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PhD Thesis

Motivic Cell Structures for Projective Spaces over Split Quaternions

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Introduction

The main goal of the present work is to endow the smooth affine homogeneous algebraic variety $\mathrm{HP}^n = \mathrm{Sp}_{2n+2} / \mathrm{Sp}_2 \times \mathrm{Sp}_{2n}$ with an explicit unstable motivic cell structure. We use methods that are likely to work also analogously for $\mathrm{OP}^1 = \mathrm{Spin}_9 / \mathrm{Spin}_8$, the *Cayley plane* $\mathrm{OP}^2 = \mathrm{F}_4 / \mathrm{Spin}_9$ and other isotropic Grassmannians. As a side-product, we see how these methods work for $\mathrm{CP}^n = \mathrm{PGL}_{n+1} / \mathrm{GL}_n$ as well.

The interest in these spaces is five-fold:

1. The complex analytifications are of an interesting homotopy type:

$$\mathrm{CP}^n(\mathbb{C})^{an} \simeq \mathbb{CP}^n, \ \mathrm{HP}^n(\mathbb{C})^{an} \simeq \mathbb{HP}^n, \ \mathrm{OP}^n(\mathbb{C})^{an} \simeq \mathbb{OP}^n.$$

The gluing maps for the second nontrivial cell onto the first nontrivial cell are exactly the famous Hopf invariant one *Hopf maps*, which are much studied in classical algebraic topology. A motivic cell structure on the variety which is compatible with this structure therefore provides a lift of these Hopf maps, indeed motivic Hopf invariant one elements. In Chapter 1 we will go into more detail about this classical story. The projective lines over split composition algebras have been used already by Hasebe to define such Hopf maps [Has10], and Dugger and Isaksen use a more homotopical construction [DI13] to define motivic Hopf elements.

2. The spaces CP^n, HP^n, OP^1, OP^2 are rank one homogeneous spherical varieties. In the classification theory of spherical varieties (normal algebraic varieties with a reductive group action such that a Borel group has a dense orbit), one has the *rank* invariant. The rank of a homogeneous spherical variety (while definable for arbitrary G-varieties) can be seen as the number of irreducible components of a boundary divisor of a wonderful completion, cf. , if one exists (compare [Tim11] for general definitions and [Pez10, Proposition 3.3.1, p.48] for a proof that this agrees with the usual definition of rank). It is easy to see that every spherical variety admits a stable motivic cell structure, cf. Fact 2.3.15. To explicitly describe (unstable) motivic cell structures, it is sensible to proceed from the easy to the more difficult cases. Rank 0 spherical varieties are complete, and we already have methods to endow these generalized flag varieties with motivic cell structures [Wen10], using an algebraic variant of Morse theory developed by Białynicki-Birula [BB73]. The rank 1 primitive spherical varieties are classified into seven infinite families and nine exceptional cases, in characteristic 0 by Ahiezer [Ahi83] (see also [CF03]) and in characteristic p by Knop [Kno14]. Surprisingly, the completion of a rank 1 primitive spherical variety is again a homogeneous space, although under a different group. We devote Remark 2.2.6 to a computation of the motive of some rank 1 spherical varieties by interpreting the relationship between these two reductive groups as a relation of the Dynkin diagrams. This is intended to explain why the motivic cell structure is more complicated than the topological one.

3. We have an algebraic vector bundle classification theory by work of Morel [Mor12], generalized by Asok, Hoyois and Wendt in a series of papers [AHW16, AHW15] to more general G-torsors. Information on the homotopy type of homogeneous spaces can be used to infer information on algebraic G-torsors on any space using obstruction theory. We briefly discuss the topological component of this application in Section 5.2.

4. Hermitian K-theory is representable in the stable $\mathbb{A}^1$-homotopy category by a spectrum constructed from quaternionic Grassmannians $\mathrm{HGr}(m, n+m)$ of which $\mathrm{HGr}(1, n+1) = \mathrm{HP}^n$ is a special case. A motivic cell structure similar to the one described in the present work endows the hermitian K-theory spectrum with a cell structure as well. We also describe a stable cell structure on $\mathrm{HGr}(m, n+m)$. This has applications in work of Hornbostel [Hor15] (where Spitzweck's proof of a stable cell structure is employed).

5. A motivic cell structure on HP^n has implications on the values of oriented cohomology theories. This was described without the language of motivic cell structures in a preprint of Panin and Walter [PW10b]. We do not discuss this in the present article, although it has been an important influence.

Main Result.

Theorem (Theorem 4.4.8)**.** *The split quaternionic projective space* HP^n *over a field* k *has an unstable motivic cell structure built as mapping cone over* HP^{n-1}*. In particular,* HP^1 *is a motivic sphere* $\mathrm{S}^{4,2}$*.*

Idea of the Proof. We use a hermitian matrix model for the projective space DP^n over a split composition algebra D. By the choice of a Lagrangian of the norm form, we define a vector bundle $V \twoheadrightarrow \mathrm{DP}^{n-1}$ with total space $V \hookrightarrow \mathrm{DP}^n$ and open complement X. By homotopy purity, the homotopy quotient DP^n / X is weakly equivalent to the Thom space over the normal bundle of $V \hookrightarrow \mathrm{DP}^n$, for which we can find explicit cells. By the canonical projection $p\colon \mathrm{DS}^n \twoheadrightarrow \mathrm{DP}^n$, we find an affine space $\widetilde{X}'$ such that p maps it to X. For C and H we prove that this restricted projection is an affine bundle, hence X is $\mathbb{A}^1$-contractible.

