (4.52) that

$$\nabla_i X^j = \frac{u_{ij}}{F(\nabla u)} - \frac{u_j}{F^2(\nabla u)} F_k u_{ki} = 0.$$

Thus $X = \overrightarrow{c}$ in Ω^*.

Step 4: The maximum point and the minimum point are unique.

We already knew that $f(t) = u(\gamma(t)) = -\cos\sqrt{\lambda_1}t$ and $\nabla u(\gamma(t)) \neq 0$ for $t \in (0, l)$. Hence u is C^2 along $\gamma(t)$ for $t \in (0, l)$ and it follows that

$$D^2 u\left(\dot\gamma(t), \dot\gamma(t)\right)\Bigg|_{\gamma(t)} = \lambda_1 \cos t \text{ for any } t \in (0, l). \tag{4.53}$$

On the other hand, we deduce from (4.47) that

$$D^2 u\left(\dot\gamma(t), \dot\gamma(t)\right)\Bigg|_{\gamma(t)} = -\lambda_1 u(\gamma(t))\langle X, \dot\gamma(t)\rangle^2. \tag{4.54}$$

Combining (4.53) and (4.54), taking $t \to 0$, we get

$$|\langle X, \dot\gamma(t)\rangle| = 1 = F(X)F^0(\dot\gamma(t)),$$

which means equality in Cauchy-Schwarz inequality (1.2) holds. Hence $X = \pm F^0_\xi(\dot\gamma(t))$. Noting that $\dot\gamma(t) = \frac{x_2 - x_1}{F^0(x_2 - x_1)}$, we have

$$X = F^0_\xi(x_2 - x_1).$$

Suppose there is some point x_3 with $u(x_3) = 1$, using the same argument, we obtain $X = F^0_\xi(x_3 - x_1)$. In view of $F^0(x_3 - x_1) = F^0(x_2 - x_1)$, we conclude $x_3 = x_2$. Therefore, there is only one maximum point as well as one minimum point.

Step 5: Ω is a segment in $\mathbb{R}$.

Suppose $\Omega \subset \mathbb{R}^n$ for $n \geq 2$. We see from Step 4 that for most of points of $\partial\Omega$, $\nabla u \neq 0$, and at these points $X = \frac{\nabla u}{F(\nabla u)}$ lies in the tangent spaces due to the Neumann boundary condition, which is impossible because X is a constant vector, a contradiction. We complete the proof. □

4.3 Estimate for the first Dirichlet eigenvalue

In this section we will prove a rough lower bound of the first Dirichlet eigenvalue, by using Li-Yau's method of gradient estimates.

Theorem 4.19. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$. Assume that λ_1^D are the first Dirichlet eigenvalue of the anisotropic Laplacian. Assume further that $\partial\Omega$ is weakly F-mean convex (see section 1.3). Then λ_1^N satisfies*

$$\lambda_1^D \geq \frac{\pi^2}{4i_F^2}. \tag{4.55}$$

In the theorem, i_F is the *inscribed radius* of Ω with respect to the norm F on $\mathbb{R}^n$, which is defined as the radius r of the biggest Wulff ball $\mathcal{W}_r(x)$ that can be enclosed in $\overline{\Omega}$.

As before, for simplicity, we write λ_1 instead of λ_1^D through this section.

It is clear that the existence of first Dirichlet eigenfunction can be easily proved by using the direct method in the calculus of variations. Moreover, we may assume u is non-negative. It was proved in [KaNo08] that u is positive. By multiplying u by a constant, we can also assume that $\sup_\Omega u = 1$ and $\inf_\Omega u = 0$ without loss of generality.

For any $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0, \beta^2 > \sup(\alpha + u)^2$, consider function

$$P(x) = \frac{F^2(\nabla u)}{2(\beta^2 - (\alpha + u)^2)}.$$

Suppose that $P(x)$ attains its maximum at $x_0 \in \overline{\Omega}$.

With the assumption that Ω is weakly F-mean convex, we first exclude the possibility $x_0 \in \partial\Omega$ with $\nabla u(x_0) \neq 0$. Indeed, suppose we have $x_0 \in \partial\Omega$ with $\nabla u(x_0) \neq 0$. Define $\nu_F := F_\xi(\nu)$ on $\partial\Omega = \{x \in \overline{\Omega} | u(x) = 0\}$. In view of $\langle \nu_F, \nu \rangle = F(\nu) > 0$, ν_F must point outward. From the Dirichlet boundary condition, we know $\nu = -\frac{\nabla u}{|\nabla u|}$ for $\nabla u \neq 0$. Hence $\nu_F = -F_\xi(\nabla u)$. Since P attains maximum at x_0, we have

$$0 \leq \frac{\partial P}{\partial \nu_F}(x_0) \quad = \quad \frac{F F_i u_{ij} \nu_F^j}{\beta^2 - (\alpha + u)^2} + F^2 \frac{\alpha \frac{\partial u}{\partial \nu_F}}{(\beta^2 - (\alpha + u)^2)^2}$$

Hence

$$-\frac{\partial^2 u}{\partial \nu_F^2} + \frac{F \alpha \frac{\partial u}{\partial \nu_F}}{\beta^2 - \alpha^2} \geq 0.$$

Note that $\frac{\partial u}{\partial \nu_F} = -F(\nabla u)$. Since $\partial\Omega$ itself is a level set of u, we can apply Theorem 1.11 to obtain

$$\frac{\partial^2 u}{\partial \nu_F^2} = Qu + FH_F.$$

In view of $Qu(x_0) = -\lambda_1 u(x_0) = 0$, we obtain that

$$-FH_F - F^2\frac{\alpha}{\beta^2 - \alpha^2} \geq 0.$$

This contradicts the fact that $H_F(\partial\Omega) \geq 0$.

On the other hand, if $\nabla u(x_0) = 0$, then $F(\nabla u)(x_0) = 0$ and $P(x_0) = 0$ which implies $F(\nabla u) = 0$, i.e., u is constant, a contradiction.

Therefore we may assume $x_0 \in \Omega$ and $\nabla u(x_0) \neq 0$. Since a_{ij} is positively definite on $\overline{\Omega} \setminus \mathcal{C}$, where $\mathcal{C} := \{x|\nabla u(x) = 0\}$, it follows from the maximum principle that

$$P_i(x_0) = 0, \tag{4.56}$$

$$a_{ij}(x_0)P_{ij}(x_0) \leq 0. \tag{4.57}$$

From now on we will compute at the point x_0. Equality (4.56) gives

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2(\nabla u)\right) = -\frac{F^2(\nabla u)(\alpha+u)u_i}{\beta^2 - (\alpha+u)^2}. \tag{4.58}$$

Then we compute $a_{ij}(x_0)P_{ij}(x_0)$.

$$\begin{aligned} a_{ij}(x_0)P_{ij}(x_0) = \quad & \frac{1}{\beta^2 - (\alpha+u)^2} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{2}F^2(\nabla u)\right) \\ & + 2a_{ij}\frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2(\nabla u)\right)\frac{\partial}{\partial x_j}\left(\frac{1}{\beta^2 - (\alpha+u)^2}\right) \\ & + a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{\beta^2 - (\alpha+u)^2}\right)\frac{1}{2}F^2(\nabla u) \\ = \quad & I + II + III. \end{aligned}$$

By using (4.58), (4.32), Bochner formula (2.7) and equation (0.3), we obtain

$$I = \frac{1}{\beta^2 - (\alpha+u)^2}\left[a_{ij}a_{kl}u_{ik}u_{jl} - \lambda_1 F^2\right], \tag{4.59}$$

$$II = -\frac{4F^4(\alpha+u)^2}{(\beta^2 - (\alpha+u)^2)^3}, \tag{4.60}$$

$$III = \frac{F^4}{(\beta^2 - (\alpha+u)^2)^2} + \frac{4F^4(\alpha+u)^2}{(\beta^2 - (\alpha+u)^2)^3} - \frac{\lambda_1 F^2 u(\alpha+u)}{(\beta^2 - (\alpha+u)^2)^2}. \tag{4.61}$$

We now apply Lemma 4.11 to (4.59) and obtain

$$\begin{aligned} a_{ij}a_{kl}u_{ik}u_{jl} &\geq a_{ij}F_kF_lu_{ik}u_{jl} \\ &= \frac{1}{F^2}a_{ij}\frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2(\nabla u)\right)\frac{\partial}{\partial x_j}\left(\frac{1}{2}F^2(\nabla u)\right) \\ &= \frac{F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^2}. \end{aligned}$$

Here we have used (4.58) and (4.32) again in the last equality. Therefore, we have

$$I \geq \frac{F^4(\alpha+u)^2}{(\beta^2-(\alpha+u)^2)^3} - \frac{\lambda_1 F^2}{\beta^2-(\alpha+u)^2}. \tag{4.62}$$

Combining (4.57), (4.60), (4.61) and (4.62), we obtain

$$0 \geq a_{ij}P_{ij} \geq \frac{F^4\beta^2}{(\beta^2-(\alpha+u)^2)^3} - \frac{\lambda_1 F^2}{\beta^2-(\alpha+u)^2} - \frac{\lambda_1 F^2 u(\alpha+u)}{(\beta^2-(\alpha+u)^2)^2}.$$

It follows that

$$\frac{F^2(\nabla u)}{\beta^2-(\alpha+u)^2}(x_0) \leq \frac{\lambda_1}{\beta^2}(\beta^2-\alpha(\alpha+u)). \tag{4.63}$$

Noting that $\sup_\Omega u = 1$ we choose $\alpha > 0$ and $\beta = \alpha+1$. Then estimate (4.63) becomes

$$\frac{F^2(\nabla u)}{(\alpha+1)^2-(\alpha+u)^2}(x_0) \leq \lambda_1\left(1-\frac{\alpha(\alpha+u)}{(\alpha+1)^2}\right) \leq \lambda_1.$$

Hence we conclude, for any $x \in \overline{\Omega}$,

$$\frac{F^2(\nabla u)}{(\alpha+1)^2-(\alpha+u)^2} \leq \lambda_1. \tag{4.64}$$

Choose $x_1 \in \Omega$ with $u(x_1) = \sup u = 1$ and $x_2 \in \partial\Omega$ with $d_F(x_1,x_2) = d_F(x_1,\partial\Omega) \leq i_F$ and $\gamma(t) : [0,1] \to \overline{\Omega}$ the minimal geodesic connected x_1 with x_2. Using the gradient estimates (4.64), we have

$$\begin{aligned} \frac{\pi}{2} - \arcsin\left(\frac{\alpha}{\alpha+1}\right) &= \int_0^1 \frac{1}{\sqrt{(\alpha+1)^2-(\alpha+u)^2}}du \leq \sqrt{\lambda_1}\int_0^1 \frac{1}{F(\nabla u)}du \\ &\leq \sqrt{\lambda_1}\int_0^1 \frac{1}{F(\nabla u(\gamma(t)))}\langle \nabla u(\gamma(t)), \dot\gamma(t)\rangle dt \\ &\leq \sqrt{\lambda_1}\int_0^1 F^0(\dot\gamma(t))dt \leq \sqrt{\lambda_1}\, i_F. \end{aligned}$$

Here we have used the Cauchy-Schwarz inequality (1.2). Letting $\alpha \to 0$, we obtain

$$\lambda_1 \geq \frac{\pi^2}{4i_F^2}.$$

Thus we finish the proof of Theorem 4.19.

Chapter 5

Anisotropic Liouville equations in two dimensions

In this chapter we will study the blow-up analysis for anisotropic Liouville equations

$$-Qu = V(x)e^u,$$

and

$$-Qu = \lambda \frac{e^u}{\int_\Omega e^u}.$$

Thoughout this chapter, we assume that the norm $F \in C^2(\mathbb{R}^n \backslash \{0\})$ and F^2 is strongly convex in $\mathbb{R}^n \backslash \{0\}$.

5.1 A sharp Moser-Trudinger type inequality

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $u \in W_0^{1,n}(\Omega)$ and $\int_\Omega F(\nabla u)^n dx \leq 1$. Then there exists a constant $C(n)$, such that*

$$\int_\Omega \exp[\lambda u^{\frac{n}{n-1}}]dx \leq C(n)|\Omega|, \tag{5.1}$$

where $\lambda \leq \lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$. λ_n is optimal in the sense that if $\lambda > \lambda_n$ we can find a sequence (u_k) such that $\int_\Omega exp[\lambda u_k^{\frac{n}{n-1}}]dx$ diverges.

Proof. As Moser did in [Mo71], we use the convex symmetrization to reduce the problem to one dimensional case. Set $F^0(x) = r(t) = Re^{-\frac{t}{n}}$, $w(t) = n\kappa_n^{\frac{1}{n}} u^\star(x)$. Here $R > 0$ is the constant such that $|\Omega| = \kappa_n R^n$. It is easy to verify that

$$\int_{\Omega^\star} F(\nabla u^\star)^n dx = \int_0^\infty w'(t)^n dt,$$

$$\int_{\Omega^\star} \exp[\lambda u^\star(x)^p]dx = \int_0^\infty \exp[\beta w(t)^{\frac{n}{n-1}} - t]dt,$$

where $\beta = \frac{\lambda}{\lambda_n}$. In views of (1.13), it suffices to prove:

If $w(t)$ is a C^1 function on $0 \le t \le \infty$ satisfying

$$w(0) = 0, w'(t) \ge 0, \int_0^\infty w'(t)^n dt \le 1,$$

then

$$\int_0^\infty \exp[\beta w(t)^{\frac{n}{n-1}} - t]dt \le C, \text{ provided } \beta \le 1.$$

For $\beta > 1$, the integral $\int_0^\infty \exp[\beta w(t)^{\frac{n}{n-1}} - t]dt$ can be made arbitrarily large. This was proved in [Mo71].

□

A direct consequence of Theorem 5.1 is a slightly weaker, but more applicable form in two dimensions.

Corollary 5.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $u \in W_0^{1,2}(\Omega)$. Define the anisotropic Moser-Trudinger functional $J_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R}$ by*

$$J_\lambda(u) = \frac{1}{2}\int_\Omega F(\nabla u)^2 - \lambda \log \int_\Omega e^u.$$

Then J_λ has a lower bound if and only if $\lambda \le 8\kappa$.

Proof. The "if" part follows directly from Theorem 5.1. For $\lambda > 8\kappa$, assume that Ω contains a Wulff ball $\mathcal{W}_\epsilon$ for some small $\epsilon > 0$. We construct the following functions in $W_0^{1,2}(\Omega)$,

$$u_a(x) = \begin{cases} -2\log \dfrac{1+\kappa a\epsilon^2}{1+\kappa a F^0(x)^2} & \text{in} \quad \mathcal{W}_\epsilon \\ 0 & \text{in} \quad \Omega \setminus \mathcal{W}_\epsilon. \end{cases}$$

A direct computation gives

$$\frac{1}{2}\int_\Omega F(\nabla u_a)^2 = 8\kappa \log a + O(1),$$

$$\log \int_\Omega e^{u_a} = \log a + O(1).$$

Hence

$$\lim_{a\to\infty} J_\lambda(u_a) = \lim_{a\to\infty} (8\kappa - \lambda)\log a = -\infty,$$

which means that J_λ has no lower bound when $\lambda > 8\kappa$. □

5.2 Brezis-Merle type concentration-compactness phenomena

In this section we first use a level set method as in [ReWe95] to generalize a Brezis-Merle inequality for the anisotropic operator Q. For simplicity, we write κ instead of κ_2.

Theorem 5.3. *Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let u be a weak solution of*

$$\begin{cases} -Qu = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $f \in L^1(\Omega)$. Then for every $\delta \in (0, 4\kappa)$ we have

$$\int_\Omega \exp\left[\frac{(4\kappa - \delta)}{\|f\|_{L^1}}|u(x)|\right] dx \leq \frac{4\kappa}{\delta}|\Omega|, \tag{5.3}$$

where $|\Omega|$ denotes the volume of Ω.

Proof. Consider the unique solution v of the symmetrized Dirichlet problem (1.38). It follows from Theorem 1.16 that

$$u^\star \leq v \text{ in } \Omega^\star.$$

It is clear that $v(x) = v(F^0(x))$ is symmetric with respect to F, and satisfies the following ODE:

$$\begin{cases} \frac{1}{r}\left(-rv'(r)\right)' = f^\star(x) \\ v(R) = 0, \qquad v'(0) = 0, \end{cases}$$

where $r = F^0(x)$, $R > 0$ is the constant such that $|\Omega| = \kappa R^2$. Therefore

$$-v'(r) = \frac{1}{r}\int_0^r tf^*(\kappa t^2)dt \leq \frac{1}{2\kappa r}\|f^\star\|_{L^1(\Omega^\star)} = \frac{1}{2\kappa r}\|f\|_{L^1(\Omega)},$$

where we used that

$$\int_{\Omega^\star} f^\star(x)dx = \int_0^R 2\kappa t f^*(\kappa t^2)dt,$$

which follows from the coarea formula with respect to F^0. It then follows that

$$v(r) = -\int_r^R v'(t)dt \leq \int_r^R \frac{1}{2\kappa t}\|f\|_{L^1(\Omega)}dt = \frac{\|f\|_{L^1}}{2\kappa}\log\frac{R}{r}.$$

Hence we have

$$\begin{aligned}\int_\Omega \exp\left[\frac{2\kappa(2-\epsilon)|u(x)|}{\|f\|_{L^1}}\right]dx &= \int_{\Omega^\star} \exp\left[\frac{2\kappa(2-\epsilon)u^\star(x)}{\|f\|_{L^1}}\right]dx \\ &\le \int_{\Omega^\star} \exp\left[\frac{2\kappa(2-\epsilon)v}{\|f\|_{L^1}}\right]dx \\ &\le \int_0^R 2\kappa r \exp\left[(2-\epsilon)\log\frac{R}{r}\right]dr = \frac{2\kappa}{\epsilon}R^2 = \frac{2}{\epsilon}|\Omega|.\end{aligned}$$

Let $\delta = 2\kappa\epsilon$, we obtain (5.3). □

Corollary 5.4. *Let u be a solution of (5.2) with $f \in L^1(\Omega)$. Then for every constant $k > 0$, we have $\exp(k|u|) \in L^1(\Omega)$.*

Proof. for any $\epsilon > 0$, we split f as $f = f_1 + f_2$ with $\|f_1\|_{L^1(\Omega)} < \epsilon$ and $f_2 \in L^\infty(\Omega)$. Let u_1 be the solution of

$$\begin{cases} -Qu_1 = f(x) & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

We know from Theorem 5.3 that $\int_\Omega \exp\left[\frac{|u_1|}{\|f_1\|_{L^1}}\right] < \infty$ and thus $\int_\Omega \exp\left[k|u_1|\right] < \infty$ with $k < 1/\epsilon$. On the other hand, by the mean value Theorem,

$$f_2 = -(Qu - Qu_1) = -\widetilde{Q}(u - u_1),$$

where

$$\widetilde{Q}(u-u_1) = \frac{\partial}{\partial x_i}\left(\frac{1}{2}F^2_{\xi_i\xi_j}\big(t\nabla u + (1-t)\nabla u_1\big)\frac{\partial}{\partial x_j}\big(u-u_1\big)\right).$$

Since $Hess(F^2)$ is positive definite, $\widetilde{Q}$ is also an elliptic operator. From elliptic theory we have $\|u - u_1\|_{L^\infty} \le C\|f_2\|_{L^\infty}$. The conclusion easily follows. □

Corollary 5.5. *Let u be a weak solution of*

$$\begin{cases} -Qu = V(x)e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $V \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \le \infty$, $p' = \frac{p}{p-1}$. Then $u \in L^\infty(\Omega)$.

