

Graded and Geometric Parabolic Induction

DISSERTATION ZUR ERLANGUNG DES DOKTORGRADES

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Chapter 1

Introduction

Graded Parabolic Induction Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a complex reductive Lie algebra with a Borel and Cartan subalgebra. Fix a parabolic subalgebra $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b}$ and denote its reductive Levi factor by $\mathfrak{p} \rightarrow \mathfrak{l}$. Denote by $\mathcal{W}_{\mathfrak{g}} \supset \mathcal{W}_{\mathfrak{l}}$ the Weyl groups of \mathfrak{g} and \mathfrak{l} .

The goal of this thesis is to construct a *graded* and *geometric* version of *parabolic induction* for modules in the BGG-category \mathcal{O} :

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}(\mathfrak{l}) \rightarrow \mathcal{O}(\mathfrak{g}), M \mapsto \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathrm{Res}_{\mathfrak{l}}^{\mathfrak{p}} M.$$

We will, amongst other things, prove:

Theorem (Theorem 4.2.2.1). *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and w a shortest coset representative in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$. There is a functor $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$ making the following diagram commute (up to natural isomorphism)*

$$\begin{array}{ccc} \mathcal{O}_{w \cdot \lambda}^{\mathbb{Z}}(\mathfrak{l}) & \xrightarrow{\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g}) \\ \downarrow v & & \downarrow v \\ \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) & \xrightarrow{\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_{\lambda}(\mathfrak{g}) \end{array}$$

and fulfilling $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}} \langle n \rangle \cong \langle n \rangle \widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$, where $\langle - \rangle$ is the shift of grading.

Here $v : \mathcal{O}^{\mathbb{Z}} \rightarrow \mathcal{O}$ denotes the *graded category* \mathcal{O} , as constructed in [Soe90] and [BGS96]. In the words of [Str03], where similar questions for translation functors are discussed, this means that parabolic induction (at least for integral blocks) is *gradable*. We construct $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$ with geometric methods, which we will explain now.

Geometric Parabolic Induction As envisioned in [BG86], the grading of category \mathcal{O} is deeply related to the *mixed geometry* of flag varieties: There should be a derived equivalence between each block of $\mathcal{O}^{\mathbb{Z}}$ and a category of *mixed sheaves* on an associated flag variety. This vision was realized in [Soe90], [BGS96] and finally [SW16], where an equivalence of categories (up to adding a root of the Tate twist)

$$\mathrm{MTDer}_{(B)}(G/Q) \xrightarrow{\sim} \mathrm{Der}^b(\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g}))$$

between *stratified mixed Tate motives* on a (partial) flag variety G/Q for the Langlands dual algebraic group G/\mathbb{C} and a derived (singular) block $\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g})$ was constructed. This equivalence is indeed a form of *Koszul duality*: the perverse t-structure on the left hand side corresponds to the Koszul dual t-structure on the right, and vice versa.

Stratified mixed Tate motives are certain constructible motivic sheaves. They behave similarly to mixed ℓ -adic sheaves and mixed Hodge modules (with the advantage that they have no extensions between Tate motives $\mathbb{C}(n)$). In particular, they are equipped with a full six functor formalism, which we can use to construct a geometric version of parabolic induction as follows.

Theorem (Theorem 4.2.1.1 and 4.2.2.1). *Let $\lambda \in \mathfrak{h}^*$ be a regular dominant integral weight and w be a shortest coset representative in $\mathcal{W}_1 \setminus \mathcal{W}_{\mathfrak{g}}$. Then the following diagram commutes up to natural isomorphism*

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/B) & \xrightarrow[\mathrm{GInd}_w]{h_{w,*} \mathrm{pr}_w^!} & \mathrm{MTDer}_{(B)}(G/B) \\ \downarrow & & \downarrow \\ \mathrm{Der}^b(\mathcal{O}_{w,\lambda}^{\mathbb{Z}}(\mathfrak{l})) & & \mathrm{Der}^b(\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g})) \\ \downarrow v & & \downarrow v \\ \mathrm{Der}^b(\mathcal{O}_{w,\lambda}(\mathfrak{l})) & \xrightarrow{\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathrm{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})) \end{array} \quad \begin{array}{c} v \left(\right. \\ \left. \right) v \end{array}$$

Here $G \supset P \supset B$ corresponds to $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b}$ and the functor $\mathrm{GInd}_w = h_{w,*} \mathrm{pr}_w^!$, which we call **geometric parabolic induction**, is defined via maps

$$P/B \xleftarrow{\mathrm{pr}_w} P_w B/B \xleftarrow{h_w} G/B.$$

We actually show a stronger statement which also holds for singular weights $\lambda \in \mathfrak{h}^*$ and allows us to prove that parabolic induction is also gradable in this case.

Soergel modules In order to prove these theorems, we use the combinatorial description of derived blocks of category \mathcal{O} and stratified mixed Tate motives on flag varieties in terms of the homotopy category of *Soergel modules*. Let $C = H^*(G/B, \mathbb{C}) \rightarrow C' = H^*(P/B, \mathbb{C})$ be the cohomology rings of the flag varieties $G/B \supset P/B$. Then for a reduced expression $w = s_n \dots s_1 \in \mathcal{W}_{\mathfrak{g}}$ define the following complex of Soergel bimodules (it is in fact an instance of a *Rouquier complex*) over C

$$\underline{R}_w \stackrel{def}{=} R_{s_1} \otimes_C \dots \otimes_C R_{s_n}, \text{ where}$$

$$R_s \stackrel{def}{=} \dots \rightarrow 0 \rightarrow C \rightarrow C \otimes_{C^s} C\langle 2 \rangle \rightarrow 0 \rightarrow \dots$$

With this notation we show:

Theorem (Theorem 2.3.4.1, 3.3.3.1 and 4.2.1.1). *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral regular weight and w a shortest coset representative in $\mathcal{W}_1 \setminus \mathcal{W}_{\mathfrak{g}}$. Then the following diagram of functors commutes (up to natural isomorphism)*

$$\begin{array}{ccc}
\text{MTDer}_{(B)}(P/B) & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/B) \\
\Delta \downarrow \wr & & \wr \downarrow \Delta \\
\text{Hot}^b(\text{MTDer}_{(B)}(P/B)_{w=0}) & & \text{Hot}^b(\text{MTDer}_{(B)}(G/B)_{w=0}) \\
\mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\
\text{Hot}^b(C' \text{-Smod}^{\mathbb{Z}, ev}) & \xrightarrow{\text{SInd}_w} & \text{Hot}^b(C \text{-Smod}^{\mathbb{Z}, ev}) \\
v \downarrow & & \downarrow v \\
\text{Hot}^b(C' \text{-Smod}) & \xrightarrow{\text{SInd}_w} & \text{Hot}^b(C \text{-Smod}) \\
\mathbb{V}_1^{w \cdot \lambda} \uparrow \wr & & \wr \uparrow \mathbb{V}_{\mathfrak{g}}^{\lambda} \\
\text{Hot}^b(\text{Proj } \mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & & \text{Hot}^b(\text{Proj } \mathcal{O}_{\lambda}(\mathfrak{g})) \\
\wr \downarrow & & \downarrow \wr \\
\text{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \text{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})).
\end{array}$$

(Large curved arrows labeled v connect the left and right sides of the diagram.)

Here by $\text{SMod}^{(\mathbb{Z})}$ we denote the categories of (graded) Soergel modules and

$$\text{SInd}_w : C' \text{-Smod}^{\mathbb{Z}, ev} \rightarrow C \text{-Smod}^{\mathbb{Z}, ev}, M \mapsto \underline{R}_w \otimes_C \text{Res}_{C'}^C M.$$

Again, we prove a more general version which also applies to singular weights $\lambda \in \mathfrak{h}^*$. Our proof strategy is the following: Firstly, we show the

statement for $w = e$, which is the easiest case, since then $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ maps projectives to projectives and GInd_e is weight exact. We then carefully analyse how (geometric) parabolic induction interacts with (geometric) *wall crossing* functors. Comparing the results, we are able to prove the general case by an induction on the length of w .

Structure In the *second chapter* we treat the representation theoretic side. After introducing the necessary notation, we first discuss basic functorial properties of parabolic induction. We then analyse the interaction of parabolic induction and translation functors. This constitutes the main part of this chapter. Using these results we are then able to explicitly describe the effect of parabolic induction on Soergel modules.

The *third chapter* is concerned with geometry. We start by recalling several results about motivic sheaves and their six functor formalism (following [Ayo07] and [CD12]) and about mixed stratified Tate motives as introduced in [SW16]. We then proceed to describe the P -orbits on a partial flag variety G/Q in terms of affine bundles over smaller partial flag varieties P/Q_w . We use this description to define the geometric parabolic induction functors GInd_w . As in the second chapter, we then analyse the interaction of geometric wall crossing functors $\pi^! \pi_!$ with geometric parabolic induction. This allows us to ultimately describe the effect of GInd_w on the level of Soergel modules.

In the *fourth and last chapter* we compare our results from representation theory and geometry in order to prove the theorems presented in the introduction.

Conventions By a \mathbb{C} -algebra A we always mean a (not necessarily commutative) \mathbb{C} -algebra with unit. By $A\text{-mod}$ we denote the category of *finitely generated* A -modules. If $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is additionally \mathbb{Z} -graded, we denote by $A\text{-mod}^{\mathbb{Z}}$ the category of graded A -modules and by $A\text{-mod}^{\mathbb{Z}, \text{ev}}$ the category of evenly graded modules, i.e. those modules which are concentrated in even degrees.

For an abelian category \mathcal{A} , we denote by $\text{Der}(\mathcal{A})$ and $\text{Der}^b(\mathcal{A})$ its (bounded) derived category and by $\text{Proj } \mathcal{A}$ the full additive subcategory of projective objects in \mathcal{A} . For an additive category \mathcal{A} , we denote by $\text{Hot}(\mathcal{A})$ and $\text{Hot}^b(\mathcal{A})$ its (bounded) homotopy category of chain complexes.

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Chapter 2

Parabolic Induction

2.1 Setup

2.1.1 Lie algebras and root systems

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a reductive complex Lie algebra together with a Borel and Cartan subalgebra. Denote by

$$\mathfrak{h}^* \supset \Phi_{\mathfrak{g}} \supset \Phi_{\mathfrak{g}}^+ \supset \Delta_{\mathfrak{g}}$$

the space of weights, set of roots, positive and simple roots corresponding to $\mathfrak{g} \supset \mathfrak{b}$. By a superscript minus as in $\Phi_{\mathfrak{g}}^- = \Phi_{\mathfrak{g}} \setminus \Phi_{\mathfrak{g}}^+$ or \mathfrak{b}^- we always denote the corresponding negative or opposite. For a root $\alpha \in \Phi$ denote by $\alpha^\vee \in \mathfrak{h}$ its coroot and by s_α the corresponding reflection. Let

$$\begin{aligned} \mathcal{W}_{\mathfrak{g}} &= \langle s_\alpha \mid \alpha \in \Delta_{\mathfrak{g}} \rangle \\ \mathcal{S}_{\mathfrak{g}} &= \{ s_\alpha \mid \alpha \in \Delta_{\mathfrak{g}} \} \end{aligned}$$

be the Weyl group and set of simple reflections. Denote by $\langle -, - \rangle$ the natural evaluation pairing on $\mathfrak{h}^* \otimes \mathfrak{h}$ and by

$$\begin{aligned} \Lambda_{\mathfrak{g}} &= \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta_{\mathfrak{g}} \} \\ \Lambda_{\mathfrak{g}}^+ &= \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta_{\mathfrak{g}} \} \end{aligned}$$

the integral weight lattice and the set positive integral weights. For an integral weight $\lambda \in \Lambda_{\mathfrak{g}}$ denote the unique weight in $\mathcal{W}_{\mathfrak{g}}\lambda \cap \Lambda_{\mathfrak{g}}^+$ by $\bar{\lambda}_{\mathfrak{g}}$.

Let $\rho \in \mathfrak{h}^*$ be the half-sum of positive roots and denote by

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

the dot-action of $\mathcal{W}_{\mathfrak{g}}$ on \mathfrak{h}^* . We denote the stabilizer of a weight $\lambda \in \mathfrak{h}^*$ with respect to the dot-action by $\mathcal{W}_{\mathfrak{g},\lambda}$.

There is a partial ordering on the set of weights given by

$$\lambda \geq \mu \stackrel{\text{def}}{\Leftrightarrow} \lambda - \mu \in \Lambda_{\mathfrak{g}}^+.$$

A weight $\lambda \in \mathfrak{h}^*$ is called *dominant* (for $\Phi_{\mathfrak{g}}^+$) if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Phi_{\mathfrak{g}}^+$. The set of integral dominant weights is hence $\Lambda_{\mathfrak{g}}^+ - \rho$.

Now let $\mathfrak{g} \supset \mathfrak{p} \rightarrow \mathfrak{l}$ be a parabolic and Levi factor of \mathfrak{g} such that $\mathfrak{p} \supset \mathfrak{b}$, for simplicity we choose a splitting $\mathfrak{l} \subset \mathfrak{p}$. We denote by $\mathfrak{n}_{\mathfrak{p}} \subset \mathfrak{p}$ the nilpotent radical of \mathfrak{p} and by $\mathfrak{z}_{\mathfrak{l}} \subset \mathfrak{l}$ the center of \mathfrak{l} . Then we have decompositions of \mathfrak{p} and \mathfrak{g} into

$$\begin{aligned} \mathfrak{p} &= \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{p}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{l}}} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}^+ \setminus \Phi_{\mathfrak{l}}^+} \mathfrak{g}_{\alpha} \\ \mathfrak{g} &= \mathfrak{n}_{\mathfrak{p}}^- \oplus \mathfrak{p} = \mathfrak{n}_{\mathfrak{p}}^- \oplus \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{p}} \end{aligned}$$

For $\alpha \in \Delta$, let $\varpi_{\alpha} \in \mathfrak{h}^*$, respectively $\varpi_{\alpha}^{\vee} \in \mathfrak{h}$, be the fundamental weights; they form a dual basis to Δ^{\vee} , respectively Δ , and are well-defined if we additionally require $\varpi_{\alpha}(\mathfrak{z}_{\mathfrak{g}}) = \{0\}$ and $\varpi_{\alpha}^{\vee} \in [\mathfrak{g}, \mathfrak{g}]$. Then

$$\mathfrak{z}_{\mathfrak{l}} = \{H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ for all } \alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{l}}\} = \langle \varpi_{\alpha}^{\vee} \mid \alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{l}} \rangle_{\mathbb{C}} \oplus \mathfrak{z}_{\mathfrak{g}}$$

and there is also a partial ordering on the set of $\mathfrak{z}_{\mathfrak{l}}$ -weights, namely

$$\nu \geq \nu' \stackrel{\text{def}}{\Leftrightarrow} \nu - \nu' \in \mathbb{Z}_{\geq 0} \{ \alpha|_{\mathfrak{z}_{\mathfrak{l}}} \mid \alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{l}} \} \text{ for } \nu, \nu' \in \mathfrak{z}_{\mathfrak{l}}.$$

A priori there are two different dot-actions of $\mathcal{W}_{\mathfrak{l}}$ on \mathfrak{h}^* . They coincide since

$$w(\lambda + \rho) - \rho = w(\lambda + \rho_{\mathfrak{l}}) - \rho_{\mathfrak{l}}$$

for all $w \in \mathcal{W}_{\mathfrak{l}}$, where we use that

$$w(\rho - \rho_{\mathfrak{l}}) = \sum_{\alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{l}}} w(\varpi_{\alpha}) = \sum_{\alpha \in \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{l}}} \varpi_{\alpha} = \rho - \rho_{\mathfrak{l}}.$$

Lemma 2.1.1.1. *The weights in $\Lambda_{\mathfrak{g}}$ which are dominant for $\Phi_{\mathfrak{l}}^+$ are precisely the weights of the form $w \cdot \lambda$ for $\lambda \in \Lambda_{\mathfrak{g}}^+$ dominant integral and $w \in \mathcal{W}$ a shortest coset representative for $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$.*

See also Figure 2.1.3 for an example.

2.1.2 Category \mathcal{O}

The BGG-category \mathcal{O} (see [BGG71]) associated to a complex reductive Lie algebra with chosen Borel and Cartan subalgebra $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ is the full subcategory of the category of \mathfrak{g} -modules, $\mathfrak{g}\text{-mod}$, given by

$$\mathcal{O}(\mathfrak{g}) \stackrel{def}{=} \left\{ M \in \mathfrak{g}\text{-mod} \left| \begin{array}{l} \mathfrak{h} \text{ acts semisimply on } M, \\ \mathfrak{b} \text{ acts locally finitely on } M, \\ M \text{ is finitely generated under } \mathfrak{g} \end{array} \right. \right\}$$

For a complex Lie algebra \mathfrak{n} , denote its universal enveloping algebra by $\mathcal{U}(\mathfrak{n})$. For $\lambda \in \mathfrak{h}^*$ let

$$M_{\mathfrak{g}}(\lambda) \stackrel{def}{=} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda}$$

be the Verma module with highest weight λ and

$$P_{\mathfrak{g}}(\lambda) \twoheadrightarrow M_{\mathfrak{g}}(\lambda) \twoheadrightarrow L_{\mathfrak{g}}(\lambda)$$

its projective cover and unique simple quotient in $\mathcal{O}(\mathfrak{g})$. For λ dominant, denote by

$$\mathcal{O}_{\lambda}(\mathfrak{g}) = \langle M_{\mathfrak{g}}(w \cdot \lambda) \mid w \in \mathcal{W}_{\mathfrak{g}, [\lambda]} \rangle_{\text{Serre}} \subset \mathcal{O}(\mathfrak{g})$$

the full Serre subcategory of $\mathcal{O}(\mathfrak{g})$ generated by the Verma modules $M_{\mathfrak{g}}(w \cdot \lambda)$, where by $\mathcal{W}_{\mathfrak{g}, [\lambda]} \subset \mathcal{W}_{\mathfrak{g}}$ we denote the integral Weyl group of λ . Then $\mathcal{O}(\mathfrak{g})$ decomposes into blocks

$$\mathcal{O}(\mathfrak{g}) = \bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ \text{dominant}}} \mathcal{O}_{\lambda}(\mathfrak{g})$$

and we denote the functor projecting on a block $\mathcal{O}_{\lambda}(\mathfrak{g})$ by pr_{λ} .

2.1.3 Generalities on parabolic induction

In the notation of Section 2.1.1, let $\mathfrak{g} \supset \mathfrak{p} \twoheadrightarrow \mathfrak{l}$ be a reductive complex Lie algebra with parabolic subalgebra and Levi factor. Then the *parabolic induction* functor is given by

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \stackrel{def}{=} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \text{Res}_{\mathfrak{l}}^{\mathfrak{p}}(-) : \mathcal{O}(\mathfrak{l}) \rightarrow \mathcal{O}(\mathfrak{g}).$$

We often drop the $\text{Res}_{\mathfrak{l}}^{\mathfrak{p}}$ from the notation. Since $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is exact and $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(\mu) = M_{\mathfrak{g}}(\mu)$ for all $\mu \in \mathfrak{h}^*$ (see below) it respects the block decomposition of category $\mathcal{O}(\mathfrak{g})$, namely restricts to

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) \rightarrow \mathcal{O}_{\lambda}(\mathfrak{g}),$$

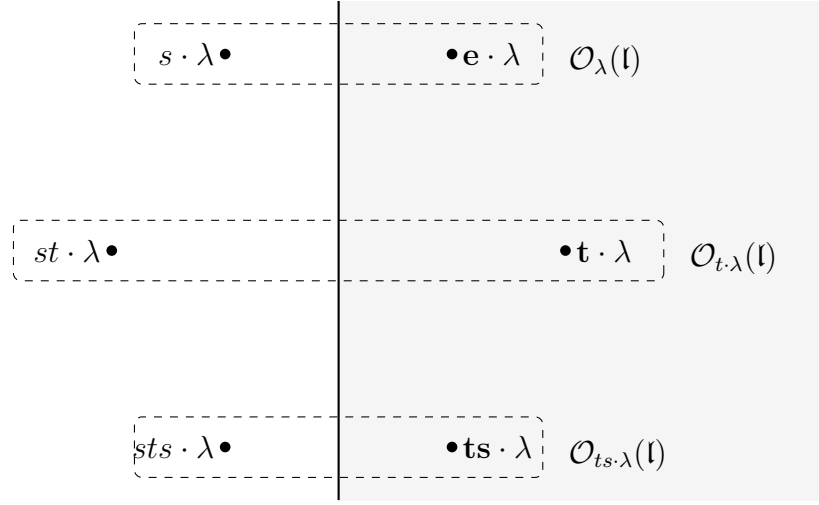


Figure 2.1: **The case $\mathfrak{sl}_2 \subset \mathfrak{sl}_3$:** Here $\mathcal{S}_{\mathfrak{g}} = \{s, t\}$, $\mathcal{S}_{\mathfrak{l}} = \{s\}$ and $\lambda \in \mathfrak{h}^*$ denotes an some regular integral weight. The shortest coset representatives $\{e, s, ts\}$ of $\mathcal{W}_{\mathfrak{l}} \backslash \mathcal{W}_{\mathfrak{g}}$ parameterize the blocks of $\mathcal{O}(\mathfrak{l})$ which map into $\mathcal{O}_{\lambda}(\mathfrak{g})$.

for $\lambda \in \mathfrak{h}^*$ dominant integral and $w \in \mathcal{W}$ a shortest coset representative for $\mathcal{W}_{\mathfrak{l}} \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}$ or in other words

$$(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{-1}(\mathcal{O}_{\mathfrak{g}}(\lambda)) = \bigoplus_{w \in \mathcal{W}_{\mathfrak{l}} \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}} \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}),$$

by Lemma 2.1.1.1. This is visualized in the example \mathfrak{sl}_3 in Figure 2.1.3. We now state some general functorial properties of parabolic induction.

Lemma 2.1.3.1. *The adjoint action of $\mathfrak{z}_{\mathfrak{l}}$ (the center of \mathfrak{l}) on $\mathcal{U}(\mathfrak{n}_{\mathfrak{p}})$, respectively $\mathcal{U}(\mathfrak{n}_{\mathfrak{p}}^-)$, is semisimple with finite dimensional weight spaces of positive, respectively negative, weight. Furthermore*

$$\mathcal{U}(\mathfrak{n}_{\mathfrak{p}})^{\text{ad}(\mathfrak{z}_{\mathfrak{l}})} = \mathcal{U}(\mathfrak{n}_{\mathfrak{p}}^-)^{\text{ad}(\mathfrak{z}_{\mathfrak{l}})} = \langle 1 \rangle_{\mathbb{C}}.$$

Proof. By the PBW theorem $\mathcal{U}(\mathfrak{n}_{\mathfrak{p}})$ is generated by monomials in X_{α} for $\alpha \in \Phi_{\mathfrak{g}}^+ \setminus \Phi_{\mathfrak{l}}^+$, and X_{α} a generator of \mathfrak{g}_{α} . Furthermore $\mathfrak{z}_{\mathfrak{l}}$ contains ϖ_{α}^{\vee} for $\alpha \in \Phi_{\mathfrak{g}}^+ \setminus \Phi_{\mathfrak{l}}^+$ and

$$[\varpi_{\alpha}^{\vee}, X_{\beta}] = \delta_{\alpha, \beta} X_{\beta}$$

for $\alpha, \beta \in \Phi_{\mathfrak{g}}^+ \setminus \Phi_{\mathfrak{l}}^+$. The statement follows. \square

Let $\lambda \in \mathfrak{h}^*$ be dominant for $\Phi_{\mathfrak{l}}^+$. We define the *parabolic restriction functor* for category \mathcal{O} by

$$\text{Res}_{\lambda}(-) : \mathcal{O}_{\lambda}(\mathfrak{g}) \rightarrow \mathcal{O}_{\lambda}(\mathfrak{l}), M \mapsto \text{pr}_{\lambda}(M_{\lambda|_{\mathfrak{z}_{\mathfrak{l}}}}^{\mathfrak{n}_{\mathfrak{p}}}),$$

where by definition

$$M_{\lambda|_{\mathfrak{sl}}}^{\mathfrak{np}} = \{m \in M \mid \mathfrak{n}_{\mathfrak{p}}m = 0 \text{ and } Zm = \lambda(Z)m \text{ for all } Z \in \mathfrak{sl}\},$$

and $\text{pr}_{\lambda} : \mathcal{O}(\mathfrak{l}) \rightarrow \mathcal{O}_{\lambda}(\mathfrak{l})$ is the projection. This is indeed well-defined by the next theorem, where we list important properties of parabolic induction.

Theorem 2.1.3.2. *Let $\lambda \in \mathfrak{h}^*$ be an integral weight which is dominant for $\Phi_{\mathfrak{l}}^+$. Then the following statements hold.*

1. *The functor $\text{Res}_{\lambda}(-)$ is well-defined and*

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}_{\lambda}(\mathfrak{l}) \rightleftarrows \mathcal{O}_{\lambda}(\mathfrak{g}) : \text{Res}_{\lambda}(-)$$

are adjoint.

2. *$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is exact and $\text{Res}_{\lambda}(-)$ is left exact.*
3. *Moreover $\text{Res}_{\lambda}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M) \cong M$ for all M in $\mathcal{O}_{\lambda}(\mathfrak{l})$.*
4. *For all $\mu \in \mathcal{W}_{\mathfrak{l}} \cdot \lambda$ we have*

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(\mu) = M_{\mathfrak{g}}(\mu) \text{ and } \text{Res}_{\lambda}(M_{\mathfrak{g}}(\mu)) = M_{\mathfrak{l}}(\mu).$$

5. *The functor $\text{Res}_{\lambda}(-)$ is exact and $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ sends (indecomposable) projective modules in $\mathcal{O}_{\lambda}(\mathfrak{l})$ to (indecomposable) projectives in $\mathcal{O}_{\lambda}(\mathfrak{g})$, if and only if λ is dominant for $\Phi_{\mathfrak{g}}^+$.*

Proof. (1) Let us first show that $\text{Res}_{\lambda}(-)$ is well-defined. For this we need to show that for $M \in \mathcal{O}_{\lambda}(\mathfrak{g})$, $M_{\lambda|_{\mathfrak{sl}}}^{\mathfrak{np}}$ is really in $\mathcal{O}(\mathfrak{l})$. Firstly, $M_{\lambda|_{\mathfrak{sl}}}^{\mathfrak{np}} \subset M$ is an \mathfrak{l} -submodule of M , hence clearly \mathfrak{h} acts semisimply and $\mathfrak{b}_{\mathfrak{l}}$ acts locally finitely on it. We need to show that $M_{\lambda|_{\mathfrak{sl}}}^{\mathfrak{np}}$ is finitely generated as an \mathfrak{l} -module. For this we show that already $N = M_{\lambda|_{\mathfrak{sl}}} \supset M_{\lambda|_{\mathfrak{sl}}}^{\mathfrak{np}}$ is finitely generated. Choose a finite set $\{x_i\}$ of \mathfrak{g} -generators of M . Without loss of generality, we can assume that each x_i is an highest weight vector and hence $\{x_i\}$ is even a set of \mathfrak{b}^- -generators, where \mathfrak{b}^- denotes the opposite Borel. We decompose

$$\mathfrak{b}^- = \mathfrak{b}_{\mathfrak{l}}^- \oplus \mathfrak{n}_{\mathfrak{p}}^-$$

where $\mathfrak{n}_{\mathfrak{p}}^-$ and $\mathfrak{b}_{\mathfrak{l}}^-$ are the opposites of $\mathfrak{n}_{\mathfrak{p}}$ and $\mathfrak{b}_{\mathfrak{l}}$. Now

$$\mathcal{U}(\mathfrak{n}_{\mathfrak{p}}^-)\{x_i\} \cap N$$

is finite dimensional, since the $\mathfrak{z}_{\mathfrak{l}}$ weight spaces of $\mathcal{U}(\mathfrak{n}_{\mathfrak{p}}^-)$ are finite dimensional (Lemma 2.1.3.1). By the PBW theorem a basis of this space provides a finite

set of \mathfrak{l} -generators of N . Since $\mathcal{U}(\mathfrak{l})$ is Noetherian, also $M_{\lambda|_{\mathfrak{z}_\mathfrak{l}}}^{\mathfrak{n}_\mathfrak{p}} \subset N$ will be finitely generated and hence in category $\mathcal{O}(\mathfrak{l})$. That $\text{Res}_\lambda(M) \in \mathcal{O}_\lambda(\mathfrak{l})$ follows from (4). Hence the first statement follows. Generally, there are natural isomorphisms

$$\text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(-), -) \cong \text{Hom}_{\mathfrak{p}}(-, -) \cong \text{Hom}_{\mathfrak{l}}(-, -^{\mathfrak{n}_\mathfrak{p}})$$

of functors on $\mathfrak{l}\text{-mod}^{\text{opp}} \times \mathfrak{g}\text{-mod}$ and hence an adjunction

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathfrak{l}\text{-mod} \rightleftarrows \mathfrak{g}\text{-mod} : -^{\mathfrak{n}_\mathfrak{p}}.$$

One easily sees that this induces the stated adjunction $(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}, \text{Res}_\lambda(-))$.

(2) $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is exact by the PBW theorem and $\text{Res}_\lambda(-)$ is left exact since it is right adjoint.

(3) See also [SS15, Lemma 5.10]. Let $M \in \mathcal{O}_\lambda(\mathfrak{l})$. We want to show

$$\text{Res}_\lambda(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M)) \cong M.$$

By the PBW theorem as a vector space (and even as a $\mathfrak{z}_\mathfrak{l}$ -module)

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M \cong \mathcal{U}(\mathfrak{n}_\mathfrak{p}^-) \otimes_{\mathbb{C}} M$$

Lemma 2.1.3.1 shows that $\mathcal{U}(\mathfrak{n}_\mathfrak{p}^-)^{\text{ad}(\mathfrak{z}_\mathfrak{l})} = \langle 1 \rangle_{\mathbb{C}}$ and hence

$$\text{Res}_\lambda(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M)) \subseteq 1 \otimes M \cong M.$$

But every vector in $1 \otimes M$ is $\mathfrak{n}_\mathfrak{p}$ -invariant and the inclusion is actually an equality. The second statement follows.