We review briefly the table of contents:

Chapter 1 is an introductory text, in which we recall the cell structure for the smooth manifold $\mathbb{DP}^n$ built from a division algebra $\mathbb{D}$ and the corresponding Hopf elements and their relevance. We explain the hermitian matrix model of projective space and why octonions do not allow projective spaces of dimension higher than two. Some classical applications of cell structures to bundle theory are discussed. The chapter closes with two constructions of cell structures on $\mathbb{CP}^n$.

The second chapter briefly recalls the motivic homotopy theory of Morel and Voevodsky and its relation to motives. It is by no means intended as an introduction and serves mostly to fix our notation. In the last section (Section 2.3), we discuss the concept of motivic cell structures of Dugger and Isaksen, which is central to this work. We also prove a crucial theorem on obtaining unstable cell structures for Thom spaces, see Theorem 2.3.20.

In the third chapter we discuss composition algebras over rings, the Cayley–Dickson construction and Zorn vector matrix notation for split composition algebras D. While this is classical material over a field, we see that the theory works flawlessly over a ring as well. Section 3.2 contains some computations that are used in the following chapter.

The fourth chapter introduces the main object of study, the projective space DP^n over a split composition algebra D. In Section 4.1 we introduce the relevant Jordan algebra of hermitian matrices over D, the projective space DP^n, the corresponding sphere DS^n and the canonical projection $p\colon \mathrm{DS}^n \twoheadrightarrow \mathrm{DP}^n$. Section 4.2 constructs the candidate X_n for the "big cell" of DP^n. In subsequent sections, we compute examples and prove theorems in the specific situations $D = \mathrm{C}, \mathrm{H}, \mathrm{O}$ (complex, quaternionic, octonionic).

The last chapter (Chapter 5) discusses some applications.

To the best knowledge of the author, the existence and description of the unstable motivic cell structure of HP^n in Theorem 4.4.8 was not available in the literature before, although the 2010 preprint of Panin and Walter almost obtains a similar motivic cell structure by different means. The work of Panin and Walter clearly was an important predecessor of the present work. The main difference of the present proof to that one is that we intended to find a common proof for all split composition algebras, methods that work for other Jordan algebras and all characteristics to obtain *explicit unstable cell structures*. Other new results in this work include the method of obtaining unstable cell structures for certain Thom spaces, Theorem 2.3.20. It is likely that it was well-known to the experts that the Cayley-Dickson process and the theory of D-projective spaces works over any commutative ring, but this was not available in the literature. We prove this in Chapter 3 and Section 4.1. The Observation 2.2.5 on the structure of rank 1 spherical varieties was discovered by the author, and used to compute the motives of rank 1 spherical varieties. The author discovered later that Landsberg and Manivel had independently made the same observation [LM01, Section 4.2]. Two technical results not due to the author were not easily available in the literature, which is why we provide a proof here: the folklore Lemma 2.1.3 that affine Zariski bundles are $\mathbb{A}^1$-equivalences and Wendt's result Lemma 2.1.14 that vector bundle projections are sharp maps.

This work leaves open many questions which might be settled soon, for example a more explicit description of the cell structure and a generalization to the Cayley plane. We intend to publish a more detailed account of Remark 2.2.6 together with an investigation of the corresponding Jordan algebras separately later.

The original approach was to obtain unstable motivic cell structures via Białynicki-Birula decompositions, which seems to be difficult, compare Remark 2.3.28.

We advise the reader to think about HP^n over a field of characteristic 0 on the first reading to get an overview over the main argument.

Assumption. We assume the reader to have some familiarity with homotopy theory, algebraic geometry and the theory of reductive groups and their representations. Some knowledge of motivic homotopy theory is required, and we only repeat some definitions to fix notation. We use the theory of motives as a black box to motivate our construction, but we do not use motives in the main argument. Both the theory of spherical varieties and of Jordan algebras are not prerequisites for reading this thesis, and all necessary parts are explained or referenced in the corresponding sections. References for the prerequisites and for further research are given in the corresponding sections.

Convention 0.0.1. Unless explicitly stated otherwise, we will deal with separated finite type schemes, not necessarily projective, defined over the integers. We call a separated finite type scheme over $\operatorname{Spec}\mathbb{Z}$ a *variety*. In some theorems, we work over a field k, which is explicitly mentioned then. All varieties that appear explicitly in this work are smooth over the base scheme, which we mention explicitly when appropriate. The algebraic groups which appear are linear groups, either additive groups or reductive linear groups split over the base. We use the notation $Z \hookrightarrow X$ for a closed immersion of Z into X and $U \overset{\circ}{\hookrightarrow} X$ for an open immersion of U into X throughout.