Proof. The conclusion follows from Corollary 5.4 and the standard elliptic theory for quasilinear equations. □

We introduce the following number

$$\begin{aligned} d_{X,Y} &= \frac{\langle F(X)F_\xi(X) - F(Y)F_\xi(Y), X - Y\rangle}{F^2(X-Y)}, \\ d_0 &= \inf\{d_{X,Y} : X, Y \in \mathbb{R}^n, X \ne 0, Y \ne 0, X \ne Y\}. \end{aligned}$$

It is clear that $d_{X,Y} = 1$ if $F(\xi) = |\xi|$. In general one can show that d_0 is bounded from below and above.

Lemma 5.6. $\min\{\frac{\lambda}{b^2}, 1\} \leq d_0 \leq 1$, *where* λ *is the smallest eigenvalue of* $Hess(F^2)$.

Proof. In the case $X = tY$ for some $t \neq 1$, it's easy to see that $d_{X,Y} = 1$. In other case, the line between X and Y does not pass through 0, hence for some $t \in [0,1]$,

$$d_{X,Y} = \frac{F^2_{\xi_i\xi_j}(tX + (1-t)Y)(X_j - Y_j)(X_i - Y_i)}{F^2(X-Y)} \geq \frac{\lambda|X-Y|^2}{b^2|X-Y|^2} = \frac{\lambda}{b^2}.$$

□

We now prove a similar Brezis-Merle inequality which associates the difference of two functions.

Theorem 5.7. *Let* u *and* v *be the weak solutions of*

$$-Qu = f(x) > 0 \ \ in\ \Omega \tag{5.4}$$

and

$$\begin{cases} -Qv &= 0 & in\ \Omega \\ \quad v &= u & on\ \partial\Omega, \end{cases} \tag{5.5}$$

respectively. Then for every $\delta \in (0, 4\kappa)$ *we have*

$$\int_\Omega \exp\left[\frac{(4\kappa - \delta)d_0}{\|f\|_{L^1}}|u-v|\right] dx \leq \frac{4\kappa}{\delta}|\Omega|. \tag{5.6}$$

Proof. We follows the level set method in [ReWe95], but in an anisotropic version. Set $\Omega_t = \{x \in \Omega : |u-v| > t\}$ and $\mu(t) = |\Omega_t|$. Making the difference of (5.4) and (5.5), we have

$$-\widetilde{Q}(u-v) = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}(u-v)\right) = f > 0,$$

where

$$a_{ij} = \frac{1}{2}F^2_{\xi_i\xi_j}\big(s\nabla u(x) + (1-s)\nabla v(x)\big).$$

for some $s \in [0,1]$. Since this equation is uniformly elliptic, we may apply Hopf's boundary lemma (See [GT98], Th.9.6) to conclude

$$\frac{\partial}{\partial \nu}(u-v) < 0, \quad \nabla u - \nabla v \neq 0 \text{ on } \partial\Omega_t.$$

It follows from (5.4) and (5.5) again that

$$\begin{aligned} \int_{\Omega_t} f(x)dx &= \int_{\Omega_t} -(Qu - Qv)dx \\ &= \int_{\partial\Omega_t} \left\langle F(\nabla u)F_\xi(\nabla u) - F(\nabla v)F_\xi(\nabla v), \frac{\nabla u - \nabla v}{|\nabla u - \nabla v|}\right\rangle \\ &\geq d_0 \int_{\partial\Omega_t} \frac{F^2(\nabla u - \nabla v)}{|\nabla u - \nabla v|}. \end{aligned}$$

Here we have used Lemma 5.6. By the isoperimetric inequality (1.29), the co-area formula (1.28) and the Hölder inequality, we have

$$\begin{aligned}
2\kappa^{\frac{1}{2}}\mu(t)^{\frac{1}{2}} &\leq P_F(\Omega_t) \\
&= -\frac{d}{dt}\int_{\Omega_t} F(\nabla u - \nabla v)dx \\
&= \int_{\partial\Omega_t} \frac{F(\nabla u - \nabla v)}{|\nabla u - \nabla v|} \\
&\leq \left(\int_{\partial\Omega_t} \frac{F^2(\nabla u - \nabla v)}{|\nabla u - \nabla v|}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega_t} \frac{1}{|\nabla u - \nabla v|}\right)^{\frac{1}{2}} \\
&= \left(\int_{\partial\Omega_t} \frac{F^2(\nabla u - \nabla v)}{|\nabla u - \nabla v|}\right)^{\frac{1}{2}} \big(-\mu'(t)\big)^{\frac{1}{2}}.
\end{aligned}$$

The above two estimates give

$$-\mu'(t) \geq \frac{4\kappa d_0 \mu(t)}{\int_{\Omega_t} f(x)dx}$$

and hence

$$-\frac{dt}{d\mu} \geq \frac{\|f\|_{L^1(\Omega)}}{4\kappa d_0 \mu}.$$

Integrating the last inequality over $(\mu, |\Omega|)$, we deduce

$$t(\mu) \leq \frac{\|f\|_{L^1(\Omega)}}{4\kappa d_0} \log\left(\frac{|\Omega|}{\mu}\right),$$

$$\exp\left(\frac{4\kappa(1-\epsilon)d_0 t(\mu)}{\|f\|_{L^1(\Omega)}}\right) \leq \left(\frac{|\Omega|}{\mu}\right)^{1-\epsilon}.$$

Using the co-area formula again, we have by integrating above inequality that

$$\begin{aligned}
\int_\Omega \exp\left[\frac{4\kappa(1-\epsilon)d_0}{\|f\|_{L^1}}|u-v|\right]dx &= \int_0^\infty \exp\left(\frac{4\kappa(1-\epsilon)d_0 t}{\|f\|_{L^1}}\right)\big(-\mu'(t)\big)dt \\
&= \int_0^{|\Omega|} \exp\left(\frac{4\kappa(1-\epsilon)d_0 t(\mu)}{\|f\|_{L^1(\Omega)}}\right)d\mu \\
&\leq \frac{|\Omega|}{\epsilon}.
\end{aligned}$$

Letting $\delta = 4\kappa\epsilon$, we get (5.6). □

We now consider a sequence (u_n) of weak solutions of (5.7).

Corollary 5.8. *Let (u_n) be a sequence of weak solutions of (5.7) with $u_n = 0$ on $\partial\Omega$ with*

$$\|V_n\|_{L^p} \leq C_1 \text{ for some } 1 < p \leq \infty,$$
$$\int_\Omega |V_n|e^{u_n} \leq \epsilon_0 < \frac{4\kappa}{p'}.$$

Then

$$\|u_n\|_{L^\infty} \leq C$$

where C only depends on C_1,$|\Omega|$ and ϵ_0.

Proof. Fix $\delta > 0$ such that $4\kappa - \delta > \epsilon_0(p' + \delta)$. By Theorem 5.3 we have

$$\int_\Omega \exp[(p' + \delta)|u_n|] \leq C.$$

Therefore e^{u_n} is bounded in $L^{p'+\delta}(\Omega)$ and $V_n e^{u_n}$ is bounded in $L^q(\Omega)$ for some $q > 1$. The conclusion now follows the standard elliptic theory for quasilinear equations. □

Next we give a variant of Corollary 5.8 without a boundary condition.

Corollary 5.9. *Let (u_n) be a sequence of weak solutions of (5.7) with*

$$V_n \geq 0, \quad \|V_n\|_{L^p} \leq C_1 \text{ for some } 1 < p \leq \infty,$$

$$\|u_n^+\|_{L^2} \leq C_2,$$
$$\int_\Omega V_n e^{u_n} \leq \epsilon_0 < \frac{4\kappa d_0}{p'}.$$

Then u_n^+ is bounded in $L^\infty_{loc}(\Omega)$.

Proof. Consider the weak solution v_n of

$$\begin{cases} -Qv_n &= 0 \quad \text{in } B_R \\ v_n &= u_n \quad \text{on } \partial B_R, \end{cases}$$

By the weak comparison principle for anisotropic operator (see [Da98]), we have

$$v_n \leq u_n \text{ in } B_R.$$

Hence

$$\|v_n^+\|_{L^2(B_R)} \leq \|u_n^+\|_{L^2(B_R)} \leq C_2.$$

Serrin's local a priori estimates(See [GT98]) implies that

$$\|v_n^+\|_{L^\infty(B_{R/2})} \leq C.$$

On the other hand, by our smallness assumption, we obtain from Theorem 5.7 and $v_n^+ \geq v_n$ that

$$\int_{B_R} \exp\left[\frac{(4\kappa-\delta)d_0}{\epsilon_0}(u_n - v_n^+)\right] dx \leq \int_{B_R} \exp\left[\frac{(4\kappa-\delta)d_0}{\|V_n e^{u_n}\|_{L^1}}(u_n - v_n)\right] dx \leq \frac{4\kappa}{\delta}|B_R|.$$

Combining this with the boundedness of v_n^+ in $L^\infty(B_{R/2})$, we obtain

$$\int_{B_{R/2}} \exp\left[\frac{(4\kappa-\delta)d_0}{\epsilon_0}u_n\right] dx \leq C.$$

Choosing δ such that $\frac{(4\kappa-\delta)d_0}{\epsilon_0} \geq p' + \delta$, we deduce that e^{u_n} is bounded in $L^{p'+\delta}(B_{R/2})$ and $V_n e^{u_n}$ is bounded in $L^q(B_{R/2})$ for some $q > 1$. By Serrin's local a priori estimates again, we have

$$\|u_n^+\|_{L^\infty(B_{R/4})} \leq C\|u_n^+\|_{L^2(B_{R/2})} \leq C.$$

□

Now we are ready to prove the following Brezis-Merle type compactness-concentration result.

Theorem 5.10. *Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of weak solutions of*

$$-Qu_n = V_n(x)e^{u_n} \text{ in } \Omega, \tag{5.7}$$

with

$$V_n \geq 0, \quad \|V_n\|_{L^p} \leq C_1 \text{ for some } 1 < p \leq \infty,$$
$$\|e^{u_n}\|_{L^{p'}} \leq C_2.$$

Define the blow-up set as follows:

$$S = \{x \in \Omega : \exists x_n \in \Omega \text{ such that } x_n \to x \text{ and } u_n(x_n) \to +\infty\}.$$

Then, one of the following possibilities happens (after taking subsequences):

(i) u_n *is bounded in* $L^\infty_{loc}(\Omega)$*;*

(ii) $u_n \to -\infty$ *uniformly on any compact subsets of* Ω*;*

(iii) $S = \{p_1, \cdots, p_m\}$ *is a finite, nonempty set, and* $u_n \to -\infty$ *uniformly on any compact subset of* $\Omega \setminus S$*. In addition,* $V_n e^{u_n} \rightharpoonup \Sigma_{i=1}^m \alpha_i \delta_{p_i}$ *in the sense of measures on* Ω *with* $\alpha_i \geq \frac{4\kappa}{p'}$ *for any* i*.*

Proof of Theorem 5.10. Since $V_n e^{u_n}$ is bounded in $L^1(\Omega)$, we may assume that there exists a nonnegative bounded measure μ such that for a subsequence (still denote by $V_n e^{u_n}$),

$$\int_\Omega V_n e^{u_n}\psi \to \int_\Omega \psi d\mu$$

for every $\psi \in C_c(\Omega)$. As in [BM91], [JoWa01], we say that a point $x \in \Omega$ is a *γ-regular point* if for some $\gamma > 0$, there exists a function $\psi \in C_c(\Omega)$, $0 \le \psi \le 1$, with $\psi = 1$ in a neighborhood of x such that

$$\int_\Omega \psi d\mu < \gamma.$$

We define

$$\Sigma(\gamma) = \{x \in \Omega : x \text{ is not a } \gamma - \text{regular point}\}.$$

It is easy to see that if $\gamma_1 \le \gamma_2$, then $\Sigma(\gamma_1) \supset \Sigma(\gamma_2)$. We also have that

$$x \in \Sigma(\gamma) \Leftrightarrow \mu(\{x\}) \ge \gamma.$$

Since μ is a bounded measure, it follows that $\Sigma(\gamma)$ is finite for any positive γ.

We split our proof by four steps.

Step 1. If x_0 is a $\frac{4\kappa}{p'}$-regular point, i.e. $x_0 \in \Omega \setminus \Sigma(4\kappa/p')$, then there exists some $R_0 > 0$ such that u_n^+ is bounded in $L^\infty(B_{R_0}(x_0))$. In fact, by Lemma 5.6, $d_0 \le 1$, there are only two cases:

(i) $x_0 \in \Omega \setminus \Sigma(4\kappa d_0/p')$;

(ii) $x_0 \in \Sigma(4\kappa d_0/p') \setminus \Sigma(4\kappa/p')$.

For the first case, the conclusion follows immediately from Corollary 5.9. We now focus on the second case.

Since $\Sigma(4\kappa d_0/p')$ is finite, we can choose some $R > 0$ small enough such that x_0 is the only point of $\Sigma(4\kappa d_0/p')$ in $\overline{B_R(x_0)}$. Hence any points on $\partial B_R(x_0)$ belong to $\Omega \setminus \Sigma(4\kappa d_0/p')$. From the conclusion for case (i) and the compactness of $\partial B_R(x_0)$, we see that u_n^+ is bounded in $L^\infty(\partial B_R(x_0))$, say by C_0.

Let w_n be the weak solution of

$$\begin{cases} -Qw_n = V_n e^{u_n} \ge 0 & \text{in } B_R(x_0) \\ w_n = C_0 \ge u_n & \text{on } \partial B_R(x_0), \end{cases}$$

The comparison principle implies that

$$w_n \ge u_n \text{ a.e. in } B_R(x_0). \tag{5.8}$$

On the other hand, it follows from Theorem 5.3 that for any $\delta \in (0, 4\kappa)$

$$\int_{B_R(x_0)} \exp\left[\frac{(4\kappa - \delta)}{\|V_n e^{u_n}\|_{L^1(B_R(x_0))}} |w_n - C_0|\right] dx \le \frac{4\kappa}{\delta} |B_R(x_0)| = \frac{4\kappa\pi R^2}{\delta}. \tag{5.9}$$

Since $x_0 \notin \Sigma(4\kappa/p')$ and R can be as small as we need, by the definition of γ-regular point, there exists $R_1 > R$ such that for n big enough and some small $\delta_0 > 0$,

$$\int_{B_{R_1}(x_0)} V_n e^{u_n} dx < \frac{4\kappa - \delta_0}{p'}.$$

We now choose $\delta < \delta_0$ small such that for some $\epsilon_0 > 0$

$$p' + \epsilon_0 < \frac{4\kappa - \delta}{\|V_n e^{u_n}\|_{L^1(B_R(x_0))}}.$$

Therefore by (5.8) and (5.9), we know that u_n^+ is bounded in $L^{p'+\epsilon_0}(B_R(x_0))$ and $V_n e^{u_n}$ is bounded in $L^q(B_R(x_0))$ for some $q > 1$. By Serrin's local a priori estimates again, we have for $R_0 = R/2$,

$$\|u_n^+\|_{L^\infty(B_{R_0})} \le C\|u_n^+\|_{L^2(B_R)} \le C.$$

Here $\|u_n^+\|_{L^2(B_R)}$ is bounded from above since e^{u_n} is bounded in $L^{p'}(\Omega)$. We complete Step 1.

Step 2. $S = \Sigma(\gamma)$ provided $\gamma < \frac{4\kappa}{p'}$. The proof follows from the same argument in [BM91] since we established the crucial claim in Step 1. It is interesting to see that $\Sigma(\gamma)$ is independent of γ if $\gamma < \frac{4\kappa}{p'}$ and hence the set $\Sigma(4\kappa d_0/p') \setminus \Sigma(4\kappa/p')$ is in fact empty.

Step 3. $S = \emptyset$ implies (i) or (ii) holds. $S = \emptyset$ means that u_n is bounded in $L^\infty_{loc}(\Omega)$. Thus, e^{u_n} is bounded in $L^p_{loc}(\Omega)$, which implies that $\mu \in L^1(\Omega) \bigcap L^p_{loc}(\Omega)$. Let v_n be the weak solution of

$$\begin{cases} -Qv_n = V_n e^{u_n} & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

Clearly, $v_n \to v$ uniformly on every compact subset of Ω, where v is the weak solution of

$$\begin{cases} -Qv = \mu & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

Let $z_n = u_n - v_n$. Then $-\widetilde{Q} z_n = 0$ in Ω and z_n is also bounded in $L^\infty_{loc}(\Omega)$. Here $\widetilde{Q} z_n$ is defined as in Corollary 5.4 by mean value Theorem. Applying the Harnack inequality for the uniformly elliptic operator $\widetilde{Q}$, we have (i) or (ii) as in [BM91].

Step 4. $S \neq \emptyset$ implies (iii) holds. As in step 3, we know that either

- u_n is bounded on any compact subset of $\Omega \setminus S$, or
- $u_n \to -\infty$ on any compact subset of $\Omega \setminus S$.

In view of step 2, $S \neq \emptyset$ implies that $\int_{B_\delta(x)} V_n e^{u_n} dx \geq \frac{4\kappa}{p'}$ for any $x \in S$ and any small $\delta > 0$. Now we can follow the argument in [BM91] to exclude the first possibility. The only difference is that we use the Green function of Q in $\mathcal{W}_R(x_0)$ instead of that of the ordinary Laplacian in a ball, namely, the function $G(x) = \frac{1}{2\kappa} \log \frac{R}{F^0(x-x_0)}$.

Combining step 3 and step 4, we complete the proof of Theorem 5.10. □

Under an extra boundary condition, we can determine α_i.

Theorem 5.11. *Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of weak solutions of (5.7) with*

$$\int_\Omega e^{u_n} \leq C.$$

(V_n) is a sequence of Lipschitz continuous functions satisfying

$$V_n \geq 0, \quad V_n \to V \text{ uniformly in } C^0(\overline{\Omega}), \|\nabla V_n\|_{L^\infty(\Omega)} \leq C, \tag{5.11}$$

In addition, we assume that

$$\max_{\partial\Omega} u_n - \min_{\partial\Omega} u_n \leq C. \tag{5.12}$$

Then if blow-up happens only at one point ((iii) in Theorem 5.10), the blow-up value $\alpha = 8\kappa$.

Proof of Theorem 5.11: Without loss of generality, we assume that u_n blow up at 0. Let v_n be the weak solution of (5.10) and $w_n = u_n - \min_{\partial\Omega} u_n - v_n$. It's easy to see

$$\begin{cases} -\widetilde{Q} w_n = & 0, & \text{in } \Omega, \\ w_n = & u_n - \min_{\partial\Omega} u_n, & \text{on } \partial\Omega. \end{cases}$$

From (5.12) and standard theory for quasilinear uniform elliptic equations, we have

$$\|w_n\|_{L^\infty(\Omega)} \leq \|w_n\|_{L^\infty(\partial\Omega)} \leq C \text{ and } \|\nabla w_n\|_{L^\infty(\Omega)} \leq C. \tag{5.13}$$

Choosing subsequence if necessary, we may assume $w_n \to w$ uniformly in $C^0(\overline{\Omega}) \bigcap C^1_{loc}(\Omega)$. Set $W_n = V_n \exp\{w_n + \min_{\partial\Omega} u_n\}$. Now we have

$$-Qv_n = V_n e^{u_n} = W_n e^{v_n}. \tag{5.14}$$

We claim that

$$\|\nabla \log W_n\|_{L^\infty(B_{r_0})} \leq C$$

for some small $r_0 > 0$. In fact, since $\nabla \log W_n = \nabla \log V_n + \nabla w_n$, in view of (5.11) and (5.13), it suffices to prove that V_n have a uniformly positive lower bound near the origin. Let $x_n \to 0, u_n(x_n) = \max_{\overline{\Omega}} u_n \to \infty$. Set $\delta_n = \exp\{-u_n(x_n)/2\}$ and $\widetilde{u_n}(x) = u_n(\delta_n x + x_n) + 2\log \delta_n$. It's easy to see that $\widetilde{u_n} \to \widetilde{u}$ locally in $C^1(\mathbb{R}^2)$, where $\widetilde{u}$ is a solution of

$$\begin{cases} -Q\widetilde{u} &= V(0)e^{\widetilde{u}}, \quad \text{in } \mathbb{R}^2, \\ \widetilde{u}(0) &= 0, \\ \widetilde{u} &\leq 0, \qquad\qquad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\widetilde{u}} &< \infty. \end{cases}$$

If $V(0) = 0$, then $\widetilde{u}$ must be a constant by Liouville type Theorem, which contradicts $\int_{\mathbb{R}^2} e^{\widetilde{u}} < \infty$. Thus we have $V(0) > 0$, which implies the positive lower bound for V_n near the origin. We have proved the claim.