(4) The first statement is trivial and the second follows from the first and point (3).

(5) See also [SS15, Lemma 5.11]. As right adjoint functor $\text{Res}_\lambda(-)$ is certainly left exact as explained in (2). Let

$$M \rightarrow N$$

be a surjection in $\mathcal{O}_\lambda(\mathfrak{g})$. We have to show that

$$\text{Res}_\lambda(M) \rightarrow \text{Res}_\lambda(N)$$

is also surjective. So let $n \in \text{Res}_\lambda(N) \subset N$ and m be a preimage in M . It suffices to show that m is $\mathfrak{n}_\mathfrak{p}$ -invariant. But this is easy to see: Applying element of $\mathfrak{n}_\mathfrak{p}$ to m increases its weight $\lambda|_{\mathfrak{z}_\mathfrak{l}}$ (Lemma 2.1.3.1), but since λ is also dominant for \mathcal{W} , this is already maximal, and hence m is $\mathfrak{n}_\mathfrak{p}$ -invariant. Hence $\text{Res}_\lambda(-)$ is also right exact. Now functors left adjoint to exact functors send projectives to projectives and the first implication follows. If on the other hand λ is not dominant for $\Phi_{\mathfrak{g}}^+$, then $M_{\mathfrak{l}}(\lambda)$ is projective but $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(\lambda) = M_{\mathfrak{g}}(\lambda)$ is not. The statement about the indecomposability follows from (3). \square

2.2 Parabolic induction and translation functors

Let λ and μ be some dominant integral weights for $\mathcal{W}_{\mathfrak{g}}$ or $\mathcal{W}_{\mathfrak{l}}$ and $\nu \stackrel{def}{=} \mu - \lambda$. Then the corresponding *translation functors* are given by

$$\begin{aligned} T_{\lambda}^{\mu}: \mathcal{O}_{\lambda}(\mathfrak{l}) &\rightarrow \mathcal{O}_{\mu}(\mathfrak{l}), M \mapsto \text{pr}_{\mu}(M \otimes_{\mathbb{C}} L_{\mathfrak{l}}(\bar{\nu}_{\mathfrak{l}})) \text{ and} \\ T_{\lambda}^{\mu}: \mathcal{O}_{\lambda}(\mathfrak{g}) &\rightarrow \mathcal{O}_{\mu}(\mathfrak{g}), M \mapsto \text{pr}_{\mu}(M \otimes_{\mathbb{C}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})), \end{aligned}$$

where pr_{μ} denotes the projection to the corresponding block in \mathcal{O} .

In this section we want to understand how translation functors and parabolic induction interact. There are two different cases. Either a translation functor maps *into* a more singular block, i.e. $\mathcal{W}_{\mathfrak{g},\lambda} \subset \mathcal{W}_{\mathfrak{g},\mu}$. In this case parabolic induction and the translation functor *commute*. Or the translation functor maps *out of* a more singular block and the situation is more complicated. Most results are a direct generalization of character formulas for translation functors as in [Jan79, Kapitel 2].

2.2.1 Tensor identity

The most important tool for this section is the *tensor identity*, which describes in its most general formulation how tensor products and induction for modules over a Hopf algebra and a subalgebra interact. For us, the following formulation suffices.

Lemma 2.2.1.1 (Tensor identity). *Let $\mathfrak{n} \subset \mathfrak{m}$ be finite dimensional complex Lie algebras and M be an \mathfrak{m} -module. Then there is a natural equivalence of functors*

$$\text{Ind}_{\mathfrak{n}}^{\mathfrak{m}}(- \otimes_{\mathbb{C}} \text{Res}_{\mathfrak{m}}^{\mathfrak{n}} M) \cong (\text{Ind}_{\mathfrak{n}}^{\mathfrak{m}} -) \otimes_{\mathbb{C}} M : \mathfrak{n}\text{-mod} \rightarrow \mathfrak{m}\text{-mod},$$

such that for $X \in \mathfrak{m}$ and $m \in M$

$$X \otimes (- \otimes m) \mapsto (X \otimes -) \otimes m + (1 \otimes -) \otimes Xm$$

Proof. There are the following natural isomorphisms of functors

$$\begin{aligned} \text{Hom}_{\mathfrak{m}}((\text{Ind}_{\mathfrak{n}}^{\mathfrak{m}} -) \otimes_{\mathbb{C}} M, -) &\cong \text{Hom}_{\mathfrak{m}}(\text{Ind}_{\mathfrak{n}}^{\mathfrak{m}}(-), \text{Hom}_{\mathbb{C}}(M, -)) \\ &\cong \text{Hom}_{\mathfrak{n}}(-, \text{Res}_{\mathfrak{m}}^{\mathfrak{n}} \text{Hom}_{\mathbb{C}}(M, -)) \\ &\cong \text{Hom}_{\mathfrak{n}}(-, \text{Hom}_{\mathbb{C}}(\text{Res}_{\mathfrak{m}}^{\mathfrak{n}} M, -)) \\ &\cong \text{Hom}_{\mathfrak{n}}(- \otimes_{\mathbb{C}} \text{Res}_{\mathfrak{m}}^{\mathfrak{n}} M, -) \\ &\cong \text{Hom}_{\mathfrak{m}}(\text{Ind}_{\mathfrak{n}}^{\mathfrak{m}}(- \otimes_{\mathbb{C}} \text{Res}_{\mathfrak{m}}^{\mathfrak{n}} M), -). \end{aligned}$$

In third equality we used that \mathfrak{n} is a subalgebra of \mathfrak{m} . Now one shows that the induced isomorphism

$$\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{m}}(- \otimes_{\mathbb{C}} \mathrm{Res}_{\mathfrak{m}}^{\mathfrak{n}} M) \cong (\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{m}} -) \otimes_{\mathbb{C}} M$$

is indeed given by the stated formula. \square

In our specific case of *parabolic induction* this implies the existence of a filtration on tensor products with induced modules.

Lemma 2.2.1.2. *Let M be a \mathfrak{l} -module and E a finite dimensional \mathfrak{g} -module. Denote by ν_1, \dots, ν_n the weights of $\mathfrak{z}_{\mathfrak{l}}$ on E , ordered in a way that $\nu_i \leq \nu_j$ implies $i \leq j$. Then $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M) \otimes E$ has a filtration, natural in M ,*

$$\{0\} = N_{n+1} \subset N_n \subset \dots \subset N_1 = (\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M) \otimes E$$

with subquotients $N_i/N_{i+1} \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M \otimes E_{\nu_i})$, where $\mathfrak{n}_{\mathfrak{p}}$ act trivially on E_{ν_i} .

Proof. The tensor identity yields

$$(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M) \otimes E \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M \otimes_{\mathbb{C}} \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{p}} E).$$

Now set $M_i := \sum_{j=i}^n M \otimes E_{\nu_j}$; this is clearly a \mathfrak{l} -submodule of $M \otimes E$. Since furthermore the weights ν_i are ordered in an ascending way, M_i is also stable under $\mathfrak{n}_{\mathfrak{p}}$ and hence a \mathfrak{p} -submodule. The modules M_i give a filtration of $M \otimes E$ as a \mathfrak{p} -module with subquotients

$$M_i/M_{i+1} \cong M \otimes E_{\nu_i}.$$

Since non-zero elements of $\mathfrak{n}_{\mathfrak{p}}$ have non-zero weights with respect to $\mathfrak{z}_{\mathfrak{l}}$, they indeed act trivially on E_{ν_i} . Let $N_i := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} M_i$. Using the exactness of parabolic induction and the tensor identity we see that the N_i define a filtration with the desired property. That this is indeed natural follows directly from the explicit description of the M_i . \square

2.2.2 Restriction of finite dimensional modules

Since translation functors are built from tensor products with finite dimensional modules, we need to understand how they split when restricted to a Levi subalgebra. Although this is generally a hard question, certain *extremal* direct summands are easy to identify.

Lemma 2.2.2.1. *Let $\nu \in \mathfrak{h}^*$ be some integral weight and $\nu' \stackrel{\text{def}}{=} \nu|_{\mathfrak{z}_{\mathfrak{l}}}$. Then, as \mathfrak{l} -module, $L_{\mathfrak{l}}(\bar{\nu}_{\mathfrak{l}})$ appears with multiplicity one as direct summand of $L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{\nu'}$. Recall that $\bar{\nu}_{\mathfrak{g}}$ and $\bar{\nu}_{\mathfrak{l}}$ denote the unique elements in $\mathcal{W}_{\mathfrak{g}}\nu \cap \Lambda_{\mathfrak{g}}^+$ and $\mathcal{W}_{\mathfrak{l}}\nu' \cap \Lambda_{\mathfrak{l}}^+$.*

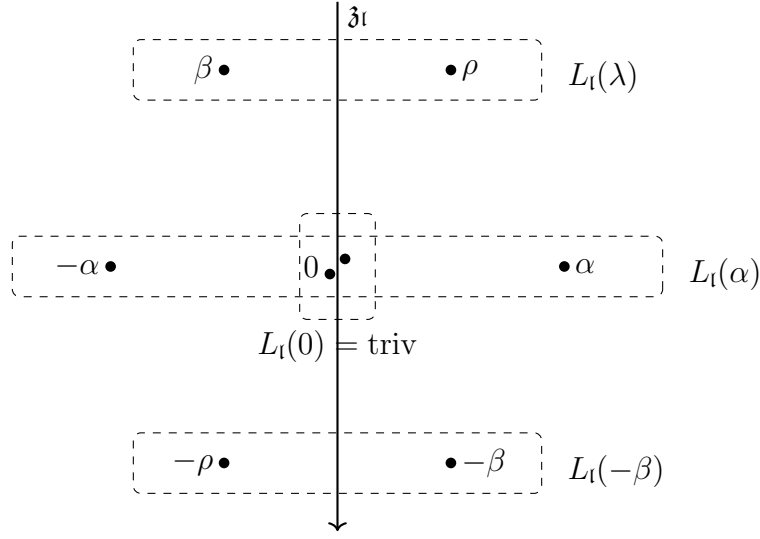


Figure 2.2: **Splitting of the adjoint representation in the case $\mathfrak{sl}_2 \subset \mathfrak{sl}_3$:** Here $\Delta_{\mathfrak{g}} = \{\alpha, \beta\}$, $\Delta_{\mathfrak{l}} = \{\alpha\}$. Dots indicate the weight spaces and the boxes surround the direct summands of the restriction to \mathfrak{l} .

Proof. Let $w \in \mathcal{W}$ such that $\bar{\nu}_{\mathfrak{g}} = w(\nu)$. Write $w = xy$ with $x \in w\mathcal{W}_{\mathfrak{l}}$ a shortest coset representative and $y \in \mathcal{W}_{\mathfrak{l}}$. Since x is a shortest coset representative, it maps positive roots for \mathfrak{l} to positive roots for \mathfrak{g} , and since $\bar{\nu}_{\mathfrak{g}}$ is dominant we get

$$\langle y(\nu), \alpha^\vee \rangle = \langle xy(\nu), x(\alpha)^\vee \rangle = \langle \bar{\nu}_{\mathfrak{g}}, x(\alpha)^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi_{\mathfrak{l}}^+.$$

Therefore

$$y(\nu) = \bar{\nu}_{\mathfrak{l}} \text{ and } xy(\nu) = \bar{\nu}_{\mathfrak{g}}.$$

Now choose some non-zero $v^+ \in L(\bar{\nu}_{\mathfrak{g}})_{y(\nu)}$. Then v^+ is a highest weight vector for \mathfrak{l} since for $\alpha \in \Phi_{\mathfrak{l}}^+$

$$\dim_{\mathbb{C}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{y(\nu)+\alpha} = \dim_{\mathbb{C}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{\bar{\nu}_{\mathfrak{g}}+x(\alpha)} = 0$$

because $x(\alpha) \in \Phi^+$ and all weights of $L(\bar{\nu}_{\mathfrak{g}})$ are in $\bar{\nu}_{\mathfrak{g}} - \mathbb{Z}_{\geq 0}\Phi^+$. So indeed v^+ generates $L_{\mathfrak{l}}(\bar{\nu}_{\mathfrak{l}})$ as \mathfrak{l} -module and $\mathfrak{z}_{\mathfrak{l}}$ acts on it via $\nu' = \nu|_{\mathfrak{z}_{\mathfrak{l}}} = y(\nu)|_{\mathfrak{z}_{\mathfrak{l}}}$. The multiplicity one statement follows from $\dim_{\mathbb{C}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{y(\nu)} = 1$. \square

2.2.3 Translation into a more singular block

Theorem 2.2.3.1. *Let μ and λ be integral dominant weights, such that $\mathcal{W}_{\mathfrak{g},\lambda} \subset \mathcal{W}_{\mathfrak{g},\mu}$. Let w be a shortest representative of a coset in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$. Let*

$M \in \mathcal{O}_{w \cdot \lambda}(\mathfrak{l})$. Then there exists a natural isomorphism

$$\mathrm{T}_\lambda^\mu \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} M.$$

Proof. Firstly, the statement is correct for Verma modules $M_{\mathfrak{l}}(xw \cdot \lambda)$, for $x \in \mathcal{W}_{\mathfrak{l}}$, since

$$\begin{aligned} \mathrm{T}_\lambda^\mu \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(xw \cdot \lambda) &\cong \mathrm{T}_\lambda^\mu M_{\mathfrak{g}}(xw \cdot \lambda) \cong M_{\mathfrak{g}}(xw \cdot \mu) \\ \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} M_{\mathfrak{l}}(xw \cdot \lambda) &\cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(xw \cdot \mu) \cong M_{\mathfrak{g}}(xw \cdot \mu) \end{aligned}$$

by Theorem 2.1.3.2 and [Hum08, Theorem 7.6]. By the exactness of the involved functors the statement is hence true on the level of characters. Let $\nu \stackrel{\text{def}}{=} \mu - \lambda$. Then the tensor identity gives a natural isomorphism

$$\mathrm{T}_\lambda^\mu \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} = \mathrm{pr}_\mu(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(-) \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})) \cong \mathrm{pr}_\mu(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{p}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}}))).$$

By Lemma 2.2.1.2 we see that, for suitable $\nu_i \in \mathfrak{z}_{\mathfrak{l}}^*$, the right hand side has a natural filtration with subquotients

$$\mathrm{pr}_\mu(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{\nu_i})).$$

We will show that this functor is zero except in the case

$$\nu_i = w(\nu)|_{\mathfrak{z}_{\mathfrak{l}}}.$$

Again by exactness, we can test this on Verma modules $M_{\mathfrak{l}}(xw \cdot \lambda)$, for $x \in \mathcal{W}_{\mathfrak{l}}$. In this case

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M_{\mathfrak{l}}(xw \cdot \lambda) \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}}))$$

has a Verma flag with subquotients of the form

$$M_{\mathfrak{g}}(xw \cdot \lambda + \xi)$$

for weights ξ of $L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})$. By [Jan79, Satz 2.10] or [Hum08, Lemma 7.5 and Theorem 7.6] and using the hypothesis $\mathcal{W}_{\mathfrak{g}, \lambda} \subset \mathcal{W}_{\mathfrak{g}, \mu}$, the only Verma module of this form which is contained in the block $\mathcal{O}_{\mu}(\mathfrak{g})$ is

$$M_{\mathfrak{g}}(xw \cdot \lambda + \xi) = M(xw \cdot \mu).$$

To not be killed by pr_μ hence ξ has to be

$$\begin{aligned} \xi &= xw \cdot \mu - xw \cdot \lambda = xw(\nu) \quad \text{and therefore} \\ \xi|_{\mathfrak{z}_{\mathfrak{l}}} &= (xw \cdot \mu - xw \cdot \lambda)|_{\mathfrak{z}_{\mathfrak{l}}} = xw(\nu)|_{\mathfrak{z}_{\mathfrak{l}}} = w(\nu)|_{\mathfrak{z}_{\mathfrak{l}}}. \end{aligned}$$

We hence have a natural isomorphism

$$\mathrm{pr}_\mu(\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})|_{\mathfrak{p}})) \cong \mathrm{pr}_\mu(\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{w(\nu)|_{\mathfrak{s}_\mathfrak{l}}})).$$

Lemma 2.2.2.1 now ensures that, as \mathfrak{l} -module, $L_{\mathfrak{l}}(\overline{w(\nu)}_{\mathfrak{l}})$ appears as a direct summand of $L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{w(\nu)|_{\mathfrak{s}_\mathfrak{l}}}$. This induces inclusions

$$\begin{aligned} \mathrm{T}_\lambda^\mu \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} &\cong \mathrm{pr}_\mu(\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{w(\nu)|_{\mathfrak{s}_\mathfrak{l}}})) \\ &\supset \mathrm{pr}_\mu \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \left(- \otimes L_{\mathfrak{l}}(\overline{w(\nu)}_{\mathfrak{l}}) \right) \\ &\supset \mathrm{pr}_\mu \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{pr}_{w \cdot \mu} \left(- \otimes L_{\mathfrak{l}}(\overline{w(\nu)}_{\mathfrak{l}}) \right) \\ &= \mathrm{pr}_\mu \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu}(-) \\ &= \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu}(-) \end{aligned}$$

The inclusions are equalities for Verma modules, and the statement follows by the exactness of all involved functors. \square

2.2.4 Translation out of a more singular block

Theorem 2.2.4.1. *Let μ and λ be integral weights, dominant for $\Phi_{\mathfrak{l}}^+$, with $\mathcal{W}_{\mathfrak{g},\mu} \subset \mathcal{W}_{\mathfrak{g},\lambda}$ and $z^{-1} \in \mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$ a shortest coset representative such that both $z \cdot \lambda$ and $z \cdot \mu$ are dominant for $\Phi_{\mathfrak{g}}^+$. Let $M \in \mathcal{O}_\lambda(\mathfrak{l})$. Then*

$$\mathrm{T}_{z \cdot \lambda}^{z \cdot \mu} \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} M$$

has a filtration, natural in M , whose successive quotients are

$$\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{T}_\lambda^{w \cdot \mu} M$$

parametrized by shortest representatives w (with respect to $z\mathcal{S}_{\mathfrak{g}}z^{-1}$, see the following Remark 2.2.4.2) of the double cosets

$$\mathcal{W}_{\mathfrak{l},\lambda} \setminus \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}$$

and ordered by the length of w : In particular $\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{T}_\lambda^\mu M$ is a submodule and $\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} \mathrm{T}_\lambda^{\tilde{w} \cdot \mu} M$ a quotient of $\mathrm{T}_{z \cdot \lambda}^{z \cdot \mu} \mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}} M$, for \tilde{w} the shortest representative of the longest word in $\mathcal{W}_{\mathfrak{g},\lambda}$.

Proof. We proceed as in the proof of Theorem 2.2.3.1. Let $x \in \mathcal{W}_{\mathfrak{l}}$. Then by [Jan79, Satz 2.17] or [Hum08, Theorem 7.12] we have the following equalities of characters:

On the one hand

$$\begin{aligned} \text{ch}(\mathbb{T}_{z,\lambda}^{z,\mu} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(x \cdot \lambda)) &= \text{ch}(\mathbb{T}_{z,\lambda}^{z,\mu} M_{\mathfrak{g}}(x \cdot \lambda)) \\ &= \sum_{w \in \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}} \text{ch} M_{\mathfrak{g}}(xw \cdot \mu) \end{aligned}$$

and on the other hand

$$\text{ch}(\mathbb{T}_{\lambda}^{w,\mu} M_{\mathfrak{l}}(x \cdot \lambda)) = \sum_{y \in \mathcal{W}_{\mathfrak{l},\lambda} / \mathcal{W}_{\mathfrak{l},w \cdot \mu}} \text{ch} M_{\mathfrak{l}}(xyw \cdot \mu)$$

and hence

$$\begin{aligned} \sum_{w \in \mathcal{W}_{\mathfrak{l},\lambda} \setminus \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}} \text{ch}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{T}_{\lambda}^{w,\mu} M_{\mathfrak{l}}(x \cdot \lambda)) &= \sum_{\substack{w \in \mathcal{W}_{\mathfrak{l},\lambda} \setminus \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu} \\ y \in \mathcal{W}_{\mathfrak{l},\lambda} / \mathcal{W}_{\mathfrak{l},w \cdot \mu}}} \text{ch} M_{\mathfrak{g}}(xyw \cdot \mu) \\ &= \sum_{w \in \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}} \text{ch} M_{\mathfrak{g}}(xw \cdot \mu). \end{aligned}$$

In the last equality we used that the stabilizer in $\mathcal{W}_{\mathfrak{l},\lambda}$ of a coset $w\mathcal{W}_{\mathfrak{g},\mu} \in \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}$ is exactly $w\mathcal{W}_{\mathfrak{g},\mu}w^{-1} \cap \mathcal{W}_{\mathfrak{l},\lambda} = \mathcal{W}_{\mathfrak{l},w \cdot \mu}$.

Putting everything together, we obtain

$$\text{ch}(\mathbb{T}_{z,\lambda}^{z,\mu} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(x \cdot \lambda)) = \sum_{w \in \mathcal{W}_{\mathfrak{l},\lambda} \setminus \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}} \text{ch}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathbb{T}_{\lambda}^{w,\mu} M_{\mathfrak{l}}(x \cdot \lambda)).$$

By the exactness of all involved functors, this shows that our theorem is at least true on the level of characters.

Now we have to take a more refined look. Let $\nu \stackrel{\text{def}}{=} \mu - \lambda$. Then the tensor identity gives a natural isomorphism

$$\mathbb{T}_{z,\lambda}^{z,\mu} \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} = \text{pr}_{z,\mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(-) \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})) \cong \text{pr}_{z,\mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes \text{Res}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}}))).$$

By Lemma 2.2.1.2 we see that, for suitable $\nu_i \in \mathfrak{J}_{\mathfrak{l}}^*$, the right hand side has a natural filtration with subquotients

$$\text{pr}_{z,\mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}})_{\nu_i})).$$

Let us analyse which of them are non-zero. By exactness, this can be tested on Verma modules $M_{\mathfrak{l}}(x \cdot \lambda)$, for $x \in \mathcal{W}_{\mathfrak{l}}$. In this case

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M_{\mathfrak{l}}(x \cdot \lambda) \otimes L_{\mathfrak{g}}(\bar{\nu}_{\mathfrak{g}}))$$

has a Verma flag with subquotients of the form

$$M_{\mathfrak{g}}(x \cdot \lambda + \xi)$$

for weights ξ of $L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})$. The only Verma modules of this form which are contained in the block $\mathcal{O}_{\mu}(\mathfrak{g})$ are of the form

$$M_{\mathfrak{g}}(x \cdot \lambda + \xi) = M_{\mathfrak{g}}(xw \cdot \mu)$$

for $w \in \mathcal{W}_{\mathfrak{g},\lambda}/\mathcal{W}_{\mathfrak{g},\mu}$. Hence

$$\begin{aligned} \xi &= xw \cdot \mu - x \cdot \lambda = x(w \cdot \mu - \lambda) \quad \text{and therefore} \\ \xi|_{\mathfrak{h}_i} &= (xw \cdot \mu - x \cdot \lambda)|_{\mathfrak{h}_i} = x(w \cdot \mu - \lambda)|_{\mathfrak{h}_i} = (w \cdot \mu - \lambda)|_{\mathfrak{h}_i}. \end{aligned}$$

Notice that the last term does not depend on x . By the above,

$$\begin{aligned} (\text{pr}_{z \cdot \mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{\nu_i})) \neq 0) &\Rightarrow \\ (\nu_i = (w \cdot \mu - \lambda)|_{\mathfrak{h}_i} \text{ for some } w \in \mathcal{W}_{\mathfrak{g},\lambda}/\mathcal{W}_{\mathfrak{g},\mu}). \end{aligned}$$

Choose such i and w . Without loss of generality we can assume that w is a shortest representative of a double coset in $\mathcal{W}_{\mathfrak{l},\lambda} \backslash \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}$, since for $\hat{w} \in \mathcal{W}_{\mathfrak{l},\lambda}$

$$(\hat{w}w \cdot \mu - \lambda)|_{\mathfrak{h}_i} = (\hat{w}w \cdot \mu - \hat{w}\lambda)|_{\mathfrak{h}_i} = \hat{w}(w \cdot \mu - \lambda)|_{\mathfrak{h}_i} = (w \cdot \mu - \lambda)|_{\mathfrak{h}_i}.$$

Now [Jan79, Satz 2.9] implies that

$$w \cdot \mu - \lambda \in \mathcal{W}_{\mathfrak{g},\lambda}\nu$$

and by Lemma 2.2.2.1, as \mathfrak{l} -module, $L_{\mathfrak{l}}(\overline{w \cdot \mu - \lambda}_{\mathfrak{l}})$ appears as a direct summand of $L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{\nu_i}$. We hence have a natural inclusion

$$\begin{aligned} \text{pr}_{z \cdot \mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes L_{\mathfrak{g}}(\overline{\nu}_{\mathfrak{g}})_{\nu_i})) &\supset \text{pr}_{z \cdot \mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(- \otimes L_{\mathfrak{l}}(\overline{w \cdot \mu - \lambda}_{\mathfrak{l}}))) \\ &\supset \text{pr}_{z \cdot \mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \text{pr}_{w \cdot \mu}(- \otimes L_{\mathfrak{l}}(\overline{w \cdot \mu - \lambda}_{\mathfrak{l}}))) \\ &\supset \text{pr}_{z \cdot \mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{\lambda}^{w \cdot \mu}(-)) \\ &= \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{\lambda}^{w \cdot \mu}(-) \end{aligned}$$

Again, by the character computation in the beginning, these are all actually equalities and the statement follows. \square

Remark 2.2.4.2. In the notation of the preceding Theorem 2.2.4.1 it makes sense to speak about shortest coset representatives with respect to $z\mathcal{S}_{\mathfrak{g}}z^{-1}$ in the double quotient $\mathcal{W}_{\mathfrak{l},\lambda} \backslash \mathcal{W}_{\mathfrak{g},\lambda} / \mathcal{W}_{\mathfrak{g},\mu}$, since all involved groups are generated by their respective intersection with $z\mathcal{S}_{\mathfrak{g}}z^{-1}$. By [Hum90, Theorem 1.12 (c)]

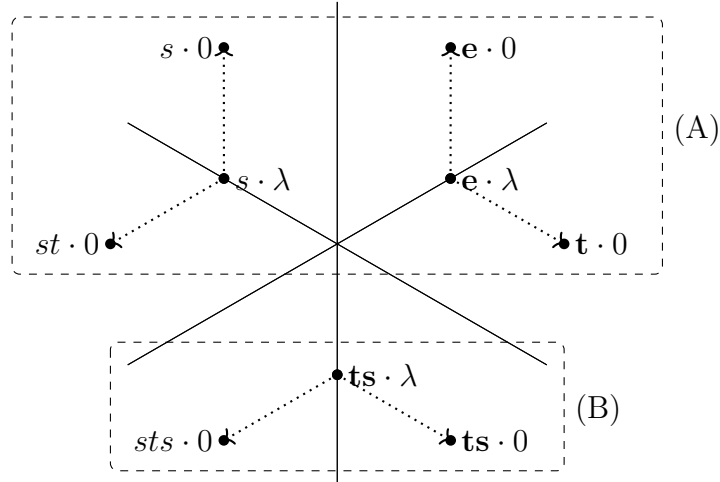
this holds for $\mathcal{W}_{\mathfrak{g},\lambda}$ and $\mathcal{W}_{\mathfrak{g},\mu}$ since λ and μ are dominant with respect to $z^{-1}\mathcal{S}_{\mathfrak{g}}z$. Since z^{-1} is a shortest coset representative in $\mathcal{W}_{\mathfrak{l}}\backslash\mathcal{W}_{\mathfrak{g}}$ one can easily see that $z^{-1}\cdot\lambda$ is dominant for $\mathcal{S}_{\mathfrak{l}}$ and hence again by [Hum90, Theorem 1.12 (c)] $\mathcal{W}_{\mathfrak{l},z^{-1}\lambda} = z^{-1}\mathcal{W}_{\mathfrak{l},\lambda}z$ is generated by its intersection with $\mathcal{S}_{\mathfrak{l}}$. But this just means that $\mathcal{W}_{\mathfrak{l},\lambda}$ is generated by its intersection with $z\mathcal{S}_{\mathfrak{l}}z^{-1} \subset \mathcal{S}_{\mathfrak{g}}z^{-1}$

Example 2.2.4.3. (1) In the case $\lambda = -\rho$, $\mathfrak{l} = \mathfrak{h}$, this recovers the well-known fact that the antidominant projective

$$P_{\mathfrak{g}}(w_0 \cdot \mu) = T_{-\rho}^{\mu} M_{\mathfrak{g}}(-\rho)$$

has a Verma flag with quotients $M_{\mathfrak{g}}(w \cdot \mu)$ of multiplicity one, where $w \in \mathcal{W}_{\mathfrak{g}}$ is a shortest coset representative in $\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{g},\mu}$.

(2) The case $\mathfrak{sl}_2 \subset \mathfrak{sl}_3 = \mathfrak{g}$: Denote by $\{s, t\}$ the simple reflections in $\mathcal{W}_{\mathfrak{g}}$ and let \mathfrak{l} be the Levi subalgebra with $\mathcal{W}_{\mathfrak{l}} = \{1, s\}$. Set furthermore $\lambda = -\varpi_{\alpha_s}$ such that $\mathcal{W}_{\mathfrak{g},\lambda} = \{1, t\}$. We are interested in the interaction of $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ and T_{λ}^0 . There are two different cases (A) for modules in $\mathcal{O}_{\lambda}(\mathfrak{l})$ and (B) for modules in $\mathcal{O}_{ts,\lambda}(\mathfrak{l})$. In the illustration we indicated the effect of T_{λ}^0 on Verma modules $M_{\mathfrak{g}}(w \cdot \lambda)$ by dotted lines and labeled the dominant weights for \mathfrak{l} with bold case letters.