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I also want to thank my family, most importantly Lotta, for supporting me even when I was often absent, physically or otherwise. My whole family is to thank for encouraging and facilitating the experiment of having kids while writing a thesis.

A story. When I was a little child, once some friends of my parents gathered in the living room. Some of them had studied mathematics when they were young and one had stayed in academia and secured a permanent position in Paris. I had read about the real and complex numbers and the quaternions superficially and was wondering about them, so I asked "What comes next?". The quick answer was of course that the next step are the octonions. Like every child, I asked the question again. This time, the reaction was wildly different. Silence. After some thought, the math professor said, as if it were totally clear: "Algebras, just algebras". At that time, the answer mystified me and I kept an interest in the classical real division algebras since this magic moment.

If only I had stuck with the division algebras!

CHAPTER 1

The Topological Story

We introduce the reader to one of the main proponents of this thesis by briefly recalling the well-known analogue in classical topology, which we aim to generalize in following chapters. The entire chapter is meant to be an introduction to this work.

For everything related to projective spaces over the classical division algebras, we can only recommend the textbook by Dray and Manogue [DM15]. The well-known survey on the octonion division algebra [Bae02] of Baez has to be praised for its many references and may provide an easy introduction, as it does not contain too many proofs.

Fact. Let $\mathbb{D}$ be a division algebra over $\mathbb{R}$ (i.e. $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$) and $\mathbb{DP}^n$ the projective space over it (with $n \leq 2$ iff $\mathbb{D} = \mathbb{O}$). Then, fixing inclusions $\mathbb{DP}^{n-1} \hookrightarrow \mathbb{DP}^n$ determines the structure of a CW complex on $\mathbb{DP}^n$, with attaching maps isomorphic to the Hopf fibrations $\mathbb{D}^n \setminus \{0\} \twoheadrightarrow \mathbb{DP}^{n-1}$.

Proof idea. The complement of the inclusion $\mathbb{DP}^{n-1} \hookrightarrow \mathbb{DP}^n$ is $\mathbb{D}^n$, which is contractible. We can view $\mathbb{D}^n \hookrightarrow \mathbb{DP}^n$ as the (mapping) cone over $\mathbb{D}^n \setminus \{0\} \to \mathbb{DP}^{n-1}$. □

In Example 1.5.3 we give a more detailed proof of this fact for the case of $\mathbb{CP}^n$.

1.1. Hopf Elements

A cell structure for projective spaces over division algebras gives rise to interesting elements in the stable homotopy groups of spheres, the Hopf elements.

Remark 1.1.1. The attaching maps for $\mathbb{DP}^{n-1} \hookrightarrow \mathbb{DP}^n$ in the case $n = 1$ are maps

$$f_{\mathbb{D}} \colon \mathbb{D}^2 \setminus \{0\} \twoheadrightarrow \mathbb{DP}^1$$

and we have

$$\begin{aligned} \mathbb{R}^2 \setminus \{0\} &\simeq \mathrm{S}^1, & \mathbb{RP}^1 &\simeq \mathrm{S}^1, \\ \mathbb{C}^2 \setminus \{0\} &\simeq \mathrm{S}^3, & \mathbb{CP}^1 &\simeq \mathrm{S}^2, \\ \mathbb{H}^2 \setminus \{0\} &\simeq \mathrm{S}^7, & \mathbb{HP}^1 &\simeq \mathrm{S}^4, \\ \mathbb{O}^2 \setminus \{0\} &\simeq \mathrm{S}^{15}, & \mathbb{OP}^1 &\simeq \mathrm{S}^8 . \end{aligned}$$

The CW complex structure is not trivial, i.e. $\mathbb{DP}^2$ is not a bouquet of spheres, the homotopy class of the attaching map is nontrivial. These attaching maps yield nonzero elements in the homotopy groups of spheres, commonly denoted

$$\begin{aligned} \epsilon &:= [f_{\mathbb{R}}] \in \pi_1(\mathrm{S}^1), \\ \eta &:= [f_{\mathbb{C}}] \in \pi_3(\mathrm{S}^2), \\ \nu &:= [f_{\mathbb{H}}] \in \pi_7(\mathrm{S}^4), \end{aligned}$$

$$\sigma := [f_{\mathbb{O}}] \in \pi_{15}(\mathrm{S}^8).$$

Non-triviality can be shown by the Hopf invariant (see the following remark), which is a cohomological, hence stable invariant. The fact that the Hopf invariant of these elements is 1 is closely related to the property of projective planes that each two distinct lines intersect in precisely one common point, as the Hopf invariant can be shown to coincide with the linking number of the pre-image of two distinct points on $\mathbb{DP}^2$. This shows that the stable homotopy type of $\mathbb{DP}^2$ is also not split as a wedge sum of spheres, which has trivial attaching maps, hence Hopf invariants 0. One calls the images in stable homotopy groups *Hopf elements* and also writes

$$\epsilon \in \pi_0^s(\mathbb{S}),\ \eta \in \pi_1^s(\mathbb{S}),\ \nu \in \pi_3^s(\mathbb{S}),\ \sigma \in \pi_7^s(\mathbb{S}).$$