From (iii) of Theorem 5.10 we know that

$$\int_\Omega -Qv_n\phi = \int_\Omega V_n e^{u_n}\phi \to \alpha\phi(0),$$

for any $\phi \in C_0^\infty(\Omega)$.

For any $1 < q < 2$, let $p = \frac{q}{q-1} > 2$. From (1.1) and Proposition 1.2, (iii), we deduce

$$\begin{aligned} \|\nabla v_n\|_{L^q(\Omega)} &\leq \sup\{\int_\Omega \nabla v_n \nabla\phi : \|\phi\|_{W_0^{1,p}} = 1\} \\ &\leq C\sup\{\int_\Omega F(\nabla v_n)F_\xi(\nabla v_n)\nabla\phi : \|\phi\|_{W_0^{1,p}} = 1\}. \end{aligned}$$

By the Sobolev embedding, $\|\phi\|_{L^\infty(\Omega)} \leq C$. Hence

$$\int_\Omega F(\nabla v_n)F_\xi(\nabla v_n)\nabla\phi = \int_\Omega -Qv_n\phi \leq \|V_n e^{u_n}\|_{L^1}\|\phi\|_{L^\infty} \leq C.$$

Therefore $\|\nabla v_n\|_{L^q} \leq C$ for any $1 < q < 2$.

By Theorem 3.4, we have a unique Green function of

$$\begin{cases} -QG(\cdot,0) &= \alpha\delta_0 \quad \text{in } \Omega \\ G(\cdot,0) &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

and G has a decomposition

$$G(x) = -\frac{\alpha}{2\kappa}\log F^0(x) + g(x), \tag{5.15}$$

where $g(x) \in C^0(\Omega)$ with

$$\lim_{x\to 0} g(x) \text{ exists, } \lim_{x\to 0} |x|\nabla g(x) = 0. \tag{5.16}$$

It follows that $v_n \rightharpoonup G$ weakly in $W^{1,q}(\Omega)$.

On the other hand, since $\int_{\widetilde{\Omega}} V_n e^{u_n} \to 0$ for any $\widetilde{\Omega} \subset\subset \Omega \setminus \{0\}$, from Corollary 5.9, we get $\|v_n^+\|_{L^\infty(\widetilde{\Omega})} \le C$. It follows that $\|v_n^+\|_{C^{1,\beta}(\widetilde{\Omega})} \le C$ for some $0 < \beta < 1$. Therefore $v_n \to G$ strongly in $C^{1,\beta}(\widetilde{\Omega})$.

Multiplying (5.14) by $\langle x, \nabla v_n\rangle$ and integrating by parts, we obtain the Pohozaev identity:

$$\begin{aligned} &\int_{\partial\mathcal{W}_\epsilon} -F(\nabla v_n)\langle F_\xi(\nabla v_n), \nu\rangle\langle x, \nabla v_n\rangle + \frac{1}{2}F^2(\nabla v_n)\langle x, \nu\rangle \\ = &\int_{\partial\mathcal{W}_\epsilon} W_n e^{v_n}\langle x, \nu\rangle - \int_{\mathcal{W}_\epsilon} 2W_n e^{v_n} + \langle x, \nabla \log W_n\rangle W_n e^{v_n}, \end{aligned} \tag{5.17}$$

where $\nu = \frac{\nabla F^0}{|\nabla F^0|}$ is the unit outward normal.

Letting $n \to \infty$, the left hand side of (5.17) converges to

$$I := \int_{\partial\mathcal{W}_\epsilon} -F(\nabla G)\langle F_\xi(\nabla G), \nu\rangle\langle x, \nabla G\rangle + \frac{1}{2}F^2(\nabla G)\langle x, \nu\rangle. \tag{5.18}$$

We calculate (5.18). Using (5.15), (5.16) and Lemma 4.11, we have that on $\partial\mathcal{W}_\epsilon$,

$$F(\nabla G) = F(-\frac{\alpha}{2\kappa}\frac{\nabla F^0}{F^0} + o(\frac{1}{F^0})) = \frac{\alpha}{2\kappa\epsilon} + o(\frac{1}{\epsilon}),$$

$$\begin{aligned} \langle F_\xi(\nabla G), \nu\rangle &= \left\langle F_\xi(\nabla G), \frac{\nabla F^0}{|\nabla F^0|}\right\rangle \\ &= \left\langle F_\xi(\nabla G), (-\frac{2\kappa}{\alpha}F^0)\frac{\nabla G - o(\frac{1}{F^0})}{|\nabla F^0|}\right\rangle \\ &= -\frac{2\kappa\epsilon}{\alpha}\left(\frac{F(\nabla G)}{|\nabla F^0|} - \frac{o(\frac{1}{\epsilon})}{|\nabla F^0|}\right) \\ &= -(1 + o(1))\frac{1}{|\nabla F^0|}, \end{aligned}$$

$$\begin{aligned} \langle x, \nabla G\rangle &= \langle x, -\frac{\alpha}{2\kappa}\frac{\nabla F^0}{F^0} + o(\frac{1}{F^0})\rangle = -\frac{\alpha}{2\kappa} + \langle x, o(\frac{1}{\epsilon})\rangle \\ &= -\frac{\alpha}{2\kappa} + \langle F^0(x), o(\frac{1}{\epsilon})\rangle = -\frac{\alpha}{2\kappa} + o(1), \end{aligned}$$

$$\langle x, \nu \rangle = \langle x, \frac{\nabla F^0}{|\nabla F^0|} \rangle = \frac{\epsilon}{|\nabla F^0|}.$$

Substituting these into (5.18), we have

$$\begin{aligned} I &= \int_{\partial \mathcal{W}_\epsilon} \left(\frac{\alpha}{2\kappa\epsilon} + o(\frac{1}{\epsilon}) \right) (1 + o(1)) \left(-\frac{\alpha}{2\kappa} + o(1) \right) \frac{1}{|\nabla F^0|} \\ &\quad + \frac{1}{2} \left(\frac{\alpha}{2\kappa\epsilon} + o(\frac{1}{\epsilon}) \right)^2 \frac{\epsilon}{|\nabla F^0|} \\ &= - \left(\frac{\alpha^2}{8\kappa^2} + o(1) \right) \frac{1}{\epsilon} \int_{\partial \mathcal{W}_\epsilon} \frac{1}{|\nabla F^0|} \\ &= -\frac{\alpha^2}{4\kappa} + o(1), \end{aligned}$$

where $o(1) \to 0$ as $\epsilon \to 0$.

On the other hand, letting $n \to \infty$ and then $\epsilon \to 0$, we easily obtain from $V_n e^{u_n} \rightharpoonup \alpha\delta_0$ and the boundedness of $\nabla \log W_n$ that the RHS of (5.17) converges to -2α. We conclude that $\alpha = 8\kappa$. □

5.3 Compactness for vanishing boundary value problem

Consider a sequence of solutions to

$$\begin{cases} -Qu_n = \lambda_n \dfrac{V_n e^{u_n}}{\int_\Omega V_n e^{u_n} dx} & \text{in } \Omega \\ u_n = 0 & \text{in } \partial\Omega, \end{cases} \tag{5.19}$$

We will prove the following compactness result.

Theorem 5.12. *Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of solutions to (5.19) with*

$$\lim_{n\to\infty} \min_\Omega V_n > 0, \quad \lim_{n\to\infty} (\max_\Omega V_n + \|\nabla V_n\|_{L^\infty(\Omega)}) < \infty.$$

Suppose, in addition, that

$$0 < \lambda_n \le C, \quad \max_\Omega u_n \to +\infty.$$

Then there exists a finite set $S = \{p_1, \cdots, p_m\} \subset \Omega$ such that

$$u_n(x) \to \sum_{i=1}^m 8\kappa G(x, p_i) \text{ in } C^{1,\beta}(\overline{\Omega} \setminus S),$$

$$\lambda_n \frac{V_n e^{u_n}}{\int_\Omega V_n e^{u_n} dx} \to \sum_{i=1}^m 8\kappa \delta_{p_i}$$

in the sense of measures in $\overline{\Omega}$, for some $0 < \beta < 1$. Here $G(x, p_i)$ and δ_{p_i} are the Green function of Q and the Dirac function with singularity p_i respectively. In particular, We have for some $m \in \mathbb{N}$,

$$\lambda_n \to 8\kappa m.$$

In this section we will frequently use the following notations

$$\max_\Omega u_n = u_n(x_n),$$

$$\widetilde{\lambda_n} = \frac{\lambda_n V_n}{\int_\Omega V_n e^{u_n} dx}, \quad \epsilon_n = \widetilde{\lambda_n}^{-1/2} e^{-u_n(x_n)/2},$$

$$\Omega_n = (\Omega - x_n)/\epsilon_n, \quad \widetilde{u_n} = u_n(\epsilon_n x + x_n) + \log \widetilde{\lambda_n} + 2\log \epsilon_n.$$

First we need some useful lemmas.

As in Section 5.2, for the solution u_n to the problem (5.19), we define the blow-up set and γ-regular point. The blow-up set is defined as

$$\widetilde{S} = \{x \in \overline{\Omega} : \exists x_n \in \Omega \text{ such that } x_n \to x \text{ and } u_n(x_n) \to +\infty\}.$$

Since $\widetilde{\lambda_n} e^{u_n}$ is bounded in $L^1(\Omega)$, we may assume that there exists a nonnegative bounded measure σ such that for a subsequence (still denoted $\widetilde{\lambda_n} e^{u_n}$),

$$\int_\Omega \widetilde{\lambda_n} e^{u_n} \psi \to \int_\Omega \psi d\sigma$$

for every $\psi \in C_0^\infty(\mathbb{R}^2)$. We say that a point $x \in \overline{\Omega}$ is a *γ-regular point* if for some $\gamma > 0$, there exists a function $\psi \in C_0^\infty(\mathbb{R}^2)$, $0 \le \psi \le 1$, with $\psi = 1$ in a neighborhood of x such that

$$\int_\Omega \psi d\sigma < \gamma.$$

We define

$$\widetilde{\Sigma}(\gamma) = \{x \in \overline{\Omega} : x \text{ is not a } \gamma\text{-regular point}\}.$$

Since σ is a bounded measure, it follows that $\widetilde{\Sigma}(\gamma)$ is finite for any positive γ.

Lemma 5.13. *There exists $\gamma_0 > 0$ such that if x_0 is a γ_0-regular point then u_n is bounded in $L^\infty(B_{R_0}(x_0) \bigcap \overline{\Omega})$ for some $R_0 > 0$. Moreover, $\widetilde{S} = \widetilde{\Sigma}(\gamma)$ provided $\gamma < \gamma_0$. In particular, u_n is bounded in $C^1_{loc}(\overline{\Omega} \setminus \widetilde{S})$.*

Proof. Let $\widetilde{\Omega} \subset \mathbb{R}^2$ be a smooth bounded open domain which contains $\overline{\Omega}$ and

$$\widehat{u_n}(x) = \begin{cases} u_n(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \widetilde{\Omega} \setminus \Omega. \end{cases}$$

Consider w_n to be the solution to

$$\begin{cases} -Qw_n &= \widetilde{\lambda_n} e^{\widehat{u_n}} & \text{in } \widetilde{\Omega} \\ w_n &= 0 & \text{in } \partial\widetilde{\Omega}. \end{cases}$$

By the weak maximum principle and the comparison principle, we get $u_n \leq w_n$ in Ω. On the other hand, since $\widetilde{\lambda_n} e^{\widehat{u_n}}$ is bounded in $L^1(\widetilde{\Omega})$, arguing as in Theorem 5.11, we know ∇w_n is bounded in $L^q(\widetilde{\Omega})$ for $1 < q < 2$. By the Sobolev embedding theorem, w_n is bounded in $L^2(\widetilde{\Omega})$. Let v_n be the solution to

$$\begin{cases} -Qv_n &= 0 & \text{in } B_{2R}(x_0) \subset \widetilde{\Omega} \\ v_n &= w_n & \text{in } \partial B_{2R}(x_0). \end{cases}$$

Using the comparison principle again, we have $v_n \leq w_n$ in $B_{2R}(x_0)$. Consequently,

$$\|v_n\|_{L^\infty(B_R(x_0))} \leq \|v_n\|_{L^2(B_{2R}(x_0))} \leq \|w_n\|_{L^2(B_{2R}(x_0))} \leq C.$$

From Theorem 5.7, we have for any $\delta \in (0, 4\kappa)$,

$$\int_{B_R(x_0)} \exp\left[\frac{(4\kappa - \delta)d_0}{\int_{B_R(x_0)} \widetilde{\lambda_n} e^{\widehat{u_n}}} |w_n - v_n|\right] dx \leq C.$$

If x_0 is a γ_0-regular point, from the definition of $\widehat{u_n}$, we easily see that

$$\int_{B_R(x_0)} \widetilde{\lambda_n} e^{\widehat{u_n}} \leq \gamma_0 + \epsilon < 4\kappa d_0$$

for small R and γ_0. It follows that e^{w_n}, and hence $e^{\widehat{u_n}}$ is bounded in $L^p(B_R(x_0))$ for some $p > 1$. Consequently, w_n, hence u_n is bounded in $L^\infty(B_{R/2}(x_0) \bigcap \overline{\Omega})$. Arguing as in Section 4, we know $\widetilde{S} = \widetilde{\Sigma}(\gamma)$ provided $\gamma < \gamma_0$. From the standard elliptic theory, u_n is bounded in $C^1_{loc}(\overline{\Omega} \setminus \widetilde{S})$. □

From Lemma 5.13, $\widetilde{S}$ is finite. Set $\widetilde{S} = \{p_1, \cdots, p_m\}$. We focus on the blow-up point x on the boundary, i.e. $x \in \widetilde{S} \bigcap \partial\Omega$.

Lemma 5.14. *If $x_0 \in \widetilde{S} \bigcap \partial\Omega$, then*

$$\lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(x_0) \bigcap \overline{\Omega}} \widetilde{\lambda_n} e^{u_n} dx \geq 8\kappa.$$

Proof. From the definition of $\widetilde{u_n}$ and (5.19), $\widetilde{u_n}$ satisfies

$$\begin{cases} -Q\widetilde{u_n} &= e^{\widetilde{u_n}} \quad \text{in } \Omega_n \\ \widetilde{u_n}(0) &= 0, \\ \widetilde{u_n} &\leq 0 \quad \text{in } \Omega_n. \end{cases}$$

For the convergence of Ω_n, we consider two cases.

Case (i). $\frac{dist(x_n,\partial\Omega)}{\epsilon_n} \to \infty$ and $\Omega_n \to \mathbb{R}^2$. By standard regularity arguments, $\widetilde{u_n} \to \widetilde{u}$ in $C^1_{loc}(\mathbb{R}^2)$, where $\widetilde{u}$ satisfies

$$\begin{cases} -Q\widetilde{u} &= e^{\widetilde{u}} \quad \text{in } \mathbb{R}^2 \\ \widetilde{u}(0) &= 0, \\ \widetilde{u} &\leq 0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{\widetilde{u}}dx &< \infty. \end{cases} \tag{5.20}$$

Case (ii). $\frac{dist(x_n,\partial\Omega)}{\epsilon_n}$ is bounded and $\Omega_n \to \mathbb{R}^2_+(t_0) := \{(x^1,x^2) \in \mathbb{R}^2 : x^1 > t_0\}$. In this case one can show that $\widetilde{u_n} \to \widetilde{v}$ in $C^1_{loc}(\mathbb{R}^2_+(t_0))$, where $\widetilde{v}$ satisfies

$$\begin{cases} -Q\widetilde{v} &= e^{\widetilde{v}} \quad \text{in } \mathbb{R}^2_+(t_0) \\ \widetilde{v}(0) &= 0, \\ \widetilde{v} &\leq 0 \quad \text{in } \mathbb{R}^2_+(t_0) \\ \widetilde{v} &= -\infty \quad \text{on } \partial\mathbb{R}^2_+(t_0) \\ \int_{\mathbb{R}^2_+(t_0)} e^{\widetilde{v}}dx &< \infty. \end{cases} \tag{5.21}$$

It follows immediately from the following Proposition 5.15 that

$$\int_{\mathbb{R}^2} e^{\widetilde{u}}dx \geq 8\kappa \quad \text{and} \quad \int_{\mathbb{R}^2_+(t_0)} e^{\widetilde{v}}dx \geq 8\kappa.$$

Consequently,

$$\begin{aligned} \lim_{r\to 0}\lim_{n\to\infty}\int_{B_r(x_0)\bigcap\overline{\Omega}} \widetilde{\lambda_n}e^{u_n}dx &\geq \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_{R\epsilon_n}(x_n)\bigcap\overline{\Omega}} \widetilde{\lambda_n}e^{u_n}dx \\ &= \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_R(0)\bigcap\overline{\Omega_n}} e^{\widetilde{u_n}}dx \\ &\geq \begin{cases} \int_{\mathbb{R}^2} e^{\widetilde{u}}dx & \text{(case(i))} \\ \int_{\mathbb{R}^2_+(t_0)} e^{\widetilde{v}}dx & \text{(case(ii))} \end{cases} \\ &\geq 8\kappa. \end{aligned}$$

□

Proposition 5.15. *(i). If u is a weak solution of (5.20), then*

$$\int_{\mathbb{R}^2} e^u dx \geq 8\kappa. \tag{5.22}$$

Moreover, equality holds if and only if u is radial symmetric with respect to F, i.e., $u(x) = u(F^0(x))$.

(ii). If v is a weak solution of (5.21), then

$$\int_{\mathbb{R}^2_+(t_0)} e^v dx \geq 8\kappa. \tag{5.23}$$

Proof. The proof follows closely the argument of Ding (see [CL91a]). For $t \in \mathbb{R}$, let $\Omega_t = \{x \in \Omega | u(x) > t\}$ and $\mu(t) = |\Omega_t|$. By the divergence Theorem,

$$\int_{\Omega_t} -Qu\, dx = \int_{\partial\Omega_t} F(\nabla u)\langle F_\xi(\nabla u), \frac{\nabla u}{|\nabla u|}\rangle = \int_{\partial\Omega_t} \frac{F^2(\nabla u)}{|\nabla u|}. \tag{5.24}$$

Using the isoperimetric inequality (1.29), the co-area formula (1.28), the Hölder inequality, (5.20) and (5.24), we obtain

$$\begin{aligned} 2\kappa^{\frac{1}{2}}\mu(t)^{\frac{1}{2}} \leq P_F(\Omega_t) = \int_{\partial\Omega_t} \frac{F(\nabla u)}{|\nabla u|} \\ \leq \left(\int_{\partial\Omega_t} \frac{F^2(\nabla u)}{|\nabla u|}\right)^{\frac{1}{2}} \left(\int_{\partial\Omega_t} \frac{1}{|\nabla u|}\right)^{\frac{1}{2}} = \left(\int_{\Omega_t} e^u\right)^{\frac{1}{2}} \big(-\mu'(t)\big)^{\frac{1}{2}}. \end{aligned}$$

It follow that

$$\begin{aligned} \int_{\mathbb{R}^2} e^u &= \int_{-\infty}^{\max u} e^t \mu(t) dt \leq \int_{-\infty}^{\max u} e^t \frac{-\mu'(t)}{4\kappa} \int_{\Omega_t} e^u dx dt \\ &= \int_{-\infty}^{\max u} \frac{1}{8\kappa} \frac{d}{dt} (\int_{\Omega_t} e^u dx)^2 dt = \frac{1}{8\kappa} (\int_{\mathbb{R}^2} e^u)^2, \end{aligned}$$

which implies (5.22).