(A) For modules in $M \in \mathcal{O}_{\lambda}(\mathfrak{l})$ parabolic induction and translation out of the wall do *not commute*, since $\mathcal{W}_{\mathfrak{l},\lambda} \neq \mathcal{W}_{\mathfrak{g},\lambda}$. We rather get a short exact sequence

$$0 \rightarrow \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{\lambda}^0 M \rightarrow T_{\lambda}^0 \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \rightarrow \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{\lambda}^{t \cdot 0} M \rightarrow 0$$

(B) For modules in $M \in \mathcal{O}_{ts,\lambda}(\mathfrak{l})$ parabolic induction and translation out of the wall do *commute*, since $\mathcal{W}_{\mathfrak{l},ts,\lambda} = \mathcal{W}_{\mathfrak{g},ts,\lambda}$:

$$T_{\lambda}^0 \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M = \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{ts,\lambda}^{ts \cdot 0} M.$$

The case (B) from the preceding example can be generalized to the following statement.

Corollary 2.2.4.4. *In the notation of Theorem 2.2.4.1 assume additionally that $\mathcal{W}_{\mathfrak{l},\lambda} = \mathcal{W}_{\mathfrak{g},\lambda}$. Let $M \in \mathcal{O}_\lambda(\mathfrak{l})$. Then there is a natural equivalence*

$$\mathrm{T}_{z,\lambda}^{z,\mu} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{\lambda}^{\mu} M.$$

2.2.5 Wall crossing functors

Composing translation functors into and out of a *wall*, i.e. a block of category \mathcal{O}_λ with $|\mathcal{W}_\lambda| = 2$, yield so called *wall crossing functors*, whose interaction with parabolic induction is described in the following. This will be an essential ingredient in the induction step of our proof that parabolic induction and geometric parabolic induction correspond to each other.

Theorem 2.2.5.1. *Let $\lambda \in \mathfrak{h}^*$ be a dominant regular weight. Let $w \in \mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$ a shortest coset representative and $s \in \mathcal{W}_{\mathfrak{g}}$ a simple reflection with $ws > w$ such that ws is also a shortest coset representative for $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}$. Denote by θ_s a wall-crossing functor through the s -wall. Namely, choose some dominant weight μ with stabilizer $\mathcal{W}_{\mathfrak{g},\mu} = \{1, s\}$ and put $\theta_s = \mathrm{T}_{\mu}^{\lambda} \mathrm{T}_{\lambda}^{\mu}$. Then for all $M \in \mathcal{O}_{w,\lambda}(\mathfrak{l})$ with a Verma flag there is a short exact sequence, natural in M ,*

$$0 \longrightarrow \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \longrightarrow \theta_s \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \longrightarrow \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w,\lambda}^{ws,\lambda} M \longrightarrow 0$$

where the first morphism is the unit of the adjunction between $\mathrm{T}_{\mu}^{\lambda}$ and $\mathrm{T}_{\lambda}^{\mu}$.

Proof. By Theorem, 2.2.3.1 we have

$$\theta_s \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \stackrel{def}{=} \mathrm{T}_{\mu}^{\lambda} \mathrm{T}_{\lambda}^{\mu} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \cong \mathrm{T}_{\mu}^{\lambda} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w,\lambda}^{w,\mu} M.$$

By Theorem 2.2.4.1, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w,\mu}^{w,\lambda} \mathrm{T}_{w,\lambda}^{w,\mu} M \longrightarrow \mathrm{T}_{\mu}^{\lambda} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w,\lambda}^{w,\mu} M \longrightarrow \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w,\mu}^{ws,\lambda} \mathrm{T}_{w,\lambda}^{w,\mu} M \longrightarrow 0$$

Now $w \cdot \mu$ is also dominant for $\Phi_{\mathfrak{l}}^+$: We have

$$\mathcal{W}_{\mathfrak{l},w,\mu} = w\mathcal{W}_{\mathfrak{g},\mu}w^{-1} \cap \mathcal{W}_{\mathfrak{l}} = \{id\}$$

since $ws w^{-1} \in \mathcal{W}_{\mathfrak{l}}$ would imply that ws and w are in the same coset in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}$ which is a contradiction to the assumption that both are shortest representatives and $ws > w$.

Hence we have

$$\mathbb{T}_{w \cdot \mu}^{w \cdot \lambda} \mathbb{T}_{w \cdot \lambda}^{w \cdot \mu} \cong \text{id} \text{ and } \mathbb{T}_{w \cdot \mu}^{ws \cdot \lambda} \mathbb{T}_{w \cdot \lambda}^{w \cdot \mu} \cong \mathbb{T}_{w \cdot \lambda}^{ws \cdot \lambda}.$$

That we can indeed choose the first morphism as the unit of the adjunction, say κ_s , follows as in [Hum98, Theorem 12.2(b)] (be aware that his notation is different, since he parametrizes blocks and translation/wall crossing functors by antidominant weights, hence everything is conjugated/multiplied by the longest element w_0). By induction on the Verma flag of M we see that the adjunction morphism is indeed injective, and then we use that $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$ is unique as submodule of $\theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$. Let us spell this out in more detail. Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

be a filtration of M such that the successive quotients are Verma modules. If $n = 0$ the statement is trivial. Else, we have the following diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{n-1} & \longrightarrow & \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_n & \longrightarrow & \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda) \longrightarrow 0 \\ & & \kappa_s \uparrow & & \kappa_s \uparrow & & \kappa_s \uparrow \\ 0 & \longrightarrow & \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{n-1} & \longrightarrow & \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_n & \longrightarrow & \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda) \longrightarrow 0 \end{array}$$

for some $x \in \mathcal{W}_l$. We can assume that the left vertical arrow is injective by induction. Now

$$\begin{aligned} \text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda), \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda)) &= \\ \text{Hom}_{\mathfrak{g}}(\mathbb{T}_{\lambda}^{\mu} M_{\mathfrak{g}}(xw \cdot \lambda), \mathbb{T}_{\lambda}^{\mu} M_{\mathfrak{g}}(xw \cdot \lambda)) &= \\ \text{Hom}_{\mathfrak{g}}(M_{\mathfrak{g}}(xw \cdot \mu), M_{\mathfrak{g}}(xw \cdot \mu)) &= \mathbb{C} \end{aligned}$$

Hence κ_s is (up to scalar) the unique non-zero morphism $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda) \rightarrow \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda)$. Since we also know that $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda)$ appears as (even unique) submodule in $\theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_l(xw \cdot \lambda)$, κ_s has to be injective. Hence also the right vertical arrow of our diagram is injective and we get that

$$\kappa_s : \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \rightarrow \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$$

is injective. That $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$ is indeed unique as a submodule of $\theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$ can also be seen by an inductive argument. Let $M_l(xw \cdot \lambda) \subseteq M$ such that no weight in M is bigger than $xw \cdot \lambda$. The assumption $ws > w$ guarantees that also in $\theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M$ no weight bigger than $xw \cdot \lambda$. Hence we have

$$\mathcal{U}(\mathfrak{g})(\theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)_{xw \cdot \lambda} = M_{\mathfrak{g}}(xw \cdot \lambda)^{\oplus(M: M_l(xw \cdot \lambda))} \subseteq \theta_s \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M.$$

This is clearly the unique submodule of this form. Now we can pass to the quotient and apply the same argument again. The statement follows by induction. \square

Corollary 2.2.5.2. *There is a natural equivalence of functors*

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{ws \cdot \lambda} \cong \mathrm{coker}(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \rightarrow \theta_s \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}) : \mathrm{Proj} \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) \rightarrow \mathrm{Proj} \mathcal{O}_{\lambda}(\mathfrak{g}).$$

Remark 2.2.5.3. The functor $\mathrm{coker}(\mathrm{id} \rightarrow \theta_s)$ is also known as *shuffling functor*. In general, it maps Verma modules to so called shuffled or twisted Verma modules, i.e. modules which have the same character as a Verma module, but a different (shuffled) composition series. See for example [Irv93], [AL03] and [Hum08, Chapter 12.1]. Since we only apply the functor in the particular situation $ws > w$, no shuffling occurs.

Lemma 2.2.5.4. *Let $\lambda \in \mathfrak{h}^*$ be a dominant regular weight and $s \in \mathcal{W}_{\mathfrak{l}}$ a simple reflection. Let $M \in \mathcal{O}_{\lambda}(\mathfrak{l})$. Then there is a natural isomorphism*

$$\theta_s \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \theta_s M.$$

Proof. Directly follows from Theorem 2.2.3.1 and Corollary 2.2.4.4. \square

2.2.6 From the singular case to the regular case

At least up to taking it to some n -fold direct sum, parabolic induction for singular blocks of category \mathcal{O} can be expressed in terms of parabolic induction for regular blocks, by translating out, then inducing, and translating into the singular block again.

Theorem 2.2.6.1. *Let λ, μ be integral dominant weights where μ is furthermore regular. Let w be a shortest representative of a coset in $\mathcal{W}_{\mathfrak{l}} \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}$ and $n = |\mathcal{W}_{\mathfrak{l}, \lambda}|$. Then there is a natural equivalence of functors*

$$\mathrm{T}_{\mu}^{\lambda} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} \cong (\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{\oplus n} : \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) \rightarrow \mathcal{O}_{\lambda}(\mathfrak{g}).$$

Proof. By Theorem 2.2.3.1 we have

$$\mathrm{T}_{\mu}^{\lambda} \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{T}_{w \cdot \mu}^{w \cdot \lambda} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} \cong \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathrm{id}^{\oplus n},$$

where the last isomorphism follows by using

$$\mathrm{T}_{w \cdot \mu}^{w \cdot \lambda} \mathrm{T}_{w \cdot \lambda}^{w \cdot \mu} M_{\mathfrak{l}}(w \cdot \lambda) = M_{\mathfrak{l}}(w \cdot \lambda)^{\oplus n}$$

and the classification of projective functors from [BG80], i.e. that projective functors (and natural transformations between them) are completely determined by their effect on a dominant Verma module (in fact any Verma module). \square

2.3 Soergel modules

In [Soe90] Soergel gives a completely combinatorial description of the bounded derived category of a block of category \mathcal{O} in terms of the bounded homotopy category of Soergel modules over the endomorphism ring of its antidominant projective module. In this section we aim to give a description of parabolic induction on the level of Soergel modules, i.e. fill out the question mark in the diagram

$$\begin{array}{ccc} \mathrm{Hot}^b(C_{\mathfrak{l}}^{w\cdot\lambda}\text{-Smod}) & \xrightarrow{?} & \mathrm{Hot}^b(C_{\mathfrak{g}}^{\lambda}\text{-Smod}) \\ \uparrow \wr & & \uparrow \wr \\ \mathrm{Der}^b(\mathcal{O}_{w\cdot\lambda}(\mathfrak{l})) & \xrightarrow{\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathrm{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})). \end{array}$$

2.3.1 Soergel's functor \mathbb{V}

Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and w a shortest representative in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g},\lambda}$. Denote by

$$P_{\mathfrak{g}}^{\lambda} \text{ and } P_{\mathfrak{l}}^{w\cdot\lambda}$$

the antidominant (self-dual) projective in $\mathcal{O}_{\lambda}(\mathfrak{g})$, respectively $\mathcal{O}_{w\cdot\lambda}(\mathfrak{l})$, and by

$$C_{\mathfrak{g}}^{\lambda} = \mathrm{End}_{\mathfrak{g}}(P_{\mathfrak{g}}^{\lambda}) \text{ and } C_{\mathfrak{l}}^{w\cdot\lambda} = \mathrm{End}_{\mathfrak{l}}(P_{\mathfrak{l}}^{w\cdot\lambda})$$

their endomorphism rings. Then *Soergel's functor* \mathbb{V} (see [Soe90]) is defined by

$$\begin{aligned} \mathbb{V}_{\mathfrak{g}}^{\lambda} &\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathfrak{g}}(P_{\mathfrak{g}}^{\lambda}, -) : \mathcal{O}_{\lambda}(\mathfrak{g}) \rightarrow \mathrm{mod}\text{-}C_{\mathfrak{g}}^{\lambda} = C_{\mathfrak{g}}^{\lambda}\text{-mod} \text{ and} \\ \mathbb{V}_{\mathfrak{l}}^{w\cdot\lambda} &\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathfrak{l}}(P_{\mathfrak{l}}^{w\cdot\lambda}, -) : \mathcal{O}_{w\cdot\lambda}(\mathfrak{l}) \rightarrow \mathrm{mod}\text{-}C_{\mathfrak{l}}^{w\cdot\lambda} = C_{\mathfrak{l}}^{w\cdot\lambda}\text{-mod}. \end{aligned}$$

Theorem 2.3.1.1 (Struktursatz [Soe90]). *Soergel's functor \mathbb{V} is fully faithful on projective modules.*

Definition 2.3.1.2. *The modules in the essential image of the restriction of \mathbb{V} to projective modules are called Soergel modules, so that \mathbb{V} induces an equivalence of categories:*

$$\begin{aligned} \mathbb{V}_{\mathfrak{g}}^{\lambda} &: \mathrm{Proj} \mathcal{O}_{\lambda}(\mathfrak{g}) \xrightarrow{\sim} C_{\mathfrak{g}}^{\lambda}\text{-Smod} \\ \mathbb{V}_{\mathfrak{l}}^{w\cdot\lambda} &: \mathrm{Proj} \mathcal{O}_{w\cdot\lambda}(\mathfrak{l}) \xrightarrow{\sim} C_{\mathfrak{l}}^{w\cdot\lambda}\text{-Smod} \end{aligned}$$

between projectives in \mathcal{O} and the category of Soergel modules over $C_{\mathfrak{g}}^{\lambda}$, respectively $C_{\mathfrak{l}}^{w\cdot\lambda}$, denoted $C_{\mathfrak{g}}^{\lambda}\text{-Smod}$, respectively $C_{\mathfrak{l}}^{w\cdot\lambda}\text{-Smod}$.

Remark 2.3.1.3. Abbreviate $C = C_{\mathfrak{g}}^{\lambda}$. Then the category C -Smod is generated by modules of the form

$$C \otimes_{C^{s_n}} \dots C \otimes_{C^{s_1}} \mathbb{C}$$

for simple reflections s_i (see Theorem 2.3.3 for the action of $\mathcal{W}_{\mathfrak{g}}$ on C), with respect to finite direct sums, taking direct summands and isomorphism. This corresponds to the fact that for regular λ all projectives in $\mathcal{O}_{\lambda}(\mathfrak{g})$ appear as direct summands in the projective modules

$$\theta_{s_n} \cdots \theta_{s_1} M_{\mathfrak{g}}(\lambda)$$

and the following Lemma.

Lemma 2.3.1.4 ([Soe90] Theorem 10). *Let $\lambda, \mu \in \mathfrak{h}^*$ be dominant integral weights such that $\mathcal{W}_{\mathfrak{g}, \lambda} \subseteq \mathcal{W}_{\mathfrak{g}, \mu}$. Then there are natural isomorphisms of functors*

$$\begin{aligned} \mathbb{V}_{\mathfrak{g}}^{\lambda} T_{\mu}^{\lambda} &\cong C_{\mathfrak{g}}^{\lambda} \otimes_{C_{\mathfrak{g}}^{\mu}} \mathbb{V}_{\mathfrak{g}}^{\mu} \text{ and} \\ \mathbb{V}_{\mathfrak{g}}^{\mu} T_{\lambda}^{\mu} &\cong \text{Res}_{C_{\mathfrak{g}}^{\lambda}}^{C_{\mathfrak{g}}^{\mu}} \mathbb{V}_{\mathfrak{g}}^{\lambda}. \end{aligned}$$

Corollary 2.3.1.5. *There are equivalences of categories*

$$\begin{aligned} \text{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})) &\xleftarrow{\sim} \text{Hot}^b(\text{Proj } \mathcal{O}_{\lambda}(\mathfrak{g})) \xrightarrow{\sim} \text{Hot}^b(C_{\mathfrak{g}}^{\lambda}\text{-Smod}), \\ \text{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) &\xleftarrow{\sim} \text{Hot}^b(\text{Proj } \mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) \xrightarrow{\sim} \text{Hot}^b(C_{\mathfrak{l}}^{w \cdot \lambda}\text{-Smod}). \end{aligned}$$

2.3.2 Harish-Chandra morphism and parabolic restriction

In this section we will discuss how the relative Harish-Chandra morphism

$$\text{HC}_{\mathfrak{g}}^{\mathfrak{p}} : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathcal{Z}(\mathfrak{l})$$

between the center of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{l})$ and the parabolic restriction functor

$$\text{Res}_{\lambda}(-) : \mathcal{O}_{\lambda}(\mathfrak{g}) \rightarrow \mathcal{O}_{\lambda}(\mathfrak{l}), M \mapsto \text{pr}_{\lambda}(M_{\lambda|_{\mathfrak{s}_{\mathfrak{l}}}}^{\mathfrak{p}})$$

interact. The results are a straightforward generalization of the fact that the action of $z \in \mathcal{Z}(\mathfrak{g})$ on a Verma module is completely determined by the action on the highest weight vector, which is completely described by its image $\text{HC}_{\mathfrak{g}}^{\mathfrak{b}}(z) \in \mathcal{S}(\mathfrak{h})$ under the Harish-Chandra morphism.

Definition 2.3.2.1. Denote by $\mathfrak{n}_{\mathfrak{p}}$ and $\mathfrak{n}_{\mathfrak{p}}^-$ the nilradical of \mathfrak{p} and its opposite \mathfrak{p}^- . The relative Harish-Chandra homomorphism

$$\mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}} : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{l})$$

is obtained by the projection on the first factor in the PBW-decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{l}) \oplus \mathfrak{n}_{\mathfrak{p}}^- \mathcal{U}(\mathfrak{p}^-) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}_{\mathfrak{p}}.$$

It restricts to a homomorphism of algebras

$$\mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}} : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathcal{Z}(\mathfrak{l}).$$

Remark 2.3.2.2. This version of the Harish-Chandra morphism depends not only on the Levi but also on the parabolic subalgebra, hence the decoration $\mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}}$.

Lemma 2.3.2.3. In the above notation let M be a \mathfrak{g} -module, $u \in \mathcal{Z}(\mathfrak{g})$ and $m \in M^{\mathfrak{n}_{\mathfrak{p}}}$. Then we have

$$um = \mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}}(u)m.$$

Proof. Let $u = u_1 + u_2 + u_3$ in the above PBW decomposition and $\mathfrak{z} \subset \mathfrak{l}$ be the center of \mathfrak{l} . Since

$$\mathfrak{n}_{\mathfrak{p}}^- \mathcal{U}(\mathfrak{p}^-) \cap \mathcal{U}(\mathfrak{g})^{\mathrm{ad}(\mathfrak{z})} = \{0\}$$

it follows that $u_2 = 0$. Hence

$$u - \mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}}(u) \in \mathcal{U}(\mathfrak{g})\mathfrak{n}_{\mathfrak{p}}$$

and $(u - \mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}}(u))m = 0$ for all $m \in M^{\mathfrak{n}_{\mathfrak{p}}}$. See also [How00]. \square

Corollary 2.3.2.4. Let $M \in \mathcal{O}_{\mathfrak{g}}(\lambda)$. Then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\mathrm{HC}_{\mathfrak{g}}^{\mathfrak{p}}} & \mathcal{Z}(\mathfrak{l}) \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathfrak{g}}(M) & \xrightarrow{(-)^{\mathfrak{n}_{\mathfrak{p}}}} & \mathrm{End}_{\mathfrak{l}}(M^{\mathfrak{n}_{\mathfrak{p}}}) \end{array}$$

We will also need this statement about antidominant projectives and parabolic restriction.

Lemma 2.3.2.5. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. Then

$$P_{\mathfrak{l}}^{\lambda} \cong \mathrm{Res}_{\lambda}(P_{\mathfrak{g}}^{\lambda}).$$

Proof. We have the following equalities:

$$P_{\mathfrak{g}}^{\lambda} \cong T_{-\rho}^{\lambda} M_{\mathfrak{g}}(-\rho) \cong T_{-\rho}^{\lambda} \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M_{\mathfrak{l}}(-\rho).$$

By Theorem 2.2.4.1 we know that the right hand side has a filtration $\{N_i\}$ with successive quotients

$$\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{-\rho}^{w \cdot \lambda} M_{\mathfrak{l}}(-\rho),$$

for w shortest representatives of the double cosets $\mathcal{W}_{\mathfrak{l}} \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}$. But for all $w \neq \operatorname{id}$ we certainly have

$$\operatorname{Res}_{\lambda}(\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{-\rho}^{w \cdot \lambda} M_{\mathfrak{l}}(-\rho)) = 0$$

since $\mathfrak{z}_{\mathfrak{l}}$ acts on them via $(w \cdot \lambda)|_{\mathfrak{z}_{\mathfrak{l}}} \neq \lambda|_{\mathfrak{z}_{\mathfrak{l}}}$. Our filtration provides us with short exact sequences of the form

$$0 \rightarrow N_{i+1} \rightarrow N_i \rightarrow \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{-\rho}^{w \cdot \lambda} M_{\mathfrak{l}}(-\rho) \rightarrow 0$$

and since $\operatorname{Res}_{\lambda}(-)$ is left exact we get exact sequences

$$0 \rightarrow \operatorname{Res}_{\lambda}(N_{i+1}) \rightarrow \operatorname{Res}_{\lambda}(N_i) \rightarrow \operatorname{Res}_{\lambda}(\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{-\rho}^{w \cdot \lambda} M_{\mathfrak{l}}(-\rho)).$$

As long as $w \neq \operatorname{id}$ the right term vanishes and hence

$$\operatorname{Res}_{\lambda}(N_{i+1}) = \operatorname{Res}_{\lambda}(N_i).$$

By induction we get

$$\operatorname{Res}_{\lambda}(P_{\mathfrak{g}}^{\lambda}) = \operatorname{Res}_{\lambda}(\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} T_{-\rho}^{\lambda} M_{\mathfrak{l}}(-\rho)) = \operatorname{Res}_{\lambda}(\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} P_{\mathfrak{l}}^{\lambda}).$$

But for modules in $M \in \mathcal{O}_{\lambda}(\mathfrak{l})$ we know that $\operatorname{Res}_{\lambda}(\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(M)) = M$ by Theorem 2.1.3.2 and the statement follows. \square

2.3.3 Endomorphismensatz

Soergel's *Endomorphismensatz* gives a completely explicit description of the endomorphism rings of antidominant projectives in category \mathcal{O} . His description is compatible with parabolic restriction in the following way.

Theorem 2.3.3.1 (Endomorphismensatz [Soe90]). *The following diagram commutes:*

$$\begin{array}{ccc}
(\mathcal{S}(\mathfrak{h})/(\mathcal{S}(\mathfrak{h})_+^{\mathcal{W}_l})^{\mathcal{W}_{l,\lambda}} & \longleftarrow & (\mathcal{S}(\mathfrak{h})/(\mathcal{S}(\mathfrak{h})_+^{\mathcal{W}_g})^{\mathcal{W}_{g,\lambda}} \\
\uparrow p \circ (+\lambda)^\# & & \uparrow p \circ (+\lambda)^\# \\
\mathcal{S}(\mathfrak{h})^{(\mathcal{W}_l \cdot)} & \longleftarrow & \mathcal{S}(\mathfrak{h})^{(\mathcal{W}_g \cdot)} \\
\uparrow \text{HC}_l^{\mathfrak{b}_l} \wr & & \uparrow \wr \text{HC}_g^{\mathfrak{b}_g} \\
\mathcal{Z}(\mathfrak{l}) & \xleftarrow{\text{HC}_g^{\mathfrak{p}}} & \mathcal{Z}(\mathfrak{g}) \\
\downarrow \text{act} & & \downarrow \text{act} \\
C_l^\lambda \stackrel{\text{def}}{=} \text{End}_l(P_l^\lambda) & \xleftarrow{\text{Res}_\lambda(-)} & \text{End}_g(P_g^\lambda) \stackrel{\text{def}}{=} C_g^\lambda
\end{array}$$

where $(+\lambda)^\#$ denotes translation of a function in $\mathcal{S}(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*)$ by λ and p the projection. Furthermore, the upward arrows $p \circ (+\lambda)^\# \circ \text{HC}_l^{\mathfrak{b}_l}$ and $p \circ (+\lambda)^\# \circ \text{HC}_g^{\mathfrak{b}_g}$ are surjective and have the same kernel as the downward arrows act , which are also surjective.

Proof. For the horizontal morphism on the bottom we use Lemma 2.3.2.5 which provides an isomorphism $P_l^\lambda \cong \text{Res}_\lambda(P_g^\lambda)$. Lemma 2.3.2.3 shows that the lower square commutes. The middle square commutes since (relative) Harish-Chandra homomorphisms are compatible, see [How00, Equation 1.12]. The upper one commutes by definition. The other statements are [Soe90, Endomorphismensatz]. \square

Corollary 2.3.3.2. *The following diagram commutes:*

$$\begin{array}{ccc}
(\mathcal{S}(\mathfrak{h})/(\mathcal{S}(\mathfrak{h})_+^{\mathcal{W}_l})^{\mathcal{W}_{l,\lambda}} & \longleftarrow & (\mathcal{S}(\mathfrak{h})/(\mathcal{S}(\mathfrak{h})_+^{\mathcal{W}_g})^{\mathcal{W}_{g,\lambda}} \\
\wr \downarrow & & \downarrow \wr \\
C_l^\lambda = \text{End}_l(P_l^\lambda) & \xleftarrow{\text{Res}_\lambda(-)} & \text{End}_g(P_g^\lambda) = C_g^\lambda
\end{array}$$

Here the vertical arrows are defined as in the preceding Theorem.

2.3.4 Parabolic induction and Soergel bimodules

The regular case

Assume that $\lambda \in \mathfrak{h}^*$ is some **regular** integral dominant weight (for example $\lambda = 0$), and abbreviate

$$C \stackrel{\text{def}}{=} C_g^\lambda \cong \mathcal{S}(\mathfrak{h})/(\mathcal{S}(\mathfrak{h})_+^{\mathcal{W}_g}).$$

For a simple reflection s , denote by C^s the s -invariants. Then $C^s \subset C$ is a Frobenius extension, and we denote by

$$R_s \stackrel{def}{=} \cdots \rightarrow 0 \rightarrow C \rightarrow C \otimes_{C^s} C \rightarrow 0 \rightarrow \cdots$$

the complex of Soergel bimodules over C known as *Rouquier complex*. Here $C \otimes_{C^s} C$ lives in cohomological degree 0, and the map is the unit of the adjunction between $\text{Res}_C^{C^s}$ and $C \otimes_{C^s}$. For a reduced expression $w = s_n \cdots s_1$ for $w \in \mathcal{W}_{\mathfrak{g}}$ we define a complex of Soergel bimodules by

$$R_w \stackrel{def}{=} R_{s_1} \otimes_C \cdots \otimes_C R_{s_n}.$$

In fact, up to homotopy, this complex does not depend on the choice of shortest expression, but this is not important for us. Also denote by

$$\text{Res} \stackrel{def}{=} \text{Res}_{C_1^\lambda}^{C_{\mathfrak{g}}^\lambda}$$

the morphism discussed in Section 2.3.2. Furthermore identify

$$C_1^{w \cdot \lambda} = \mathbb{S}(\mathfrak{h}) / (\mathbb{S}(\mathfrak{h})_+^{\mathcal{W}_i}) = C_1^\lambda$$

In the rest of this Chapter we will—among other things—prove that on the level of Soergel modules, parabolic induction for regular blocks

$$\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \text{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) \rightarrow \text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g}))$$

is given by the functor

$$\text{SInd}_w^\lambda \stackrel{def}{=} R_w \otimes_C \text{Res}(-)$$

Theorem 2.3.4.1. *Let $\lambda \in \mathfrak{h}^*$ be a regular integral dominant weight. Let w be a shortest coset representative in $\mathcal{W}_1 \setminus \mathcal{W}_{\mathfrak{g}}$ and $w = s_n \cdots s_1$ a reduced expression. Then the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \text{Hot}^b(C_1^{w \cdot \lambda}\text{-Smod}) & \xrightarrow{\text{SInd}_w^\lambda} & \text{Hot}^b(C_{\mathfrak{g}}^\lambda\text{-Smod}) \\ \wr \uparrow \mathbb{V}_1^{w \cdot \lambda} & & \mathbb{V}_{\mathfrak{g}}^\lambda \uparrow \wr \\ \text{Hot}^b(\text{Proj } \mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & & \text{Hot}^b(\text{Proj } \mathcal{O}_\lambda(\mathfrak{g})) \\ \wr \uparrow & & \wr \uparrow \\ \text{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g})). \end{array}$$

Proof. Postponed. □

The singular case

Now let $\mu \in \mathfrak{h}^*$ be a possibly **singular** integral dominant weight and let w be a shortest coset representative in $\mathcal{W}_1 \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}$, let $m = |\mathcal{W}_{1, w \cdot \mu}|$. Then there are natural maps

$$C_{\mathfrak{g}}^{\mu} \rightarrow C = C_{\mathfrak{g}}^{\lambda} \leftarrow C_{\mathfrak{t}}^{\lambda} = C_{\mathfrak{t}}^{w \cdot \lambda} \leftarrow C_{\mathfrak{t}}^{w \cdot \mu}$$

On the level of Soergel modules, parabolic induction for singular blocks (or rather an m -fold direct sum of it)

$$(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{\oplus m} : \text{Der}^b(\mathcal{O}_{w \cdot \mu}(\mathfrak{t})) \rightarrow \text{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g}))$$

is given by the functor

$$\overline{\text{SInd}}_w^{\mu} \stackrel{\text{def}}{=} \text{Res}_{C_{\mathfrak{g}}^{w \cdot \mu}} R_w \otimes_C C_{\mathfrak{t}}^{\lambda} \otimes_{C_{\mathfrak{t}}^{w \cdot \mu}} (-)$$

Theorem 2.3.4.2. *Let $\mu, \lambda \in \mathfrak{h}^*$ be integral dominant weights, where λ is furthermore regular. Let w be a shortest coset representative in $\mathcal{W}_1 \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \mu}$. Then the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \text{Hot}^b(C_{\mathfrak{t}}^{w \cdot \mu}\text{-Smod}) & \xrightarrow{\overline{\text{SInd}}_w^{\mu}} & \text{Hot}^b(C_{\mathfrak{g}}^{\mu}\text{-Smod}) \\ \wr \uparrow & & \wr \uparrow \\ \text{Der}^b(\mathcal{O}_{w \cdot \mu}(\mathfrak{t})) & \xrightarrow{(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{\oplus m}} & \text{Der}^b(\mathcal{O}_{\mu}(\mathfrak{g})). \end{array}$$

Proof. Follows from Theorem 2.3.4.1 and Theorem 2.2.6.1 using that under Soergel's functor \mathbb{V} , T_{μ}^{λ} corresponds to $\text{Res}_{C_{\mathfrak{g}}^{\mu}}$ and T_{λ}^{μ} to $C \otimes_{C_{\mathfrak{g}}^{w \cdot \mu}}$, see [Soe90, Theorem 10]. \square

Unfortunately, up until this point, we do not know how to get rid of the m -fold direct sum. In the case $w = e$ though, the following theorem gives a complete answer:

Theorem 2.3.4.3. *Let $w = e$. Then the following diagram commutes up to natural isomorphism.*

$$\begin{array}{ccc} C_{\mathfrak{t}}^{\mu}\text{-mod} & \xrightarrow{\text{Res}} & C_{\mathfrak{g}}^{\mu}\text{-mod} \\ \mathbb{V}_{\mathfrak{t}}^{\mu} \uparrow & & \uparrow \mathbb{V}_{\mathfrak{g}}^{\mu} \\ \mathcal{O}_{\mu}(\mathfrak{t}) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_{\mu}(\mathfrak{g}) \end{array}$$

Proof. Since $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is left adjoint, we have to use a different definition of Soergel's functor \mathbb{V} . By [Soe90, Lemma 9], there are equivalences of functors:

$$\begin{aligned}\mathbb{V}_{\mathfrak{g}}^{\mu} &= \text{Hom}_{\mathfrak{g}}(P_{\mathfrak{g}}^{\mu}, -) \cong d \text{Hom}_{\mathfrak{g}}(-, P_{\mathfrak{g}}^{\mu}) \\ \mathbb{V}_{\mathfrak{l}}^{\mu} &= \text{Hom}_{\mathfrak{l}}(P_{\mathfrak{l}}^{\mu}, -) \cong d \text{Hom}_{\mathfrak{l}}(-, P_{\mathfrak{l}}^{\mu}),\end{aligned}$$

where d denotes the duality. There are the following equivalences of functors $\mathcal{O}_{\mu}(\mathfrak{l}) \rightarrow \mathbb{C}\text{-mod}$:

$$\text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} -, P_{\mathfrak{g}}^{\mu}) \xrightarrow[\sim]{\text{Res}_{\mu}(-)} \text{Hom}_{\mathfrak{l}}(-, \text{Res}_{\mu}(P_{\mathfrak{g}}^{\mu})) = \text{Hom}_{\mathfrak{l}}(-, P_{\mathfrak{l}}^{\mu})$$

For the first equivalence we use the adjunction and $\text{Res}_{\mu}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} -) \cong \text{id}$ (Theorem 2.1.3.2). For the equality on the right we identify $P_{\mathfrak{l}}^{\mu} = \text{Res}_{\mu}(P_{\mathfrak{g}}^{\mu})$ which we are allowed to do by Lemma 2.3.2.5.