Remark 1.1.2. To a homotopy class of a map $\varphi\colon \mathrm{S}^{2n-1} \to \mathrm{S}^n$ (such as the $\epsilon, \eta, \nu, \sigma$ just mentioned) one can assign the *Hopf invariant* by looking at the cup product in $\mathrm{H}^n_{sing}(C_\varphi)$, the integral cohomology of the mapping cone over φ. As $\mathrm{H}^n_{sing}(C_\varphi)$ is generated by a single class α coming from S^n and $\mathrm{H}^{2n}_{sing}(C_\varphi)$ is generated by a single class β coming from the cone over S^{2n-1}, the cup product $\alpha \cup \alpha$ can be expressed as $h(\varphi)\beta$ where $h(\varphi)$ is an integer, called the Hopf invariant. For $\epsilon, \eta, \nu, \sigma$, the Hopf invariant is 1. One can show that the existence of a division algebra structure on $\mathbb{R}^{2n}$ implies the existence of a Hopf invariant 1 element $f_{\mathbb{R}^{2n}}$, and Adams famously proved [Ada60] that the only Hopf invariant 1 elements in the stable homotopy groups of spheres are in degrees $0, 1, 3, 7$, thereby proving that $\mathbb{R}^{2n}$ admits a division algebra structure only for $n \in \{1, 2, 4, 8\}$. This remarkable connection explains part of the importance of cell structures on projective spaces over division algebras.

1.2. Hermitian Matrices and Octonions

We describe the hermitian matrix model for projective spaces over division algebras.

One can model $\mathbb{DP}^n$ by understanding its points not as lines in $\mathbb{D}^{n+1}$ (which is not straightforward for the quaternions, and troubling for the octonions), but as projectors onto these lines. To avoid having multiple representatives for a single line, one can stick to hermitian matrices (which form a Jordan algebra in characteristic 0, hence the letter J will be used). A vector $v \in \mathbb{D}^{n+1} \setminus \{0\}$ determines a matrix $vv^\dagger \in \mathrm{Mat}^{n+1}(D)$ (writing $v^\dagger$ for the conjugate-transpose) which, up to a scalar, projects every vector $w \in \mathbb{D}^{n+1}$ onto the v-line in $\mathbb{D}^{n+1}$, as $vv^\dagger w = v\langle v, w\rangle$. As soon as we take a normed vector v, with $|v| = 1 = \langle v, v\rangle = v^*v$, the matrix vv^* is precisely the projector onto v. The space of hermitian projectors of rank one

$$J\mathbb{DP}^n := \{A \in \mathrm{Mat}^{n+1}(\mathbb{D}) \mid A = A^*,\ A^2 = A,\ \mathrm{rk}(A) = 1\}$$

is a homogeneous space, isomorphic to $\mathbb{DP}^n$:

$$J\mathbb{RP}^n \xleftarrow{\sim} \mathrm{SO}(n+1)/S(\mathrm{O}(1) \times \mathrm{O}(n)) \xrightarrow{\sim} \mathbb{RP}^n$$
$$J\mathbb{CP}^n \xleftarrow{\sim} \mathrm{SU}(n+1)/S(\mathrm{U}(1) \times \mathrm{U}(n)) \xrightarrow{\sim} \mathbb{CP}^n$$
$$J\mathbb{HP}^n \xleftarrow{\sim} \mathrm{Sp}(n+1)/\,\mathrm{Sp}(1) \times \mathrm{Sp}(n) \xrightarrow{\sim} \mathbb{HP}^n$$
$$J\mathbb{OP}^2 =: \mathbb{OP}^2 \xleftarrow{\sim} \mathrm{F}_4\,/\,\mathrm{B}_4$$

where, as usual, we just define the octonionic plane via the hermitian matrix model. The idea of using idempotents in hermitian matrices to construct an octonionic projective plane seems to go back to Jordan [Jor49]. The homogeneous space structure was investigated first by Borel [Bor50].

One can define $\mathrm{U}_n(\mathbb{D})$ in a way such that $\mathrm{SU}_n(\mathbb{R}) = \mathrm{SO}(n)$, $\mathrm{SU}_n(\mathbb{C}) = \mathrm{SU}(n)$, $\mathrm{U}_n(\mathbb{H}) = \mathrm{Sp}(n)$, then $\mathbb{DP}^n = \mathrm{SU}_{n+1}(\mathbb{D})/S(\mathrm{U}_1(\mathbb{D}) \times \mathrm{U}_n(\mathbb{D}))$.

Observe that the fibration $\mathbb{D}\,\mathrm{S}^n := \{v \in D^n \mid |v| = 1\} \twoheadrightarrow \mathbb{DP}^n$ sending a nonzero vector v to the line it spans (or, in the Jordan model, to the projector $vv^\dagger$ onto that line), is the quotient map after a $U_1(\mathbb{D})$-action, and $\mathbb{D}\,\mathrm{S}^n = \mathrm{U}_{n+1}(\mathbb{D})/\,\mathrm{U}_n(\mathbb{D})$.