If equality in (5.22) holds, we must have equality in isoperimetric inequality, which means Ω_t must be a Wulff ball. In other words, u is radial symmetric with respect to F. Conversely, if u is radial symmetric with respect to F, we can immediately solve an ODE to get

$$u(x) = -2\log(1 + \frac{1}{8} F^0(x)^2), \tag{5.25}$$

up to translation and scaling, which gives equality in (5.22).

For the second statement, the argument above also works since the boundary condition $v = -\infty$ implies all level sets of v are closed domain in $R^2_+(t_0)$.

□

We would like to propose

Conjecture 5.16. *Any solution (5.25) to*

$$\begin{cases} -Qu = e^u & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u dx < \infty \end{cases} \tag{5.26}$$

has the following form

$$u(x) = -2\log(1 + \frac{1}{8}F^0(x)^2),$$

up to translation and scaling.

It's not difficult to verify that $u(x) = -2\log(1 + \frac{1}{8}F^0(x)^2)$ solves (5.26). In the isotropic case, i.e., $F(\xi) = |\xi|$, this conjecture is true. This is the result of Chen-Li in [CL91a]. However, the proof in [CL91a], and also other proofs we know, does not work for the anisotropic case.

Now we use Lemma 5.13 and Lemma 5.14 to exclude the boundary blow-up.

Lemma 5.17. *There exists a neighborhood $\mathcal{N}$ of $\partial\Omega$ such that u_n is bounded in $L^\infty(\mathcal{N})$, i.e. $\widetilde{S}\bigcap\partial\Omega = \emptyset$.*

Proof. We argue by contradiction. Let $p_1 \in \widetilde{S}\bigcap\partial\Omega$. We may assume that p_1 is the only point of $\widetilde{S}$ in $\overline{\Omega}\bigcap B_{r_0}(p_1)$. We separate into two cases.

Case 1. $\int_\Omega V_n e^{u_n} dx \to \infty$. Hence $\widetilde{\lambda_n} \to 0$. We claim first that

$$u_n \to \sum_{i=1}^m \gamma_i G(\cdot, p_i) \text{ in } C^1_{loc}(\overline{\Omega} \setminus \widetilde{S}),$$

where $\gamma_i = \lim_{r\to 0}\lim_{n\to\infty}\int_{B_r(p_i)\bigcap\overline{\Omega}} \widetilde{\lambda_n} e^{u_n} dx$, $G(\cdot, p_i)$ is the unique Green function with singularity at p_i if p_i lies in the interior of Ω and 0 if p_i lies on $\partial\Omega$. In fact, since u_n is bounded in $C^1_{loc}(\overline{\Omega} \setminus \widetilde{S})$ and $\widetilde{\lambda_n} \to 0$, we have that $\int_{\widetilde{\Omega}} \widetilde{\lambda_n} e^{u_n} \to 0$ for any $\widetilde{\Omega} \subset\subset \overline{\Omega} \setminus \widetilde{S}$. In view of the definition of γ_i, we see that $d\sigma = \sum_{i=1}^m \gamma_i \delta_{p_i}$ and

$$\int_\Omega \widetilde{\lambda_n} e^{u_n} \psi \to \sum_{i=1}^m \gamma_i \psi(p_i)$$

for any $\psi \in C_0^\infty(\mathbb{R}^2)$. As $\|\widetilde{\lambda_n} e^{u_n}\|_{L^1} \le C$, we know that $\|\nabla u_n\|_{L^q} \le C$ for any $1 < q < 2$. It follows that $u_n \rightharpoonup G$ weakly in $W^{1,q}(\Omega)$. Testing equation (5.19) with $\psi \in C_0^\infty(\Omega)$, we obtain

$$\int_\Omega -Qu_n\psi dx = \int_\Omega \widetilde{\lambda_n} e^{u_n} \psi dx \to \sum_{i=1}^m \gamma_i \psi(p_i).$$

Therefore,

$$\int_\Omega -QG\psi dx = \sum_{i=1}^m \gamma_i \psi(p_i).$$

Note that $\psi(p_i) = 0$ when $p_i \in \partial\Omega$. Thus $G = \sum_{i=1}^m \gamma_i G(\cdot, p_i)$, where $G(\cdot, p_i)$ is described as before. The C^1_{loc} convergence, hence the claim, immediately follows.

Hereafter we will use a modified Pohozaev identity to get rid of some boundary term, which was used in [RoWe08].

Let $y_n = p_1 + \rho_{n,r}\nu(p_1)$ with

$$\rho_{n,r} = \frac{\int_{\partial\Omega \bigcap B_r(p_1)} F^2(\nabla u_n)\langle x - p_1, \nu(x)\rangle dx}{\int_{\partial\Omega \bigcap B_r(p_1)} F^2(\nabla u_n)\langle \nu(x_0), \nu(x)\rangle dx}$$

where r is small enough such that $\frac{1}{2} \le \langle \nu(x_0), \nu(x)\rangle \le 1$ for $x \in \partial\Omega \bigcap B_{r_0}(p_1)$. Here $\nu(x)$ is the unit outward normal. It follows that $|\rho_{n,r}| \le 2r$ and

$$\int_{\partial\Omega \bigcap B_r(p_1)} F^2(\nabla u_n)\langle x - y_n, \nu(x)\rangle dx = 0. \tag{5.27}$$

Multiplying (5.19) with $\langle x - y_n, \nabla u_n\rangle$ and integrating by parts, we obtain the modified Pohozaev identity:

$$\begin{aligned} & \int_{\Omega \bigcap B_r(p_1)} 2\widetilde{\lambda_n}(e^{u_n} - 1) \\ = & \int_{\partial(\Omega \bigcap B_r(p_1))} F(\nabla u_n)\langle F_\xi(\nabla u_n), \nu(x)\rangle\langle x - y_n, \nabla u_n\rangle \\ & -\frac{1}{2}F^2(\nabla u_n)\langle x - y_n, \nu(x)\rangle + \widetilde{\lambda_n}(e^{u_n} - 1)\langle x - y_n, \nu(x)\rangle, \end{aligned} \tag{5.28}$$

Since $u_n|_{\partial\Omega} = 0$, we have $\nabla u|_{\partial\Omega} = \frac{\partial u_n}{\partial \nu}\nu$ if it does not vanish. Hence

$$\begin{aligned} & F(\nabla u_n)\langle F_\xi(\nabla u_n), \nu(x)\rangle\langle x - y_n, \nabla u_n\rangle - \frac{1}{2}F^2(\nabla u_n)\langle x - y_n, \nu(x)\rangle \\ = & \frac{1}{2}F^2(\nabla u_n)\langle x - y_n, \nu(x)\rangle \end{aligned}$$

on $\partial\Omega \bigcap B_{r_0}(p_1)$. By (5.27) and the boundary condition $u_n|_{\partial\Omega} = 0$, (5.28) reduces to

$$\begin{aligned} & \int_{\Omega \bigcap B_r(p_1)} 2\widetilde{\lambda_n}(e^{u_n} - 1) \\ = & \int_{\Omega \bigcap \partial B_r(p_1)} F(\nabla u_n)\langle F_\xi(\nabla u_n), \nu(x)\rangle\langle x - y_n, \nabla u_n\rangle \\ & -\frac{1}{2}F^2(\nabla u_n)\langle x - y_n, \nu(x)\rangle + \widetilde{\lambda_n}(e^{u_n} - 1)\langle x - y_n, \nu(x)\rangle. \end{aligned} \tag{5.29}$$

Since $u_n \to \sum_{i=1}^m \gamma_i G(\cdot, p_i)$ in $C^1_{loc}(\overline{\Omega} \setminus \widetilde{S})$, $G(\cdot, p_1) = 0$, $G(\cdot, p_j)$ belongs to and hence is bounded in $C^1(B_{r_0}(p_1))$ for any $j = 2, \cdots, m$ (for p_1 is the only point of $\widetilde{S}$ in $\overline{\Omega} \bigcap B_{r_0}(p_1)$). let $n \to \infty$ and then $r \to 0$, the RHS of (5.29) converges to

$$\lim_{r \to 0} \int_{\Omega \bigcap \partial B_r(p_1)} O(r) dx = \lim_{r \to 0} O(r^4) = 0.$$

On the other hand, the LHS of (5.29) converges to $2\gamma_1$, which is no less than 16κ by Lemma 5.14. A contradiction.

Case 2. $\int_\Omega V_n e^{u_n} dx \leq C$. In this case, we have the three alternatives in Theorem 5.10.

We claim that $\widetilde{S} \subset \partial\Omega$. Indeed, by the maximum principle, $u_n \geq 0$ in Ω. Therefore, (ii), (iii) in Theorem 5.10 cannot happen, i.e., u_n is bounded in $L^\infty_{loc}(\Omega)$.

By Lemma 5.13 and by passing to subsequence,

$$u_n \to u \text{ in } C^1_{loc}(\overline{\Omega} \setminus \widetilde{S}),$$

where $u \in C^1(\overline{\Omega} \setminus \widetilde{S})$ satisfies

$$\begin{cases} -Qu &= \widetilde{\lambda} e^u \quad \text{in } \Omega \\ \quad u &= 0 \qquad \text{on } \partial\Omega \setminus \widetilde{S}. \end{cases}$$

Here $\widetilde{\lambda_n} \to \widetilde{\lambda}$ because $\int_\Omega V_n e^{u_n} dx$ is bounded below and then $\widetilde{\lambda_n}$ is bounded. It's easy to see that u can be extended to a $C^1(\overline{\Omega})$ function. Argue similarly as in Case 1, the RHS of (5.29) converges to 0, a contradiction to Lemma 5.14.

We complete the proof of Lemma 5.17. □

Proof of Theorem 5.12. By the assumption on V_n and λ_n, we may assume $\lambda_n \to \lambda \geq 0$, $V_n \to V$ with $\min_\Omega V > 0$. Set $\widehat{u_n} = u_n + \log \widetilde{\lambda_n}$. Thus $\widehat{u_n} \leq u_n + C$. By Lemma 5.17, we know that u_n is bounded in a neighborhood of $\partial\Omega$. It follows that $\widehat{u_n}$ is bounded above in a neighborhood of $\partial\Omega$.

Since $-Q\widehat{u_n} = e^{\widehat{u_n}}$ and $\int_\Omega e^{\widehat{u_n}} \leq C$, from Theorem 5.10, three alternatives may happen. We claim that $\max_\Omega \widehat{u_n} \to \infty$. Otherwise, if $\widehat{u_n} \leq C$, then $\|u_n\|_{L^\infty(\Omega)} \leq C$, which contradicts $\max_\Omega u_n \to \infty$. Consequently, (i) and (ii) cannot happen in Theorem 5.10. Therefore there exists a finite set $S = \{p_1, \cdots, p_m\} \subset \Omega$ such that

$$e^{\widehat{u_n}} = \lambda_n \frac{V_n e^{u_n}}{\int_\Omega V_n e^{u_n} dx} \to \sum_{i=1}^m \alpha_i \delta_{p_i}$$

in the sense of measure in $\overline{\Omega}$. Arguing as before, we easily see that

$$u_n(x) \to \sum_{i=1}^m \alpha_i G(x, p_i) \text{ in } C^{1,\beta}(\overline{\Omega} \setminus S).$$

It remains to determine α_i. Choose r_0 small enough such that p_i is the only point of S in $B_{r_0}(p_i) \subset \Omega$ for any $i = 1, \cdots, m$. It follows that

$$\max_{\partial B_{r_0}(p_i)} u_n - \min_{\partial B_{r_0}(p_i)} u_n \le C(r_0).$$

Now we can apply Theorem 5.11 in $B_{r_0}(p_i)$ to conclude that $\alpha_i = 8\kappa$. This proves Theorem 5.12. □

As a direct consequence, we obtain

Theorem 5.18. *Let Ω be a bounded domain in $\mathbb{R}^2$ and (u_n) be a sequence of solutions to*

$$\begin{cases} -Qu = \lambda \dfrac{Ve^u}{\int_\Omega Ve^u dx} & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{5.30}$$

with

$$\min_\Omega V > 0, \quad \max_\Omega V + \|\nabla V\|_{L^\infty(\Omega)} < \infty.$$

Then for any compact interval $\Lambda \subset (8\kappa(m-1), 8\kappa m)$ and $\lambda \in \Lambda$, $m \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$u(x) \le C \text{ for } x \in \Omega.$$

5.4 An existence result

In this section, we prove the following existence result.

Theorem 5.19. *Let Ω be a smooth bounded domain whose complement contains at least one bounded region and V be as in Theorem 5.18. Then (5.30) admits a solution for all $\lambda \in (8\kappa, 16\kappa)$.*

When $\lambda \in (8\kappa, 16\kappa)$, we are in a supercritical case, in the sense that the functional $J_\lambda(u)$ has no lower bound (See Corollary 5.2). Our method follows closely [DJLW99], where they constructed saddle type critical points for the isotropic case. We only give a sketch of this proof.

Proof of Theorem 5.19: The proof can be divided into several steps.

Step 1. First we define the center of mass of a function $u \in W_0^{1,2}(\Omega)$ by

$$m_c(u) = \frac{\int_\Omega x e^u}{\int_\Omega e^u}.$$

For simplicity, assume $\mathbb{R}^2 \setminus \Omega$ has a bounded component which is the unit disk $D = \{(r,\theta)|0 \le r < 1, \theta \in [0,2\pi)\}$ centered at the origin. We then define a family of functions $h : D \to W_0^{1,2}(\Omega)$ satisfying

$$J_\lambda(h(r,\theta)) \to -\infty \text{ as } r \to 1$$

and

$$m_c(h(r,\theta)) \text{ is a continuous curve enclosing } D.$$

The existence of such a family is guaranteed by $\lambda > 8\kappa$. Denote the set of all such families by $\mathcal{D}_\lambda$. We now define a minimax value

$$\alpha_\lambda := \inf_{h\in\mathcal{D}_\lambda} \sup_{u\in h(D)} J_\lambda(u).$$

Step 2. For any $\lambda \in (8\kappa, 16\kappa)$, $\alpha_\lambda > -\infty$. We need first an improved Moser-Trudinger inequality introduced by Aubin.

Lemma 5.20. *Let $\delta_0 > 0$ and $\gamma_0 \in (0, \frac{1}{2})$ be given numbers. Let S_1 and S_2 be two subset of $\overline{\Omega}$ satisfying dist$(S_1, S_2) \ge \delta_0 > 0$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \gamma_0) > 0$ such that*

$$J_{16\kappa-\epsilon}(u) > -c$$

holds for all $u \in W_0^{1,2}(\Omega)$ satisfying

$$\frac{\int_{S_1} e^u}{\int_\Omega e^u} \ge \gamma_0 \text{ and } \frac{\int_{S_2} e^u}{\int_\Omega e^u} \ge \gamma_0.$$

Proof. The lemma follows from the argument in [CL91b] and the Moser-Trudinger inequality $J_{8\kappa}(u) \ge -c$. □

We return to the proof of Step 2. Suppose by contradiction that α_λ has no lower bound, then we have sequences $h_i \in \mathcal{D}_\lambda$ and $u_i \in h_i(D)$ such that $J(u_i) \to -\infty$ and $m_c(u_i) = 0$. On the other hand, in view of Lemma 5.20, there exists $x_0 \in \overline{\Omega}$ such that

$$\frac{\int_{B_{1/2}(x_0)\bigcap\Omega} e^{u_i}}{\int_\Omega e^{u_i}} \to 1,$$

which leads to $|m_c(u_i) - x_0| < \frac{2}{3}$. This contradicts $m_c(u_i) = 0$.

Step 3. It can be easily checked that $\frac{\alpha_\lambda}{\lambda}$ is non-increasing in $(8\kappa, 16\kappa)$. Define

$$\Lambda := \{\lambda \in (8\kappa, 16\kappa)| \frac{\alpha_\lambda}{\lambda} \text{ is differentiable at } \lambda\}.$$

We can follow the method in [DJLW99] to prove that α_λ is achieved by a critical point u_λ of J_λ provided that $\lambda \in \Lambda$.

Step 4. For any $\lambda \in (8\kappa, 16\kappa)$, we find a sequence $\lambda_k \in \Lambda$ and u_{λ_k} satisfying (5.30). Since λ is not multiple of 8κ, we see from Theorem 5.18 that u_{λ_k} converges to some u, which is a solution of (5.30). This finishes the proof of Theorem 5.19. $\square$

Part II

Anisotropic geometric problems

Chapter 6

Minkowski (relative) geometry of hypersurfaces

Convention for Part II: Throughout Part II, the Latin alphabet $i, j, k, \cdots$ denotes indices from 1 to n and the Greek alphabet $\alpha, \beta, \gamma, \cdots$ denotes indices from 1 to $n+1$. We will always use the Einstein summation convention. We remark that the notations in Part I and Part II are somehow different.

In relative or Minkowski geometry, we are always given a Minkowski norm.

Definition 6.1. *A function $F : \mathbb{R}^{n+1} \to [0, +\infty)$ is called a **Minkowski norm** if*

(i) F is a norm of $\mathbb{R}^{n+1}$, i.e., F is a convex, 1-homogeneous function satisfying $F(x) > 0$ when $x \neq 0$;

(ii) $F \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;

(iii) F satisfies a uniformly elliptic condition: Hess$(\frac{1}{2}F^2)$ is positive definite in $\mathbb{R}^{n+1} \setminus \{0\}$.

In Part I, we have defined the dual norm of F,

$$F^0(\xi) := \sup_{x \neq 0} \frac{\langle x, \xi \rangle}{F(x)}, \quad \xi \in \mathbb{R}^{n+1}.$$

we remark that F^0 is also a Minkowski norm (see [Sh 01]). The following properties is quite simple consequences of 1-homogeneous of F and F^0, as proved in Chapter 1. We list them here again.

Proposition 6.2.

(i) $\frac{\partial F}{\partial x^\alpha}(x) x^\alpha = F(x), \quad \frac{\partial F^0}{\partial \xi^\alpha}(\xi) \xi^\alpha = F^0(\xi);$

(ii) $\frac{\partial^2 F}{\partial x^\alpha \partial x^\beta}(x)x^\alpha = 0, \quad \frac{\partial^2 F^0}{\partial \xi^\alpha \partial \xi^\beta}(\xi)\xi^\alpha = 0,$ *for* $x, \xi \neq 0, \quad \forall \beta = 1, \cdots, n+1;$

(iii) $F(DF^0(\xi)) = 1, \quad F^0(DF(x)) = 1,$ *for* $x, \xi \neq 0;$

(iv) $F^0(\xi)DF(DF^0(\xi)) = \xi, \quad F(x)DF^0(DF(x)) = x,$ *for* $x, \xi \neq 0.$

Here $DF = (\frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^{n+1}})$ *and* $DF^0 = (\frac{\partial F^0}{\partial \xi^1}, \cdots, \frac{\partial F^0}{\partial \xi^{n+1}})$.