By Theorem 2.3.3.1 and Corollary 2.3.3.2 this promotes to an equivalence

$$\text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} -, P_{\mathfrak{g}}^{\mu}) \xrightarrow[\sim]{} \text{Res}_{C_{\mathfrak{l}}^{\mu}}^{C_{\mathfrak{g}}^{\mu}} \text{Hom}_{\mathfrak{l}}(-, P_{\mathfrak{l}}^{\mu})$$

of functors $\mathcal{O}_{\mu}(\mathfrak{l}) \rightarrow C_{\mathfrak{g}}^{\mu}\text{-mod}$. Now we dualize on both sides and obtain the statement. \square

Corollary 2.3.4.4. *Let $w = e$. The following diagram commutes up to natural isomorphism.*

$$\begin{array}{ccc} \text{Hot}^b(C_{\mathfrak{l}}^{\mu}\text{-Smod}) & \xrightarrow{\text{Res}} & \text{Hot}^b(C_{\mathfrak{g}}^{\mu}\text{-Smod}) \\ \uparrow \wr_{\mathbb{V}_{\mathfrak{l}}^{\mu}} & & \mathbb{V}_{\mathfrak{g}}^{\mu} \uparrow \wr \\ \text{Hot}^b(\text{Proj } \mathcal{O}_{\mu}(\mathfrak{l})) & & \text{Hot}^b(\text{Proj } \mathcal{O}_{\mu}(\mathfrak{g})) \\ \uparrow \wr & & \uparrow \wr \\ \text{Der}^b(\mathcal{O}_{\mu}(\mathfrak{l})) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \text{Der}^b(\mathcal{O}_{\mu}(\mathfrak{g})). \end{array}$$

Proof. Follows from Theorem 2.3.4.3, using that here $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ maps projectives to projectives by Theorem 2.1.3.2 (5) and hence acts on the homotopy categories of projectives by pointwise application. \square

Proof of the regular case

The proof mainly relies on Theorem 2.2.5.1 and Corollary 2.2.5.2 and an induction on $l(w)$.

Proof of Theorem 2.3.4.1. First assume that $l(w) = 0$, then $w = e$ and the statement is Corollary 2.3.4.4.

Now let $ws > w$ with both ws and w shortest representatives in $\mathcal{W}_1 \setminus \mathcal{W}_g$. Assuming that the statement holds for w , we show that it holds for ws .

Denote by Δ the equivalence between a derived category and the homotopy category of projectives. Let $M \in \text{Der}^b(\mathcal{O}_{ws \cdot \lambda}(\mathfrak{t}))$. Denote $\overline{M} = T_{ws \cdot \lambda}^{w \cdot \lambda} M$. We have the following diagram of distinguished triangles:

$$\begin{array}{ccccccc}
\mathbb{V}_g^\lambda \Delta \text{Ind}_p^g \overline{M} & \longrightarrow & \mathbb{V}_g^\lambda \Delta \theta_s \text{Ind}_p^g \overline{M} & \longrightarrow & \mathbb{V}_g^\lambda \Delta \text{Ind}_p^g M & \xrightarrow{+1} & \longrightarrow \\
\parallel & & (1) \downarrow \wr & & \parallel & & \\
\mathbb{V}_g^\lambda \Delta \text{Ind}_p^g \overline{M} & \xrightarrow{(*)} & C \otimes_{C^s} \mathbb{V}_g^\lambda \Delta \text{Ind}_p^g \overline{M} & \longrightarrow & \mathbb{V}_g^\lambda \Delta \text{Ind}_p^g M & \xrightarrow{+1} & \longrightarrow \\
(2) \downarrow \wr & & (2) \downarrow \wr & & \parallel & & \\
\text{SInd}_w^\lambda \mathbb{V}_1^{w \cdot \lambda} \Delta \overline{M} & \longrightarrow & C \otimes_{C^s} \text{SInd}_w^\lambda \mathbb{V}_1^{w \cdot \lambda} \Delta \overline{M} & \longrightarrow & \mathbb{V}_g^\lambda \Delta \text{Ind}_p^g M & \xrightarrow{+1} & \longrightarrow \\
(3) \parallel & & (3) \parallel & & \parallel & & \\
\text{SInd}_w^\lambda \mathbb{V}_1^{ws \cdot \lambda} \Delta M & \longrightarrow & C \otimes_{C^s} \text{SInd}_w^\lambda \mathbb{V}_1^{ws \cdot \lambda} \Delta M & \longrightarrow & \mathbb{V}_g^\lambda \Delta \text{Ind}_p^g M & \xrightarrow{+1} & \longrightarrow
\end{array}$$

The first triangle is given by Theorem 2.2.5.1.

(1) Since θ_s is exact and maps projectives to projectives, it commutes with Δ . On Soergel modules θ_s is given by $C \otimes_{C^s}$, see [Soe90, Korollar 1].

(2) This is the induction hypothesis.

(3) Recall that we identified $C_1^{ws \cdot \lambda} = \text{S}(\mathfrak{h}) / (\text{S}(\mathfrak{h})_+^{\mathcal{W}_1}) = C_1^{w \cdot \lambda}$. The functor $T_{ws \cdot \lambda}^{w \cdot \lambda}$ is an equivalence of categories and we have $\mathbb{V}_1^{ws \cdot \lambda} = \mathbb{V}_1^{w \cdot \lambda} T_{ws \cdot \lambda}^{w \cdot \lambda}$.

(*) This is given by the adjunction homomorphism by Theorem 2.2.5.1.

We hence have the following isomorphism

$$\begin{aligned}
\mathbb{V}_g^\lambda \Delta \text{Ind}_p^g M &\cong \text{Cone}(\text{SInd}_w^\lambda \mathbb{V}_1^{ws \cdot \lambda} \Delta M \rightarrow C \otimes_{C^s} \text{SInd}_w^\lambda \mathbb{V}_1^{ws \cdot \lambda} \Delta M) \\
&= R_s \otimes_C \text{SInd}_w^\lambda \mathbb{V}_1^{ws \cdot \lambda} M \\
&= \text{SInd}_{ws}^\lambda \mathbb{V}_1^{ws \cdot \lambda} M
\end{aligned}$$

where by Cone we denote the mapping cone. In order to show that this is indeed a *natural* isomorphism, we want to apply the following Lemma 2.3.4.5. We have to show that for $M, N \in \text{Der}^b(\mathcal{O}_{ws \cdot \lambda}(\mathfrak{t}))$

$$\text{Hom}_{\text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g}))}(\text{Ind}_p^g T_{ws \cdot \lambda}^{w \cdot \lambda} N, \text{Ind}_p^g M) = 0.$$

But this follows from

$$\text{Ext}^i(M_g(xw \cdot \lambda), M_g(yws \cdot \lambda)) = 0$$

for all $x, y \in \mathcal{W}_1$, $i \in \mathbb{Z}$, see [Hum98, Theorem 6.11]. \square

Lemma 2.3.4.5. *Let $F, G : \mathcal{T} \rightarrow \mathcal{T}'$ be triangulated functors between triangulated categories \mathcal{T} and \mathcal{T}' , and let $\phi : F \Rightarrow G$ be a morphism of functors. For $X \in \mathcal{T}$ abbreviate $C(X) \stackrel{\text{def}}{=} \text{Cone}(\phi_X : F(X) \rightarrow G(X))$. Assume that for $X, Y \in \mathcal{T}$ we have*

$$\text{Hom}_{\mathcal{T}'}(F(X)[1], C(Y)) = 0.$$

Then there exists (up to natural isomorphism) a unique functor $H : \mathcal{T} \rightarrow \mathcal{T}'$ and morphisms $G \rightarrow H \rightarrow F[1]$ which induce distinguished triangles

$$F(X) \xrightarrow{\phi_X} G(X) \longrightarrow H(X) \xrightarrow{+1}$$

for all $X \in \mathcal{T}$. In particular $H(X) \cong C(X)$.

Proof. Let $f : X \rightarrow Y$ be a morphism in \mathcal{T} . We claim that there is a *unique* morphism $C(f)$ making the following diagram commute:

$$\begin{array}{ccccc} F(X) & \xrightarrow{\phi_X} & G(X) & \xrightarrow{\psi_X} & C(X) & \xrightarrow{+1} \\ \downarrow F(f) & & \downarrow G(f) & & \downarrow \exists! C(f) & \\ F(Y) & \xrightarrow{\phi_Y} & G(Y) & \xrightarrow{\psi_Y} & C(Y) & \xrightarrow{+1} \end{array}$$

To see this, consider the long exact sequence:

$$\dots \rightarrow 0 = \text{Hom}_{\mathcal{T}'}(F(X)[1], C(Y)) \rightarrow \text{Hom}_{\mathcal{T}'}(C(X), C(Y)) \rightarrow$$

$$\xrightarrow{\psi_X^*} \text{Hom}_{\mathcal{T}'}(G(X), C(Y)) \xrightarrow{\phi_X^*} \text{Hom}_{\mathcal{T}'}(F(X), C(Y)) \rightarrow \dots$$

Then $\phi_X^*(\psi_Y G(f)) = \psi_Y G(f) \phi_X = \psi_Y \phi_Y F(f) = 0$. Hence $C(f)$ exists and is uniquely determined by the equation $C(f) \psi_X = \psi_X^*(C(f)) = \psi_Y G(f)$.

Moreover if $g : Y \rightarrow Z$ is a second morphism in \mathcal{T} , the uniqueness immediately implies $C(g)C(f) = C(gf)$. Hence C defines a functor with the required properties. The same uniqueness arguments show that C is uniquely determined with these properties (up to natural isomorphism). \square

Chapter 3

Geometric Parabolic Induction

3.1 Setup

3.1.1 Motivic sheaves and six functors

In this section we recall some of the notations and constructions regarding motivic sheaves used in [SW16]. We denote by \mathcal{T} the system of triangulated categories of motives constructed from the spectrum representing the semisimplification of de Rham cohomology, see [SW16, Section 2.4]. This is defined in [Dre15] and a *motivic triangulated category* in the sense of Cisinski–Déglise, [CD12].

\mathcal{T} associates to every complex variety $X \in \text{Var}(\mathbb{C})$ a triangulated \mathbb{C} -linear monoidal category $\mathcal{T}(X)$ and to every morphism $f : X \rightarrow Y$ a symmetric triangulated functor $f^* : \mathcal{T}(Y) \rightarrow \mathcal{T}(X)$. Denote the tensor unit in $\mathcal{T}(X)$ by $\mathbb{1}_X$, we will also refer to this as *constant motive* or *constant sheaf*.

Since this system of categories is a *motivic triangulated category* it comes with a *full six functor formalism*, whose most important properties we subsume in the following list, see [CD12].

1. For every morphism $f : Y \rightarrow X$ in $\text{Var}(k)$ the functor f^* has a right adjoint the **direct image functor** f_*

$$f^* : \mathcal{T}(X) \rightleftarrows \mathcal{T}(Y) : f_*$$

2. For any morphism $f : Y \rightarrow X$ in $\text{Var}(\mathbb{C})$, one can construct a further pair of adjoint functors, the **exceptional functors**

$$f_! : \mathcal{T}(Y) \rightleftarrows \mathcal{T}(X) : f^!$$

which fit together to form a covariant (respectively contravariant) 2-functor $f \mapsto f_!$ (resp. $f \mapsto f^!$).

3. For each $X \in \mathbf{Var}(k)$, the tensor structure on $\mathcal{T}(X)$ is closed in the sense that for every $E \in \mathcal{T}(X)$, the functor $- \otimes E$ has a right adjoint

$$- \otimes E : \mathcal{T}(X) \rightleftarrows \mathcal{T}(X) : \mathcal{H}om_X(E, -),$$

the **internal Hom functor**.

4. **(Stability)** For every $X \in \mathbf{Var}(\mathbb{C})$, let $p : \mathbb{A}_X^1 \rightarrow X$ be the canonical projection with zero section s . Then the endofunctor

$$s^!p^* : \mathcal{T}(X) \rightarrow \mathcal{T}(X)$$

is invertible. For $E \in \mathcal{T}(X)$ and $n \in \mathbb{Z}$ we denote

$$E(n) := (s^!p^*)^n(E)[-2n]$$

the n -th **Tate twist** of E .

5. With X and p as above, \mathcal{T} satisfies **\mathbb{A}^1 -homotopy invariance** in the sense that the unit of the adjunction (p^*, p_*) is an isomorphism.

$$\mathrm{id} \xrightarrow{\sim} p_*p^*.$$

6. For any $f : Y \rightarrow X$ in $\mathbf{Var}(\mathbb{C})$ there exists a natural transformation

$$f_! \rightarrow f_*$$

which is an isomorphism when f is proper.

7. **(Relative purity)** For any smooth morphism $f : Y \rightarrow X$ in $\mathbf{Var}(k)$ of relative dimension d there is a canonical isomorphism

$$f^* \rightarrow f^!(-d)[-2d].$$

8. **(Base change)** For any Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

there exist natural isomorphisms of functors

$$g^*f_! \xrightarrow{\sim} f'_!g'^*, \quad g'_*f'^! \xrightarrow{\sim} f^!g_*,$$

9. (**Localization**) For $i : Z \rightarrow X$ a closed immersion with open complement $j : U \rightarrow X$, there are distinguished triangles

$$j_!j^! \rightarrow 1 \rightarrow i_*i^* \rightarrow j_!j^![1]$$

$$i_!i^! \rightarrow 1 \rightarrow j_*j^* \rightarrow i_!i^![1]$$

where the first and second maps are the counits and units of the respective adjunctions.

10. (**Projection formulae, Verdier duality**) For any morphism $f : Y \rightarrow X$ in $\text{Var}(\mathbb{C})$, there exist natural isomorphisms

$$\begin{aligned} (f_!E) \otimes_X F &\xrightarrow{\sim} f_!(E \otimes_Y f^*F), \\ \mathcal{H}om_X(E, f_*F) &\xrightarrow{\sim} f_*\mathcal{H}om_Y(f^*E, F), \\ \mathcal{H}om_X(f_!E, F) &\xrightarrow{\sim} f_*\mathcal{H}om_Y(E, f^!F), \\ f^!\mathcal{H}om_X(E, F) &\xrightarrow{\sim} \mathcal{H}om_Y(f^*E, f^!F). \end{aligned}$$

11. Define the subcategory of **constructible** objects $\mathcal{T}^c(S) \subset \mathcal{T}(S)$ to be the subcategory of compact objects. This subcategory coincides with the thick full subcategory generated by $f_!f^!\mathbb{1}(n)$ for $n \in \mathbb{Z}$ and $f : X \rightarrow S$ smooth. The six functors $f_!, f^!, f^*, f_*, \otimes, \mathcal{H}om$ preserve compact objects.
12. Let $f : X \rightarrow \text{Spec}(\mathbb{C})$ in $\text{Var}(\mathbb{C})$. For $E \in \mathcal{T}(X)$ we denote by

$$\mathbb{D}_X(E) := \mathcal{H}om_X(E, f^!(\mathbb{1}))$$

the **Verdier dual** of E . For all $E, F \in \mathcal{T}^c(X)$, there is a canonical duality isomorphism

$$\mathbb{D}_X(E \otimes \mathbb{D}_X(F)) \xrightarrow{\sim} \mathcal{H}om_X(E, F).$$

Furthermore, for any morphism $f : Y \rightarrow X$ in $\text{Var}(\mathbb{C})$ and any $E \in \mathcal{T}^c(X)$ there are natural isomorphisms

$$\begin{aligned} \mathbb{D}_X(\mathbb{D}_X(E)) &\cong E, \\ \mathbb{D}_Y(f^*(E)) &\cong f^!(\mathbb{D}_X(E)), \\ \mathbb{D}_X(f_!(E)) &\cong f_*(\mathbb{D}_Y(E)). \end{aligned}$$

Furthermore, due to the particular construction of the categories $\mathcal{T}(X)$, we have the following additional properties.

13. (**Grading condition**) There are no extensions between the Tate motives on \mathbb{A}^n , or in formulas

$$\mathrm{Hom}_{\mathcal{T}(\mathbb{A}^n)}(\mathbb{1}, \mathbb{1}(n)[m]) = \begin{cases} \mathbb{C}, & \text{for } n = m = 0 \\ 0, & \text{else.} \end{cases}$$

14. (**Realization functor**) For every $X \in \mathrm{Var}(\mathbb{C})$ there is a realization functor

$$\mathrm{Real} : \mathcal{T}(X) \rightarrow \mathrm{Der}^b(X(\mathbb{C}); \mathbb{C})$$

into the bounded derived category of sheaves on $X(\mathbb{C})$ equipped with the metric topology. The realization functor is triangulated, monoidal and compatible with the six functors.

3.1.2 Stratified mixed Tate motives

The categories $\mathcal{T}(X)$, similarly to the derived category of sheaves on $X(\mathbb{C})$, are gigantic. We will restrict ourselves to the analogue to constructible sheaves in our setting, namely to *stratified mixed Tate motives* as introduced in [SW16]. We will recall all important definitions and properties in this section.

Definitions

Let (X, S) be an affinely stratified variety over \mathbb{C} , i.e. a variety X with a finite partition into locally closed subvarieties (called the strata of X)

$$X = \bigcup_{s \in S} X_s,$$

such that each stratum X_s is isomorphic to \mathbb{A}^n for some n , and the closure \overline{X}_s is a union of strata. The embeddings are denoted by $j_s : X_s \hookrightarrow X$. The prime example we always have in mind here is the flag variety of a reductive group with its Bruhat stratification. Starting from this datum, [SW16] defines the category of *stratified mixed Tate motives* on X , which we recall in this paragraph. We start with the basic case of just one stratum.

Definition 3.1.2.1 ([SW16] 3.1). *For $X \cong \mathbb{A}^n$, denote by $\mathrm{MTDer}(X)$ the full triangulated subcategory of $\mathcal{T}(X)$ generated by motives isomorphic to $\mathbb{1}_X(p)$ for $p \in \mathbb{Z}$. Recall that by $\mathbb{1}_X$ we denote the tensor unit in $\mathcal{T}(X)$.*

By the grading property we get the following.

Proposition 3.1.2.2. *For $X \cong \mathbb{A}^n$, we have the following equivalence of monoidal \mathbb{C} -linear categories:*

$$\mathrm{MTDer}(X) \cong \mathbb{C}\text{-mod}^{\mathbb{Z} \times \mathbb{Z}} \cong \mathrm{Der}^b(\mathbb{C}\text{-mod}^{\mathbb{Z}}).$$

We can now proceed to the general case. Since our category should be closed under taking Verdier duals and other reasonable combinations of the six functors, we have to assume that (X, S) fulfills an additional condition:

Definition 3.1.2.3 ([SW16] 4.5). *(X, S) is called Whitney-Tate if for all $s, t \in S$ and $M \in \mathrm{MTDer}(X_s)$ we have $j_t^* j_{s*} M \in \mathrm{MTDer}(X_t)$.*

From now on we always assume that (X, S) is Whitney-Tate. In [SW16] it is shown that (partial) flag varieties and other examples are indeed Whitney-Tate.

Definition 3.1.2.4 ([SW16] 4.7). *The category of stratified mixed Tate motives on X , denoted by $\mathrm{MTDer}_S(X)$, is the full subcategory of $\mathcal{T}(X)$ consisting of objects M such that $j_s^* M \in \mathrm{MTDer}(X_s)$ for all $s \in S$.*

The right definition of a map between affinely stratified varieties is different to the usual definition of a stratified map, as defined for example in [GM88].

Definition 3.1.2.5. *Let (X, S) and (Y, S') be affinely stratified varieties. We call $f : X \rightarrow Y$ an affinely stratified map if*

1. *for all $s \in S'$ the inverse image $f^{-1}(Y_s)$ is a union of strata;*
2. *for each X_s mapping into $Y_{s'}$, the induced map $f : X_s \rightarrow Y_{s'}$ is a projection $\mathbb{A}^n \times \mathbb{A}^m \rightarrow \mathbb{A}^m$.*

Stratified mixed Tate motives are compatible with functors induced from affinely stratified maps.

Lemma 3.1.2.6. *Let $X \in \mathrm{Var}(\mathbb{C})$. Consider*

$$s : X \rightleftarrows \mathbb{A}_X^n : p$$

where p denotes the projection and s the zero section. Then

$$\begin{aligned} p_*(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X & p!(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X(-n)[-2n] \\ p^*(\mathbb{1}_X) &= \mathbb{1}_{\mathbb{A}_X^n} & p'(\mathbb{1}_X) &= \mathbb{1}_{\mathbb{A}_X^n}(n)[2n] \\ s^*(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X & s'(\mathbb{1}_{\mathbb{A}_X^n}) &= \mathbb{1}_X(-n)[-2n] \end{aligned}$$

Furthermore $\mathbb{D}_X(\mathbb{1}_X(m)[2m]) = \mathbb{1}_X(\dim X - m)[2 \dim X - 2m]$ if X is smooth.

Proposition 3.1.2.7. *Let (X, S) and (Y, S') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ an affinely stratified map. Then the induced functors restrict to stratified mixed Tate motives on X and Y . In formulas*

$$f_*, f^! : \text{MTDer}_S(X) \rightleftarrows \text{MTDer}_{S'}(Y) : f^*, f^!$$

Also the internal Hom, duality and tensor product restrict.

Proof. [EK16, Proposition 3.8] □

Weight structure

Weight structures—as first considered in [Bon10]—provide a very concise framework for the powerful *yoga of weights*, as applied, for example, in the proof of the Weil conjectures or the decomposition theorem for perverse sheaves.

Definition 3.1.2.8. *Let \mathcal{C} be a triangulated category. A weight structure on \mathcal{C} is a pair $(\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ of full subcategories of \mathcal{C} such that with $\mathcal{C}_{w \leq n} := \mathcal{C}_{w \leq 0}[n]$ and $\mathcal{C}_{w \geq n} := \mathcal{C}_{w \geq 0}[n]$ the following conditions are satisfied:*

1. $\mathcal{C}_{w \leq 0}$ and $\mathcal{C}_{w \geq 0}$ are closed under direct summands;
2. $\mathcal{C}_{w \leq 0} \subseteq \mathcal{C}_{w \leq 1}$ and $\mathcal{C}_{w \geq 1} \subseteq \mathcal{C}_{w \geq 0}$;
3. for all $X \in \mathcal{C}_{w \leq 0}$ and $Y \in \mathcal{C}_{w \geq 1}$, we have $\text{Hom}_{\mathcal{C}}(X, Y) = 0$
4. for any $X \in \mathcal{C}$ there is a distinguished triangle $A \rightarrow X \rightarrow B \xrightarrow{+1}$ with $A \in \mathcal{C}_{w \leq 0}$ and $B \in \mathcal{C}_{w \geq 1}$

The full subcategory $\mathcal{C}_{w=0} = \mathcal{C}_{w \leq 0} \cap \mathcal{C}_{w \geq 0}$ is called the heart of the weight structure.

A weight structure on stratified mixed Tate motives on an affinely stratified variety can be obtained by gluing of weight structures on the strata. The motive $\mathbb{1}_{\mathbb{A}^n}(p)[q]$ is defined to have weight $q - 2p$.

Definition 3.1.2.9. *Let $\text{MTDer}(\mathbb{A}^n)_{w \leq 0}$ (resp. $\text{MTDer}(\mathbb{A}^n)_{w \geq 0}$) be the full subcategory of $\text{MTDer}(\mathbb{A}^n)$ consisting of objects isomorphic to finite direct sums of $\mathbb{1}_{\mathbb{A}^n}(p)[q]$ for $q \leq 2p$ ($q \geq 2p$). This defines a weight structure on $\text{MTDer}(\mathbb{A}^n)$.*

Proof. We use Proposition 3.1.2.2 to identify $\text{MTDer}(\mathbb{A}^n)$ with the derived category of graded vector spaces. Here the axioms of a weight structure are easily checked. □

Definition 3.1.2.10 ([SW16] 5.1). *Let (X, S) be an affinely Whitney-Tate stratified variety. Then we obtain a weight structure on $\mathrm{MTDer}_S(X)$ by setting*

$$\begin{aligned} \mathrm{MTDer}_S(X)_{w \leq 0} &:= \{M \mid j_s^* M \in \mathrm{MTDer}(X_s)_{w \leq 0} \text{ for all } s \in S\} \\ \mathrm{MTDer}_S(X)_{w \geq 0} &:= \{M \mid j_s^! M \in \mathrm{MTDer}(X_s)_{w \geq 0} \text{ for all } s \in S\} \end{aligned}$$

With this definition we have the following compatibilities with the six functors.

Proposition 3.1.2.11 ([EK16] 3.12). *Let (X, S) and (Y, S') be affinely Whitney-Tate stratified varieties and $f : X \rightarrow Y$ an affinely stratified map. Then*

1. *the functors $f^*, f_!$ are weight left exact, i.e. they preserve $w \leq 0$;*
2. *the functors $f^!, f_*$ are weight right exact, i.e. they preserve $w \geq 0$;*
3. *the tensor product is weight left exact, i.e. restricts to*

$$\mathrm{MTDer}_S(X)_{w \leq n} \times \mathrm{MTDer}_S(X)_{w \leq m} \rightarrow \mathrm{MTDer}_S(X)_{w \leq n+m}$$

4. *Verdier duality reverses weights, i.e. restricts to*

$$\mathbb{D}_X : \mathrm{MTDer}_S(X)_{w \leq n}^{\mathrm{op}} \rightarrow \mathrm{MTDer}_S(X)_{w \geq -n}$$

5. *the internal Hom functor $\mathcal{H}om_X$ is weight right exact, i.e. restricts to*

$$\mathrm{MTDer}_S(X)_{w \leq n}^{\mathrm{op}} \times \mathrm{MTDer}_S(X)_{w \geq m} \rightarrow \mathrm{MTDer}_S(X)_{w \geq m-n}$$

6. *For f smooth $f^!$ and f^* are weight exact;*
7. *For f proper $f_!$ and f_* are weight exact;*
8. *If X is smooth $\mathbb{1}_X(n)[2n]$ is of weight zero for all $n \in \mathbb{Z}$.*

Tilting

We will now recall the formalism of tilting for derived (dg)-categories, as introduced in [Ric89], [Kel93] and [Kel94].

Definition 3.1.2.12. *Let \mathcal{A} be an abelian category. A complex $I \in \mathrm{Hot}(\mathcal{A})$ is called **homotopy-injective** if the natural map*

$$\mathrm{Hom}_{\mathrm{Hot}(\mathcal{A})}(A, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Der}(\mathcal{A})}(A, I)$$

is an isomorphism for all $A \in \text{Hot}(\mathcal{A})$. A complex $P \in \text{Hot}(\mathcal{A})$ is called **homotopy-projective** if the natural map

$$\text{Hom}_{\text{Hot}(\mathcal{A})}(P, A) \xrightarrow{\sim} \text{Hom}_{\text{Der}(\mathcal{A})}(P, A)$$

is an isomorphism for all $A \in \text{Hot}(\mathcal{A})$.