This story breaks down for $\mathbb{D} = \mathbb{O}$, as we have $\mathrm{U}_1(\mathbb{R}) = \mathrm{SO}(1) = \mathrm{S}^0$, $\mathrm{U}_1(\mathbb{C}) = \mathrm{SU}(1) = \mathrm{S}^1$, $\mathrm{U}_1(\mathbb{H}) = \mathrm{Sp}(2) = \mathrm{S}^3$, but S^7 does not admit a Lie group structure (see the next section for an explanation).

1.3. Why Octonions are Bad

We will now discuss why there is no octonionic projective space beyond dimension two.

The classical reason is that the theorem of Desargues holds for any projective incidence geometry of dimension greater than two by an elementary argument, and it implies that the incidence geometry can be coordinatized by an associative coordinate algebra. This must clearly fail for the non-associative octonions, so we know that there can be no such incidence geometry.

The naive definition of $\mathbb{OP}^n$ as vectors in $\mathbb{O}^{n+1} \setminus \{0\}$ modulo left scalar multiplication has the problem that the equality up to left scalar multiplication fails to be an equivalence relation. If one takes the generated equivalence relation, the quotient space becomes contractible.

If we just try to use a definition of $\mathbb{OP}^n$ via hermitian matrices as above, it turns out that the set $J_{n+1}(\mathbb{O})$ of hermitian matrices over the octonions is no longer a J-algebra (or Jordan algebra) for $n > 2$, in particular it is not power-associative, so that the projector condition does not behave correctly. This prevents us from extending this definition to higher dimensions.

Another reason is of homotopical nature: if we require any serious candidate for an octonionic projective space of dimension 3 to contain OP^2 and an additional cell, such that the cohomology is $\mathbb{Z}[x]/x^4$ with $\deg(x) = 8$, then there is no such space. This can be shown using Steenrod operations [Hat02, Corollary 4L.10, p. 498].

Another homotopical oddity appears with the octonions: The fiber of a Hopf fibration coming from a division algebra (or split composition algebra) consists of the units in the algebra, which is a Lie group (resp. algebraic group) structure on a sphere (resp. affine quadric) for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (resp. C and H, to be introduced in Chapter 3), but for the octonions $\mathbb{O}$ it is an H-space structure on S^7, which does not admit a Lie group structure at all. This can be shown by a cohomological argument: for any non-abelian Lie group G,

$$\mathrm{H}^3_{sing}(G;\mathbb{R}) \xrightarrow{\sim} \mathrm{H}^3_{dR}(G;\mathbb{R}) \xrightarrow{\sim} \mathrm{H}^3(\mathrm{Lie}(G);\mathbb{R}).$$

The (bi-invariant, hence closed) non-vanishing Cartan 3-form $(x, y, z) \mapsto \langle [x, y], z\rangle$ shows that $\mathrm{H}^3(\mathrm{Lie}(G)) \neq 0$. Since $\mathrm{H}^3_{sing}(\mathrm{S}^7; \mathbb{R}) = 0$, the 7-sphere is not a Lie group.

After this discussion, we may rightly ask: Why are there a projective line and plane over the octonions? This question was answered by Stasheff, who showed the equivalence of the existence of a projective space over an algebra of a given dimension n to an associativity condition A_n [Sta63, Section 5]. The octonions are an alternative algebra, which implies that any sub-algebra generated by 2 elements is associative. This is enough to define a projective plane.

Stasheff's theorem relies on the following observation: The infinite projective space $\mathbb{P}^\infty$ is the classifying space of the multiplicative group $\mathbb{G}_m$, denoted $\mathrm{B}\,\mathbb{G}_m = \mathbb{P}^\infty$. We can regard other classifying spaces as generalizations of the infinite projective space and may also filter it by a flag of finite-dimensional subspaces like $\mathbb{P}^n$. If we use the unit group of a division algebra, we get the corresponding projective spaces over the division algebra. Here the fact that S^7 is just an H-group with the A_2-property, explains why there is no $\mathrm{B}\,\mathrm{S}^7$, but only the first and second stage of approximation.

For the octonionic projective line $\mathbb{OP}^1$, the non-associativity has very little consequences, as $p\colon \mathbb{OS}^1 \twoheadrightarrow \mathbb{OP}^1,\ v \mapsto vv^\dagger$ involves only multiplications of two elements of $\mathbb{O}$ each and the same holds for the property $(vv^\dagger)^2 = vv^\dagger$.

The latter fails for $\mathbb{OP}^2$, where for $v \in \mathbb{O}^3$ with $|v| = 1$, in general $(vv^\dagger)^2 \neq vv^\dagger$. This problem can be ameliorated by looking at the subspace $\mathbb{OS}^{2,a} := \{v \in \mathbb{O}^3 \mid |v| = 1,\ \mathrm{assoc}(v) = 0\}$, where $\mathrm{assoc}(v)$ is the 3-dimensional vector consisting of the associators $\{v_i, v_j^*, v_j v_k^*\}$ for $(i, j, k) \in ((2, 0, 1), (0, 1, 2), (1, 2, 0))$. Since $\mathbb{OS}^{2,a}$ is no longer a sphere, this introduces other problems.