A smooth convex hypersurface in $\mathbb{R}^{n+1}$ corresponding to F is the Wulff shape

$$\mathcal{W}_F := \{x \in \mathbb{R}^{n+1} | F^0(x) = 1\}.$$

Conversely, a smooth convex hypersurface $\mathcal{M}$ in $\mathbb{R}^{n+1}$ determines uniquely a convex function F such that the Wulff shape corresponding to such F is $\mathcal{M}$. Wulff shape plays the fundamental role as a comparison body in the relative differential geometry.

For an oriented n-dimensional hypersurface M in $\mathbb{R}^{n+1}$, The function F defines an anisotropic area functional

$$|M|_F := \int_M F(\bar{\nu}) d\mathcal{H}^n,$$

where $\bar{\nu}$ and $\mathcal{H}^n$ denote the standard unit outer normal to M and the n-dimensional Hausdorff measure respectively. We denote by $d\mu_F = F(\bar{\nu})d\mathcal{H}^n$ and call it the **anisotropic measure**.

The unit **anisotropic outer normal** is defined by

$$\nu_F := DF(\bar{\nu}).$$

It is easy to see from Proposition 6.2 (iii) that $\nu_F \in \mathcal{W}_F$. We call $\nu_F : M \to \mathcal{W}_F$ the **anisotropic Gauss map**.

From now on we will omit the subscript F for simplicity, i.e., $d\mu = d\mu_F$, $\nu = \nu_F$ and $\mathcal{W} = \mathcal{W}_F$, etc..

Since $\mathrm{Hess}(\frac{1}{2}F^2)$ is positive definite, we can consider $\mathbb{R}^{n+1}$ as a Riemannian manifold equipped with a metric

$$G(x)(\xi, \eta) := \sum_{\alpha,\beta=1}^{n+1} G_{\alpha\beta}(x)\xi^\alpha \eta^\beta = \sum_{\alpha,\beta=1}^{n+1} \frac{\partial^2(\frac{1}{2}(F^0)^2)}{\partial x^\alpha \partial x^\beta}(x)\xi^\alpha \eta^\beta \tag{6.1}$$

for $x \in \mathbb{R}^{n+1}$, $\xi, \eta \in T_x\mathbb{R}^{n+1}$. It is easy to see from Proposition 6.2 (i) and (ii) that

$$G(x)(x, x) = 1, \quad G(x)(x, \xi) = 0 \text{ for any } x \in \mathcal{W} \subset \mathbb{R}^{n+1}, \quad \xi \in T_x\mathcal{W}. \tag{6.2}$$

Similarly, for an oriented n-dimensional hypersurface M in $\mathbb{R}^{n+1}$, we have

$$G(\nu)(\nu, \nu) = 1, \quad G(\nu)(\nu, \xi) = 0 \text{ for } x \in M, \quad \nu = \nu(x), \quad \xi \in T_xM.$$

We define

$$g(x) := G(\nu(x))|_{T_xM}, \quad x \in M$$

as a Riemannian metric on $M \subset \mathbb{R}^{n+1}$. This Riemannian metric will play a fundamental role in relative geometry of hypersurface. Note that in classical Euclidean geometry the metric on a hypersurface is the restriction of the Euclidean metric. This is the major difference between the relative geometry and Euclidean geometry.

Since F^0 is not quadratic, the third derivative of F^0 does not vanish. We denote

$$Q(x)(\xi,\eta,\zeta) := \sum_{\alpha,\beta,\gamma=1}^{n+1} Q_{\alpha\beta\gamma}(x)\xi^\alpha\eta^\beta\zeta^\gamma = \sum_{\alpha,\beta,\gamma=1}^{n+1} \frac{\partial^3(\frac{1}{2}(F^0)^2)}{\partial x^\alpha \partial x^\beta \partial x^\gamma}(x)\xi^\alpha\eta^\beta\zeta^\gamma,$$

for $x, \xi, \eta, \zeta \in \mathbb{R}^{n+1}$. It follows again from Proposition 6.2 (i) and (ii) that

$$Q(x)(x,\xi,\eta) = 0. \tag{6.3}$$

To study the relative geometry of hypersurface, it is indispensable to define the anisotropic second fundamental form.

For $X \in M \subset \mathbb{R}^{n+1}$, choose local coordinate $\{y^\alpha\}_{\alpha=1}^{n+1}$ in $\mathbb{R}^{n+1}$, such that $\{\frac{\partial}{\partial y^\alpha}\}_{\alpha=1}^{n+1}$ are tangent to M and $\frac{\partial}{\partial y^{n+1}} = \nu$ is the unit anisotropic outer normal of M. By identification $\partial_i = \frac{\partial}{\partial y^i} = \partial_i X$ for $i = 1, \cdots, n$. Let $g_{ij}(X) = g(\nu)(\partial_i X, \partial_j X)$ be the Riemannian metric on M. Denote by g^{ij} the inverse of g_{ij}.

Given the standard volume form Ω (Lebesgue measure) in $\mathbb{R}^{n+1}$, the anisotropic measure on M can be interpreted as

$$d\mu = \Omega(\nu, \partial_1, \cdots, \partial_n)dy^1 \cdots dy^n. \tag{6.4}$$

This follows from the fact that $d\mu = F^0(\bar\nu)\Omega(\bar\nu, \partial_1, \cdots, \partial_n)dy^1 \cdots dy^n$ and $\nu = F^0(\bar\nu)\bar\nu +$ tangent part, due to Proposition 6.2 (i).

The **anisotropic second fundamental form** is defined by

$$h_{ij}(X) = -G(\nu)(\nu, \partial_i\partial_j X).$$

It is a symmetric 2-tensor on M.

With the Riemannian metric g and the anisotropic second fundamental form, we can define the anisotropic principle curvatures and the Gauss-Kronecker curvature.

Definition 6.3. *The eigenvalues of the anisotropic second fundamental form h_{ij} with respect to the metric g_{ij} (i.e. the eigenvalues of the matrix $g^{jk}h_{ij}$) are called the **anisotropic principle curvatures**. The inverse of the anisotropic principle curvatures are called the **anisotropic principle radii**.*

Definition 6.4. *The* ***anisotropic mean curvature*** *is* $H = g^{ij}h_{ij}$. *The* ***anisotropic Gauss-Kronecker curvature*** *is* $K = \det(g^{jk}h_{ij})$.

Remark 6.5. *The definition of the anisotropic mean curvature or principle curvatures are different with that in Part I. The definition here interprets more geometric meaning, while that in Part I is more analytic. However, they are essentially the same, because (1) The anisotropic mean curvature can both defined from the first variation of anisotropic area functional. The difference is which normal we choose to deform the hypersurface. Here we choose the anisotropic normal, while for that in Part I we choose the standard normal; (2) The anisotropic principle curvatures of Wulff shape are both the identical matrix from the different definitions; (3) The convexity of a hypersurface is independent of which definition we choose.*

Proposition 6.6. *A hypersurface M is* ***convex*** *(****strongly convex*** *resp.) if and only if $(h_{ij}) \geq 0$ (> 0 resp.).*

Proof. we just need to observe that

$$h_{ij} = \frac{1}{F(\bar{\nu})}\bar{h}_{ij},$$

where $\bar{h}_{ij}$ is the standard second fundamental form $\bar{h}_{ij} = \langle \bar{\nu}, \partial_i\partial_j X\rangle_{\mathbb{R}^{n+1}}$.

Indeed, in view of $\nu = DF(\bar{\nu})$, we have

$$\begin{aligned} h_{ij} &= -G(\nu)(\nu, \partial_i\partial_j X) = -DF^0(DF(\bar{\nu}))\cdot\partial_i\partial_j X \\ &= -\frac{\bar{\nu}}{F(\bar{\nu})}\cdot\partial_i\partial_j X = \frac{1}{F(\bar{\nu})}\bar{h}_{ij}. \end{aligned}$$

Here we have used Proposition 6.2 (iii) and (iv). □

Similarly as in the classical theory of hypersurface, we have the following Gauss and Weingarten formulas, Gauss and Codazzi equations.

Lemma 6.7.

$$\partial_i\partial_j X = -h_{ij}\nu + \nabla_{\partial_i}\partial_j + A_{ij}^k\partial_k X; \ \textit{(Gauss formula)} \tag{6.5}$$

$$\partial_i \nu = g^{jk}h_{ij}\partial_k X; \ \textit{(Weingarten formula)} \tag{6.6}$$

$$\begin{aligned} R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} + \nabla_{\partial_l}A_{jki} - \nabla_{\partial_k}A_{jli} \\ &\quad + A_{jk}^m A_{mli} - A_{jl}^m A_{mki}; \ \textit{(Gauss equation)} \end{aligned} \tag{6.7}$$

$$h_{ijk} + h_j^l A_{lki} = h_{ikj} + h_k^l A_{lji}. \ \textit{(Codazzi equation)} \tag{6.8}$$

Here ∇ is the Levi-Civita connection of the Riemannian metric g, R is the Riemannian curvature tensor of g, A is a 3-tensor

$$A_{ijk} = -\frac{1}{2}\left(h_i^l Q_{jkl} + h_j^l Q_{ilk} - h_k^l Q_{ijl}\right), \tag{6.9}$$

where $Q_{ijk} = Q(\nu)(\partial_i X, \partial_j X, \partial_k X)$.

Proof. Taking derivative of the equation $G(\nu)(\nu,\nu) = 1$, $G(\nu)(\nu, \partial_j X) = 0$ and using (6.3) we have

$$G(\nu)(\partial_i \nu, \nu) = 0,$$

$$G(\nu)(\partial_i \nu, \partial_i X) + G(\nu)(\nu, \partial_i \partial_j X) = 0,$$

which implies the Weingarten formula (6.6).

To verify the Gauss formula (6.5), it is sufficient to give the explicit formula (6.9) for A. Denote Γ_{ij}^k the Christoffel symbol with respect to ∇. Taking derivative of the equation $g_{ij} = G(\nu)(\partial_i X, \partial_j X)$, we have

$$\begin{aligned} \partial_k g_{ij} &= G(\nu)(\partial_k \partial_i X, \partial_j X) + G(\nu)(\partial_i X, \partial_k \partial_j X) + Q(\nu)(\partial_k \nu, \partial_i X, \partial_j X) \\ &= (\Gamma_{ik}^l + A_{ik}^l) g_{jl} + (\Gamma_{jk}^l + A_{jk}^l) g_{il} + h_k^l Q_{ijl}. \end{aligned} \tag{6.10}$$

Note that $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$. Then (6.9) follows easily from (6.10).

Taking covariant derivative of the Weingarten formula (6.6) , we have

$$\nabla_{\partial_j} \nabla_{\partial_i} \nu = (h_{i,j}^k + h_i^l A_{jl}^k) \partial_k X + \text{ anisotropic normal part }.$$

Then the Codazzi equation (6.8) follows from the symmetry $\nabla_{\partial_j} \nabla_{\partial_i} \nu = \nabla_{\partial_i} \nabla_{\partial_j} \nu$.

We are remained with the verification of the Gauss equation (6.7). We choose the normal coordinate at some point p_0 with respect to g, such that $g_{ij}(p_0) = \delta_{ij}$ and $\Gamma_{ij}^k(p_0) = 0$. Taking derivative of the Gauss formula (6.5), we have at p_0,

$$\begin{aligned} &\partial_l \partial_j \partial_i X = \nabla_{\partial_l}\left[-h_{ij}\nu + (\Gamma_{ij}^k + A_{ij}^k)\partial_k X\right] \\ = &\left[-h_{ij} h_l^k + \partial_l(\Gamma_{ij}^k + A_{ij}^k) + A_{ij}^m A_{lm}^k\right] \partial_k X + \text{ anisotropic normal part }. \end{aligned}$$

Hence

$$\begin{aligned} &0 = \partial_l \partial_j \partial_i X - \partial_j \partial_l \partial_i X \\ = &\left[h_{il} h_j^k - h_{ij} h_l^k + \partial_l \Gamma_{ij}^k - \partial_j \Gamma_{il}^k + \partial_l A_{ij}^k - \partial_j A_{il}^k + A_{ij}^m A_{lm}^k - A_{il}^m A_{jm}^k\right] \partial_k X \\ &+ \text{ anisotropic normal part }. \end{aligned} \tag{6.11}$$

By definition of Riemannian curvature, at p_0,

$$\begin{aligned} R_{ijkl} &= g(\nu)(\nabla_{\partial_k}\nabla_{\partial_l}\partial_j X - \nabla_{\partial_l}\nabla_{\partial_k}\partial_j X, \partial_i X) \\ &= \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj}. \end{aligned}$$

Now it follows from (6.11) that

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \partial_l A_{jki} - \partial_k A_{jli} + A^m_{jk}A_{mli} - A^m_{jl}A_{mki}.$$

Since both sides are tensors, (6.7) holds at every point. We complete the proof. □

Remark 6.8. *We see from* (6.9) *that the* 3*-tensor A is symmetric in the first two indices, namely, $A_{ijk} = A_{jik}$. However, it is not totally symmetric in general. Indeed, it can be shown that $A_{ijk} + A_{ikj} = -h^l_i Q_{ljk}$.*

We will compute in the following example the geometry quantities defined above for the special case $M = \mathcal{W}$. It shows that $\mathcal{W}$ plays the same role in relative geometry as the standard sphere in classical Euclidean geometry.

Example 6.9. *Consider the hypersurface $M = \mathcal{W} \subset \mathbb{R}^{n+1}$, the Wulff shape. In this case, the position vector and the unit anisotropic outer normal coincide, i.e., $X = \nu(X)$ for $X \in \mathcal{W}$. Hence $\partial_i X = \partial_i \nu = h^j_i \partial_j X$, which implies that $h^j_i = \delta_{ij}$ and in turn, $h_{ij} = g_{ij}$. In this case $A_{ijk} = -\frac{1}{2}Q_{ijk}$ and it is totally symmetric for all the indices. It is easy to see that*

$$\nabla_{\partial_l} Q_{ijk} = \nabla_{\partial_k} Q_{ijl}. \tag{6.12}$$

Hence the Gauss equation (6.7) *can be easier written as*

$$R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk} + \frac{1}{4}Q^m_{jk}Q_{mli} - \frac{1}{4}Q^m_{jl}Q_{mki}. \tag{6.13}$$

The following lemma states that the difference between anisotropic volume form and induced volume form by g has a natural relationship with the tensor A. It is a simple but quite important observation in the relative geometry.

Lemma 6.10. *Let dV_g be the induced volume form of M equipped with g. Assume that $d\mu = \varphi dV_g$. Then*

$$\partial_i \log \varphi = A^j_{ij} = g^{jk}A_{ijk}.$$

Proof. In local coordinates,

$$\varphi = \frac{\Omega(\nu, \partial_1 X, \cdots, \partial_n X)}{\sqrt{\det(g)}}.$$

We compute that

$$\begin{aligned}
& \partial_i \log \Omega(\nu, \partial_1 X, \cdots, \partial_n X) \\
= & \frac{1}{\Omega(\nu, \partial_1 X, \cdots, \partial_n X)} \sum_{j=1}^{n} \Omega(\nu, \partial_1 X, \cdots, \partial_i \partial_j X, \cdots, \partial_n X) \\
= & (\Gamma_{ij}^j + A_{ij}^j).
\end{aligned}$$

On the other hand, since g is a Riemannian metric on M,

$$\partial_i \log \sqrt{\det(g)} = \frac{1}{2} g^{jk} \partial_i g_{jk} = \Gamma_{ij}^j.$$

Therefore, we have

$$\partial_i \log \varphi = A_{ij}^j.$$

□

Chapter 7

Anisotropic Minkowski problem

7.1 Formulation of the anisotropic Minkowski problem

Let M be an n-dimensional closed, strongly convex hypersurface in $\mathbb{R}^{n+1}$. Since the map $\mathbb{S}^n \to \mathcal{W} : \bar{\nu} \mapsto \nu = DF(\bar{\nu})$ defines a nondegenerate diffeomorphism between $\mathbb{S}^n$ and $\mathcal{W}$, we easily see that the anisotropic Gauss map $\nu : M \to \mathcal{W}$ is everywhere nondegenerate diffeomorphism. We can use it to reparametrize the convex hypersurface, i.e.

$$X : \mathcal{W} \to M \subset \mathbb{R}^{n+1}, \quad X(z) = X(\nu^{-1}(z)), \quad z \in \mathcal{W}.$$

By virtue of Proposition 6.6, the anisotropic Gauss-Kronecker curvature of M is positive. With this parametrization, it can be viewed as a positive function $K(\nu^{-1}(z))$ on the Wulff shape $\mathcal{W}$.

The anisotropic Minkowski problem is the anisotropic version of Minkowski problem in classical geometry. Namely, it is a problem of prescribing the anisotropic Gauss-Kronecker curvature on the anisotropic normals of a closed strongly convex hypersurface. We state this problem as follows:

Anisotropic Minkowski problem: Given a positive function K on $\mathcal{W}$, is there a closed strongly convex hypersurface whose anisotropic Gauss-Kronecker curvature is K as a function on its anisotropic normals?

A necessary condition to this problem is that K must satisfy

$$\int_{\mathcal{W}} G(z)(z, E^\alpha)\frac{1}{K(z)}d\mu = 0, \quad \forall \alpha = 1, \cdots, n+1, \tag{7.1}$$

where E^α denote the standard α-th coordinate vector in $\mathbb{R}^{n+1}$. In fact, in view of

(6.4) and by using divergence theorem, we have

$$\begin{aligned}
& \int_{\mathcal{W}} G(z)(z, E^\alpha)\frac{1}{K(z)}d\mu \\
= & \int_M G(\nu)(\nu, E^\alpha)d\mu(M) = \int_M G(\nu)(\nu, E^\alpha)\Omega(\nu, \partial_1, \cdots, \partial_n)dy^1 \cdots dy^n \\
= & \int_M \Omega(E^\alpha, \partial_1, \cdots, \partial_n)dy^1 \cdots dy^n = \int_M \langle \bar{\nu}, E^\alpha \rangle_{\mathbb{R}^{n+1}} dvol(M) \\
= & \int_{\bar{M}} \operatorname{div}(E^\alpha) d\mathcal{H}^{n+1} = 0.
\end{aligned}$$

Here $\bar{M}$ is the body enclosed by M, and $dvol(M) = \Omega(\bar{\nu}, \partial_1, \cdots, \partial_n)dy^1 \cdots dy^n$ is the induced volume form of the Euclidean metric in $\mathbb{R}^{n+1}$.

In this chapter, we solve the anisotropic Minkowski problem. The main result is the following

Theorem 7.1. *Let F be a Minkowski norm in $\mathbb{R}^{n+1}$. Let K be a positive function in $C^k(\mathcal{W})$ with $k \geq 2$ and satisfy the condition* (7.1)*. Then there is a $C^{k+1,\alpha}(\forall 0 < \alpha < 1)$ closed strongly convex hypersurface M in $\mathbb{R}^{n+1}$ whose anisotropic Gauss-Kronecker curvature is K as a function on its anisotropic normals. Moreover, M is unique up to translations.*

As in the classical Minkowski problem, we will reduce the solvability of the anisotropic Minkowski problem to that of a fully nonlinear elliptic equation of a suitable support function. First of all, let us introduce the anisotropic support function.

The anisotropic support function of M is defined as

$$S(z) = \sup_{y \in M} G(z)(z, y) = G(z)(z, X(z)), \text{ for } z \in \mathcal{W}.$$

We will compute the metric g and the anisotropic second fundamental form h of M in terms of the anisotropic support function S.