Definition 3.1.2.13. Let \mathcal{A} be an abelian category. A collection $\{T_i\}$ of complexes $T_i \in \text{Hot}(\mathcal{A})$ is called **tilting** if for all i, j and $n \in \mathbb{Z}$ the natural map

$$\text{Hom}_{\text{Hot}(\mathcal{A})}(T_i, T_j[n]) \xrightarrow{\sim} \text{Hom}_{\text{Der}(\mathcal{A})}(T_i, T_j[n])$$

is an isomorphism and

$$\text{Hom}_{\text{Der}(\mathcal{A})}(T_i, T_j[n]) = 0$$

for all $n \neq 0$.

For complexes $M, N \in \text{Hot}(\mathcal{P})$ in some additive category \mathcal{P} , we denote by $\text{Hom}_{\mathcal{P}}(M, N) \in \text{Hot}(\mathcal{P})$ their Hom-complex.

Theorem 3.1.2.14 (Tilting [Kel93]). Let \mathcal{A} be an abelian category and $\{T_i\}$ a tilting collection. Then there is an equivalence of triangulated categories

$$\Delta : \text{Hot}^b(\langle \{T_i\} \rangle_{\oplus}^{\text{Der}(\mathcal{A})}) \xrightarrow{\sim} \langle \{T_i\} \rangle_{\Delta}^{\text{Der}(\mathcal{A})} \subset \text{Der}(\mathcal{A})$$

called **tilting**. Here by $\langle - \rangle_{\oplus}^{\mathcal{B}}$ we denote closure under finite direct sums in an additive category \mathcal{B} and by $\langle - \rangle_{\Delta}^{\mathcal{B}}$ closure under distinguished triangles in a triangulated category \mathcal{B} .

Proof. This is copied almost word by word from [SW16, Appendix B]. We just sketch a proof for $\{T_i\} = \{T\}$. Since by assumption

$$\text{Hom}_{\text{Hot}(\mathcal{A})}(T, T[n]) \xrightarrow{\sim} \text{Hom}_{\text{Der}(\mathcal{A})}(T, T[n])$$

we have by dévissage

$$\langle T \rangle_{\oplus}^{\text{Der}} \cong \langle T \rangle_{\oplus}^{\text{Hot}} \quad \text{and} \quad \langle T \rangle_{\Delta}^{\text{Der}} \cong \langle T \rangle_{\Delta}^{\text{Hot}}.$$

So it suffices to proof that there is an equivalence

$$\Delta : \text{Hot}^b(\langle T \rangle_{\oplus}^{\text{Hot}(\mathcal{A})}) \xrightarrow{\sim} \langle T \rangle_{\Delta}^{\text{Hot}(\mathcal{A})}.$$

Let $E \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{A}}(T, T)$ be the endomorphism complex of T . This is a differential graded algebra (dg-algebra). Let $Z \stackrel{\text{def}}{=} \mathcal{Z}^0(E) \oplus E^{\leq 0}$ be the truncation

of E and $H \stackrel{\text{def}}{=} \mathcal{H}^0(E)$ be the 0-th cohomology of E . By the tilting property the cohomology of E is concentrated in degree zero and hence the natural morphisms

$$E \leftarrow Z \rightarrow H$$

are quasi-isomorphisms of dg-algebras and furthermore

$$H = \text{Hom}_{\text{Hot}(\mathcal{A})}(T, T) \cong \text{Hom}_{\text{Der}(\mathcal{A})}(T, T).$$

For a dg-algebra R we denote by

$$\text{dgHot-}R \supset \text{dgFree-}R$$

the homotopy category of right R -dg-modules and the triangulated subcategory generated by the free module R . Then there is the following chain of equivalences of triangulated categories

$$\text{Hot}^b(\langle T \rangle_{\oplus}^{\text{Hot}(\mathcal{A})}) \xrightarrow{\sim (1)} \text{dgFree-}H \xleftarrow{\sim (2)} \text{dgFree-}Z \xrightarrow{\sim (3)} \text{dgFree-}E \xleftarrow{\sim (4)} \langle T \rangle_{\Delta}^{\text{Hot}(\mathcal{A})}$$

1. The following functor induces an equivalence of categories

$$\text{Hom}_{\text{Hot}(\mathcal{A})}(T, -) : \langle T \rangle_{\oplus}^{\text{Hot}(\mathcal{A})} \xrightarrow{\sim} \langle H \rangle_{\oplus}^{\text{mod-}H}.$$

Since H is a dg-algebra concentrated in degree 0, dg-modules over H are just complexes of H -modules and we have an equivalence

$$\text{Hot}^b(\langle T \rangle_{\oplus}^{\text{Hot}(\mathcal{A})}) \xrightarrow{\sim} \text{Hot}^b(\langle H \rangle_{\oplus}^{\text{mod-}H}) = \text{dgFree-}H.$$

2. This is $- \otimes_Z H$, which is an equivalence since $Z \rightarrow H$ is a quasi-isomorphism.
3. This is $- \otimes_Z E$, which is an equivalence since $Z \hookrightarrow E$ is a quasi-isomorphism.
4. This is given by the functor

$$\text{Hom}_{\mathcal{A}}(T, -) : \langle T \rangle_{\Delta}^{\text{Hot}(\mathcal{A})} \xrightarrow{\sim} \text{dgFree-}E.$$

Our tilting functor Δ is defined as the composition of those equivalences (or their inverse functors). \square

Remark 3.1.2.15. We could have also used the functors $\text{Hom}_{\text{Hot } \mathcal{A}}(-, T)$ and $\text{Hom}_{\mathcal{A}}(-, T)$. Then our tilting equivalence would be of the form

$$\begin{array}{ccc}
\mathrm{Hot}^b(\langle T \rangle_{\oplus}^{\mathrm{Hot}(\mathcal{A})}) & \xrightarrow{\sim} & (H\text{-dgFree})^{\mathrm{op}} \\
& \nearrow \sim & \\
(Z\text{-dgFree})^{\mathrm{op}} & \xrightarrow{\sim} & (E\text{-dgFree})^{\mathrm{op}} \xleftarrow{\sim} \langle T \rangle_{\Delta}^{\mathrm{Hot}(\mathcal{A})}
\end{array}$$

But we rather avoid opposite categories.

Proposition 3.1.2.16 (Tilting and functors). *Let \mathcal{A}, \mathcal{B} be abelian categories. Let $\{T_i\} \subset \mathrm{Hot}(\mathcal{A})$ and $\{U_i\} \subset \mathrm{Hot}(\mathcal{B})$ be tilting collections. Assume furthermore that all U_i are homotopy-projective.*

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor. Assume that for all T_i there exists a U_j and a quasi-isomorphism $c_i : U_j \rightarrow F(T_i)$. Then the following diagram commutes up to natural isomorphism

$$\begin{array}{ccccc}
\mathrm{Hot}^b(\langle \{T_i\} \rangle_{\oplus}^{\mathrm{Der}(\mathcal{A})}) & \xrightarrow{\sim \Delta} & \langle \{T_i\} \rangle_{\Delta}^{\mathrm{Der}(\mathcal{A})} & \hookrightarrow & \mathrm{Der}(\mathcal{A}) \\
\downarrow F & & \downarrow F & & \downarrow F \\
\mathrm{Hot}^b(\langle \{U_i\} \rangle_{\oplus}^{\mathrm{Der}(\mathcal{B})}) & \xrightarrow{\sim \Delta} & \langle \{U_i\} \rangle_{\Delta}^{\mathrm{Der}(\mathcal{B})} & \hookrightarrow & \mathrm{Der}(\mathcal{B})
\end{array}$$

where on the left side F acts by pointwise application or in other words, it is induced by an sequence of functors

$$\langle \{T_i\} \rangle_{\oplus}^{\mathrm{Der}(\mathcal{A})} \xleftarrow{\sim} \langle \{T_i\} \rangle_{\oplus}^{\mathrm{Hot}(\mathcal{A})} \xrightarrow{F} \langle \{F(T_i)\} \rangle_{\oplus}^{\mathrm{Hot}(\mathcal{B})} \rightarrow \langle \{F(T_i)\} \rangle_{\oplus}^{\mathrm{Der}(\mathcal{B})} \hookrightarrow \langle \{U_i\} \rangle_{\oplus}^{\mathrm{Der}(\mathcal{B})}$$

Proof. This is copied almost word by word from [SVW, Appendix]. Again, we restrict ourselves to the case $\{T_i\} = \{T\}$ and $\{U_i\} = \{U\}$ and a quasi-isomorphism $c : U \rightarrow F(T)$. We abbreviate $S = F(T)$ and $E_T = \mathrm{Hom}_{\mathcal{A}}(T, T)$, $E_S = \mathrm{Hom}_{\mathcal{B}}(S, S)$, $E_U = \mathrm{Hom}_{\mathcal{B}}(U, U)$. Let Z_T, Z_S, Z_U and H_T, H_S, H_U their degree 0 truncation and 0-th cohomology, respectively.

We consider a diagram where the top and bottom row are defining the tilting equivalences Δ as in Theorem 3.1.2.14 and fill it up with natural isomorphisms.

$$\begin{array}{ccccccccc}
\text{Hot}^b(\langle T \rangle_{\oplus}^{\text{Hot}(\mathcal{A})}) & \xrightarrow{\sim} & \text{Hom}_{\text{Hot}(\mathcal{A})}(T, -) & \xrightarrow{\sim} & \text{dgFree-} H_T & \xleftarrow{\sim} & \text{dgFree-} Z_T & \xrightarrow{\sim} & \text{dgFree-} E_T & \xleftarrow{\sim} & \text{Hom}_{\mathcal{A}}(T, -) & \xrightarrow{\sim} & \langle T \rangle_{\Delta}^{\text{Hot}(\mathcal{A})} \\
\downarrow F & & (1) \quad -\otimes_{H_T} H_S & & \downarrow & & (2) \quad -\otimes_{Z_T} Z_S & & \downarrow & & (3) \quad -\otimes_{E_T} E_S & & (4) & \downarrow F \\
\text{Hot}^b(\langle S \rangle_{\oplus}^{\text{Hot}(\mathcal{A})}) & \xrightarrow{\sim} & \text{Hom}_{\text{Hot}(\mathcal{B})}(S, -) & \xrightarrow{\sim} & \text{dgFree-} H_S & \xleftarrow{\sim} & \text{dgFree-} Z_S & \xrightarrow{\sim} & \text{dgFree-} E_S & \xleftarrow{\sim} & \text{Hom}_{\mathcal{B}}(S, -) & \xrightarrow{\sim} & \langle S \rangle_{\Delta}^{\text{Hot}(\mathcal{B})} \\
\downarrow & & (5) \quad -\otimes_{H_S} H_X & & \downarrow & & (6) \quad -\otimes_{Z_S} Z_X & & \downarrow & & (7) \quad -\otimes_{E_S} X & & (8) & \downarrow \\
\text{Hot}^b(\langle U \rangle_{\oplus}^{\text{Der}(\mathcal{B})}) & \xrightarrow{\sim} & \text{Hom}_{\text{Der}(\mathcal{B})}(U, -) & \xrightarrow{\sim} & \text{dgFree}^{\text{Der-}} H_U & \xleftarrow{\sim} & \text{dgFree}^{\text{Der-}} Z_U & \xrightarrow{\sim} & \text{dgFree}^{\text{Der-}} E_U & \xleftarrow{\sim} & \text{Hom}_{\mathcal{B}}(U, -) & \xrightarrow{\sim} & \langle U \rangle_{\Delta}^{\text{Der}(\mathcal{B})} \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & \\
\text{Hot}^b(\langle U \rangle_{\oplus}^{\text{Hot}(\mathcal{B})}) & \xrightarrow{\sim} & \text{Hom}_{\text{Hot}(\mathcal{B})}(U, -) & \xrightarrow{\sim} & \text{dgFree-} H_U & \xleftarrow{\sim} & \text{dgFree-} Z_U & \xrightarrow{\sim} & \text{dgFree-} E_U & \xleftarrow{\sim} & \text{Hom}_{\mathcal{B}}(U, -) & \xrightarrow{\sim} & \langle U \rangle_{\Delta}^{\text{Hot}(\mathcal{B})}
\end{array}$$

1. Here we use the map

$$F : H_T = \text{Hom}_{\text{Hot}(\mathcal{A})}(T, T) \rightarrow \text{Hom}_{\text{Hot}(\mathcal{B})}(F(T), F(T)) = H_S.$$

The natural transformation is given by

$$\text{Hom}_{\text{Hot}(\mathcal{A})}(T, -) \otimes_{H_T} H_S \xrightarrow{\text{comp}(F \otimes \text{id})} \text{Hom}_{\text{Hot}(\mathcal{B})}(F(T), F(-))$$

where by comp we denote composition. This is clearly an isomorphism when applied to T and hence restricts to a natural isomorphism by devissage.

2. The morphism $Z_T \rightarrow Z_S$ is induced by

$$F : E_T = \text{Hom}_{\mathcal{A}}(T, T) \rightarrow \text{Hom}_{\mathcal{B}}(F(T), F(T)) = E_S.$$

The following diagram commutes.

$$\begin{array}{ccc}
Z_T & \longrightarrow & H_T \\
\downarrow F & & \downarrow F \\
Z_S & \longrightarrow & H_S
\end{array}$$

We hence get a natural isomorphism

$$-\otimes_{Z_T} H_T \otimes_{H_T} H_S \cong -\otimes_{Z_T} Z_S \otimes_{Z_S} H_S.$$

3. As in the last point.
4. The natural transformation is given by

$$\mathrm{Hom}_{\mathcal{A}}(T, -) \otimes_{E_T} E_S \xrightarrow{\mathrm{comp}(F \otimes \mathrm{id})} \mathrm{Hom}_{\mathcal{B}}(F(T), F(-)).$$

Again this restricts to a natural isomorphism by devissage.

For a dg-algebra R we denote by $\mathrm{dgFree}^{\mathrm{Der}}\text{-}R$ the full triangulated subcategory of $\mathrm{dgDer}\text{-}R$ generated by the free module R . Localization induces an equivalence of categories

$$\mathrm{dgFree}\text{-}R \xrightarrow{\sim} \mathrm{dgFree}^{\mathrm{Der}}\text{-}R.$$

We denote $X = \mathrm{Hom}_{\mathcal{B}}(U, S)$. This is a $E_S\text{-}E_U$ -dg-bimodule. We denote by Z_X and H_X its degree 0 truncation and 0-th cohomology, respectively. Since U is homotopy-projective by assumption, the cohomology of X is concentrated in degree 0 and

$$H_X = \mathrm{Hom}_{\mathrm{Hot}\mathcal{B}}(U, S) = \mathrm{Hom}_{\mathrm{Der}\mathcal{B}}(U, S) \xrightarrow[\sim]{\xi^*} \mathrm{Hom}_{\mathrm{Der}\mathcal{B}}(U, U) = H_U$$

is freely generated by $[c]$ as right (dg-)module over H_U . Furthermore, the maps

$$H_X \leftarrow Z_X \hookrightarrow X$$

are quasi-isomorphisms of $Z_S\text{-}Z_U$ -dg-bimodules.

5. The left vertical arrow is defined via

$$\langle S \rangle_{\oplus}^{\mathrm{Hot}(\mathcal{B})} \rightarrow \langle S \rangle_{\oplus}^{\mathrm{Der}(\mathcal{B})} = \langle U \rangle_{\oplus}^{\mathrm{Der}(\mathcal{B})}.$$

The right vertical arrow restricts to $\mathrm{dgFree}^{\mathrm{Der}}\text{-}H_U$ since H_X is free. The natural isomorphism is defined similarly as in (1) by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Hot}(\mathcal{B})}(S, -) \otimes_{H_S} H_X &\xrightarrow{\mathrm{comp}} \mathrm{Hom}_{\mathrm{Hot}(\mathcal{B})}(U, -) \\ &= \mathrm{Hom}_{\mathrm{Der}(\mathcal{B})}(U, -). \end{aligned}$$

6. The right vertical and bottom horizontal arrow restrict to $\mathrm{dgFree}^{\mathrm{Der}}\text{-}Z_U$, since H_U and Z_X are quasi-isomorphic to Z_U as Z_U -dg-modules. It is easy to see that the following diagram commutes up to natural isomorphism.

$$\begin{array}{ccc}
\mathrm{dgFree}\text{-}H_S & \xleftarrow{-\otimes_{Z_S} H_S} & \mathrm{dgFree}\text{-}Z_S \\
\downarrow -\otimes_{H_S} H_X & & \downarrow -\otimes_{Z_S} Z_X \\
\langle H_X \rangle_{\Delta}^{\mathrm{dgHot}\text{-}H_U} & \xleftarrow{-\otimes_{Z_S} H_S} & \langle Z_X \rangle_{\Delta}^{\mathrm{dgHot}\text{-}Z_U} \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{dgFree}^{\mathrm{Der}}\text{-}H_U & \xrightarrow{\mathrm{Res}_{Z_U}^{H_U}} & \mathrm{dgFree}^{\mathrm{Der}}\text{-}Z_U
\end{array}$$

Here we use $\mathrm{Res}_{Z_U}^{H_U}$, which goes in the wrong direction, to avoid using its inverse functor, which is the derived functor $-\otimes_{Z_U}^{\mathbb{L}} H_U$.

7. As in the last point.
8. Similar to (5).

The squares in the bottom are easily filled with natural isomorphisms and we are finished with this proof. \square

Remark 3.1.2.17. The completely dual statements hold when we instead require homotopy-injective resolutions $c : F(T) \rightarrow U$ and use the tilting equivalence discussed in Remark 3.1.2.15.

Tilting and motives

We apply the tilting formalism to stratified mixed Tate motives. Under an additional *pointwise purity* condition this allows us to identify the category of stratified mixed Tate motives with the homotopy category of its weight zero objects.

Definition 3.1.2.18 ([SW16] 6.1). *Let (X, S) be an affinely Whitney-Tate stratified variety and $? \in \{*, !\}$. A motive $M \in \mathrm{MTDer}_S(X)$ is called pointwise $? \text{-pure}$ if for all $s \in S$*

$$i_s^? M \in \mathrm{MTDer}(X_s)_{w=0}.$$

If both conditions are satisfied, the motive is called pointwise pure.

Proposition 3.1.2.19 ([SW16] 6.3). *Let (X, S) be an affinely Whitney-Tate stratified variety and $M, N \in \mathrm{MTDer}_S(X)$ such that M is pointwise $* \text{-pure}$ and N is pointwise $! \text{-pure}$. Then $\mathrm{Hom}_{\mathcal{T}(X)}(M, N[a]) = 0$ for all $a > 0$.*

Theorem 3.1.2.20 (Tilting for motives, [SW16] 9.2). *Let (X, S) be an affinely Whitney-Tate stratified variety, such that all objects of $\mathrm{MTDer}_S(X)_{w=0}$ are additionally pointwise pure. Then there is an equivalence of categories, called tilting,*

$$\Delta : \mathrm{MTDer}_S(X) \xrightarrow{\sim} \mathrm{Hot}^b(\mathrm{MTDer}_S(X)_{w=0}).$$

Proof. The category $\mathcal{T}(X)$ can be embedded in a derived category of a Grothendieck abelian category. We can hence take a system of homotopy injective resolutions of representatives of the isomorphism classes of indecomposable objects in $\mathrm{MTDer}_S(X)_{w=0}$. This forms a tilting collection by Proposition 3.1.2.19 and the pointwise purity assumption. We can hence apply Theorem 3.1.2.14 and the statement follows. \square

Remark 3.1.2.21. The last theorem can also be stated differently. Namely, let $\{L_s\}_{s \in S} \subseteq \mathrm{MTPer}_S(X)_{w=0} \subseteq \mathrm{MTDer}_S(X)_{w=0}$ be a set of representative of indecomposable weight zero perverse stratified mixed Tate motives on X . Then $\{L_s(i)[2i]\}_{s \in S, i \in \mathbb{Z}}$ generates $\mathrm{MTDer}_S(X)_{w=0}$ as an additive subcategory by [SW16, Corollary 11.11]. Assume without loss of generality that the $L_s(i)[2i]$ are homotopy projective. They hence form a tilting collection as considered in the proof. Now let $L = \bigoplus_{s \in S} L_s$ and

$$H = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}(X)}(L, L(n)[2n]).$$

Then we have

$$\begin{aligned} \mathrm{Der}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}H) &\cong \mathrm{Hot}^b(\langle \{L_s(i)[2i]\}_{s \in S, i \in \mathbb{Z}} \rangle_{\oplus}^{\mathcal{T}(X)}) \\ &= \mathrm{Hot}^b(\mathrm{MTDer}_S(X)_{w=0}) \cong \mathrm{MTDer}_S(X) \end{aligned}$$

where by $\mathrm{mod}^{\mathbb{Z}}\text{-}H$ we denote the category of finitely generated graded H right modules. See Theorem 3.1.2.14 and [RSW14, Proposition 5.4.2.] for the first equivalence.

Theorem 3.1.2.22 (Tilting for motives and functors). *Let $(X, S), (Y, S')$ be affinely Whitney-Tate stratified varieties, such that all objects of $\mathrm{MTDer}_S(X)_{w=0}$ and $\mathrm{MTDer}_{S'}(Y)_{w=0}$ are additionally pointwise pure. Let $f : X \rightarrow Y$ be an affinely stratified morphism. Assume that either f is smooth and proper or that f a closed immersion with X and Y smooth. Then tilting commutes with $f^* = f^!(d)[2d]$ and $f_* = f_!$. So for example the diagram*

$$\begin{array}{ccc} \mathrm{Hot}^b(\mathrm{MTDer}_S(X)_{w=0}) & \xrightarrow{\sim} \Delta & \mathrm{MTDer}_S(X) \\ \downarrow f_* & & \downarrow f_* \\ \mathrm{Hot}^b(\mathrm{MTDer}_{S'}(Y)_{w=0}) & \xrightarrow{\sim} \Delta & \mathrm{MTDer}_{S'}(Y) \end{array}$$

commutes up to natural isomorphism, where on the left side f_* acts by pointwise application. Furthermore tilting commutes with shifts of the form $(n)[2n]$ for $n \in \mathbb{Z}$.

Proof. First of all, the functors $f^* = f^!(d)[2d]$ and $f_* = f_!$ are weight exact, hence really restrict to weight zero motives by Proposition 3.1.2.11. It suffices to show the statement for one of the functors, since the other functors are adjoints (or shifts of it).

Assume that f is smooth and proper. The tilting equivalence was constructed by embedding $\mathcal{T}(X)$ and $\mathcal{T}(Y)$ in a derived category of a Grothendieck abelian category, say $\mathcal{A}_X, \mathcal{A}_Y$. The functor $f^* : \mathcal{T}(Y) \rightarrow \mathcal{T}(X)$ is actually just pointwise application of an exact functor $f^* : \mathcal{A}_Y \rightarrow \mathcal{A}_X$, since f is smooth (see [CD12, 5.1.16]). We can furthermore assume that our tilting collections for $\{T_i\} \subset \text{MTDer}_S(X)_{w=0}$ and $\{U_i\} \subset \text{MTDer}_{S'}(Y)_{w=0}$ are given by homotopy-injective complexes and chosen in a way, that for every T_i there is a U_j and a quasi-isomorphism $f^*(T_i) \rightarrow U_j$. We are hence exactly in the setting of Proposition 3.1.2.16 (see also Remark 3.1.2.17) which shows that f^* commutes with tilting. By adjunction also $f_* = f_!$ have to commute.

The exact same argument works for $f_! = f_*$ when f is a closed immersion.

Denote by $\pi : \mathbb{P}_X^1 \rightarrow X$ the projection. For every $M \in \mathcal{T}(X)$ we have a canonical splitting $\pi_*\pi^*M = M \oplus M(-1)[-2]$, hence also shifts of the form $(n)[2n]$ commute with tilting. \square

3.1.3 Flag varieties

In this section we introduce a lot of notation for reductive algebraic groups and partial flag varieties. We also introduce the maps we will later use to construct a geometric version of parabolic induction.

Generalities

Let $G \supset B \supset T$ be a reductive algebraic group over the complex numbers together with a Borel subgroup and maximal torus. Denote by

$$X(T) \supset \Phi \supset \Phi^+ \supset \Delta$$

the character lattice of T , the root system, set of positive and simple roots associated to the choice of B . Denote by

$$\mathcal{W} = N_G(T)/T \supset \mathcal{S}$$

the Weyl group and the set of simple reflections. By abuse of notation, we will let elements of \mathcal{W} act on cosets, subgroups, etc. whenever the action does not

depend on a choice of representative mod T . For $\alpha \in \Phi$ we denote the root subgroup of G on which the conjugation action by T is described by α by

$$\mathbb{G}_\alpha \cong U_\alpha \subset G.$$

Denote by U and U^- the unipotent radical of B and B^- , where by B^- we denote the opposite Borel. For $x \in \mathcal{W}$ we define

$$U_x \stackrel{\text{def}}{=} U \cap xU^-x^{-1} = \langle U_\alpha \mid \alpha \in x(\Phi^-) \cap \Phi^+ \rangle \subset B.$$

By a *standard parabolic subgroup* we mean a subgroup $G \supset Q \supset B$. We denote its Weyl group and simple reflections by

$$\mathcal{S}_Q = \mathcal{S} \cap \mathcal{W}_Q \subset \mathcal{W}_Q \subset \mathcal{W}.$$

From here on out, we always fix one particular standard parabolic subgroup

$$B \subset P \subset G.$$

We denote its Levi factor by

$$P \twoheadrightarrow P/\text{Rad}_u(P) \stackrel{\text{def}}{=} L$$

and for convenience choose a splitting of this map to interpret L as a subgroup of P . We denote by

$$A = B/\text{Rad}_u(P) \subset L$$

its Borel subgroup. In this chapter we will be interested in (partial) flag varieties associated to G and L . But for convenience, we will always prefer to work with quotients of P instead of quotients of L using

$$L/A \cong P/B.$$

Partial flag varieties associated to G or P are always affinely Whitney-Tate stratified varieties with respect to their stratification by Bruhat cells (these are precisely the B -orbits) [SW16, 4.10] and we can hence look at the associated categories of stratified mixed Tate motives

$$\text{MTDer}_{(B)}(G/B), \text{MTDer}_{(B)}(G/P), \text{MTDer}_{(B)}(P/B) \dots$$

Maps for geometric parabolic induction

We will now introduce all necessary notations and maps we will later use to define a geometric version of parabolic induction. If $B \subset Q \subset G$ is a standard parabolic subgroup, then there is a well known *generalized Bruhat decomposition* of G into $P \times Q$ orbits, given by

$$G = \bigsqcup_{\bar{w} \in \mathcal{W}_P \backslash \mathcal{W} / \mathcal{W}_Q} P\bar{w}Q$$

and an associated stratification of the partial flag variety

$$G/Q = \bigsqcup_{\bar{w} \in \mathcal{W}_P \backslash \mathcal{W} / \mathcal{W}_Q} P\bar{w}Q/Q.$$

As it turns out, those strata $P\bar{w}Q/Q$ are affine bundles over partial flag varieties associated to P , let say P/Q_w , for $P \supset Q_w \supset B$ a standard parabolic depending on Q and w . Geometric parabolic induction will be constructed by passing between sheaves on P/Q_w and G/Q , using the maps and notation from the following Theorem.

Theorem 3.1.3.1. *Let $w \in \mathcal{W}$ be a shortest coset representative in $\mathcal{W}_P \backslash \mathcal{W} / \mathcal{W}_Q$ and set*

$$\begin{aligned} \mathcal{W}_{Q,w} &\stackrel{\text{def}}{=} \mathcal{W}_P \cap w\mathcal{W}_Qw^{-1} \subset \mathcal{W}_P, \\ Q_w &\stackrel{\text{def}}{=} B\mathcal{W}_{Q,w}B \subset P \text{ and} \\ A_w &\stackrel{\text{def}}{=} L \cap Q_w \end{aligned}$$

Let x be a shortest representative in $\mathcal{W}_P / \mathcal{W}_{Q,w}$ (it makes sense to talk about shortest representatives here, since $\mathcal{W}_{Q,w}$ is generated by simple reflections, see (1)). Consider the diagram

$$\begin{array}{ccccccc} L/Q_w \cap L & \xleftarrow{\sim} & P/Q_w & \xleftarrow{\text{pr}_w} & P \times^{Q_w} BwQ/Q & \xrightarrow{\sim} & PwQ/Q \xrightarrow{\text{hw}} G/Q \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ AxA_w/A_w & \xleftarrow{\sim} & BxQ_w/Q_w & \xleftarrow{\text{pr}_w} & BxQ_w \times^{Q_w} BwQ/Q & \xrightarrow{\sim} & BxwQ/Q \\ & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr \\ & & U_x \dot{x} & \xleftarrow{\text{pr}_1} & U_x \dot{x} \times U_w \dot{w} & \xrightarrow{\sim} & U_{xw} \dot{x} \dot{w} \end{array}$$

where by $\dot{y} \in N_G(T)$ we denote a representative in $y = \dot{y}T$. Then the following statements hold.

1. $\mathcal{W}_{Q,w}$ is generated by simple reflections,
2. Q_w acts on BwQ by left multiplication and
3. Q_w contains the stabilizer of the action of P on wQ/Q

$$Q_w \supset P \cap wQw^{-1}.$$

4. The diagram is well defined and all squares are commutative and Cartesian.
5. The arrows marked by \sim are isomorphisms.

Proof. (1) Since Q is a standard parabolic subgroup, \mathcal{W}_Q is the isotropy group of some dominant weight λ . Since w is a shortest coset representative $w \cdot \lambda$ is still dominant for \mathcal{S}_P . Now $\mathcal{W}_{Q,w}$ is the isotropy group of λ in \mathcal{W}_P and hence generated by simple reflections, see [Hum90, Theorem 1.12 (c)].

(2) Let $s \in \mathcal{W}_{Q,w}$ be a simple reflection. Write $s = wqw^{-1}$ for $q \in \mathcal{W}_Q$. Let $y \in \mathcal{W}_Q$, then

$$BsBwByB \subset BswyB \cup BwyB = BwqyB \cup BwyB \subset BwQ.$$

The statement follows since $\mathcal{W}_{Q,w}$ is generated by simple reflections.