1.4. Vector Bundles

Let G be a Lie group, e.g. $\mathrm{GL}(n, \mathbb{C})$.

Definition 1.4.1. A *classifying space* $\mathrm{B}\,G$ of G is a homotopy quotient $\mathrm{B}\,G := * /\!\!/ G$ (defined up to weak equivalence).

Given any contractible free G-space EG, the ordinary quotient EG/G is a classifying space $\mathrm{B}\,G$ for G.

Theorem 1.4.2 (Pontryagin–Steenrod)**.** *Let X be a paracompact Hausdorff space and G a Lie group. We have a natural isomorphism between homotopy classes and isomorphism classes:*

$$[X, \mathrm{B}\,G] \xrightarrow{\sim} \{V \to X \text{ a principal } G\text{-bundle}\}/_{\simeq}.$$

This was proved by Steenrod [Ste51, 19.3].

Remark 1.4.3. Given a monomorphism of connected Lie groups $H \to G$, we have an induced fibration of classifying spaces $\mathrm{B}\,H \to \mathrm{B}\,G$ with homotopy fiber G/H. Applying the Hom-functor $[X, -]$ in the homotopy category to this fiber sequence, we get an exact sequence (for connected X)

$$[X, G/H] \to [X, \mathrm{B}\,H] \to [X, \mathrm{B}\,G],$$

where exactness is taken in the sense of pointed sets, with the distinguished point given by the constant map to the basepoint of $\mathrm{B}\,H$, $\mathrm{B}\,G$ or G/H, each of which is induced by the identity element of G. Applying Steenrod's theorem, we see that obstructions for a principal H-bundle to be trivial as a G-bundle live in $[X, G/H]$.

In *obstruction theory*, one can define obstruction classes in the $\pi_*(G/H)$-valued cohomology of X, whose vanishing answers the question whether a given G-bundle admits reduction of the structure group to H, i.e. whether it is induced from an H-bundle.

Remark 1.4.4. If X admits a cell structure with no cells above dimension n (e.g. if X is a $\leq n$-dimensional smooth manifold), and G/H is connected and admits a cell structure with no cells in dimensions 1 to $n+1$, then by cellular approximation all continuous maps $X \to G/H$ are homotopic to cellular ones, which map d-cells to d-cells or lower, hence all of X to 0-cells in G/H. In that case, $[X, G/H] = 1$.

Example 1.4.5. Knowledge on the homotopy type of a homogeneous space G/H implies statements on principal bundles:

(1) For $\mathrm{U}(n, \mathbb{C}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$, we have $\mathrm{GL}(n, \mathbb{C}) / \mathrm{U}(n, \mathbb{C})$ contractible by polar decomposition of matrices (Iwasawa decomposition), hence every complex vector bundle admits a unitary structure, unique up to isomorphism.
(2) For $\mathrm{O}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$, we have $\mathrm{GL}(n, \mathbb{R}) / \mathrm{O}(n, \mathbb{R})$ contractible by polar decomposition of matrices again, hence every real vector bundle admits a metric, unique up to isomorphism.
(3) For $\mathrm{SO}(n, \mathbb{R}) \hookrightarrow \mathrm{O}(n, \mathbb{R})$, we have $\mathrm{O}(n, \mathbb{R}) / \mathrm{SO}(n, \mathbb{R}) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$, the obstruction for reduction of the structure group from $\mathrm{O}(n, \mathbb{R})$ to $\mathrm{SO}(n, \mathbb{R})$ is precisely orientability, with $\mathbb{Z}/2\mathbb{Z}$ corresponding to the two possible orientations of an orientable bundle with a metric.

Remark 1.4.6. The obstructions for G-bundles on a space X to admit reduction of the structure group to H are in $H^{n+1}(X, \pi_n(G/H))$ (as one can construct a Moore-Postnikov-tower for $X \to BG$). If G/H admits a cell structure with lowest cell of dimension $n+1$ (so G/H is n-connected), and X is a smooth manifold of dimension n (so that all cohomology in degrees above n vanishes), these obstruction classes are always vanishing, so that every G-bundle on X is induced from an H-bundle on X.

Example 1.4.7. A concrete application is given by rank r vector bundles on dimension d smooth manifolds: if $r > d$, then every rank r vector bundle splits as direct sum of a rank d vector bundle with a trivial rank $r - d$ vector bundle. This can be seen by associating an $O(r)$-bundle to a rank r vector bundle (by choice of a metric) and observing that $O(r)/O(r-1) = S^{r-1}$ is $(r-2)$-connected, so that by the previous remark, all obstructions vanish.

1.5. Cell Structures on Projective Spaces

We will briefly introduce the two main ways of constructing CW complex structures on projective spaces in topology.