Let $z \in \mathcal{W}$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of $T_z\mathcal{W} = T_{X(z)}M$ with respect to the Riemannian metric g. Denote by ∇ the covariant derivative with respect to g on $\mathcal{W}$. Taking the first covariant derivative of S, we have

$$\begin{aligned}
\nabla_{e_i} S(z) &= G(z)(\nabla_{e_i} z, X(z)) + G(z)(z, \nabla_{e_i} X(z)) + Q(z)(\nabla_{e_i} z, z, X(z)) \\
&= G(z)(\nabla_{e_i} z, X(z)),
\end{aligned}$$

The last two terms vanish due to (6.2) and (6.3).

Taking the second covariant derivative of S, by using Gauss formula (6.5) we have (we compute at normal coordinate of g, namely, $\nabla_{e_i} e_j = 0$)

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_j} S(z) &= e_i e_j G(z)(z, X(z)) = e_i(G(z)(e_j z, X)) \\
&= G(z)(e_i e_j z, X(z)) + G(z)(e_j z, e_i X) + Q(e_i z, e_j z, X(z)) \\
&= -\delta_{ij} G(z)(z, X(z)) - \frac{1}{2} Q(e_i z, e_j z, e_k z) G(z)(e_k, X(z)) \\
&\quad - G(z)(z, (-h_{ij}(X)z)) + Q(e_i z, e_j z, e_k z) G(z)(e_k, X(z)) \\
&= -\delta_{ij} S(z) + h_{ij}(X(z)) + \frac{1}{2} Q_{ijk} \nabla_{e_k} S(z).
\end{aligned}$$

Here we also used the observation in Example 6.9.

For simplicity, we use the abbreviation S_i, S_{ij} to denote the covariant derivative of g. Thus it follows from previous computation that the anisotropic second fundamental form of M has the formula

$$h_{ij}(X(z)) = S_{ij}(z) - \frac{1}{2} Q_{ijk} S_k(z) + \delta_{ij} S(z), \quad \forall z \in \mathcal{W}.$$

To compute the metric g of M, we use the Weingarten formula (6.6),

$$e_i z = g^{jk}(X) h_{ij}(X) \nabla_{e_k} X,$$

from which we obtain

$$\delta_{ij} = g(z)(e_i z, e_j z) = h_{ik} g^{kl} h_{jl}.$$

In turn, we have

$$g_{ij} = h_{ik} h_{jk}.$$

Therefore, the anisotropic principal radii of M are the eigenvalues of

$$g_{ik} h^{jk} = h_{ij} = S_{ij}(z) - \frac{1}{2} Q_{ijk} S_k(z) + \delta_{ij} S(z), \quad \forall z \in \mathcal{W}.$$

The Gauss-Kronecker curvature is

$$K(z) = \frac{1}{\det(S_{ij}(z) - \frac{1}{2} Q_{ijk} S_k(z) + \delta_{ij} S(z))}, \quad \forall z \in \mathcal{W}.$$

In summary, we have proved the following proposition.

Proposition 7.2. *Parametrizing a C^2 strongly convex hypersurface M by the inverse anisotropic Gauss map over $\mathcal{W}$, we have that the eigenvalue of $S_{ij} - \frac{1}{2} Q_{ijk} S_k + S\delta_{ij}$ is the anisotropic principle radii of M. In particular, the anisotropic Gauss-Kronecker curvature of M satisfies*

$$\det(S_{ij} - \frac{1}{2} Q_{ijk} S_k + S\delta_{ij}) = \frac{1}{K} \quad \text{on } \mathcal{W}. \tag{7.2}$$

Conversely, given S a C^2 function on $\mathcal{W}$ with $(S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S) > 0$, we are able to find a strongly convex hypersurface such that its anisotropic support function is S.

Proposition 7.3. *Any function $S \in C^2(\mathcal{W})$ with $(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}) > 0$ is an anisotropic support function of a C^2 strongly convex hypersurface M in $\mathbb{R}^{n+1}$.*

Proof. We extend S to be a homogeneous function of degree one in $\mathbb{R}^{n+1} \setminus \{0\}$ by setting $S(x) = F^0(x)S\left(\frac{x}{F^0(x)}\right)$. Denote by $\nabla_{(\mathbb{R}^{n+1},G)}$ be the covariant derivative of $\mathbb{R}^{n+1}$ equipped with the metric G. Define

$$M = \left\{\nabla_{(\mathbb{R}^{n+1},G)}S(x) | x \in \mathbb{R}^{n+1} \setminus \{0\}\right\}.$$

Let $e_{n+1} = z$ be the position vector of $\mathcal{W}$ and $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field with respect to g on $\mathcal{W}$ such that $\{e_1, \cdots, e_{n+1}\}$ is a positive oriented orthonormal frame field with respect to G in $\mathbb{R}^{n+1}$. Then it follows from the homogeneity of S that for $y \in M$, there exists $z \in \mathcal{W}$, such that

$$y = y(z) = \nabla_{(\mathbb{R}^{n+1},G)}S(z) = \nabla_{e_i}S(z)e_i(z) + S(z)e_{n+1}(z). \tag{7.3}$$

It is clear that

$$e_i(e_{n+1}) = e_i, \quad e_i(e_j) = \nabla_{e_i}e_j - \frac{1}{2}Q_{ijk}e_k - \delta_{ij}e_{n+1}. \tag{7.4}$$

Using (7.4), we compute the derivative of y on $\mathcal{W}$ (at normal coordinates, namely, $\nabla_{e_i}e_j = 0$),

$$\begin{aligned} e_j(y) &= e_je_i(S)e_i + v_ie_j(e_i) + e_j(S)e_{n+1} + Se_j(e_{n+1}) \\ &= S_{ij}e_i + S_i(-\frac{1}{2}Q_{ijk}e_k - \delta_{ij}e_{n+1}) + S_je_{n+1} + Se_j \\ &= (S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S)e_i. \end{aligned} \tag{7.5}$$

Since $(S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S) > 0$ by assumption, (7.5) implies that the tangent space of M at $y(z)$ is span$\{e_1(z), \cdots, e_n(z)\}$. Hence $e_{n+1}(z) = z$ is the anisotropic normal at $y(x)$. Now $\{e_1, \cdots, e_n, z\}$ gives an orientation of M. Also the map $y(z) = \nabla_{e_i}S(z)e_i(z) + S(z)e_{n+1}(z)$ is globally invertible and M is an embedded hypersurface in $\mathbb{R}^{n+1}$.

In view of (7.3), $S(z) = G(z)(z, y(z))$. It follows from the previous computation that the anisotropic principle curvatures at $y(z)$ are the reciprocals of the eigenvalues of $(S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S) > 0$. Therefore, M is strongly convex and S is its anisotropic support function. □

By virtue of Proposition 7.2 and 7.3, the solvability of the anisotropic Minkowski problem is reduced to that of the equation (7.2) on the Wulff shape $\mathcal{W}$ under the condition that $K \in C^k(\mathcal{W})$, $k \geq 2$, $K > 0$ and satisfies the equation (7.1). Therefore, to prove Theorem 7.1, it is equivalent to prove the solvability of the equation (7.2).

Definition 7.4. *We call a solution S of (7.2) is an admissible solution if the $n \times n$ matrix $S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}$ is positive definite.*

Theorem 7.5. *Let F be a Minkowski norm in $\mathbb{R}^{n+1}$. Let K be a positive function in $C^k(\mathcal{W})$ with $k \geq 2$ and satisfy the condition (7.1). Then we can find an admissible solution $S \in C^{k+1,\alpha}(\mathcal{W})(\forall 0 < \alpha < 1)$ to the equation (7.2). If there exist two admissible solutions S and $\tilde{S}$ to (7.2), then there exist some constants $c_1, \cdots, c_{n+1}$, such that*

$$S(z) - \tilde{S}(z) = \sum_{\alpha=1}^{n+1} c_\alpha G(z)(z, E^\alpha).$$

We will use the method of continuity to find an admissible solution of (7.2). In the next section, we prove the a priori estimates for the equation (7.2).

7.2 A priori estimates

In this section, we shall establish the a priori estimates for the admissible solution to the equation (7.2). We will frequently use the symmetric function $\sigma_n(u_{ij}) = \det(u_{ij})$.

In view of the Gauss formula (6.5), we have for $z \in \mathcal{W} \subset \mathbb{R}^{n+1}$,

$$z_{ij} = -z\delta_{ij} - \frac{1}{2}Q_{ijk}z_k.$$

Hence

$$(G(z)(z, E^\alpha))_{ij} = \frac{1}{2}Q_{ijk}(G(z)(z, E^\alpha))_k - G(z)(z, E^\alpha)\delta_{ij},$$

which implies

$$L_S(G(z)(z, E^\alpha)) = 0. \tag{7.6}$$

Hence for constants $\{a_\alpha\}_{\alpha=1}^{n+1}$, the function $L(z) = a_\alpha G(z)(z, E^\alpha)$, $z \in \mathcal{W} \subset \mathbb{R}^{n+1}$, satisfies

$$L_{ij} - \frac{1}{2}Q_{ijk}L_k + L\delta_{ij} = 0.$$

Thus for a solution S of (7.2), $S+L$ is also a solution. Such an observation allows us to restrict S to satisfy the following orthogonal condition

$$\int_{\mathcal{W}} G(z)(z, E^{\alpha})S(z)d\mu = 0 \quad \forall \alpha = 1, 2, \cdots, n+1. \tag{7.7}$$

This orthogonal condition means that the origin lies in the interior of the convex body enclosed by M.

Under the restriction (7.7), we are able to prove the following a priori estimates.

Theorem 7.6. *For each integral $k \geq 1$ and $\alpha \in (0,1)$, there exist a constant C, depending on $n, k, \alpha, \|K\|_{C^k(\mathcal{W})}, inf_{\mathcal{W}} K, \|F\|_{C^{k+3}(\mathcal{W})}$, such that*

$$\|S\|_{C^{k+1,\alpha}(\mathcal{W})} \leq C,$$

for all admissible solutions of (7.2) *satisfying the condition* (7.7).

We remark that the norm of S in this chapter are all with respect to the metric $g = G|_{\mathcal{W}}$ on $\mathcal{W}$.

7.2.1 C^0 estimate

We first establish a uniform positive lower bound and upper bound for S. Since (7.7) means that the origin lies in the interior of the convex body enclosed by M, in order to obtain the bounds for S, it is sufficient to find the bound for the anisotropic inner and outer radius of M relative to $\mathcal{W}$.

The anisotropic inner radius of M relative to $\mathcal{W}$ is defined as

$$r(M) := \sup\{t > 0 : t\mathcal{W} + y \subset \mathcal{K} \text{ for some } y \in \mathbb{R}^{n+1}\},$$

and the anisotropic outer radius of $\mathcal{K}$ relative to $\mathcal{W}$ is defined as

$$R(M) := \inf\{t > 0 : \mathcal{K} \subset t\mathcal{W} + y \text{ for some } y \in \mathbb{R}^{n+1}\}.$$

Lemma 7.7. *Let M be a compact convex C^2 hypersurface in $\mathbb{R}^{n+1}$ and K be its anisotropic Gauss-Kronecker curvature function defined on $\mathcal{W}$. Then*

$$\frac{1}{2}m_1 \leq r(M) \leq R(M) \leq \frac{1}{2}m_2,$$

where

$$m_1 = 2|\mathcal{W}|^{-\frac{1}{n}} \left(\int_{\mathcal{W}} \frac{1}{K(z)} d\mu \right)^{\frac{n+1}{n}} \left(\inf_{y \in \mathcal{W}} \int_{\mathcal{W}} \frac{1}{K(z)} \max\{0, G(z)(z,y)\} d\mu \right)^{-1},$$

$$m_2 = 2\frac{1}{n+1}|\mathcal{W}|^{-\frac{1}{n}}\left(\int_{\mathcal{W}}\frac{1}{K(z)}d\mu\right)^{-1}\cdot\left(\inf_{y\in\mathcal{W}}\int_{\mathcal{W}}\frac{1}{K(z)}\max\{0,G(z)(z,y)\}d\mu\right)^{\frac{n+1}{n}},$$

and $|\mathcal{W}|$ is the standard n-dimensional volume of $\mathcal{W}$. In particular, if S is an admissible solution of (7.2) *on $\mathcal{W}$ and satisfies* (7.7)*, then*

$$0 < m_1 \le S \le m_2.$$

Proof. Since the origin lies in the interior of M, we can find $p_0 \in M$ with $R_0 := F^0(p_0) = \max_{p\in M} F^0(p)$, set $\bar{p}_0 = \frac{p_0}{F^0(p_0)} \in \mathcal{W}$. It is easy to see that $M \subset R_0\mathcal{W}$. Hence $R(M) \le R_0$. The support function S at $z \in \mathcal{W}$ satisfies

$$S(z) = \sup_{y\in M} G(z)(z,y) \ge \max\{0, G(z)(z,p_0)\} = R_0 \max\{0, G(z)(z,\bar{p}_0)\}.$$

Denote $f = \frac{1}{K}$. By multiplying f and integrating over $\mathcal{W}$, we have

$$R(M) \le R_0 \le \left(\int_{\mathcal{W}} f(z)S(z)d\mu\right)\left(\int_{\mathcal{W}} f(z)\max\{0, G(z)(z,\bar{p}_0)\}d\mu\right)^{-1} \tag{7.8}$$

In view of (6.4), we have a Minkowski formula,

$$\begin{aligned}\int_{\mathcal{W}} f(z)S(z)d\mu &= \int_M S(\nu(X))d\mu(M) &(7.9)\\ = \int_M G(\nu)(\nu,X)\Omega(\nu,\partial_1,\cdots,\partial_n)dy^1\cdots dy^n\\ = \int_M \Omega(X,\partial_1,\cdots,\partial_n)dy^1\cdots dy^n = (n+1)Vol(\bar{M}).\end{aligned}$$

The anisotropic isoperimetric inequality (see Busemann [Bu49]) tells that

$$Vol(\bar{M}) \le \frac{1}{n+1}|\mathcal{W}|^{-\frac{1}{n}}|M|_F^{\frac{n+1}{n}} = \frac{1}{n+1}|\mathcal{W}|^{-\frac{1}{n}}\left(\int_{\mathcal{W}} f(z)d\mu\right)^{\frac{n+1}{n}}. \tag{7.10}$$

Combining (7.8), (7.9) and (7.10), we obtain the upper bound of $R(M)$,

$$R(M) \le |\mathcal{W}|^{-\frac{1}{n}}\left(\int_{\mathcal{W}} f(z)d\mu\right)^{\frac{n+1}{n}}\left(\inf_{y\in\mathcal{W}}\int_{\mathcal{W}} f(z)\max\{0,G(z)(z,y)\}d\mu\right)^{-1}.$$

Next we want to find the positive lower bound of $r(M)$. Since $\bar{M}$ is enclosed in a rescaled Wulff shape with radius $R(M)$, we have

$$Vol(\bar{M}) \le \frac{1}{n+1}|\mathcal{W}|R(M)^{n+1}. \tag{7.11}$$

It follows from (7.8), (7.9) and (7.11) that

$$R(M) \geq |\mathcal{W}|^{-\frac{1}{n}} \left(\inf_{y\in\mathcal{W}} \int_{\mathcal{W}} f(z) \max\{0, G(z)(z,y)\} d\mu \right)^{\frac{1}{n}}, \tag{7.12}$$

and then

$$Vol(\bar{M}) \geq \frac{1}{n+1} |\mathcal{W}|^{-\frac{1}{n}} \left(\inf_{y\in\mathcal{W}} \int_{\mathcal{W}} f(z) \max\{0, G(z)(z,y)\} d\mu \right)^{\frac{n+1}{n}}. \tag{7.13}$$

Recalling an inequality by Ben Andrews [An01], Proposition 5.1, which is a consequence of the Diskant inequalities,

$$r(M) \geq \frac{Vol(\bar{M})}{|M|_F} = \frac{Vol(\bar{M})}{\int_{\mathcal{W}} f(z) d\mu}. \tag{7.14}$$

Combining (7.13) and (7.14), we get the positive lower bound of $r(M)$,

$$\begin{aligned} r(M) \geq \quad & \frac{1}{n+1} |\mathcal{W}|^{-\frac{1}{n}} \left(\int_{\mathcal{W}} f(z) d\mu \right)^{-1} \\ & \cdot \left(\inf_{y\in\mathcal{W}} \int_{\mathcal{W}} f(z) \max\{0, G(z)(z,y)\} d\mu \right)^{\frac{n+1}{n}}. \end{aligned}$$

Now it is easy to derive the upper and positive lower bound of S in terms of $r(M)$ and $R(M)$. In fact, it follows from Schwarz inequality that

$$S(z) = G(z)(z, X(z)) \leq F^0(z) F^0(X(z)) \leq 2R(M), \quad \forall z \in \mathcal{W}. \tag{7.15}$$

On the other hand, for any $z \in \mathcal{W}$, let $t(z) > 0$ be the number such that $t(z)z \in M$. It follows from the difinition of $r(M)$ that $2r(M) \leq \sup_{z\in\mathcal{W}} t(z)$. Consequently,

$$S(z) = \sup_{y\in M} G(z)(z,y) \geq G(z)(z, t(z)z) = t(z) \geq 2r(M), \quad \forall z \in \mathcal{W}. \tag{7.16}$$

□

7.2.2 C^1 estimate

The next step is a priori C^1 estimate for S. Such estimate is not necessary in the classical Minkowski problem, since there the C^2 estimate is more direct. However, for the equation (7.2), the gradient term in the determinant causes problem, which cannot be solved until we have the C^1 estimate first. Therefore, in the anisotropic Minkowski problem, the C^1 estimate seems necessary.

Lemma 7.8. *Let S be an admissible solution of (7.2) on $\mathcal{W}$. Let $f = \frac{1}{K}$. Then there exists a constant C, depending on $n, m_1, m_2, \max f, \max|\nabla f^{\frac{1}{n}}|, \|F^0\|_{C^4(\mathcal{W})}$, such that*

$$|\nabla S| \le C. \tag{7.17}$$

Proof. Suppose that $|\nabla S| \ge 1$, otherwise, we are done. Denote $u_{ij} = S_{ij} - \frac{1}{2}Q_{ijk}S_k + \delta_{ij}S$ and $\mathcal{F}^{ij} = \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})$. We know that $\mathcal{F}^{ij}$ is an elliptic operator at an admissible solution.

Let $W = \log|\nabla S|^2 + e^{\alpha(m_2 - S)}$, with $\alpha > 0$ to be chosen later. Suppose that W attains its maximum at point $z_0 \in \mathcal{W}$. Choose an orthonormal basis at z_0 such that u_{ij} is diagonal. It is clear that $\mathcal{F}^{ij}$ is also diagonal at z_0.