(3) We show that in fact $B(P \cap wQw^{-1})B \subset Q_w$. We have

$$\begin{aligned} B(P \cap wQw^{-1})B &= B\mathcal{W}_P B \cap BwB\mathcal{W}_Q Bw^{-1}B \\ &= B\mathcal{W}_P B \cap Bw\mathcal{W}_Q Bw^{-1}B \\ &\subset \bigcup_I B(\mathcal{W}_P \cap w\mathcal{W}_Q w_I^{-1})B \end{aligned}$$

where the second equality holds since w is reduced with respect to \mathcal{W}_Q and the w_I denote subexpressions of w (see [Bou02, IV.2.1 Lemma 1]). But now assume that there are $p \in \mathcal{W}_P$, $q \in \mathcal{W}_Q$ such that $p = wqw_I^{-1}$. Then $pw_I = wq$ and both represent the same coset in $\mathcal{W}_P \backslash \mathcal{W} / \mathcal{W}_Q$. Since $l(w_I) \leq l(w)$ and w is the shortest representative, we have $w = w_I$ and hence

$$B(P \cap wQw^{-1})B \subset B(\mathcal{W}_P \cap w\mathcal{W}_Q w^{-1})B = Q_w.$$

(4) Follows from (2).

(5) The multiplication map in the first row is an isomorphism by (3). The maps from the bottom to the middle row are isomorphisms since x, w and xw are shortest coset representatives in $\mathcal{W}_P / \mathcal{W}_{Q,w}$, $\mathcal{W} / \mathcal{W}_Q$ and $\mathcal{W} / \mathcal{W}_Q$, respectively. Here we use that for $x \in \mathcal{W}_P$, xw is a shortest representative in

$\mathcal{W}/\mathcal{W}_Q$ if and only if x is a shortest representative in $\mathcal{W}_P/(\mathcal{W}_P \cap w\mathcal{W}_Qw^{-1})$ (see [He07, 1.2]). The multiplication map in the bottom row is an isomorphism since $l(xw) = l(x) + l(w)$ (see [Bou02, IV Exercise §1.3]). All other statements follow.

See also Chapter 3 in [BT72] for a good reference on BN-pairs and parabolic subgroups. \square

Remark 3.1.3.2. In the regular case $Q = B$ the notation dramatically simplifies to

$$\begin{array}{ccccccc}
 L/A & \xleftarrow{\sim} & P/B & \xleftarrow{\text{pr}_w} & P \times^B BwB/B & \xrightarrow{\sim} & PwB/B & \xrightarrow{h_w} & G/B \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 AxA/A & \xleftarrow{\sim} & BxB/B & \xleftarrow{\text{pr}_w} & BxB \times^B BwB/B & \xrightarrow{\sim} & BxwB/B & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & U_x\dot{x} & \xleftarrow{\text{pr}_1} & U_x\dot{x} \times U_w\dot{w} & \xrightarrow{\sim} & U_{xw}\dot{x}\dot{w} & &
 \end{array}$$

for $x \in W_P$.

Example 3.1.3.3. Since admittedly the notation and all the involved maps can be quite confusing, we illustrate everything in the case $G = \text{GL}_3$. Let s, t be the simple reflections in G .

(1) The case of **disjoint parabolic subgroups**. Let $P = B \cup BsB$, $Q = B \cup BtB$ be minimal parabolic subgroups. Then $G/Q = \mathbb{P}^2$ has two P -orbits corresponding to the decomposition $\mathbb{P}^2 = \mathbb{P}^1 \cup \mathbb{A}^2$.

$$\begin{array}{ccc}
 PtsQ/Q = & \begin{array}{c} \text{[shaded box]} \\ BtsQ/Q \end{array} & \xrightarrow{\text{pr}_{ts}} & P/Q_{ts} = \bullet \\
 PQ/Q = & \begin{array}{c} \bullet \\ \text{---} \\ Q/Q \quad BsQ/Q \end{array} & \xrightarrow{\text{pr}_e} & P/Q_e = \begin{array}{c} \bullet \\ \text{---} \\ Q_e/Q_e \quad BsQ_e/Q_e \end{array}
 \end{array}$$

Here $\text{pr}_e : PQ/Q \cong \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1 \cong P/Q_e$ and $\text{pr}_{ts} : PtsQ/Q \cong \mathbb{A}^2 \rightarrow pt \cong P/Q_{ts}$.

(2) The case of **meeting parabolic subgroups**. Let $P = Q = B \cup BsB$. Then $G/Q = \mathbb{P}^2$ has two P -orbits corresponding to the decomposition $\mathbb{P}^2 = pt \cup \mathcal{O}(1)$.

$$\begin{array}{ccc}
\begin{array}{c}
\text{Pt}Q/Q = \begin{array}{|c} \hline BtQ/Q \\ \hline BstQ/Q \\ \hline \end{array} \\
PQ/Q = \bullet \\
Q/Q
\end{array} & \xrightarrow{\text{pr}_t} & \begin{array}{c}
P/Q_t = \bullet \\
Q_t/Q_t \quad BstQ_t/Q_t
\end{array} \\
& \xrightarrow{\text{pr}_e} & \begin{array}{c}
P/Q_e = \bullet \\
Q_e/Q_e
\end{array}
\end{array}$$

Here $\text{pr}_e : Q/Q \cong pt \xrightarrow{\sim} pt \cong P/Q_e$ and $\text{pr}_t : PtQ/Q \cong \mathcal{O}(1) \rightarrow \mathbb{P}^1 \cong P/Q_t$, where $\mathcal{O}(1)$ is the *hyperplane bundle* or *Serre's twisting sheaf* on \mathbb{P}^1 .

Compatibilities

All maps constructed in the preceding section are well-behaved with respect to passing between different standard parabolic subgroups, as described in the following.

Lemma 3.1.3.4. *In the notation of Theorem 3.1.3.1 let $Q \subset Q'$ be another standard parabolic containing Q and denote all objects associated to Q' by $-'$. Let $w \in \mathcal{W}$ be a shortest coset representative in $\mathcal{W}_P \backslash \mathcal{W}/\mathcal{W}_{Q'}$ and $x \in \mathcal{W}_{Q'}$ be a shortest coset representative in*

$$w^{-1}\mathcal{W}_{Q',w} \backslash \mathcal{W}_{Q'}/\mathcal{W}_Q = (w^{-1}\mathcal{W}_{Pw} \cap \mathcal{W}_{Q'}) \backslash \mathcal{W}_{Q'}/\mathcal{W}_Q.$$

Then $Q_{wx} \subset Q'_w$ and the following diagram commutes

$$\begin{array}{ccccc}
PwxQ/Q & \xleftarrow[\sim]{\text{mult}} & P \times^{Q_{wx}} BwxQ/Q & \xrightarrow[\text{pr}_{wx}]{\gg} & P/Q_{wx} \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
PwQ'/Q' & \xleftarrow[\sim]{\text{mult}} & P \times^{Q'_w} BwQ'/Q' & \xrightarrow[\text{pr}'_w]{\gg} & P/Q'_w
\end{array}$$

If $x = e$, the diagram is moreover Cartesian.

Lemma 3.1.3.5. *In the notation of Lemma 3.1.3.4, denote by Z the pullback of the diagram*

$$\begin{array}{ccc}
Z & \hookrightarrow & G/Q \\
\downarrow \pi & & \downarrow \pi \\
PwQ'/Q' & \xrightarrow{h'_w} & G/Q'
\end{array}$$

Then

$$Z = \bigsqcup_x PwxQ/Q \subset G/Q$$

where x runs over the shortest representatives of the double cosets

$$w^{-1}\mathcal{W}_{Q',w} \setminus \mathcal{W}_{Q'}/\mathcal{W}_Q = (w^{-1}\mathcal{W}_{Pw} \cap \mathcal{W}_{Q'}) \setminus \mathcal{W}_{Q'}/\mathcal{W}_Q.$$

Corollary 3.1.3.6. *In the notation of Lemma 3.1.3.5 assume additionally that $\mathcal{W}_{Q'} \subset w^{-1}\mathcal{W}_{Pw}$, then the following diagram is Cartesian.*

$$\begin{array}{ccc} PwQ/Q & \xleftarrow{h_w} & G/Q \\ \downarrow \pi & & \downarrow \pi \\ PwQ'/Q' & \xleftarrow{h'_w} & G/Q' \end{array}$$

3.2 Geometric parabolic induction and translation functors

In this section we will introduce *geometric parabolic induction* and then study its interaction with the geometric versions of translation functors, i.e. the functors

$$\pi_! : \text{MTDer}_{(B)}(G/Q) \longleftarrow \text{MTDer}_{(B)}(G/Q') : \pi^!$$

associated to the projections $G/Q \rightarrow G/Q'$ for $Q \subset Q'$.

There will be two different cases. Passing *into* a smaller flag variety, i.e. applying the functors $\pi_!$, will *commute* with geometric parabolic induction. However, the case of passing *out of* a smaller flag variety, i.e. applying $\pi^!$, will be more complicated.

3.2.1 Geometric parabolic induction

Definition 3.2.1.1. *Let $B \subset Q \subset G$ be a standard parabolic subgroup and let $w \in \mathcal{W}$ be a shortest coset representative in $\mathcal{W}_P \setminus \mathcal{W}/\mathcal{W}_Q$. We then call the functor*

$$\text{MTDer}_{(B)}(P/Q_w) \xrightarrow[\text{GInd}_w]{h_w, * \text{pr}_w^!} \text{MTDer}_{(B)}(G/Q)$$

geometric parabolic induction and denote it by GInd_w . The maps

$$P/Q_w \xleftarrow{\text{pr}_w} PwQ/Q \xleftarrow{h_w} G/Q$$

are defined as in Theorem 3.1.3.1.

Lemma 3.2.1.2. 1. GInd_w is well defined, i.e. it restricts to stratified mixed Tate motives.

2. If $w = e$, then $\mathrm{GInd}_e = \mathrm{h}_{e,*}$ is weight-exact.

3. Denote by $i_x : BxQ_w/Q_w \rightarrow P/Q_w$ and $i'_{xw} : BxwQ/Q \rightarrow G/Q$ the inclusions, then

$$\mathrm{GInd}_w i_{x,*} \mathbb{1} = i'_{xw,*} \mathbb{1}(l(w))[2l(w)]$$

where by $\mathbb{1}$ we denote the constant motive on BxQ_w/Q_w and $BxwQ/Q$, respectively.

Proof. (1) By Theorem 3.1.3.1 both pr_w and h_w are affinely stratified maps, compatible with the Bruhat stratification of P/Q_w and G/Q . The statement follows using Lemma 3.1.2.7.

(2) In this case $P/Q_w = P/P \cap Q \cong PQ/Q$. Hence pr_e is an isomorphism and furthermore h_e is a closed embedding and hence weight exact.

(3) By Theorem 3.1.3.1, the diagram

$$\begin{array}{ccccc} BxwQ/Q & \xleftarrow{i''_{xw}} & PwQ/Q & \xleftarrow{\mathrm{h}_w} & G/Q \\ \downarrow \mathrm{pr}_w & & \downarrow \mathrm{pr}_w & & \\ BxQ_w/Q_w & \xleftarrow{i_x} & P/Q_w & & \end{array}$$

is Cartesian. Hence by base change and Lemma 3.1.2.6

$$\mathrm{GInd}_w i_{x,*} \mathbb{1} = \mathrm{h}_{w,*} \mathrm{pr}_w^! i_{x,*} \mathbb{1} = \mathrm{h}_{w,*} i''_{xw,*} \mathrm{pr}_w^! \mathbb{1} = i'_{xw,*} \mathbb{1}(l(w))[2l(w)]. \quad \square$$

3.2.2 Into a smaller flag variety

Theorem 3.2.2.1. Let $B \subset Q \subset Q'$ be standard parabolic subgroups. Denote all objects associated to Q' by $-'$. Let w be a shortest coset representative of $\mathcal{W}_{Q'} \setminus \mathcal{W}/\mathcal{W}_P$. Then the following diagram of functors commutes (up to natural isomorphism).

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\mathrm{GInd}_w} & \mathrm{MTDer}_{(B)}(G/Q) \\ \downarrow \pi_* & & \downarrow \pi_* \\ \mathrm{MTDer}_{(B)}(P/Q'_w) & \xrightarrow{\mathrm{GInd}'_w} & \mathrm{MTDer}_{(B)}(G/Q') \end{array}$$

Proof. Follows immediately from Lemma 3.1.3.4 and base change. \square

3.2.3 Out of a smaller flag variety

Theorem 3.2.3.1. *Let $B \subset Q \subset Q'$ be standard parabolic subgroups. Denote all objects associated to Q' by $-'$. Consider the composition*

$$\mathrm{MTDer}_{(B)}(P/Q'_w) \xrightarrow{\mathrm{GInd}'_w} \mathrm{MTDer}_{(B)}(G/Q') \xrightarrow{\pi^!} \mathrm{MTDer}_{(B)}(G/Q)$$

Choose an ordering by length $\{x_k\}_{1 \leq k \leq n}$ on the set of shortest coset representatives in

$$w^{-1}\mathcal{W}_{Q',w} \setminus \mathcal{W}_{Q'} / \mathcal{W}_Q = (w^{-1}\mathcal{W}_{Pw} \cap \mathcal{W}_{Q'}) \setminus \mathcal{W}_{Q'} / \mathcal{W}_Q.$$

Then for all $M \in \mathrm{MTDer}_{(B)}(P/Q_w)$, there exists a family of distinguished triangles

$$M_{k-1} \longrightarrow M_k \longrightarrow \mathrm{GInd}_{wx_k} \pi_k^! M \xrightarrow{+1}$$

in $\mathrm{MTDer}_{(B)}(G/Q)$ where

$$M_n = \pi^! \mathrm{GInd}'_w M \text{ and } M_0 = 0$$

and the right hand side is given by

$$\mathrm{MTDer}_{(B)}(P/Q'_w) \xrightarrow{\pi_k^!} \mathrm{MTDer}_{(B)}(P/Q_{wx_k}) \xrightarrow{\mathrm{GInd}_{wx_k}} \mathrm{MTDer}_{(B)}(G/Q)$$

and the map π_k is induced by the inclusion $Q_{wx_k} \subset Q'_w$.

Proof. By Lemma 3.1.3.5 there is a Cartesian square

$$\begin{array}{ccc} Z & \xleftarrow{h} & G/Q \\ \downarrow \pi & & \downarrow \pi \\ PwQ'/Q' & \xleftarrow{h_w} & G/Q' \end{array}$$

with

$$Z = \bigsqcup_k Pwx_kQ/Q \subset G/Q.$$

Denote by

$$\begin{aligned} i_k &: Pwx_kQ/Q \hookrightarrow Z \\ i_{\leq k} &: \bigcup_{l \leq k} Pwx_lQ/Q \hookrightarrow Z \end{aligned}$$

the inclusions. For $N \in \mathrm{MTDer}_{(B)}(Z)$ define $N_k \stackrel{\text{def}}{=} i_{\leq k, !} i_{\leq k}^* N$. Then there exists a family of distinguished triangles

$$N_{k-1} \longrightarrow N_k \longrightarrow i_{k,*} i_k^! N \xrightarrow{+1}$$

with $N_0 = 0$ and $N_n = N$, using the localisation sequence and induction on n . We apply this to $N \stackrel{def}{=} \pi^! \text{pr}_w^! M$ and set $M_k \stackrel{def}{=} h_* N_k$. We hence obtain distinguished triangles

$$M_{k-1} = h_* N_{k-1} \longrightarrow M_k = h_* N_k \longrightarrow h_* i_{k,*} i_k^! \pi^! \text{pr}_w^! M \xrightarrow{+1}$$

We furthermore have

$$M_n = h_* N = h_* \pi^! \text{pr}_w^! M = \pi^! h_{w,*} \text{pr}_w^! M = \pi^! \text{GInd}_w M$$

and for all k by Lemma 3.1.3.4

$$h_* i_{k,*} i_k^! \pi^! \text{pr}_w^! M = h_{wx_k,*} \text{pr}_{wx_k}^! \pi_k^! M = \text{GInd}_{wx_k} \pi_k^! M$$

and the statement follows. \square

Example 3.2.3.2. Keep the notation from the last Theorem.

(1) Assume that the set $\{x_k\} = \{1\}$ has just one element. Then there is an isomorphism of functors

$$\pi^! \text{GInd}'_w \cong \text{GInd}_w \pi^!$$

(2) Assume that the set $\{x_n\} = \{1, s\}$ has just two elements. Then the theorem yields a distinguished triangle

$$\text{GInd}_w \pi^! M \longrightarrow \pi^! \text{GInd}'_w M \longrightarrow \text{GInd}_{ws} \pi_s^! M \xrightarrow{+1}$$

3.2.4 Geometric wall crossing

We discuss the interaction of the geometric version of the wall crossing functors θ_s for category \mathcal{O} and geometric parabolic induction. As in the Category \mathcal{O} case, this will be an essential ingredient in the induction step of our proof that parabolic induction and geometric parabolic induction correspond to each other.

Theorem 3.2.4.1. *Let $Q = (B \cup BsB)$. Let $w \in \mathcal{W}_P \setminus \mathcal{W}$ a shortest coset representative and $s \in \mathcal{W}$ a simple reflection with $ws > w$ such that ws is also a shortest coset representative for $\mathcal{W}_P \setminus \mathcal{W}$. Let $\pi : G/B \rightarrow G/Q$. Then there is a distinguished triangle of functors*

$$\text{GInd}_w \longrightarrow \pi^! \pi_! \text{GInd}_w \longrightarrow \text{GInd}_{ws} \xrightarrow{+1}$$

from $\text{MTDer}_{(B)}(P/B)$ to $\text{MTDer}_{(B)}(G/B)$. The map on the left hand side is the unit of the adjunction $(\pi_!, \pi^!)$.

Proof. This is more or less Theorem 3.2.2.1 and 3.2.3.1 combined. Consider the Cartesian diagram

$$\begin{array}{ccc} PwB/B \uplus PwsB/B & \xrightarrow{i \uplus j} & Z \xleftarrow{h} G/B \\ \downarrow \pi & & \downarrow \pi \\ PwQ/Q & \xleftarrow{h'_w} & G/Q \end{array}$$

Then there are the following natural isomorphisms of functors

$$\pi^! \pi_! \text{GInd}_w = \pi^! \pi_! h_{w,*} \text{pr}_w^! \cong \pi^! \pi_! h_* i_* \text{pr}_w^! \cong \pi^! h'_{w,*} \pi_! i_! \text{pr}_w^! \cong h_* \pi^! \pi_! i_! \text{pr}_w^!$$

We apply the localization triangle associated to (i, j) to the term on the right hand side and obtain

$$h_* i_! i^! \pi^! \pi_! i_! \text{pr}_w^! \longrightarrow h_* \pi^! \pi_! i_! \text{pr}_w^! \longrightarrow h_* j_* j^! \pi^! \pi_! i_! \text{pr}_w^! \xrightarrow{+1}$$

Since $ws > w$ the map $\pi_i : PwB/B \rightarrow PwQ/Q$ is an isomorphism. Hence

$$h_* i_! i^! \pi^! \pi_! i_! \text{pr}_w^! = h_* i_! \text{pr}_w^! = \text{GInd}_w .$$

Furthermore $ws > w$ implies that $Q_w = B$, since

$$\mathcal{W}_P \cap w\mathcal{W}_Q w^{-1} = \mathcal{W}_P \cap w\{1, s\}w^{-1} = \{1\} .$$

By Lemma 3.1.3.4 the diagrams

$$\begin{array}{ccc} PwB/B & \xrightarrow{\text{pr}_w} & P/B \\ \downarrow \wr \pi_i & & \parallel \pi' \\ PwQ/Q & \xrightarrow{\text{pr}'_w} & P/Q_w = P/B \end{array} \quad \begin{array}{ccc} PwsB/B & \xrightarrow{\text{pr}_{ws}} & P/B \\ \downarrow \pi_j & & \parallel \pi' \\ PwQ/Q & \xrightarrow{\text{pr}'_w} & P/Q_w = P/B \end{array}$$

commute, and the left hand diagram is cartesian. Hence

$$h_* j_* j^! \pi^! \pi_! i_! \text{pr}_w^! \cong h_* j_* j^! \pi^! \text{pr}'_w \pi'_! \cong h_* j_* \text{pr}'_{ws} \pi^! \pi'_! \cong h_{ws,*} \text{pr}'_{ws} = \text{GInd}_{ws} .$$

Putting everything together, our distinguished triangle reads

$$\text{GInd}_w \longrightarrow \pi^! \pi_! \text{GInd}_w \longrightarrow \text{GInd}_{ws} \xrightarrow{+1}$$

But a priori the first map is induced by the counit of the adjunction $(i_!, i^!)$. That this coincides with the unit of the adjunction $(\pi_!, \pi^!)$ follows from the following general Lemma. \square

Lemma 3.2.4.2. *Let $i : Z \hookrightarrow X : \pi$ be two morphisms in $\text{Var}(\mathbb{C})$, such that i is a closed embedding and $\pi i = \text{id}_Z$. Then the following diagram of functors $\mathcal{T}(Z) \rightarrow \mathcal{T}(X)$ commutes*

$$\begin{array}{ccc} i_! & \longrightarrow & \pi^! \pi_! i_! \\ \parallel & & \parallel \\ i_! & \longrightarrow & \pi^! \\ \parallel & & \parallel \\ i_! \pi^! & \longrightarrow & \pi^! \end{array}$$

where the top row is the unit of the adjunction $(\pi_!, \pi^!)$ and the bottom row the counit of $(i_!, i^!)$.

Proof. The two base change morphisms (called exchange morphisms in [CD12]) $i_! \text{id}^* \rightarrow \pi^! \text{id}_*$ associated to the diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow \text{id} & & \downarrow \pi \\ Z & \xrightarrow{\text{id}} & Z \end{array}$$

coincide. □

The theorem implies that for every $M \in \text{MTDer}_{(B)}(P/B)$ there is an isomorphism

$$\text{GInd}_{ws} M \cong \text{Cone}(\text{GInd}_w M \rightarrow \pi^! \pi_! \text{GInd}_w M)$$

in $\text{MTDer}_{(B)}(G/B)$, where by Cone we denote the mapping cone. In general however, mapping cones are *not functorial*. But as in the proof of Theorem 2.3.4.1, in our particular situation Lemma 2.3.4.5 applies and the mapping cone is indeed functorial.

Lemma 3.2.4.3. *For all $M, N \in \text{MTDer}_{(B)}(P/B)$ we have*

$$\text{Hom}_{\mathcal{T}(G/B)}(\text{GInd}_w(M), \text{GInd}_{ws}(N)) = 0.$$

Proof. This is simply a matter of their support. Let

$$U \stackrel{\text{def}}{=} PwsB/B \xleftarrow{j} Z \stackrel{\text{def}}{=} U \cup W \xleftarrow{i} W \stackrel{\text{def}}{=} PwB/B$$

and denote by k the inclusion of Z in X . Notice that U is open in Z .

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{T}(G/B)}(\mathrm{GInd}_w M, \mathrm{GInd}_{w_s} N) \\
&= \mathrm{Hom}_{\mathcal{T}(G/B)}(\mathfrak{h}_{w_s, !} \mathrm{pr}_{w_s}^* N, \mathfrak{h}_{w, !} \mathrm{pr}_w^* M) \quad (\text{duality}) \\
&= \mathrm{Hom}_{\mathcal{T}(Z)}(k^* \mathfrak{h}_{w_s, !} \mathrm{pr}_{w_s}^* N, k^* \mathfrak{h}_{w, !} \mathrm{pr}_w^* M) \quad (\text{support} \subseteq Z) \\
&= \mathrm{Hom}_{\mathcal{T}(Z)}(j_! j^* k^* \mathfrak{h}_{w_s, !} \mathrm{pr}_{w_s}^* N, i_! i^* k^* \mathfrak{h}_{w, !} \mathrm{pr}_w^* M) \quad (\text{support} \subseteq W, \text{ resp. } U) \\
&= \mathrm{Hom}_{\mathcal{T}(Z)}(j^* k^* \mathfrak{h}_{w_s, !} \mathrm{pr}_{w_s}^* N, j^* i_! i^* k^* \mathfrak{h}_{w, !} \mathrm{pr}_w^* M) \quad (\text{adjunction and } j^* = j^!) \\
&= 0 \quad (\text{since } j^* i_! = 0)
\end{aligned}$$

and the claim follows. \square

Corollary 3.2.4.4. *There is a natural equivalence of functors*

$$\mathrm{GInd}_{w_s} \cong \mathrm{Cone}(\mathrm{GInd}_w \rightarrow \pi^! \pi_! \mathrm{GInd}_w).$$

Proof. Follows from Theorem 3.2.4.1, Lemma 3.2.4.3 and Lemma 2.3.4.5. \square

3.2.5 From the singular case to the regular case

Up to direct sums and shifts, geometric parabolic induction for partial flag varieties G/Q can be expressed in terms of geometric parabolic induction for the regular flag variety G/B and geometric translation functors.

Theorem 3.2.5.1. *Let $B \subset Q \subset G$ and w be a shortest representative of a coset in $\mathcal{W}_P \setminus \mathcal{W}/\mathcal{W}_Q$. Let $\pi : G/B \rightarrow G/Q$ and $\pi' : P/B \rightarrow P/Q_w$ be the projections. Then there is a natural equivalence of functors*

$$\begin{aligned}
\pi_* \mathrm{GInd}_w \pi^! &\cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathrm{GInd}'_w(l(x))[2l(x)] : \\
&\mathrm{MTDer}_{(B)}(P/Q_w) \rightarrow \mathrm{MTDer}_{(B)}(G/Q)
\end{aligned}$$

Proof. By Theorem 3.2.2.1 we have

$$\pi_* \mathrm{GInd}_w \pi^! = \mathrm{GInd}_w \pi'_* \pi'^!.$$

Now we argue as in [BGS96, Lemma 3.5.4]. The decomposition theorem yields

$$\pi'_* \mathbb{1}_{P/B} \cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathbb{1}_{P/Q_w}(-l(x))[-2l(x)]$$

and using Verdier duality we get

$$\begin{aligned}
\pi'_* \pi'^! &\cong \pi'_* \mathcal{H}om_{P/B}(\mathbb{1}, \pi'^!(-)) \cong \mathcal{H}om_{P/Q_w}(\pi'_* \mathbb{1}, -) \\
&\cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathcal{H}om_{P/Q_w}(\mathbb{1}_{P/Q_w}(-l(x))[-2l(x)], -) \\
&\cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \text{id}(l(x))[2l(x)].
\end{aligned}$$

The statement follows. \square

Remark 3.2.5.2. Let us explain why we were allowed to use the decomposition theorem for perverse sheaves here. After all, the decomposition theorem is a statement about constructible sheaves and not motives.

By [Dre15], for all $X \in \text{Var}(\mathbb{C})$, there is a Hodge realization functor

$$\text{Real}_H : \mathcal{T}(X) \rightarrow \text{Der}(X(\mathbb{C}), \mathbb{C})$$

from motives on X into the derived category of sheaves on $X(\mathbb{C})$ (equipped with its metric topology). This is compatible with the six operations.

For an affinely Whitney-Tate stratified variety (X, S) , Real_H restricts to a functor

$$\text{Real}_H : \text{MTDer}_S(X) \rightarrow \text{Der}_S^b(X, \mathbb{C}).$$

By [SW16, Theorem 11.3] it induces isomorphisms

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{MTDer}_S(X)}(M, N(i)) \xrightarrow{\sim} \text{Hom}_{\text{Der}_S^b(X, \mathbb{C})}(\text{Real}_H(M), \text{Real}_H(N))$$

which are compatible with composition. In the notation of the last proof, the decomposition theorem yields

$$\pi'_* \mathbb{C}_{P/B} \cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathbb{C}_{P/Q_w}[-2l(x)] \in \text{Der}_{(B)}^b(P/Q_w, \mathbb{C}).$$

Actually, the decomposition theorem in its full strength is not needed here. We can also use that π is a fibration with typical fibre Q_w/B , and apply the Leray-Serre spectral sequence, which degenerates on page two by parity vanishing: The cohomology of both Q_w/B and P/B is concentrated in even degrees. This implies

$$\pi'_* \mathbb{C}_{P/B} = \bigoplus_{i \in \mathbb{Z}} H^i(Q_w/B, \mathbb{C}) \otimes \mathbb{C}_{P/Q_w}[-i].$$

Combined with the equality

$$H^i(Q_w/B, \mathbb{C}) = \bigoplus_{\substack{x \in \mathcal{W}_{Q,w} \\ 2l(x)=i}} \mathbb{C}$$

we obtain our statement. By [SW16, Lemma 6.6.], $\pi'_* \mathbb{1}_{P/B}$ and $\mathbb{1}_{P/Q_w}$ are pointwise pure of weight zero. This implies

$$\begin{aligned} \mathrm{Hom}_{\mathrm{MTDer}_S(X)}(\pi'_* \mathbb{1}_{P/B}, \mathbb{1}_{P/Q_w}(n)[2n](i)) &= 0 \text{ and} \\ \mathrm{Hom}_{\mathrm{MTDer}_S(X)}(\mathbb{1}_{P/Q_w}(n)[2n](i), \pi'_* \mathbb{1}_{P/B}) &= 0 \end{aligned}$$

for all $i \neq 0, n \in \mathbb{Z}$, using [SW16, Corollary 6.3]. Hence there are isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{MTDer}_S(X)}(\pi'_* \mathbb{1}_{P/B}, \mathbb{1}_{P/Q_w}(n)[2n]) \\ &\cong \mathrm{Hom}_{\mathrm{Der}_{(B)}^b(P/Q_w, \mathbb{C})}(\pi'_* \underline{\mathbb{C}}_{P/B}, \underline{\mathbb{C}}_{P/Q_w}(n)[2n]) \text{ and} \\ &\mathrm{Hom}_{\mathrm{MTDer}_S(X)}(\mathbb{1}_{P/Q_w}(n)[2n], \pi'_* \mathbb{1}_{P/B}) \\ &\cong \mathrm{Hom}_{\mathrm{Der}_{(B)}^b(P/Q_w, \mathbb{C})}(\underline{\mathbb{C}}_{P/Q_w}(n)[2n], \pi'_* \underline{\mathbb{C}}_{P/B}). \end{aligned}$$

So we can transport the projections and embeddings from the direct sum decomposition of $\pi'_* \underline{\mathbb{C}}_{P/B}$ to $\pi'_* \mathbb{1}_{P/B}$ and obtain

$$\pi'_* \mathbb{1}_{P/B} \cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathbb{1}_{P/Q_w}(-l(x))[-2l(x)].$$

Admittedly, this argument is awkward, and there should be a much more direct proof using a motivic version of the Leray-Serre spectral sequence.