Definition 1.5.1. We compare two concepts of cellularity for Hausdorff spaces:

(1) Call a discrete space 0-cellular. Inductively a space X is n-cellular for $n \in \mathbb{N}$ if it is homotopy equivalent to the mapping cone of a continuous map $\bigvee_{i \in I} \mathrm{S}^{n_i} \to Y$ for a set of non-negative numbers $\{n_i\}_{i \in I}$ and an $n-1$-cellular space Y. We call a space X *CW complex* if it is the direct limit over an increasing filtration of X

by n-cellular spaces, with increasing n. The topology on such a direct limit is the closure-finite weak topology, hence the name CW for such spaces. Any fixed filtration by n-cellular spaces is called a *CW structure*.

(2) Let $\mathcal{C}$ be the smallest class of spaces, closed under weak homotopy equivalence and homotopy colimits that contains all discrete spaces and all spheres. We call $X \in \mathcal{C}$ *cellular* and any explicit diagram D consisting of D_i weakly equivalent to a sphere or a discrete space such that $X = \operatorname{hocolim}(D)$ a *cell structure*. If we replace "closed under weak homotopy equivalence" by "closed under homotopy equivalence" we speak of *strictly cellular* and *strict cell structures*. If the diagram D of a cell structure is finite, the cell structure is called *finite*.

Remark 1.5.2. Inductively, every n-cellular space is cellular. Given a CW complex X, the filtration by n-cellular spaces consists of cofibrations, hence the direct limit presentation is in fact a homotopy colimit, so X is cellular. One may more brutally also apply CW approximation, which gives a CW model Y and a weak equivalence $Y \to X$. In the other direction, cellular spaces are a more general concept, as by CW approximation, every Hausdorff space is cellular in this weak sense. For strictly cellular spaces, it is not immediately clear whether the class of cellular spaces might be larger. However, given a strict cell structure for a cellular space, one may easily construct a filtration on the diagram and therefore a CW structure.

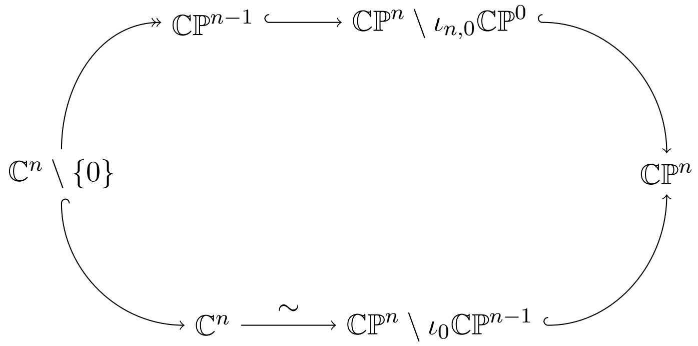
Example 1.5.3. We take two inclusions

$$\begin{aligned} \iota_0 \colon \mathbb{CP}^{n-1} &\hookrightarrow \mathbb{CP}^n, & [x_1 : \cdots : x_n] &\mapsto [0 : x_1 : \cdots : x_n] \\ \iota_{n,0} \colon \mathbb{CP}^0 &\hookrightarrow \mathbb{CP}^n, & [1] &\mapsto [1 : 0 : \cdots : 0] \end{aligned}$$

We may cover $\mathbb{CP}^n$ by the two open complements

$$U := \mathbb{CP}^n \setminus \iota_0 \mathbb{CP}^{n-1}, \qquad V := \mathbb{CP}^n \setminus \iota_{n,0} \mathbb{CP}^0.$$

We may identify U with $\mathbb{C}^n$ by the map $[1 : x_1 : \cdots : x_n] \mapsto (x_1, \ldots, x_n)$ with obvious inverse. There is a retraction $V \twoheadrightarrow \mathbb{CP}^{n-1}$ by the map $[x_0 : \cdots : x_n] \mapsto [x_1 : \cdots : x_n]$. Inside U, there is a subset isomorphic to $\mathbb{C}^n \setminus \{0\}$, which admits a surjection onto $\mathbb{CP}^{n-1}$ by quotient after the $\mathbb{C}^\times$-action (which we take as the definition of $\mathbb{CP}^{n-1}$), commonly called *Hopf map*. We subsume these constructions in a diagram:



The horizontal arrows are weak homotopy equivalences, the arrow $\mathbb{C}^n \setminus \{0\} \hookrightarrow \mathbb{C}^n$ is a cofibration, $\mathbb{C}^n$ is contractible, hence the diagram is a homotopy cofiber diagram. In other words

$$\mathbb{C}^n \setminus \{0\} \twoheadrightarrow \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$$

is a homotopy cofiber sequence that endows $\mathbb{CP}^n$ with an explicit inductive cell structure by attaching a cell onto $\mathbb{CP}^{n-1}$ as cone over the Hopf map.

If we analyze this construction closely, we see that there was a choice of filtration for $\mathbb{CP}^n$, which corresponds to the choice of a full flag of sub-vector spaces in $\mathbb{C}^{n+1}$, and an application of the Weyl group S_n, namely in using an opposite $\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^n$ than the one from the chosen filtration. It is also interesting that one uses the 0-cell to attach the n-cell.