Then at z_0, we have

$$0 = W_i = \frac{|\nabla S|_i^2}{|\nabla S|^2} - \alpha e^{\alpha(m_2 - S)}S_i = \frac{2S_kS_{ki}}{|\nabla S|^2} - \alpha e^{\alpha(m_2-S)}S_i, \tag{7.18}$$

$$\begin{aligned} 0 \ge \mathcal{F}^{ij}W_{ij} = \mathcal{F}^{ij}\frac{2S_lS_{lij} + 2S_{ki}S_{kj}}{|\nabla S|^2} - \frac{\mathcal{F}^{ij}|\nabla S|_i^2|\nabla S|_j^2}{|\nabla S|^4} \\ -\alpha e^{\alpha(m_2-S)}\mathcal{F}^{ij}S_{ij} + \alpha^2 e^{\alpha(m_2-S)}\mathcal{F}^{ij}S_iS_j. \end{aligned} \tag{7.19}$$

Notice that

$$\mathcal{F}^{ii}u_{ii} = nf, \quad \mathcal{F}^{ii}u_{iik} = f_k. \tag{7.20}$$

Using (7.18) and (7.20), we estimate the second term in the RHS of (7.19) as follows,

$$\begin{aligned} &-\frac{\mathcal{F}^{ij}|\nabla S|_i^2|\nabla S|_j^2}{|\nabla S|^4} = -\mathcal{F}^{ii}\alpha e^{\alpha(m_2-S)}S_i\frac{2S_kS_{ki}}{|\nabla S|^2} \\ = \;& -2\alpha e^{\alpha(m_2-S)}\frac{\mathcal{F}^{ii}S_iS_k(u_{ki} + \frac{1}{2}Q_{ikl}S_l - \delta_{ki}S)}{|\nabla S|^2} \\ \ge \;& -2\alpha e^{\alpha(m_2-S)}(nf + C_1\mathcal{F}^{ii}S_i) + 2\alpha e^{\alpha(m_2-S)}\frac{S\mathcal{F}^{ii}S_i^2}{|\nabla S|^2} \\ \ge \;& -2\alpha e^{\alpha(m_2-S)}(nf + \frac{\alpha}{4}\mathcal{F}^{ii}S_i^2 + \frac{C_2}{\alpha}\sum_i \mathcal{F}^{ii}), \end{aligned} \tag{7.21}$$

where we dropped the term $2\alpha e^{\alpha(m_2-S)}\frac{S\mathcal{F}^{ii}S_i^2}{|\nabla S|^2}$ since it is positive, while used the Hölder inequality for the term $\mathcal{F}^{ii}S_i$.

Observe that

$$S_{lij} = S_{ijl} + R_{mijl}S_m, \tag{7.22}$$

$$\mathcal{F}^{ij}S_{ki}S_{kj} \geq 0. \tag{7.23}$$

By employing (7.18), (7.20), (7.21), (7.22) and (7.23) in (7.19), we obtain that

$$\begin{aligned} 0 &\geq \mathcal{F}^{ij}W_{ij} \\ &\geq \frac{2\mathcal{F}^{ij}S_l(u_{ij}+\frac{1}{2}Q_{ijk}S_k-\delta_{ij}S)_l}{|\nabla S|^2} - C_3e^{\alpha(m_2-S)}\sum_i \mathcal{F}^{ii} \\ &\quad -\alpha e^{\alpha(m_2-S)}\mathcal{F}^{ij}(u_{ij}+\frac{1}{2}Q_{ijk}S_k-\delta_{ij}S) \\ &\quad +\frac{1}{2}\alpha^2 e^{\alpha(m_2-S)}\mathcal{F}^{ii}S_i^2 - 2\alpha e^{\alpha(m_2-S)}nf \\ &\geq -2\frac{|\nabla f|}{|\nabla S|} - 3n\alpha e^{\alpha(m_2-S)}nf + (\alpha S - C_4)e^{\alpha(m_2-S)}\sum_i \mathcal{F}^{ii} \\ &\quad +\frac{1}{2}\alpha^2 e^{\alpha(m_2-S)}\mathcal{F}^{ii}S_i^2. \end{aligned} \tag{7.24}$$

Note that by Newton-Maclaurin's inequality, we have (see Guan [Gu04])

$$\sum_i \mathcal{F}^{ii} \geq C(n)\sigma_n^{\frac{n-1}{n}} \geq C > 0. \tag{7.25}$$

Choose α large, such that $\alpha m_1 - C_4 \geq 1$. It follows from (7.24) and (7.25) that

$$\begin{aligned} \sum_i \mathcal{F}^{ii}(1+S_i^2) &\leq 2\frac{|\nabla f|}{|\nabla S|} + 3\alpha e^{\alpha(m_2-S)}f \\ &\leq 2\max|\nabla f| + 3n\alpha e^{\alpha m_2}\max f. \end{aligned} \tag{7.26}$$

On the other hand, by using Gårding inequality (see Gårding [Ga59]), we see that

$$\begin{aligned} \sum_i \mathcal{F}^{ii}(1+S_i^2) &\geq n\left(\prod_i u_{ii}\right)^{\frac{n-1}{n}}\left(\prod_i(1+S_i^2)\right)^{\frac{1}{n}} \\ &\geq nf^{\frac{n-1}{n}}(1+|\nabla S|^2)^{\frac{1}{n}}. \end{aligned} \tag{7.27}$$

Putting (7.27) into (7.26), we conclude that

$$|\nabla S| \leq C(n, m_1, m_2, \max f, \max|\nabla f^{\frac{1}{n}}|, \|F^0\|_{C^4(\mathcal{W})}).$$

□

7.2.3 C^2 estimate

The C^2 estimate for the classical Minkowski problem is somehow direct and fine due to the structure of its equation. In particular, it involves the exact formula of Gauss equation. In our problem the gradient term in the determinant also brings troubles. We brings here some idea from Yau's proof [Ya78] in Calabi conjecture and Guan-Li's proof [GL10] for more general complex Monge-Ampére equations to our equation. It avoids the use of explicit formula for Gauss equation.

Lemma 7.9. *Let S be an admissible solution of (7.2) on $\mathcal{W}$. Let $f = \frac{1}{K}$. Then there exists a constant C, depending on $n, m_1, m_2, \|f\|_{C^2}, \min f, \|F^0\|_{C^5(\mathcal{W})}$, such that*

$$|\nabla^2 S| \leq C. \tag{7.28}$$

Proof. Let $\bar{\mathcal{F}}(u_{ij}) = \log \sigma_n(u_{ij})$ and u^{ij} be the inverse matrix of u_{ij}. Then

$$\bar{\mathcal{F}}^{ij} = f u^{ij}, \quad \bar{\mathcal{F}}^{ij,kl} = -u^{ik}u^{jl}. \tag{7.29}$$

For an admissible solution S, $0 < 2\sigma_2(u_{ij}) = (\sum_i u_{ii})^2 - \sum_{i,j} |u_{ij}|^2$. In view of Lemma 7.7 and 7.8, to bound $|\nabla^2 S|$, it is sufficient to bound ΔS from above. Here Δ denotes the Laplace operator with respect to g. We may assume that $\Delta S \geq C$ for some $C > 0$.

Let $\Phi = \log(a + \Delta S) + e^{\beta(m_2 - S)}$ with $a = \sup_{\mathcal{W}} |-\frac{1}{2}Q_{iik}S_k + nS|$, $\beta > 0$ to be chosen later. Suppose that Φ attains its maximum at point $z_0 \in \mathcal{W}$. Choose the orthonormal basis at z_0 such that u_{ij} is diagonal. Clearly $\bar{\mathcal{F}}^{ij}$ is also diagonal at z_0. Then we have

$$\Phi_i = \frac{(\Delta S)_i}{a + \Delta S} - \beta e^{\beta(m_2 - S)} S_i = 0, \tag{7.30}$$

$$\begin{aligned} 0 \geq \bar{\mathcal{F}}^{ij}\Phi_{ij} \ &= \ \frac{\bar{\mathcal{F}}^{ii}(\Delta S)_{ii}}{a + \Delta S} - \frac{\bar{\mathcal{F}}^{ii}(\Delta S)_i^2}{(a + \Delta S)^2} \\ &\quad -\beta e^{\beta(m_2 - S)}\bar{\mathcal{F}}^{ii}S_{ii} + \beta^2 e^{\beta(m_2 - S)}\bar{\mathcal{F}}^{ii}S_i^2. \end{aligned} \tag{7.31}$$

We estimate the term $\bar{\mathcal{F}}^{ii}(\Delta S)_{ii}$ and $\bar{\mathcal{F}}^{ii}(\Delta S)_i^2$ by using (7.30) as follows.

$$\begin{aligned} &\bar{\mathcal{F}}^{ii}(\Delta S)_{ii} = \bar{\mathcal{F}}^{ii}(S_{iikk} + Riem * \nabla^2 S + \nabla Riem * \nabla S) \\ \geq\ & \bar{\mathcal{F}}^{ii}(u_{ii} + \frac{1}{2}Q_{iil}S_l - S)_{kk} - C_5 \sum_i \bar{\mathcal{F}}^{ii}(|\nabla^2 S| + |\nabla S|) \\ \geq\ & \Delta f - \bar{\mathcal{F}}^{ij,rs}u_{ijk}u_{rsk} + \frac{1}{2}\bar{\mathcal{F}}^{ii}Q_{iil}(\Delta S)_l - C_6 \sum_i \bar{\mathcal{F}}^{ii}(|\nabla^2 S| + |\nabla S|) \\ \geq\ & u^{ii}u^{jj}u_{ijk}^2 + \beta e^{\beta(m_2 - S)}\bar{\mathcal{F}}^{ii}\frac{1}{2}Q_{iil}S_l(a + \Delta S) - C_7 \sum_i \bar{\mathcal{F}}^{ii}\Delta S - C_8(1 + \sum_i \bar{\mathcal{F}}^{ii}). \end{aligned} \tag{7.32}$$

Here and in the following we use *Riem* to denote the Riemannian tensor of g and the notation $*$ to denote scalar contraction of two tensors by g.

$$\begin{aligned}
&\bar{\mathcal{F}}^{ii}(\Delta S)_i^2 = \bar{\mathcal{F}}^{ii}(S_{ikk} + Riem * \nabla S)^2 \qquad (7.33)\\
\leq\ & \bar{\mathcal{F}}^{ii}S_{ikk}^2 + C_9\bar{\mathcal{F}}^{ii}|S_{ikk}| + C_{10}\sum_i \bar{\mathcal{F}}^{ii}\\
\leq\ & \bar{\mathcal{F}}^{ii}S_{ikk}^2 + C_9\bar{\mathcal{F}}^{ii}\left((a+\Delta S)|S_i|\beta e^{\beta(m_2-S)} + |Riem * \nabla S|\right) + C_{10}\sum_i \bar{\mathcal{F}}^{ii}\\
\leq\ & \bar{\mathcal{F}}^{ii}S_{ikk}^2 + C_{11}(\beta e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}|S_i|\Delta S + \sum_i \bar{\mathcal{F}}^{ii}).
\end{aligned}$$

Since

$$S_{ikk} = u_{ikk} + (\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k,$$

we have

$$\begin{aligned}
&S_{ikk}^2 - u_{ikk}^2 = 2u_{ikk}(\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k + (\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k^2 \qquad (7.34)\\
=\ & 2S_{ikk}(\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k - (\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k^2\\
=\ & 2\left((a+\Delta S)\beta e^{\beta(m_2-S)}S_i + Riem * \nabla S\right)(\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k - (\frac{1}{2}Q_{ikl}S_l - S\delta_{ik})_k^2\\
\leq\ & C_{12}(1+\beta e^{\beta(m_2-S)}|S_i|)\left((\Delta S)^2 + \Delta S + 1\right).
\end{aligned}$$

Now using (7.33) and (7.34), we obtain

$$\begin{aligned}
&\frac{\bar{\mathcal{F}}^{ii}(\Delta S)_i^2}{(a+\Delta S)^2} \leq \frac{\bar{\mathcal{F}}^{ii}S_{ikk}^2}{(a+\Delta S)^2} + C_{13}(\beta e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}|S_i| + \sum_i \bar{\mathcal{F}}^{ii}) \qquad (7.35)\\
\leq\ & \frac{\bar{\mathcal{F}}^{ii}u_{ikk}^2}{(a+\Delta S)^2} + C_{14}(\beta e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}|S_i| + \sum_i \bar{\mathcal{F}}^{ii})\\
\leq\ & \frac{\bar{\mathcal{F}}^{ii}u_{ikk}^2}{(a+\Delta S)^2} + \frac{1}{2}C_{14}\beta^{\frac{3}{2}}e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}S_i^2 + \frac{1}{2}C_{14}\beta^{\frac{1}{2}}e^{\beta(m_2-S)}\sum_i \bar{\mathcal{F}}^{ii} + C_{14}\sum_i \bar{\mathcal{F}}^{ii}.
\end{aligned}$$

In the last inequality, we used the Hölder inequality.

We estimate the term $\bar{\mathcal{F}}^{ii}u_{ikk}^2$ by Cauchy-Schwarz inequality,

$$\begin{aligned}
\bar{\mathcal{F}}^{ii}u_{ikk}^2 &\leq \sum_{i,j,k} u^{ii}(u_{jj}u^{jj}u_{ijk}^2) \qquad (7.36)\\
&\leq \sum_j u_{jj}\sum_{i,j,k} u^{ii}u^{jj}u_{ijk}^2\\
&\leq (a+\Delta S)\sum_{i,j,k} u^{ii}u^{jj}u_{ijk}^2.
\end{aligned}$$

It follows from (7.32), (7.35) and (7.36) that

$$\begin{aligned}\frac{\bar{\mathcal{F}}^{ii}(\Delta S)_{ii}}{a+\Delta S} - \frac{\bar{\mathcal{F}}^{ii}(\Delta S)_i^2}{(a+\Delta S)^2} \geq\ & \beta e^{\beta(m_2-S)}\frac{1}{2}\bar{\mathcal{F}}^{ii}Q_{iil}S_l - C_{15}\beta^{\frac{3}{2}}e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}S_i^2 \\ & -C_{16}\beta^{\frac{1}{2}}e^{\beta(m_2-S)}\sum_i \bar{\mathcal{F}}^{ii} - C_{17}\sum_i \bar{\mathcal{F}}^{ii}.\end{aligned} \tag{7.37}$$

We also have

$$\begin{aligned} -\beta e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ij}S_{ij} &= -\beta e^{\beta(m_2-S)}\bar{\mathcal{F}}^{ii}(u_{ii} + \frac{1}{2}Q_{iil}S_l - S) \\ \geq\ & -\beta e^{\beta(m_2-S)}f - \beta e^{\beta(m_2-S)}\frac{1}{2}\bar{\mathcal{F}}^{ii}Q_{iil}S_l + m_1\beta e^{\beta(m_2-S)}\sum_i \bar{\mathcal{F}}^{ii},\end{aligned} \tag{7.38}$$

where m_1 is the uniform positive bound of S in Lemma 7.7.

Therefore, by combining (7.31), (7.37) and (7.38), we get

$$\begin{aligned} 0 \geq\ & e^{\beta(m_2-S)}(m_1\beta - C_{16}\beta^{\frac{1}{2}} - C_{18})\sum_i \bar{\mathcal{F}}^{ii} \\ & + e^{\beta(m_2-S)}(\beta^2 - C_{15}\beta^{\frac{3}{2}})\bar{\mathcal{F}}^{ii}S_i^2 - C_{19}.\end{aligned} \tag{7.39}$$

Choose β large enough in (7.39), we obtain

$$\sum_i \bar{\mathcal{F}}^{ii} \leq C, \tag{7.40}$$

where C depends on $m_1, m_2, \|f\|_{C^2}, \|S\|_{C^1}, \|F^0\|_{C^5(\mathcal{W})}$.

Recall the following elementary inequality (see Yau [Ya78]),

$$\frac{\sum_i \lambda_i}{\prod_i \lambda_i} \leq \left(\sum_i \lambda_i^{-1}\right)^{n-1} \quad \text{for } \lambda_i > 0, \forall i = 1, \cdots, n.$$

Since $\bar{\mathcal{F}}^{ii} = u^{ii} = u_{ii}^{-1}$, $\det(u_{ii}) = f$ we use the previous inequality by $\lambda_i = u_{ii}$ and (7.40) to get

$$\sum_i u_{ii} \leq C_{20}\prod_i u_{ii} \leq C_{21}$$

which implies

$$\Delta S \leq C$$

where C depends on $m_1, m_2, \|f\|_{C^2}, \|S\|_{C^1}, \|F\|_{C^5(\mathcal{W})}$. Note that our computation is only valid at the maximum point z_0 of Φ. Nevertheless, we have

$$\Phi \leq \Phi(z_0) = \log(a + \Delta S(z_0)) + \beta e^{\beta(m_2 - S(z_0))} \leq C,$$

which yields the C^2 estimate for S at every point. □

Proof of Theorem 7.6: Once we have C^2 estimate, Theorem 7.6 follows from the Evans-Krylov theorem and the standard elliptic theory. □

7.3 Openness and proof of Theorem 7.1

By virtue of Theorem 7.6, we can assume that $K \in C^\infty(\mathcal{W})$. We will use the method of continuity to prove Theorem 7.5. To be precise, let

$$\frac{1}{K_t} = \frac{t}{K} + (1-t) \quad \forall 0 \le t \le 1.$$

It is easy to see that $K_t > 0$ and satisfies (7.1). Define

$$\mathcal{S} = \{t \in [0,1] | \det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}) = \frac{1}{K_t} \text{ has an admissible solution on } \mathcal{W}\}.$$

Clearly, $0 \in \mathcal{S}$ since $\mathcal{W}$ has anisotropic Gauss curvature 1 (see Example 6.9). We will apply the implicit function theorem to (7.2) to prove the openness of the set $\mathcal{S}$.

Proposition 7.10. *The set $\mathcal{S}$ is open in $[0,1]$.*

To prove the openness of $\mathcal{S}$, we shall study the linearized operator of $S \mapsto \det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij})$.

Denote $u_{ij} = S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}$. Let L_S be the linearized operator of $S \mapsto \det(u_{ij})$, namely,

$$L_S(v) = \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})(v_{ij} - \frac{1}{2}Q_{ijk}v_k + v\delta_{ij})$$

for any $v \in C^\infty(\mathcal{W})$. The following proposition shows that L_S is a self-adjoint operator.

Lemma 7.11. *For any $S, v, w \in C^2(\mathcal{W})$, we have*

$$\int_{\mathcal{W}} wL_S(v)d\mu = \int_{\mathcal{W}} vL_S(w)d\mu.$$

Proof. From the Gauss equation (6.13), we have

$$S_{ijk} - S_{ikj} = R_{pijk}S_p = S_j\delta_{ik} - S_k\delta_{ij} + \frac{1}{4}(Q_{ijm}Q_{mkp} - Q_{ikm}Q_{mjp})S_p. \quad (7.41)$$

By using (6.12), namely $Q_{ijk,l} = Q_{ilk,j}$ and (7.41), we see that

$$\begin{aligned} & u_{ijk} - u_{ikj} \qquad (7.42) \\ = \ & \frac{1}{4}(Q_{ijm}Q_{mkp} - Q_{ikm}Q_{mjp})S_p - \frac{1}{2}Q_{ijm}S_{mk} + \frac{1}{2}Q_{ikm}S_{mj} \\ = \ & -\frac{1}{2}Q_{ijm}(u_{mk} - S_{mk} - S\delta_{mk}) + \frac{1}{2}Q_{ikm}(u_{mj} - S_{mj} - S\delta_{mj}) \\ & -\frac{1}{2}Q_{ijm}S_{mk} + \frac{1}{2}Q_{ikm}S_{mj} \\ = \ & \frac{1}{2}Q_{ikm}u_{mj} - \frac{1}{2}Q_{ijm}u_{mk}. \end{aligned}$$

By definition,

$$\sigma_n(u_{ij}) = \frac{1}{n!} \sum_{\substack{i_1,\cdots i_n=1,\\ j_1,\cdots,j_n=1}}^{n} \delta^{i_1\cdots i_n}_{j_1\cdots j_n} u_{i_1j_1}\cdots u_{i_nj_n}.$$

$$\frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) = \frac{1}{(n-1)!} \sum_{\substack{i_1,\cdots i_{n-1}=1,\\ j_1,\cdots,j_{n-1}=1}}^{n} \delta^{i_1\cdots i_{n-1}i}_{j_1\cdots j_{n-1}j} u_{i_1j_1}\cdots u_{i_{n-1}j_{n-1}}.$$

Here $\delta^{i_1\cdots i_n}_{j_1\cdots j_n}$ denotes the Kronecker symbols, i.e., it equals to 1 (-1 reps.) if $(i_1\cdots i_n)$ is an even (odd reps.) permutation of $(j_1\cdots j_n)$ and it equals to 0 in other cases.