3.3 Soergel modules

The category of stratified mixed Tate motives on a flag variety has a completely combinatorial description as the bounded homotopy category of Soergel modules. In this section we aim to give a description of geometric parabolic induction on the level of Soergel modules, i.e. fill out the question mark in the diagram

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\mathrm{GInd}_w} & \mathrm{MTDer}_{(B)}(G/Q) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hot}^b(H(P/Q_w)\text{-Smod}^{\mathbb{Z}, ev}) & \xrightarrow{?} & \mathrm{Hot}^b(H(G/Q)\text{-Smod}^{\mathbb{Z}, ev}) \end{array}$$

3.3.1 Erweiterungssatz, Tilting and Soergel modules

Let (X, S) be an affinely Whitney-Tate stratified variety. Then the *hypercohomology functor* is defined by

$$\mathbb{H} : \text{MTDer}_S(X) \rightarrow H(X)\text{-mod}^{\mathbb{Z} \times \mathbb{Z}}, M \mapsto \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}(X)}(\mathbb{1}_X, M(i)[j]),$$

where $H(X) \stackrel{\text{def}}{=} \mathbb{H}(\mathbb{1}_X)$.

Theorem 3.3.1.1 (Erweiterungssatz). *Let $X \in \text{Var}(\mathbb{C})$ be a partial flag variety. Then the hypercohomology functor*

$$\mathbb{H} : \text{MTDer}_{(B)}(X)_{w=0} \hookrightarrow H(X)\text{-mod}^{\mathbb{Z}}$$

is fully faithful on weight zero stratified mixed Tate motives.

Proof. See [Gin91] for a proof using mixed Hodge modules. All the proof really relies on is a six functor formalism and a theory of weights. It hence also holds in our setting as spelled out in [SW16, Theorem 8.4]. \square

Definition 3.3.1.2. *The modules in the essential image of \mathbb{H} are called (graded) Soergel modules, so that \mathbb{H} induces an equivalence of categories:*

$$\mathbb{H} : \text{MTDer}_{(B)}(X)_{w=0} \xrightarrow{\sim} H(X)\text{-Smod}^{\mathbb{Z}, \text{ev}}$$

between weight zero stratified mixed Tate motives and the category of evenly graded Soergel modules over $H(X)$ denoted $H(X)\text{-Smod}^{\mathbb{Z}, \text{ev}}$.

Remark 3.3.1.3. Let $X = G/Q$. Abbreviate $C = H(G/Q)$. Then the category $C\text{-Smod}^{\mathbb{Z}}$ is generated by modules of the form

$$C \otimes_{C^{s_n}} \dots C \otimes_{C^{s_1}} C$$

for simple reflections s_i , with respect to finite direct sums, taking direct summands, shifts and isomorphism. This corresponds to the next Lemma and the fact that for regular $X = G/B$ all simple perverse motives in $\text{MTDer}_{(B)}(G/B)$ can appear as (shifts of) direct summands in the motives modules

$$\pi_n^! \pi_{n,!} \cdots \pi_1^! \pi_{1,!} i_{pt,!} \mathbb{1}$$

where $\pi_i : G/B \rightarrow G/(B \cup Bs_i B)$ is the projection and $i_{pt} : B/B \rightarrow G/B$ the inclusion of the point.

Lemma 3.3.1.4 ([Soe90] Theorem 14). *Let $B \subset Q' \subset Q \subset G$ be parabolic subgroups and let $\pi : G/Q' \rightarrow G/Q$ be the projection. Then there are natural isomorphisms of functors*

$$\begin{aligned} \mathbb{H}\pi^* &\cong H(G/Q') \otimes_{H(G/Q)} \mathbb{H} \text{ and} \\ \mathbb{H}\pi_* &\cong \text{Res}_{H(G/Q')}^{H(G/Q)} \mathbb{H}. \end{aligned}$$

Corollary 3.3.1.5 (Corollary 9.4 [SW16]). *Let $X = G/Q$ be a flag variety. There are equivalences of triangulated categories*

$$\text{MTDer}_{(B)}(X) \xrightarrow[\sim]{\Delta} \text{Hot}^b(\text{MTDer}_{(B)}(X)_{w=0}) \xrightarrow[\sim]{\mathbb{H}} \text{Hot}^b(H(X)\text{-Smod}^{\mathbb{Z},ev})$$

where Δ denotes the tilting equivalence, see Theorem 3.1.2.20.

Proof. The proof uses that for partial flag varieties all objects in $\text{MTDer}_{(B)}(X)_{w=0}$ are additionally pointwise pure. \square

3.3.2 Cohomology rings of flag varieties

There is a completely explicit description of the cohomology ring of flag varieties, due to Borel. In this section we describe how this is compatible with respect to the inclusion $P/Q_e \hookrightarrow G/Q$.

Lemma 3.3.2.1. *Denote by $X(T)$ the character lattice of the torus $T \subset G$ and by*

$$S = S(X(T) \otimes_{\mathbb{Z}} \mathbb{C})$$

the symmetric algebra of its complexification. Then the following diagram of short exact sequences commutes

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S(S_+^{\mathcal{W}}) & \hookrightarrow & S(S_+^{\mathcal{W}_P}) \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \\ \downarrow^{c_1} & & \downarrow^{c_1} \\ H(G/B) & \xrightarrow{h_e^*} & H(P/B) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where c_1 denotes the map induced by the first Chern class of a line bundle induced by a character of T .

Proof. $H(G/B)$ and $H(P/B)$ are the de Rham cohomology groups of G/B and P/B and this is the classical Borel image. The diagram commutes since pullback of line bundles and taking Chern classes commutes. \square

Lemma 3.3.2.2. *Let $Q \subset G$ be a standard parabolic. Then the following diagram commutes*

$$\begin{array}{ccc} C_G^Q \stackrel{\text{def}}{=} (S/S(S_+^{\mathcal{W}}))^{\mathcal{W}_Q} & \longrightarrow & (S/S(S_+^{\mathcal{W}_P}))^{\mathcal{W}_{Q,e}} \stackrel{\text{def}}{=} C_{Q_e}^P \\ \downarrow \wr & & \downarrow \wr \\ H(G/Q) & \xrightarrow{h_e^*} & H(P/Q_e) \end{array}$$

Proof. The isomorphism $C_G^Q \cong H(G/Q)$ is established by identifying the image of the injection

$$\pi^* : H(G/Q) \hookrightarrow H(G/B)$$

with the \mathcal{W}_Q -invariants in $S/S(S_+^{\mathcal{W}})$, where $\pi : G/B \rightarrow G/Q$ denotes the projection, see [BGG73]. The same holds for $H(P/Q_e)$ and the statement follows by $h_e \pi = \pi h_e$ and Lemma 3.3.2.1. \square

3.3.3 Geometric parabolic induction and Soergel bi-modules

The regular case

Recall that $B \subset P \subset G$ was a parabolic subgroup. Abbreviate

$$C = C_G^B = H(G/B) \cong S(X(T) \otimes_{\mathbb{Z}} \mathbb{C}) / (S(X(T) \otimes_{\mathbb{Z}} \mathbb{C})_+^{\mathcal{W}}).$$

This is a graded ring, living in *even* and positive degrees. Denote for a graded module M its n -th shift by $M\langle n \rangle$, such that

$$(M\langle n \rangle)^i = M^{i+n}.$$

For a simple reflection s , denote by C^s the s -invariants. Then $C^s \subset C$ is a Frobenius extension, and we denote by

$$R_s \stackrel{\text{def}}{=} \cdots \rightarrow 0 \rightarrow C \rightarrow C \otimes_{C^s} C\langle 2 \rangle \rightarrow 0 \rightarrow \cdots$$

the complex of graded Soergel bimodules over C known as *Rouquier complex*. Here $C \otimes_{C^s} C\langle 2 \rangle$ lives in cohomological degree 0, and the map is the unit of the adjunction between $\text{Res}_C^{C^s}(-\langle 1 \rangle)$ and $C \otimes_{C^s} -\langle 1 \rangle$. For a reduced expression $w = s_n \cdots s_1$ of $w \in \mathcal{W}$, we define a complex of graded Soergel bimodules by

$$\underline{R}_w \stackrel{\text{def}}{=} R_{s_1} \otimes_C \cdots \otimes_C R_{s_n}.$$

Denote by

$$\text{Res} \stackrel{\text{def}}{=} \text{Res}_{C_P^B}^C$$

the functor associated to the morphism $C \rightarrow C_P^B$ discussed in Section 3.3.2. In the rest of this Chapter we will—among other things—prove that on the level of graded Soergel modules, geometric parabolic induction for regular flag variety

$$\text{GInd}_w : \text{MTDer}_{(B)}(P/B) \rightarrow \text{MTDer}_{(B)}(G/B)$$

is given by the functor

$$\text{SInd}_w \stackrel{\text{def}}{=} \underline{R}_w \otimes_C \text{Res}(-).$$

Theorem 3.3.3.1. *Let w be a shortest coset representative in $\mathcal{W}_P \backslash \mathcal{W}$ and $w = s_n \cdots s_1$ a reduced expression. Then the following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \text{MTDer}_{(B)}(P/B) & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/B) \\ \Delta \downarrow \wr & & \wr \downarrow \Delta \\ \text{Hot}^b(\text{MTDer}_{(B)}(P/B)_{w=0}) & & \text{Hot}^b(\text{MTDer}_{(B)}(G/B)_{w=0}) \\ \mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\ \text{Hot}^b(C_P^B\text{-Smod}^{\mathbb{Z},ev}) & \xrightarrow{\text{SInd}_w} & \text{Hot}^b(C\text{-Smod}^{\mathbb{Z},ev}). \end{array}$$

Proof. Postponed. □

The singular case

Now let $B \subset Q \subset G$ be another parabolic subgroup. Let w be a shortest coset representative in $\mathcal{W}_P \backslash \mathcal{W}/\mathcal{W}_Q$. Recall that $\mathcal{W}_{Q,w} = \mathcal{W}_P \cap w\mathcal{W}_Q w^{-1}$ and let $m = |\mathcal{W}_{P,w}|$. Then there are natural maps

$$C_G^Q \rightarrow C = C_G^B \leftarrow C_P^B \leftarrow C_P^{Q_w}$$

On the level of graded Soergel modules, geometric parabolic induction for partial flag varieties (or better an m -fold direct sum of shifted copies of it)

$$\bigoplus_{x \in \mathcal{W}_{P,w}} \mathrm{GInd}_w(-)(-l(x))[-2l(x)] : \mathrm{MTDer}_{(B)}(P/Q_w) \rightarrow \mathrm{MTDer}_{(B)}(G/Q)$$

is given by the functor

$$\overline{\mathrm{SInd}}_w^Q \stackrel{\text{def}}{=} \mathrm{Res}_C^{C_G^Q} \mathbf{R}_w \otimes_C C_P^B \otimes_{C_P^{Q_w}} (-)$$

Theorem 3.3.3.2. *let $B \subset Q \subset G$ be another parabolic subgroup. Let w be a shortest coset representative in $\mathcal{W}_P \setminus \mathcal{W}/\mathcal{W}_Q$. Let $w = s_n \cdots s_1$ a reduced expression. Abbreviate*

$$\overline{\mathrm{GInd}}_w = \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathrm{GInd}_w(-)(-l(x))[-2l(x)]$$

Then the following diagram of functors commutes up to natural isomorphism

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\overline{\mathrm{GInd}}_w} & \mathrm{MTDer}_{(B)}(G/Q) \\ \Delta \downarrow \wr & & \wr \downarrow \Delta \\ \mathrm{Hot}^b(\mathrm{MTDer}_{(B)}(P/Q_w)_{w=0}) & & \mathrm{Hot}^b(\mathrm{MTDer}_{(B)}(G/Q)_{w=0}) \\ \mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\ \mathrm{Hot}^b(C_P^{Q_w}\text{-Smod}^{\mathbb{Z},ev}) & \xrightarrow{\overline{\mathrm{SInd}}_w^Q} & \mathrm{Hot}^b(C_G^Q\text{-Smod}^{\mathbb{Z},ev}). \end{array}$$

Proof. Denote by $\pi : G/B \rightarrow G/Q$ and $\pi' : P/B \rightarrow P/Q_w$ the projections. Then on the one hand we have

$$\pi_! \mathrm{GInd}_w \pi'^! \cong \mathrm{GInd}_w \pi'_! \pi'^! \cong \overline{\mathrm{GInd}}_w = \bigoplus_{x \in \mathcal{W}_{Q,w}} \mathrm{GInd}_w(-)(l(x))[2l(x)]$$

by Theorem 3.2.5.1. On the other hand

$$\begin{aligned} \mathbb{H}\Delta \pi_* \mathrm{GInd}_w \pi'^* &\cong \mathrm{Res}_C^{C_G^Q} \mathbb{H}\Delta \mathrm{GInd}_w \pi'^* \\ &\cong \mathrm{Res}_C^{C_G^Q} \mathbf{R}_w \otimes_C \mathbb{H}\Delta \pi'^* \\ &\cong \mathrm{Res}_C^{C_G^Q} \mathbf{R}_w \otimes_C C_P^B \otimes_{C_P^{Q_w}} \mathbb{H}\Delta \\ &= \overline{\mathrm{SInd}}_w^Q \mathbb{H}\Delta \end{aligned}$$

where we use that π_* and π'_* commute with tilting, since π' is proper and smooth (Theorem 3.1.2.22), and that under the hypercohomology functor \mathbb{H} , π_* corresponds to $\text{Res}_{C_G^Q}$ and π^* to $C \otimes_{C_G^Q}$, see [Soe90, Theorem 14]. Now $\pi_! = \pi_*$ since π is proper and $\pi'^* = \pi'^!(-d)[-2d]$ since π' is smooth, where d denotes the relative dimension of π' . But d is exactly the length of the longest word in $\mathcal{W}_{Q,w}$, hence

$$\begin{aligned} \pi_* \text{GInd}_w \pi'^* &\cong \bigoplus_{x \in \mathcal{W}_{Q,w}} \text{GInd}_w(-)(l(x) - d)[2l(x) - 2d] \\ &= \bigoplus_{x \in \mathcal{W}_{Q,w}} \text{GInd}_w(-)(-l(x))[-2l(x)] \end{aligned}$$

and the statement follows. \square

Unfortunately, up until this point, we do not know how to get rid of the m -fold direct sum. In the case $w = e$ though, the following theorem gives a complete answer:

Theorem 3.3.3.3. *Let $w = e$. The following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc} \text{MTDer}_{(B)}(P/Q_e) & \xrightarrow{\text{GInd}_{e=h_e,*}} & \text{MTDer}_{(B)}(G/Q) \\ \downarrow \mathbb{H} & & \downarrow \mathbb{H} \\ C_P^{Q_e}\text{-mod} & \xrightarrow{\text{Res}} & C_G^Q\text{-mod} \end{array}$$

Proof. Let $M \in \text{MTDer}_{(B)}(P/Q_e)$. By definition we have

$$\begin{aligned} \mathbb{H}(h_{e,*}M) &= \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}(G/Q)}(\mathbb{1}_{G/Q}, h_{e,*}M(i)[j]) \\ &= \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}(P/Q_e)}(h_e^* \mathbb{1}_{G/Q}, M(i)[j]) \\ &= \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}(P/Q_e)}(\mathbb{1}_{P/Q_e}, M(i)[j]) \\ &= \mathbb{H}(M) \end{aligned}$$

and the statement follows from Lemma 3.3.2.2. \square

Corollary 3.3.3.4. *The following diagram of functors commutes up to natural isomorphism*

$$\begin{array}{ccc}
\mathrm{MTDer}_{(B)}(P/Q_e) & \xrightarrow{\mathrm{GInd}_e = \mathfrak{h}_{e,*}} & \mathrm{MTDer}_{(B)}(G/Q) \\
\Delta \downarrow \wr & & \Delta \downarrow \wr \\
\mathrm{Hot}^b(\mathrm{MTDer}_{(B)}(P/Q_w)_{w=0}) & & \mathrm{Hot}^b(\mathrm{MTDer}_{(B)}(G/Q)_{w=0}) \\
\mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\
\mathrm{Hot}^b(C_P^{Q_e}\text{-Smod}^{\mathbb{Z}}) & \xrightarrow{\mathrm{Res}} & \mathrm{Hot}^b(C_G^Q\text{-Smod}^{\mathbb{Z}})
\end{array}$$

Proof. Follows from Theorem 3.3.3.3. We use that \mathfrak{h}_e is a closed embedding, and hence $\mathfrak{h}_{e,*} = \mathfrak{h}_{e,!}$ acts on the homotopy categories of weight zero motives by pointwise application, see Theorem 3.1.2.22. \square

Proof of the regular case

The proof mainly relies on Theorem 3.2.4.1 and an induction on $l(w)$.

Proof of Theorem 3.3.3.1. First assume that $l(w) = 0$, then $w = e$ and the statement is Corollary 3.3.3.4.

Now let $ws > w$ with both ws and w shortest representatives in $\mathcal{W}_P \setminus \mathcal{W}$. Assuming that the statement holds for w , we show that it holds for ws .

Denote by Δ the tilting equivalence. Let $\pi : G/B \rightarrow G/(B \cup BsB)$ be the projection. Let $M \in \mathrm{MTDer}_{(B)}(P/B)$. We have the following diagram of distinguished triangles:

$$\begin{array}{ccccc}
\mathbb{H}\Delta \mathrm{GInd}_w M & \longrightarrow & \mathbb{H}\Delta \pi^! \pi_! \mathrm{GInd}_w M & \longrightarrow & \mathbb{H}\Delta \mathrm{GInd}_{ws} M \xrightarrow{+1} \\
\parallel & & (1) \downarrow \wr & & \parallel \\
\mathbb{H}\Delta \mathrm{GInd}_w M & \xrightarrow{(*)} & C \otimes_{C^s} \mathbb{H}\Delta \mathrm{GInd}_w M \langle 2 \rangle & \longrightarrow & \mathbb{H}\Delta \mathrm{GInd}_{ws} M \xrightarrow{+1} \\
(2) \downarrow \wr & & (2) \downarrow \wr & & \parallel \\
\mathrm{SInd}_w \mathbb{H}\Delta M & \longrightarrow & C \otimes_{C^s} \mathrm{SInd}_w \mathbb{H}\Delta M \langle 2 \rangle & \longrightarrow & \mathbb{H}\Delta \mathrm{GInd}_{ws} M \xrightarrow{+1}
\end{array}$$

The first triangle is given by Theorem 3.2.4.1.

(1) Since $\pi^! \pi_!$ commutes with Δ by 3.1.2.22. On Soergel modules $\pi^! \pi_!$ is given by $C \otimes_{C^s} \langle 2 \rangle$, see [Soe90, Korollar 2].

(2) This is the induction hypothesis.

(*) This is given by the adjunction homomorphism by Theorem 3.2.4.1.

We hence have the following isomorphism

$$\begin{aligned}
\mathbb{H}\Delta \mathrm{GInd}_w M &\cong \mathrm{Cone}(\mathrm{SInd}_w \mathbb{H}\Delta M \rightarrow C \otimes_{C^s} \mathrm{SInd}_w \mathbb{H}\Delta M \langle 2 \rangle) \\
&= \mathrm{R}_s \otimes_C \mathrm{SInd}_w \mathbb{H}\Delta M \\
&= \mathrm{SInd}_{ws} \mathbb{H}\Delta M
\end{aligned}$$

where by Cone we denote the mapping cone. This is indeed a *natural* isomorphism by the discussion in Section 3.2.4 and Lemma 2.3.4.5. \square

Chapter 4

Main Results

4.1 Setup

4.1.1 Lie algebras and groups

We recall and compare some of the notations of Chapter 2 and 3. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a reductive Lie algebra with Borel and Cartan subalgebra. Denote by $G \supset B \supset T$ a *Langlands dual* algebraic group over \mathbb{C} , i.e a group such that the root system with simple roots associated to $\text{Lie}(G) \supset \text{Lie}(B) \supset \text{Lie}(T)$ is dual to the one of $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$. The Weyl group \mathcal{W} and simple roots \mathcal{S} corresponding to $\mathfrak{g} \supset \mathfrak{b}$ and $G \supset B$ are hence identified. Now let

$$\begin{aligned} \mathfrak{b} \subset \mathfrak{p} &\rightarrow \mathfrak{l} \\ B \subset P &\rightarrow L \end{aligned}$$

be corresponding parabolic subgroups/algebras with their Levi factor, i.e. $\mathcal{W}_P = \mathcal{W}_{\mathfrak{l}}$. Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. Then the stabilizer of λ with respect to the dot action $\mathcal{W}_{\mathfrak{g},\lambda}$ is generated by simple roots. Hence λ corresponds to a standard parabolic subgroup

$$B \subset Q \subset G$$

with $\mathcal{W}_{\mathfrak{g},\lambda} = \mathcal{W}_Q$. We also have equalities $\mathcal{W}_{Q,w} = \mathcal{W}_P \cap w\mathcal{W}_Qw^{-1} = \mathcal{W}_{\mathfrak{l},w\cdot\lambda}$, for $w \in \mathcal{W}$ shortest coset representatives of $\mathcal{W}_P \backslash \mathcal{W} / \mathcal{W}_Q$.

We can naturally identify

$$\begin{aligned} H(P/Q_w) &= C_P^{Q_w} = (\mathbb{S}(X(T) \otimes_Z \mathbb{C}) / (\mathbb{S}(X(T) \otimes_Z \mathbb{C})_+^{\mathcal{W}_P}))^{\mathcal{W}_{Q,w}} \\ &= (\mathbb{S}(\mathfrak{h}) / (\mathbb{S}(\mathfrak{h})_+^{\mathcal{W}_{\mathfrak{l}}}))^{\mathcal{W}_{\mathfrak{l},w\cdot\lambda}} \\ &= C_{\mathfrak{l}}^{w\cdot\lambda} = \text{End}_{\mathfrak{l}}(P_{\mathfrak{l}}^{w\cdot\lambda}) \end{aligned}$$

and similarly

$$\begin{aligned} H(G/Q) &= C_G^Q = (\mathbb{S}(X(T) \otimes_{\mathbb{Z}} \mathbb{C}) / (\mathbb{S}(X(T) \otimes_{\mathbb{Z}} \mathbb{C})_+^{\mathcal{W}}))^{\mathcal{W}_Q} \\ &= (\mathbb{S}(\mathfrak{h}) / (\mathbb{S}(\mathfrak{h})_+^{\mathcal{W}_{\mathfrak{g}}}))^{\mathcal{W}_{\mathfrak{g}, \lambda}} \\ &= C_{\mathfrak{g}}^{\lambda} = \text{End}_{\mathfrak{g}}(P_{\mathfrak{g}}^{w \cdot \lambda}) \end{aligned}$$

where we use the natural identification $X(T) \otimes_{\mathbb{Z}} \mathbb{C} = (\mathfrak{h}^{\vee})^* = \mathfrak{h}$.

Furthermore, their categories of (graded) Soergel modules, which we had defined as the essential image of projective modules and weight zero stratified mixed Tate motives under Soergel's functor \mathbb{V} and the hypercohomology functor \mathbb{H} , respectively, coincide. By this we mean, that functor v , forgetting the grading, restricts to a functor

$$v : H(P/Q_w)\text{-Smod}^{\mathbb{Z}, ev} \rightarrow \text{End}_{\mathfrak{t}}(P_{\mathfrak{t}}^{w \cdot \lambda})\text{-Smod}$$

and every module on the right hand side can be lifted, i.e. has a preimage under v . The same statement holds for G/Q and \mathfrak{g} .

4.2 Results

4.2.1 Geometric Parabolic Induction and Parabolic Induction

Combining the results from Chapter 2 and 3, we obtain our main theorem.

Theorem 4.2.1.1. *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and $Q \subset G$ the corresponding standard parabolic subgroup. Let $w \in \mathcal{W}$ be a shortest coset representative in $\mathcal{W}_P \backslash \mathcal{W} / W_Q = \mathcal{W}_{\mathfrak{t}} \backslash \mathcal{W}_{\mathfrak{g}} / \mathcal{W}_{\mathfrak{g}, \lambda}$. Let $n = |\mathcal{W}_{\mathfrak{t}, w \cdot \lambda}|$ and*

$$\overline{\text{GInd}}_w = \bigoplus_{x \in \mathcal{W}_{Q, w}} \text{GInd}_w(-)(-l(x))[-2l(x)]$$

Then the following diagram commutes up to natural isomorphism

$$\begin{array}{ccc}
\text{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\overline{\text{GInd}}_w} & \text{MTDer}_{(B)}(G/Q) \\
\Delta \downarrow \wr & & \wr \downarrow \Delta \\
\text{Hot}^b(\text{MTDer}_{(B)}(P/Q_w)_{w=0}) & & \text{Hot}^b(\text{MTDer}_{(B)}(G/Q)_{w=0}) \\
\mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\
\text{Hot}^b(C_P^{Q_w}\text{-Smod}^{\mathbb{Z},ev}) & \xrightarrow{\overline{\text{SInd}}_w^Q} & \text{Hot}^b(C_G^Q\text{-Smod}^{\mathbb{Z},ev}) \\
v \downarrow & & \downarrow v \\
\text{Hot}^b(C_{\mathfrak{t}}^{w,\lambda}\text{-Smod}) & \xrightarrow{\overline{\text{SInd}}_w^\lambda} & \text{Hot}^b(C_{\mathfrak{g}}^\lambda\text{-Smod}) \\
\mathbb{V}_{\mathfrak{t}}^{w,\lambda} \uparrow \wr & & \wr \uparrow \mathbb{V}_{\mathfrak{g}}^\lambda \\
\text{Hot}^b(\text{Proj } \mathcal{O}_{w,\lambda}(\mathfrak{t})) & & \text{Hot}^b(\text{Proj } \mathcal{O}_\lambda(\mathfrak{g})) \\
\wr \uparrow & & \wr \uparrow \\
\text{Der}^b(\mathcal{O}_{w,\lambda}(\mathfrak{t})) & \xrightarrow{(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{\oplus n}} & \text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g}))
\end{array}$$

(Large curved arrows labeled v connect the left and right sides of the diagram.)

Proof. The upper and lower rectangles are Theorem 3.3.3.2 and 2.3.4.2. By definition $v\overline{\text{SInd}}_w^Q = \overline{\text{SInd}}_w^\lambda v$ and the statement follows. \square

Remark 4.2.1.2. Unfortunately, we will not prove that the corresponding diagram with just $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ and GInd_w and without the direct sum commutes. But let us sketch a possible approach. In the next Section 4.2.2 we will show how a Krull-Remak-Schmidt argument allows to get rid of the direct sum for the restrictions of the (geometric) parabolic to the heart of a t-structure on the categories and show that the following diagram of functors commutes (up to natural isomorphism):

$$\begin{array}{ccc}
\text{MTDer}_{(B)}(P/Q_w)^\heartsuit & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/Q)^\heartsuit \\
\downarrow v & & \downarrow v \\
\mathcal{O}_{w,\lambda}(\mathfrak{t}) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_\lambda(\mathfrak{g}).
\end{array}$$

Now it would suffice to show that the following diagram of functors commutes

$$\begin{array}{ccc}
\text{Der}^b(\text{MTDer}_{(B)}(P/Q_w)^\heartsuit) & \xrightarrow{\text{GInd}_w} & \text{Der}^b(\text{MTDer}_{(B)}(G/Q)^\heartsuit) \\
\wr \downarrow \text{real} & & \wr \downarrow \text{real} \\
\text{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/Q)
\end{array}$$

where the upper horizontal arrow is given by pointwise application of the (t-exact) functor GInd_w . This is true by using for example [Bei87, Lemma A 7.1] or [Lur, Theorem 1.3.3.2]. Both results require the existence of a lift of GInd_w to some upgraded category of motives; an f -category in the former and a stable ∞ -category in the latter. Since $\mathrm{GInd}_w = \mathfrak{h}_{w,*} \mathrm{pr}_w^!$ is defined using the six functors, this lift exists. Introducing the necessary notation would go beyond the scope of this thesis. We hence omit a formal proof.

For regular weights everything just works fine.

Corollary 4.2.1.3. *Let $\lambda \in \mathfrak{h}^*$ be a regular integral dominant weight and $w \in \mathcal{W}$ be a shortest coset representative in $\mathcal{W}_\mathfrak{l} \setminus \mathcal{W}_\mathfrak{g}$ then the following diagram commutes up to natural isomorphism.*

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/B) & \xrightarrow{\mathrm{GInd}_w} & \mathrm{MTDer}_{(B)}(G/B) \\ \downarrow v & & \downarrow v \\ \mathrm{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & \xrightarrow{\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}}} & \mathrm{Der}^b(\mathcal{O}_\lambda(\mathfrak{g})) \end{array}$$

4.2.2 Graded Parabolic Induction

The main goal of this section is to use our results to show that parabolic induction for integral blocks of category \mathcal{O} is *gradable* (see [Str03, Definition 3.3]), which means that there is a functor $\widehat{\mathrm{Ind}}_\mathfrak{p}^{\mathfrak{g}}$ making the following diagram commute up to natural isomorphism

$$\begin{array}{ccc} \mathcal{O}_{w \cdot \lambda}^{\mathbb{Z}}(\mathfrak{l}) & \xrightarrow{\widehat{\mathrm{Ind}}_\mathfrak{p}^{\mathfrak{g}}} & \mathcal{O}_\lambda^{\mathbb{Z}}(\mathfrak{g}) \\ \downarrow v & & \downarrow v \\ \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) & \xrightarrow{\mathrm{Ind}_\mathfrak{p}^{\mathfrak{g}}} & \mathcal{O}_\lambda(\mathfrak{g}) \end{array}$$

and fulfilling $\widehat{\mathrm{Ind}}_\mathfrak{p}^{\mathfrak{g}} \langle n \rangle = \langle n \rangle \widehat{\mathrm{Ind}}_\mathfrak{p}^{\mathfrak{g}}$, where $\mathcal{O}^{\mathbb{Z}}$ denotes the *graded category* \mathcal{O} as defined in [BGS96] and $\langle - \rangle$ denotes the shift of grading.