Thus using the antisymmetry of the Kronecker symbols and (7.42), we obtain

$$\begin{aligned} & \sum_{j=1}^{n}\left(\frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})\right)_j \qquad (7.43) \\ = \ & \frac{1}{2(n-1)!} \sum_{\substack{i_1,\cdots i_{n-1}=1,\\ j,j_1,\cdots,j_{n-1}=1}}^{n} (n-1)\delta^{i_1\cdots i_{n-1}i}_{j_1\cdots j_{n-1}j}(u_{i_1j_1j} - u_{i_1jj_1})u_{i_2j_2}\cdots u_{i_{n-1}j_{n-1}} \\ = \ & \frac{1}{2(n-2)!} \sum_{\substack{i_1,\cdots i_{n-1}=1,\\ j,j_1,\cdots,j_{n-1}=1}}^{n} \delta^{i_1\cdots i_{n-1}i}_{j_1\cdots j_{n-1}j}\left(\frac{1}{2}Q_{i_1jm}u_{mj_1} - \frac{1}{2}Q_{i_1j_1m}u_{mj}\right)u_{i_2j_2}\cdots u_{i_{n-1}j_{n-1}}. \end{aligned}$$

We assume u_{ij} is diagonal at some point p_0, i.e., $u_{ij} = \lambda_i\delta_{ij}$. We will use the notation $P(1,\cdots,n)$ to denote the permutation group of $\{1,\cdots,n\}$ and similarly, $P(1,\cdots,\hat{i},\cdots,n)$ the permutation group of $\{1,\cdots,n\}$ without index i.

Then at the point p_0, (7.43) reduces to

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij})\right)_j \qquad (7.44)\\
= &\frac{1}{2(n-2)!}\sum_{j,i_1,j_1=1}^{n}\delta^{i_1 i}_{j_1 j}\left(\frac{1}{2}Q_{i_1 j j_1}\lambda_{j_1}-\frac{1}{2}Q_{i_1 j_1 j}\lambda_j\right)\sum_{\substack{(i_2,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\hat{i_1},\cdots,n)}}\lambda_{i_2}\cdots\lambda_{i_{n-1}}\\
= &\frac{1}{2(n-2)!}\sum_{j=i,j_1=i_1\neq i}^{n}\left(\frac{1}{2}Q_{i_1 i i_1}\lambda_{i_1}-\frac{1}{2}Q_{i_1 i_1 i}\lambda_i\right)\sum_{\substack{(i_2,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\hat{i_1},\cdots,n)}}\lambda_{i_2}\cdots\lambda_{i_{n-1}}\\
&+\frac{1}{2(n-2)!}\sum_{j_1=i,j=i_1\neq i}(-1)\left(\frac{1}{2}Q_{i_1 i_1 i}\lambda_i-\frac{1}{2}Q_{i_1 i i_1}\lambda_{i_1}\right)\sum_{\substack{(i_2,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\hat{i_1},\cdots,n)}}\lambda_{i_2}\cdots\lambda_{i_{n-1}}\\
= &\frac{1}{(n-2)!}\sum_{\substack{(i_2,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\hat{i_1},\cdots,n)}}\left(\frac{1}{2}Q_{i_1 i i_1}\lambda_{i_1}\cdots\lambda_{i_{n-1}}-\frac{1}{2}Q_{i_1 i_1 i}\lambda_i\lambda_{i_2}\cdots\lambda_{i_{n-1}}\right).
\end{aligned}$$

On the other hand, at p_0, we have

$$\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij}) = \frac{1}{(n-1)!}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\lambda_{i_1}\cdots\lambda_{i_{n-1}}\delta_{ij}. \qquad (7.45)$$

Recall from Lemma 6.10 that $d\mu=\varphi dV_g$ and $\varphi_i=\frac{1}{2}\sum_{k=1}^{n}Q_{ikk}\varphi$. By combining (7.44) and (7.45), we obtain at p_0,

$$\begin{aligned}
&\left(\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij})\right)_j v_i\varphi+\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij})\left(v_i\varphi_j+\frac{1}{2}Q_{ijk}v_k\varphi\right) \qquad (7.46)\\
= &\frac{1}{(n-2)!}\sum_{i=1}^{n}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\left(\frac{1}{2}Q_{i_1 i i_1}\lambda_{i_1}\cdots\lambda_{i_{n-1}}-\frac{1}{2}Q_{i_1 i_1 i}\lambda_i\lambda_{i_2}\cdots\lambda_{i_{n-1}}\right)v_i\varphi\\
&+\frac{1}{(n-1)!}\sum_{i=1}^{n}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\lambda_{i_1}\cdots\lambda_{i_{n-1}}\left(v_i(-\frac{1}{2}\sum_{k=1}^{n}Q_{ikk}\varphi)+\sum_{k=1}^{n}\frac{1}{2}Q_{iik}v_k\varphi\right).
\end{aligned}$$

It is easy to see that the second term in the right hand side of (7.46) can be viewed as

$$\frac{1}{(n-1)!}\sum_{i=1}^{n}\sum_{k\neq i}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}} \lambda_{i_1}\cdots\lambda_{i_{n-1}}\cdot \left(v_i(-\frac{1}{2}\sum_{k=1}^{n}Q_{ikk}\varphi)+\sum_{k=1}^{n}\frac{1}{2}Q_{iik}v_k\varphi\right). \quad (7.47)$$

A simple computation shows that

$$(n-1)\sum_{i=1}^{n}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\frac{1}{2}Q_{i_1 i i_1}\lambda_{i_1}\cdots\lambda_{i_{n-1}}v_i\varphi \quad (7.48)$$
$$=\sum_{i=1}^{n}\sum_{k\neq i}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\frac{1}{2}Q_{ikk}\lambda_{i_1}\cdots\lambda_{i_{n-1}}v_i\varphi.$$

$$(n-1)\sum_{i=1}^{n}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\frac{1}{2}Q_{i_1 i_1 i}\lambda_i\lambda_{i_2}\cdots\lambda_{i_{n-1}}v_i\varphi \quad (7.49)$$
$$=\sum_{i=1}^{n}\sum_{k\neq i}\sum_{\substack{(i_1,\cdots,i_{n-1})\in\\ P(1,\cdots,\hat{i},\cdots,n)}}\frac{1}{2}Q_{iik}\lambda_{i_1}\cdots\lambda_{i_{n-1}}v_k\varphi.$$

Substituting (7.47), (7.48) and (7.49) into (7.46), we see that

$$\left(\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij})\right)_j v_i\varphi+\frac{\partial\sigma_n}{\partial u_{ij}}(u_{ij})\left(v_i\varphi_j+\frac{1}{2}Q_{ijk}v_k\varphi\right)=0. \quad (7.50)$$

Since at every point we can choose a local normal coordinate such that u_{ij} is diagonal, (7.50) holds for any points in $\mathcal{W}$.

With the help of (7.50), we are easy to achieve the lemma. Indeed, integrating by parts, we have

$$\begin{aligned}
&\int_{\mathcal{W}} w L_S(v) d\mu \\
= &\int_{\mathcal{W}} w \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})(v_{ij} - \frac{1}{2}Q_{ijk}v_k + v\delta_{ij})\varphi dV_g \\
= &-\int_{\mathcal{W}} \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) w_i v_j \varphi dV_g + \int_{\mathcal{W}} \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})\delta_{ij} w v \varphi dV_g \\
&-\int_{\mathcal{W}} w \left\{ \left(\frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) \right)_j v_i \varphi + \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) \left(v_i \varphi_j + \frac{1}{2} Q_{ijk} v_k \varphi \right) \right\} dV_g \\
= &-\int_{\mathcal{W}} \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) w_i v_j d\mu + \int_{\mathcal{W}} \frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij})\delta_{ij} w v d\mu,
\end{aligned}$$

which is symmetric in w and v. We finish the proof of Lemma 7.11. □

The following is an immediate consequence of Lemma 7.11. It also generalizes the necessary condition (7.1) to any functions $S \in C^2(\mathcal{W})$, not just the anisotropic support function for some convex hypersurface.

Corollary 7.12. *Let $S \in C^2(\mathcal{W})$. Let E^α denote the standard α-th coordinate vector of $\mathbb{R}^{n+1}$. For the position vector $z \in \mathcal{W} \subset \mathbb{R}^{n+1}$, we have*

$$\int_{\mathcal{W}} G(z)(z, E^\alpha) \det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}) d\mu = 0, \forall \alpha = 1, \cdots, n+1.$$

Proof. Recall the equation (7.6),

$$L_S(G(z)(z, E^\alpha)) = 0.$$

Then it follows from Proposition 7.11 that

$$\begin{aligned}
&\int_{\mathcal{W}} G(z)(z, E^\alpha) \det(S_{ij} - \frac{1}{2}Q_{ijk}S_k + S\delta_{ij}) d\mu \\
= &\frac{1}{n}\int_{\mathcal{W}} G(z)(z, E^\alpha) L_S(S) d\mu = \frac{1}{n}\int_{\mathcal{W}} L_S(G(z)(z, E^\alpha)) S d\mu = 0.
\end{aligned}$$

□

We observe from (7.6) that the function space $\mathrm{span}\{G(z)(z, E^1), \cdots, G(z)(z, E^{n+1})\}$ lies in the kernel of L_S. Next we show that the kernel of L_S contains only the functions in $\mathrm{span}\{G(z)(z, E^1), \cdots, G(z)(z, E^{n+1})\}$.

Lemma 7.13. *Let $v \in C^2(\mathcal{W})$ be a function such that $L_S(v) = 0$. Then, $v = \sum_{\alpha=1}^{n+1} a_\alpha G(z)(z, E^\alpha)$ for some constants $a_1, \cdots, a_{n+1}$.*

Proof. We follow the idea of Cheng-Yau's proof. Let $e_{n+1} = z$ be the position vector of $\mathcal{W}$ and $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field with respect to g on $\mathcal{W}$ such that $\{e_1, \cdots, e_{n+1}\}$ is a positive oriented orthonormal frame field with respect to G in $\mathbb{R}^{n+1}$. Let $\{\omega^1, \cdots, \omega^{n+1}\}$ be the dual 1-form of $\{e_1, \cdots, e_{n+1}\}$, i.e., $\omega^\alpha(e_\beta) = \delta_{\alpha\beta}$. Clearly we have

$$\omega^{n+1}|_{\mathcal{W}} = 0, \quad e_i(e_{n+1}) = e_i,$$

$$e_i(e_j) = \nabla_{e_i} e_j - \frac{1}{2} Q_{ijk} e_k - \delta_{ij} e_{n+1}.$$

Consider the vector valued function $Z = \sum_{i=1}^n v_i e_i + v e_{n+1}$. Then $v = G(z)(z, Z)$ and on $\mathcal{W}$ (we compute at normal coordinates, namely, $\nabla_{e_i} e_j = 0$),

$$\begin{aligned} dZ &= e_j(Z)\omega^j = [e_j e_i(v) e_i + v_i e_j(e_i) + e_j(v) e_{n+1} + v e_j(e_{n+1})]\, \omega^j \qquad (7.51) \\ &= \left[v_{ij} e_i + v_i(-\frac{1}{2} Q_{ijk} e_k - \delta_{ij} e_{n+1}) + v_j e_{n+1} + v e_j \right] \omega^j \\ &= \left[(v_{ij} - \frac{1}{2} Q_{ijk} v_k + \delta_{ij} v) e_i \right] \omega^j. \end{aligned}$$

Let $X = \sum_{i=1}^n S_i e_i + S e_{n+1}$, then the same computation as (7.51) gives

$$dX = \left[(S_{ij} - \frac{1}{2} Q_{ijk} S_k + \delta_{ij} S) e_i \right] \omega^j.$$

Consider the $(n-1)$-form $\bar{\Omega} = X \wedge Z \wedge dZ \wedge dX \wedge \cdots \wedge dX$, where dX appears $(n-2)$ times. Since $L_S(v) = 0$, we see that

$$\begin{aligned} & dX \wedge Z \wedge dZ \wedge dX \wedge \cdots \wedge dX \\ = & \left[\frac{\partial \sigma_n}{u_{ij}}(u_{ij})(v_{ij} - \frac{1}{2} Q_{ijk} v_k + \delta_{ij} v) \right] (Z \wedge e_1 \wedge \cdots \wedge e_n) \otimes (\omega^1 \wedge \cdots \wedge \omega^n) = 0. \end{aligned}$$

Hence we have

$$0 = \int_{\mathcal{W}} d\bar{\Omega} = \int_{\mathcal{W}} X \wedge dZ \wedge dZ \wedge dX \wedge \cdots \wedge dX.$$

The same argument as in [CY76], Page 507, leads to the conclusion that

$$v_{ij} - \frac{1}{2} Q_{ijk} v_k + \delta_{ij} v = 0 \quad \forall i, j = 1, \cdots, n.$$

Thus Z is constant due to (7.51) and can be written as $Z = a_\alpha E^\alpha$ for some constants a_α. Consequently,

$$v = G(z)(z, Z) = a_\alpha G(z)(z, E^\alpha).$$

□

Now we are ready to prove Proposition 7.10 and then Theorem 7.5 and 7.1.

Proof of Proposition 7.10: Without loss of generality, we assume that S satisfies (7.7). By virtue of Proposition 7.6 we may further assume that $K \in C^\infty(\mathcal{W})$. Let $H^m(\mathcal{W})$ be the Sobolev space of $\mathcal{W}$ with the Riemannian metric g. Choose m sufficient large such that $H^m(\mathcal{W}) \subset C^4(\mathcal{W})$. Consider L_S as a bounded linear map from $H^{m+2}(\mathcal{W})$ to $H^m(\mathcal{W})$. It follows from Lemma 7.13 that $\mathrm{Ker}(L_S) = \mathrm{span}\{G(z)(z, E^1), \cdots, G(z)(z, E^{n+1})\}$. On the other hand, L_S is self-adjoint due to Lemma 7.11. Hence by the standard Hilbert space theory, we have

$$\mathrm{Image}(L_S) = \mathrm{Ker}(L_S^*)^\perp = \mathrm{span}\{G(z)(z, E^1), \cdots, G(z)(z, E^{n+1})\}^\perp.$$

Consequently, for any $f \in H^m(\mathcal{W})$ with $\int G(z)(z, E^\alpha) f(z) d\mu = 0, \quad \forall \alpha = 1, \cdots, n+1$, we have $f \in \mathrm{Image}(L_S)$, which means $L_S : H^{m+2}(\mathcal{W}) \to H^m(\mathcal{W})$ is surjective. The standard implicit function theorem yields that the operator $S \mapsto \det(S_{ij} - \frac{1}{2} Q_{ijk} S_k + S\delta_{ij})$ is locally invertible near S, which implies the set $\mathcal{S}$ is open. □

Proof of Theorem 7.5: We see from Theorem 7.6 that $\mathcal{S}$ is closed. Since $\mathcal{S}$ is also open and non-empty, we conclude that $\mathcal{S} = [0, 1]$. In particular, (7.2) has an admissible solution on $\mathcal{W}$.

We now turn to the uniqueness part. Assume S and $\tilde{S}$ are two solutions to (7.2). Denote by $U = (u_{ij}) = (S_{ij} - \frac{1}{2} Q_{ijk} S_k + S\delta_{ij})$ and $\tilde{U} = (\tilde{u}_{ij}) = (\tilde{S}_{ij} - \frac{1}{2} Q_{ijk} \tilde{S}_k + \tilde{S}\delta_{ij})$.

For any $n \times n$ symmetric matrices $W_1, \cdots, W_n$. Let $\sigma_n(W_1, \cdots, W_n)$ denote the complete polarization of σ_n, i.e.,

$$\sigma_n(W_1, \cdots, W_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \big|_{\lambda_1 = \cdots = \lambda_n = 0} \left(\sigma_n(\lambda_1 W_1 + \cdots + \lambda_n W_n)\right).$$

Clearly,

$$\frac{\partial \sigma_n}{\partial u_{ij}}(u_{ij}) \tilde{u}_{ij} = n\sigma_n(\underbrace{U, \cdots, U,}_{(n-1)\text{ times}} \tilde{U}).$$

It follows from Lemma 7.11 that

$$\begin{aligned} \int_{\mathcal{W}} S\sigma_n(\tilde{U}) d\mu &= \int_{\mathcal{W}} S \cdot \frac{1}{n} L_{\tilde{S}}(\tilde{S}) d\mu \\ = \quad &\int_{\mathcal{W}} \frac{1}{n} L_{\tilde{S}}(S) \tilde{S} d\mu = \int_{\mathcal{W}} \tilde{S}\sigma_n(\tilde{U}, \cdots, \tilde{U}, U) d\mu. \end{aligned} \tag{7.52}$$

In the same way, we have

$$\int_{\mathcal{W}} \tilde{S}\sigma_n(U) d\mu = \int_{\mathcal{W}} S\sigma_n(U, \cdots, U, \tilde{U}) d\mu. \tag{7.53}$$

Combining (7.52) and (7.53), we obtain

$$2\int_{\mathcal{W}} S\left(\sigma_n(\tilde{U}) - \sigma_n(U,\cdots,U,\tilde{U})\right) d\mu \tag{7.54}$$
$$= \int_{\mathcal{W}} \left[\tilde{S}\left(\sigma_n(U,\tilde{U},\cdots,\tilde{U}) - \sigma_n(U)\right) - S\left(\sigma_n(U,\cdots,U,\tilde{U}) - \sigma_n(\tilde{U})\right)\right] d\mu.$$

Recall the Gårding inequality for the polarizations of σ_n (see Gårding [Ga59]),

$$\sigma_n(U,\cdots,U,\tilde{U}) \geq \sigma_n^{\frac{1}{n}}(\tilde{U})\sigma_n^{\frac{n-1}{n}}(U), \tag{7.55}$$

with the equality holds if and only if U and $\tilde{U}$ are proportional. In view of the assumption that $\sigma_n(U) = \sigma_n(\tilde{U})$, we see from (7.55) that the left hand side of (7.54) is non-positive, whence the right hand side is also non-positive. However, the right hand side of (7.54) is anti-symmetric with respect to S and $\tilde{S}$, we conclude that it vanishes. This implies the equality holds in (7.55), and in turn, U and $\tilde{U}$ are proportional. Since $\sigma_n(U) = \sigma_n(\tilde{U})$, we obtain that $U = \tilde{U}$ for every point in $\mathcal{W}$. In particular,

$$L_S(S - \tilde{S}) = n(\sigma_n(U) - \sigma_n(U,\cdots,U,\tilde{U})) = 0.$$

By Lemma 7.13, we conclude that $S - \tilde{S} = c_\alpha G(z)(z, E^\alpha)$. The proof is completed. □

Proof of Theorem 7.1: The existence part follows directly from Theorem 7.5 and Proposition 7.2 and 7.3. Notice that for two hypersurfaces M and $\tilde{M}$ with the same anisotropic Gauss-Kronecker curvature,

$$G(z)(z, M(z) - \tilde{M}(z)) = S(M(z)) - \tilde{S}(M(z)) = c_\alpha G(z)(z, E^\alpha).$$

Therefore, $M(z) - \tilde{M}(z) = c_\alpha E^\alpha$, which is a constant vector in $\mathbb{R}^{n+1}$, namely, M and $\tilde{M}$ coincide up to a translation. □

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