Graded category \mathcal{O} (for a fixed block) is constructed by establishing a grading on the ring

$$A = \mathrm{End}_{\mathcal{O}}(P)$$

where P denotes a (minimal) projective generator of the given block and then defining graded category \mathcal{O} as the category of finitely generated graded modules over A .

$$\begin{array}{ccc}
\mathcal{O}_\lambda^{\mathbb{Z}} & \xrightarrow{\text{def}} & \text{mod}^{\mathbb{Z}}\text{-}A \\
\downarrow v & & \downarrow v \\
\mathcal{O}_\lambda & \xrightarrow[\sim]{\text{Hom}_{\mathcal{O}}(P, -)} & \text{mod-}A
\end{array}$$

This grading on A is established by realizing it as Ext-ring of a certain complex of sheaves on the Langlands dual flag variety.

Let us explain what this concretely means in our setting. See [SW16, Section 11] for a reference. Fix a dominant integral weight λ and the corresponding standard parabolic Q . Denote by

$$\begin{aligned}
P &= \bigoplus_{x \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{g}, \lambda}} P_{\mathfrak{g}}(x \cdot \lambda) \in \mathcal{O}_\lambda(\mathfrak{g}) \text{ and} \\
L &= \bigoplus_{x \in \mathcal{W}/\mathcal{W}_Q} \mathcal{IC}_x \in \text{MTPer}_{(B)}(G/Q)_{w=0}
\end{aligned}$$

the sum of the indecomposable projectives and the sum of simple weight zero perverse stratified mixed Tate motives \mathcal{IC}_x supported on $\overline{BxQ/Q}$, respectively. Denote by

$$\begin{aligned}
A &= \text{End}_{\mathfrak{g}}(P) \text{ and} \\
A' &= \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{T}(G/Q)}(L, L(i)[2i]).
\end{aligned}$$

By showing $\mathbb{V}(P_{\mathfrak{g}}(x \cdot \lambda)) \cong \mathbb{H}(\mathcal{IC}_x)$ as $C_G^Q = C_{\mathfrak{g}}^\lambda$ -modules and using Soergel's Erweiterungssatz and Struktursatz, one sees that in fact $A \cong A'$. This puts a grading on A . To be compatible with [BGS96], we redefine this grading to be even, i.e. we want the shift $(i)[2i]$ to correspond to $\langle 2i \rangle$, or in other words $A'_{2i} = \text{Hom}_{\mathcal{T}(G/Q)}(L, L(i)[2i])$ and $A'_{2i+1} = 0$. Then the graded category \mathcal{O} is defined by

$$\mathcal{O}_\lambda^{\mathbb{Z}}(\mathfrak{g}) \stackrel{\text{def}}{=} \text{mod}^{\mathbb{Z}}\text{-}A'$$

Denote by $B \cong B'$ the algebras analogously defined for P/Q_w and $\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})$.

The tilting equivalence from Theorem 3.1.2.20 can also be stated as an equivalence

$$\Delta : \text{MTDer}_{(B)}(G/Q) \xrightarrow{\sim} \text{Der}^b(\text{mod}^{\mathbb{Z}, \text{ev}}\text{-}A')$$

as discussed in Remark 3.1.2.21. This equivalence equips $\text{MTDer}_{(B)}(G/Q)$ with a t -structure, which is the Koszul dual of the perverse t -structure, see [SW16, Section 1.4]. We denote its heart by $\text{MTDer}_{(B)}(G/Q)^\heartsuit$. Hence mixed stratified Tate motives on the flag variety provide a geometric realization of

the evenly graded category \mathcal{O}

$$\begin{aligned} \text{MTDer}_{(B)}(G/Q)^\heartsuit &\cong \text{mod}^{\mathbb{Z},\text{ev}} - A' \cong \mathcal{O}_\lambda^{\mathbb{Z},\text{ev}}(\mathfrak{g}) \\ \text{MTDer}_{(B)}(P/Q_w)^\heartsuit &\cong \text{mod}^{\mathbb{Z},\text{ev}} - B' \cong \mathcal{O}_{w\cdot\lambda}^{\mathbb{Z},\text{ev}}(\mathfrak{l}) \end{aligned}$$

and we can use our geometric construction to show that parabolic induction is gradable.

Theorem 4.2.2.1. *Let λ be a dominant integral weight. Then there is a functor $\widehat{\text{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$ compatible with the shift of grading $\langle n \rangle$, making the following diagram commute*

$$\begin{array}{ccc} \mathcal{O}_{w\cdot\lambda}^{\mathbb{Z}}(\mathfrak{l}) & \xrightarrow{\widehat{\text{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_\lambda^{\mathbb{Z}}(\mathfrak{g}) \\ \downarrow v & & \downarrow v \\ \mathcal{O}_{w\cdot\lambda}(\mathfrak{l}) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_\lambda(\mathfrak{g}) \end{array}$$

Proof. In the notation of Theorem 4.2.1.1 consider the following diagram.

$$\begin{array}{ccc} \text{MTDer}_{(B)}(P/Q_w) & \xrightarrow{\overline{\text{GInd}}_w} & \text{MTDer}_{(B)}(G/Q) \\ \downarrow v & \begin{array}{c} \searrow \Delta \\ \text{Der}^b(\text{mod}^{\mathbb{Z},\text{ev}} - B') \xrightarrow{F} \text{Der}^b(\text{mod}^{\mathbb{Z},\text{ev}} - A') \\ \downarrow v \qquad \qquad \downarrow v \\ \text{Der}^b(\text{mod} - B) \xrightarrow{G} \text{Der}^b(\text{mod} - A) \end{array} & \downarrow v \\ \text{Der}^b(\mathcal{O}_{w\cdot\lambda}(\mathfrak{l})) & \xrightarrow{(\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}})^{\oplus n}} & \text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g})) \end{array}$$

The functors v on the very left and right are defined as in Theorem 4.2.1.1 and the functors v in the middle are forgetting the grading and using the isomorphisms $A \cong A'$ and $B \cong B'$. In fact, the trapezia on the left and the right commute (up to natural isomorphism). See Remark 4.2.2.2 for an expanded version. Both F and G denote the functors induced by the equivalences. Then by definition

$$\begin{aligned} F &= \overline{F}\langle i_1 \rangle \oplus \cdots \oplus \overline{F}\langle i_n \rangle \\ G &= \overline{G}^{\oplus n}. \end{aligned}$$

split into direct summands. G is clearly t -exact since $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is, hence we are precisely in the setting of Proposition 4.2.2.3, which gives us a natural equivalence of functors $v\overline{F}_0 \cong \overline{G}_0 v$ and thereby a commutative diagram

$$\begin{array}{ccc}
\text{MTDer}_{(B)}(P/Q_w)^\heartsuit & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/Q)^\heartsuit \\
\downarrow v & \swarrow \Delta \sim & \searrow \Delta \sim \\
\mathcal{O}_{w \cdot \lambda}^{\mathbb{Z}, \text{ev}}(\mathfrak{l}) = \text{mod}^{\mathbb{Z}, \text{ev}}-B' & \xrightarrow{\bar{F}_0} & \text{mod}^{\mathbb{Z}, \text{ev}}-A' = \mathcal{O}_\lambda^{\mathbb{Z}, \text{ev}}(\mathfrak{g}) \\
\downarrow v & & \downarrow v \\
\text{mod-}B & \xrightarrow{\bar{G}_0} & \text{mod-}A \\
\swarrow \sim & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \searrow \sim \\
\mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) & & \mathcal{O}_\lambda(\mathfrak{g}).
\end{array}$$

Clearly \bar{F}_0 commutes with the shift of grading, since it is induced by $\text{GInd}_w = \mathfrak{h}_{w,*} \text{pr}_w^!$, which commutes with (i)[2i]. So \bar{F}_0 is a grading of parabolic induction for the evenly graded category \mathcal{O} . This can be easily extended to the whole graded category \mathcal{O} since $\mathcal{O}^{\mathbb{Z}} = \mathcal{O}^{\mathbb{Z}, \text{ev}} \oplus \mathcal{O}^{\mathbb{Z}, \text{ev}}\langle 1 \rangle$. \square

Remark 4.2.2.2. The following diagram commutes (up to natural isomorphism)

$$\begin{array}{ccc}
\text{Hot}^b(\text{MTDer}_{(B)}(G/Q)_{w=0}) & \xleftarrow[\sim]{\Delta} & \text{MTDer}_{(B)}(G/Q) \\
\parallel & & \downarrow \wr \Delta \\
\text{Hot}^b(\langle \mathcal{IC}_x \mid x \in \mathcal{W} \rangle_{\oplus, (1)[2]}^{\mathcal{T}(G/Q)}) & \xrightarrow{\sim} & \text{Der}^b(\text{mod}^{\mathbb{Z}}-A') \\
\mathbb{H} \downarrow \wr & & \downarrow \wr \\
\text{Hot}^b(\langle \mathbb{H}(\mathcal{IC}_x) \mid x \in \mathcal{W} \rangle_{\oplus, (2)}^{C_G^Q\text{-mod}^{\mathbb{Z}}}) & \xrightarrow{\sim} & \text{Der}^b(\text{mod}^{\mathbb{Z}}-\mathbb{H}(A')) \\
v \downarrow & & \downarrow v \\
\text{Hot}^b(\langle v\mathbb{H}(\mathcal{IC}_x) \mid x \in \mathcal{W} \rangle_{\oplus}^{C_G^Q\text{-mod}}) & \xrightarrow{\sim} & \text{Der}^b(\text{mod-}v\mathbb{H}(A')) \\
\downarrow \wr & & \downarrow \wr \\
\text{Hot}^b(\langle \mathbb{V}P_{\mathfrak{g}}(x \cdot \lambda) \mid x \in \mathcal{W} \rangle_{\oplus}^{C_{\mathfrak{g}}^\lambda\text{-mod}}) & \xrightarrow{\sim} & \text{Der}^b(\text{mod-}\mathbb{V}(A)) \\
\wr \uparrow \mathbb{V} & & \uparrow \wr \\
\text{Hot}^b(\langle P_{\mathfrak{g}}(x \cdot \lambda) \mid x \in \mathcal{W} \rangle_{\oplus}^{\mathcal{O}_\lambda(\mathfrak{g})}) & \xrightarrow{\sim} & \text{Der}^b(\text{mod-}A) \\
\parallel & & \uparrow \wr \\
\text{Hot}^b(\text{Proj } \mathcal{O}_\lambda(\mathfrak{g})) & \xrightarrow{\sim} & \text{Der}^b(\mathcal{O}_\lambda(\mathfrak{g}))
\end{array}$$

where the horizontal arrows are the obvious equivalences and we denote

$$\begin{aligned}\mathbb{H}(A') &\stackrel{def}{=} \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{C_G^Q\text{-mod}^{\mathbb{Z}}}(\mathbb{H}L, \mathbb{H}L\langle 2i \rangle), \\ v\mathbb{H}(A') &\stackrel{def}{=} \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{C_G^Q\text{-mod}}(v\mathbb{H}L, v\mathbb{H}L\langle 2i \rangle) \\ &= \mathrm{End}_{C_G^Q\text{-mod}}(v\mathbb{H}L) \text{ and} \\ \mathbb{V}(A) &\stackrel{def}{=} \mathrm{End}_{C_{\mathfrak{g}}^\lambda\text{-mod}}(\mathbb{V}P).\end{aligned}$$

Proposition 4.2.2.3. *Let A and B be finite dimensional graded \mathbb{C} -algebras and*

$$v : A\text{-mod}^{\mathbb{Z}} \rightarrow A\text{-mod} \text{ and } B\text{-mod}^{\mathbb{Z}} \rightarrow B\text{-mod}$$

be the functors forgetting the grading. Assume that there is a diagram

$$\begin{array}{ccc}\mathrm{Der}^b(B\text{-mod}^{\mathbb{Z}}) & \xrightarrow{F} & \mathrm{Der}^b(A\text{-mod}^{\mathbb{Z}}) \\ \downarrow v & & \downarrow v \\ \mathrm{Der}^b(B\text{-mod}) & \xrightarrow{G} & \mathrm{Der}^b(A\text{-mod})\end{array}$$

commuting up to natural isomorphism, and that F commutes with the shift of grading. Then the following statements hold.

1. *F is exact with respect to the standard t -structure if and only if G is.*
2. *Assume that F splits into a direct sum*

$$F = \overline{F}\langle i_1 \rangle \oplus \cdots \oplus \overline{F}\langle i_n \rangle.$$

of shifted versions of a functor \overline{F} . Then F is exact if and only if \overline{F} is exact.

3. *Assume that F and G are exact and that there are functors \overline{F} , \overline{G} such that*

$$\begin{aligned}F &= \overline{F}\langle i_1 \rangle \oplus \cdots \oplus \overline{F}\langle i_n \rangle \\ G &= \overline{G}^{\oplus n}.\end{aligned}$$

Denote the induced functors on the heart of the t -structure by F_0 , G_0 , \overline{F}_0 , \overline{G}_0 . Then the following diagram commutes up to natural isomorphism

$$\begin{array}{ccc}
B\text{-mod}^{\mathbb{Z}} & \xrightarrow{\overline{F}_0} & A\text{-mod}^{\mathbb{Z}} \\
\downarrow v & & \downarrow v \\
B\text{-mod} & \xrightarrow{\overline{G}_0} & A\text{-mod}
\end{array}
.$$

Proof. (1) Denote by $\mathcal{H}^i(C)$ the i -th cohomology of a complex C . Then clearly $\mathcal{H}^i(vC) = v\mathcal{H}^i(C)$ for all $C \in \text{Der}^b(A\text{-mod}^{\mathbb{Z}}), \text{Der}^b(B\text{-mod}^{\mathbb{Z}})$. Since the standard t -structure is defined by vanishing conditions on cohomology, the statement follows.

(2) As in (1).

(3) Since by assumption and points (1) and (2) \overline{F} and \overline{G} are exact, they restrict to the hearts of the t -structure, which are naturally isomorphic to $A\text{-mod}^{\mathbb{Z}}, B\text{-mod}^{\mathbb{Z}}, A\text{-mod}$ and $B\text{-mod}$, respectively. So the diagram makes sense. Now clearly $\overline{G}_0, G_0, \overline{F}_0$ and F_0 are exact functors. By a graded version of the Eilenberg-Watts Theorem [Wat60], this implies that there are natural isomorphisms

$$\begin{aligned}
\overline{F}_0 &\cong M \otimes_B - \text{ and} \\
\overline{G}_0 &\cong N \otimes_B -
\end{aligned}$$

for the (graded) B - A -bimodules $M = F_0(B), N = G_0(B)$ and hence

$$\begin{aligned}
F_0 &\cong \bigoplus_j M\langle i_j \rangle \otimes_B - \text{ and} \\
G_0 &\cong N^{\oplus n} \otimes_B -.
\end{aligned}$$

By assumption, there is a natural equivalence $G_0v \cong vF_0$ and hence

$$v(M)^{\oplus n} \cong v\left(\bigoplus_j M\langle i_j \rangle\right) \cong N^{\oplus n}.$$

Now decomposing both $v(M)$ and N into a finite direct sum of indecomposables and applying the Krull–Remak–Schmidt theorem implies that there is an isomorphism

$$v(M) \cong N$$

and hence a natural isomorphism

$$\overline{G}_0v \cong v\overline{F}_0.$$

The statement follows. \square

Appendix A

Deutsche Zusammenfassung

Graduierte Parabolische Induktion Sei $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ eine reductive Lie-Algebra mit einer Borelschen and Cartanschen Untereralgebra. Fixiere eine parabolische Untereralgebra $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b}$ und nenne ihren reductive Levi-Faktor $\mathfrak{p} \twoheadrightarrow \mathfrak{l}$. Bezeichne mit $\mathcal{W}_{\mathfrak{g}} \supset \mathcal{W}_{\mathfrak{l}}$ die Weyl-Gruppen von \mathfrak{g} und \mathfrak{l} .

Das Ziel dieser Arbeit ist die Konstruktion einer *graduerten* und *geometrischen* Version von *parabolischer Induktion* für Moduln in der BGG-Kategorie \mathcal{O} :

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}(\mathfrak{l}) \rightarrow \mathcal{O}(\mathfrak{g}), M \mapsto \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathrm{Res}_{\mathfrak{l}}^{\mathfrak{p}} M.$$

Wir werden unter anderem beweisen:

Theorem (Theorem 4.2.2.1). *Sei $\lambda \in \mathfrak{h}^*$ ein dominantes integrales Gewicht und w der kürzester Repräsentant einer Nebenklasse in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$. Dann existiert ein Funktor $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$, sodass das folgende Diagramm (bis auf natürlichen Isomorphismus) kommutiert*

$$\begin{array}{ccc} \mathcal{O}_{w \cdot \lambda}^{\mathbb{Z}}(\mathfrak{l}) & \xrightarrow{\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g}) \\ \downarrow v & & \downarrow v \\ \mathcal{O}_{w \cdot \lambda}(\mathfrak{l}) & \xrightarrow{\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathcal{O}_{\lambda}(\mathfrak{g}) \end{array}$$

und sodass $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}} \langle n \rangle \cong \langle n \rangle \widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$, wobei $\langle - \rangle$ die Graduierung verschiebt.

Hier bezeichnet $v : \mathcal{O}^{\mathbb{Z}} \rightarrow \mathcal{O}$ die *graduerte Kategorie* \mathcal{O} , (siehe [Soe90] und [BGS96]). In den Worten von [Str03], wo ähnliche Fragen für Verschiebungsfunktoren diskutiert werden, impliziert dieses Theorem, dass parabolische Induktion (zumindest für integrale Blöcke) *graduierbar* ist. Wir konstruieren $\widehat{\mathrm{Ind}}_{\mathfrak{p}}^{\mathfrak{g}}$ mit geometrischen Methoden, die wir nun beschreiben.

Geometrische Parabolische Induktion Beilinson und Ginzburg erklären in [BG86], wie die *gemischte Geometrie* von Fahnenvarietäten eine Graduierung von Kategorie \mathcal{O} induzieren sollte: Sie vermuten eine derivierte Äquivalenz zwischen jedem einzelnen Block von Kategorie $\mathcal{O}^{\mathbb{Z}}$ und einer Kategorie von *gemischten Garben* auf einer assoziierten Fahnenvarietät. Diese Vision wurde in [Soe90], [BGS96] und schlussendlich [SW16] realisiert, wo eine Äquivalenz von Kategorien (bis auf das Hinzufügen einer Wurzel des Tate-Twists)

$$\mathrm{MTDer}_{(B)}(G/Q) \xrightarrow{\sim} \mathrm{Der}^b(\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g}))$$

zwischen *stratifizierten gemischten Tate-Motiven* auf einer (partiellen) Fahnenvarietät G/Q für die Langlands-duale algebraische Gruppe G/\mathbb{C} und einem (singulären) Block $\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g})$ konstruiert wird. Diese Äquivalenz ist in der Tat eine Form von *Koszul-Dualität*: die perverse t-Struktur auf der linken Seite korrespondiert zu der Koszul-dualen t-Struktur auf der rechten Seite, und umgekehrt.

Stratifizierte gemischte Tate-Motive sind bestimmte konstruierbare motivische Garben. Sie verhalten sich ähnlich wie gemischte ℓ -adische Garben und gemischte Hodge-Moduln (mit dem Vorteil, dass die Tate-Motive $\mathbb{C}(n)$ nicht erweitern). Sie sind insbesondere mit einem Sechs-Funktor-Formalismus ausgestattet, den wir für die Konstruktion einer geometrischen Version von parabolischer Induktion benutzen können.

Theorem (Theorem 4.2.1.1 and 4.2.2.1). *Sei $\lambda \in \mathfrak{h}^*$ ein reguläres integrales Gewicht und w der kürzeste Repräsentant einer Nebenklasse in $\mathcal{W}_{\mathfrak{l}} \setminus \mathcal{W}_{\mathfrak{g}}$. Dann kommutiert das folgende Diagramm (bis auf natürlichen Isomorphismus)*

$$\begin{array}{ccc} \mathrm{MTDer}_{(B)}(P/B) & \xrightarrow[\mathrm{GInd}_w]{h_{w,*} \mathrm{pr}_w^!} & \mathrm{MTDer}_{(B)}(G/B) \\ \downarrow & & \downarrow \\ \mathrm{Der}^b(\mathcal{O}_{w \cdot \lambda}^{\mathbb{Z}}(\mathfrak{l})) & & \mathrm{Der}^b(\mathcal{O}_{\lambda}^{\mathbb{Z}}(\mathfrak{g})) \\ \downarrow v & & \downarrow v \\ \mathrm{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & \xrightarrow{\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \mathrm{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})) \end{array} \quad \begin{array}{c} v \\ \curvearrowright \\ v \end{array}$$

Hier korrespondiert $G \supset P \supset B$ zu $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{b}$ und der Funktor $\mathrm{GInd}_w = h_{w,*} \mathrm{pr}_w^!$, den wir **geometrische parabolische Induktion** nennen, ist definiert durch

$$P/B \xleftarrow{\mathrm{pr}_w} PwB/B \xrightarrow{h_w} G/B.$$

Tatsächlich beweisen wir eine allgemeinere Aussage, die auch singuläre Gewichte $\lambda \in \mathfrak{h}^*$ beinhaltet. Mit ihr können wir beweisen, dass parabolische Induktion auch in diesem Fall graduierbar ist.

Soergelsche Moduln Zum Beweis dieser Theoreme benutzen wir die kombinatorische Beschreibung von derivierten Blöcken der Kategorie \mathcal{O} und stratifizierten gemischten Tate-Motiven auf Fahnenvarietäten durch die Homotopiekategorie von *Soergelschen Moduln*.

Seien $C = H^*(G/B, \mathbb{C}) \rightarrow C' = H^*(P/B, \mathbb{C})$ die Kohomologieringe von $G/B \supset P/B$. Für einen reduzierten Ausdruck $w = s_n \dots s_1 \in \mathcal{W}_{\mathfrak{g}}$, definiere den folgenden Komplex von Soergelschen Bimoduln über C

$$\underline{R}_w \stackrel{def}{=} R_{s_1} \otimes_C \dots \otimes_C R_{s_n}, \text{ wobei}$$

$$R_s \stackrel{def}{=} \dots \rightarrow 0 \rightarrow C \rightarrow C \otimes_{C^s} C\langle 2 \rangle \rightarrow 0 \rightarrow \dots$$

Wir werden zeigen:

Theorem (Theorem 2.3.4.1, 3.3.3.1 und 4.2.1.1). *Sei $\lambda \in \mathfrak{h}^*$ ein dominantes reguläres Gewicht und w der kürzeste Repräsentant einer Nebenklasse in $\mathcal{W}_1 \setminus \mathcal{W}_{\mathfrak{g}}$. Dann kommutiert das folgende Diagramm (bis auf natürlichen Isomorphismus)*

$$\begin{array}{ccc}
 \text{MTDer}_{(B)}(P/B) & \xrightarrow{\text{GInd}_w} & \text{MTDer}_{(B)}(G/B) \\
 \Delta \downarrow \wr & & \wr \downarrow \Delta \\
 \text{Hot}^b(\text{MTDer}_{(B)}(P/B)_{w=0}) & & \text{Hot}^b(\text{MTDer}_{(B)}(G/B)_{w=0}) \\
 \mathbb{H} \downarrow \wr & & \wr \downarrow \mathbb{H} \\
 \text{Hot}^b(C' \text{-Smod}^{\mathbb{Z}, ev}) & \xrightarrow{\text{SInd}_w} & \text{Hot}^b(C \text{-Smod}^{\mathbb{Z}, ev}) \\
 v \downarrow & & \downarrow v \\
 \text{Hot}^b(C' \text{-Smod}) & \xrightarrow{\text{SInd}_w} & \text{Hot}^b(C \text{-Smod}) \\
 \mathbb{V}_1^{w \cdot \lambda} \uparrow \wr & & \wr \uparrow \mathbb{V}_{\mathfrak{g}}^{\lambda} \\
 \text{Hot}^b(\text{Proj } \mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & & \text{Hot}^b(\text{Proj } \mathcal{O}_{\lambda}(\mathfrak{g})) \\
 \wr \uparrow & & \wr \uparrow \\
 \text{Der}^b(\mathcal{O}_{w \cdot \lambda}(\mathfrak{l})) & \xrightarrow{\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}} & \text{Der}^b(\mathcal{O}_{\lambda}(\mathfrak{g})).
 \end{array}$$

Hier bezeichnet $\text{SMod}^{\mathbb{Z}}$ die Kategorie der (graduierten) Soergelschen Moduln und

$$\text{SInd}_w : C' \text{-Smod}^{\mathbb{Z}, ev} \rightarrow C \text{-Smod}^{\mathbb{Z}, ev}, M \mapsto \underline{R}_w \otimes_C \text{Res}_{C'}^C M.$$

Auch hier beweisen wir eine allgemeinere Aussage, die auch singuläre Gewichte $\lambda \in \mathfrak{h}^*$ beinhaltet. Unsere Beweisstrategie ist die folgende: Zuerst zeigen wir die Aussage für $w = e$. Das ist der einfachste Fall, da hier $\text{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ projektive Moduln auf ebensolche abbildet und GInd_e exakt bezüglich der Gewichtsstruktur ist. Dann analysieren wir die Interaktion von (geometrischer) parabolischer Induktion und der (geometrischen) *Verschiebung durch die Wand*. Ein Vergleich der Resultate beweist dann den allgemeinen Fall mit einer Induktion über die Länge von w .

Bibliography

- [AL03] Henning Haahr Andersen and Niels Lauritzen, *Twisted verma modules*, Studies in memory of Issai Schur, Springer, 2003, pp. 1–26.
- [Ayo07] Joseph Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I*, Astérisque (2007), no. 314, x+466 pp. (2008). MR 2423375
- [Bei87] Alexander Beilinson, *On the derived category of perverse sheaves*, K-theory, Arithmetic and Geometry, Springer, 1987, pp. 27–41.
- [BG80] J. N. Bernstein and S. I. Gelfand, *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*, Compositio Math. **41** (1980), no. 2, 245–285. MR 581584
- [BG86] A. A. Beilinson and V. A. Ginzburg, *Mixed categories, Ext-duality and representations (results and conjectures)*, unpublished (1986).
- [BGG71] J. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Structure of representations that are generated by vectors of highest weight*, Funkcional. Anal. i Priložen. **5** (1971), no. 1, 1–9. MR 0291204 (45 #298)
- [BGG73] ———, *Schubert cells and cohomology of the spaces g/p* , Russian Mathematical Surveys **28** (1973), no. 3, 1.
- [BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527. MR 1322847 (96k:17010)
- [Bon10] Mikhail V. Bondarko, *Weight structures vs. t -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory **6** (2010), no. 3, 387–504. MR 2746283

- [Bou02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR 1890629
- [BT72] Armand Borel and Jacques Tits, *Compléments à l'article: “Groupes réductifs”*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 253–276. MR 0315007
- [CD12] Denis-Charles Cisinski and Frédéric Déglise, *Triangulated categories of mixed motives*, Arxiv preprint arXiv:0912.2110 (2012).
- [Dre15] Brad Drew, *Rectification of deligne’s mixed hodge structures*, 2015.
- [EK16] Jens Niklas Eberhardt and Shane Kelly, *Mixed motives and geometric representation theory in equal characteristic*, 2016.
- [Gin91] Victor Ginsburg, *Perverse sheaves and \mathbb{C}^* -actions*, Journal of the American Mathematical Society (1991), 483–490.
- [GM88] Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR 932724
- [He07] Xuhua He, *Minimal length elements in some double cosets of Coxeter groups*, Advances in Mathematics **215** (2007), no. 2, 469 – 503.
- [How00] Roger E. Howe, *Harish-Chandra homomorphisms*, The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998), Proc. Sympos. Pure Math., vol. 68, Amer. Math. Soc., Providence, RI, 2000, pp. 321–332. MR 1767901
- [Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460
- [Hum98] J. Humphreys, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 21, Springer, 1998.
- [Hum08] James E. Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008. MR 2428237 (2009f:17013)

- [Irv93] Ronald S Irving, *Shuffled verma modules and principal series modules over complex semisimple lie algebras*, Journal of the London Mathematical Society **2** (1993), no. 2, 263–277.
- [Jan79] J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, Springer, 1979.
- [Kel93] Bernhard Keller, *A remark on tilting theory and dg algebras*, manuscripta mathematica **79** (1993), no. 1, 247–252.
- [Kel94] ———, *Deriving dg categories*, Annales scientifiques de l’Ecole normale supérieure, vol. 27, 1994, pp. 63–102.
- [Lur] Jacob Lurie, *Higher algebra*.
- [Ric89] Jeremy Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456. MR 1002456
- [RSW14] Simon Riche, Wolfgang Soergel, and Geordie Williamson, *Modular koszul duality*, Compositio Mathematica **150** (2014), no. 2, 273332.
- [Soe90] Wolfgang Soergel, *Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), no. 2, 421–445. MR 1029692 (91e:17007)
- [SS15] Antonio Sartori and Catharina Stroppel, *Categorification of tensor product representations of \mathfrak{sl}_k and category \mathcal{O}* , Journal of Algebra **428** (2015), 256 – 291.
- [Str03] Catharina Stroppel, *Category \mathcal{O} : gradings and translation functors*, Journal of Algebra **268** (2003), no. 1, 301–326.
- [SVW] Wolfgang Soergel, Rahbar Virk, and Matthias Wendt, *Equivariant motives and representation theory*, (To Appear).
- [SW16] Wolfgang Soergel and Matthias Wendt, *Perverse motives and graded derived category \mathcal{O}* , Journal of the Institute of Mathematics of Jussieu (2016), 1–49.
- [Wat60] Charles E Watts, *Intrinsic characterizations of some additive functors*, Proceedings of the American Mathematical Society **11** (1960), no. 1, 5–8.