Example 1.5.4. As we noted in the previous example, the subset $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ is contractible. Let $N \twoheadrightarrow \mathbb{CP}^{n-1}$ be the normal bundle of the embedding of the complement $\mathbb{CP}^{n-1} \hookleftarrow\!\!\rightarrow \mathbb{CP}^n$. There is a homotopy cofiber sequence

$$\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} \overset{\circ}{\hookrightarrow} \mathbb{CP}^n \to \mathrm{Th}(N)$$

which shows that $\mathbb{CP}^n \to \mathrm{Th}(N)$ is a weak equivalence. As Thom spaces over cellular spaces are cellular, this shows that $\mathbb{CP}^n$ is cellular. Furthermore, it exhibits $\mathbb{CP}^n$ as iterated Thom space.

This explains why one would use the 0-cell to attach the n-cell, if one were to write down an explicit cell structure for $\mathbb{CP}^n$, as we will see in the next example.

The iterated Thom space structure is a more canonical cell structure than the previous one, as we did not use the Weyl group. On the other hand, we do not immediately see attaching maps for the cells, which is a drawback for applications.

Example 1.5.5. We now look at an iterated Thom space over trivial bundles, which is locally (but not globally) the situation for $\mathbb{CP}^n$. Let $M^0 := \{1\}$ a one-point space and M^i a series of smooth manifolds with $M^{i-1} \hookleftarrow\!\!\rightarrow M^i$ a closed embedding, $N^i \twoheadrightarrow M^{i-1}$ a series of trivial rank n_i bundles where each N^i is the normal bundle of $M^{i-1} \hookleftarrow\!\!\rightarrow M^i$. We know that

$$M^i \simeq \mathrm{Th}(N^i) \simeq \Sigma_+^{n_i} M^{i-1} \simeq \mathrm{S}^{n_i} \wedge \left(M^{i-1} \vee \mathrm{S}^0\right).$$

By recurrence and distributing $\wedge$ over $\vee$, applying $\mathrm{S}^n \wedge \mathrm{S}^m \simeq \mathrm{S}^{n+m}$, we get

$$M^i \simeq \bigvee_{j=0}^{i} \mathrm{S}^{\left(\sum_{k=0}^{j} n_k\right)}.$$

We see that the top-dimensional cell of M^i arises from the top-dimensional cell of M^{i-1}, and all other cells are made larger. If we remove the 0-cell from M^i, the remaining cells are suspensions of the cells of M^{i-1} with one point removed, which retracts onto M^{i-1}.

The difference between this situation and the space $\mathbb{CP}^n$ is that the attaching maps for $\mathbb{CP}^n$ are not homotopy-trivial, so the smash products are twisted, i.e. nontrivial Thom spaces instead. However, away from a single point, the Hopf map is a globally trivializable bundle, which is why we still get a cell structure.

CHAPTER 2

Motivic Homotopy Theory

In this chapter, we motivate $\mathbb{A}^1$-homotopy theory (also known as motivic homotopy theory) and the philosophy of motives and give the basic definitions (originally due to Morel and Voevodsky). The focus lies on the concept of a cell structure, an analogue of the structure of a CW complex in algebraic geometry.

2.1. Motivic Spaces

One can give two main motivations for the subject of motivic homotopy theory: Grothendieck proposed the theory of motives, a universal cohomology theory with a very strong connection to the theory of algebraic cycles, hence K-theory and several important and hard questions in number theory and algebraic geometry. Just as ordinary singular cohomology is representable in the classical homotopy category, one may expect the same for this universal cohomology theory. As far as our current understanding of categories and functors of motives has advanced, a similar representability holds in the algebraic setting. The second reason, as given by Voevodsky, was to use more methods from algebraic topology in the realm of algebraic varieties, without restricting attention to the complex or real points and the analytic topology.

On a more technical level, the motivation for the specific construction of Morel and Voevodsky [MV99] can be described as the wish for a homotopy purity theorem Theorem 2.3.8 and representability of algebraic K-theory by $\mathbb{Z}$ times the infinite Grassmannian [MV99, Proposition 4.3.9] as well as representability of motivic cohomology and with it a way towards the solution of the Bloch–Kato conjecture (e.g. [Voe03]).

Fact 2.1.1. At the heart of motivic homotopy theory lies a canonical construction that one can define on any Grothendieck site with an interval object, in our case the site of smooth schemes Sm_S over a base scheme S equipped with the Nisnevich topology and the interval object $\mathbb{A}^1$. We presume the base scheme S to be separated and Noetherian of finite Krull dimension. We define the Nisnevich topology on Sm_S to be generated by finite families of étale maps $U_i \to X$ which admit for any $x \in X$ a $u \in U_i$ for some i such that the induced morphism of residue fields at u and x is an isomorphism. The construction proceeds by first enlarging the category Sm_S to its universal cocompletion, the category of presheaves $\mathrm{PShv}(\mathrm{Sm}_S)$, and taking simplicial objects in this category. Equivalently, we take presheaves with values in simplicial sets, so-called simplicial presheaves. This category is equipped with a class of weak equivalences , and a compatible model category structure (i.e. a class of fibrations and cofibrations satisfying the axioms for a simplicial monoidal model structure). The resulting ($\mathbb{A}^1$-local) model category of *motivic spaces*

$\mathrm{Spc}(S)$ can be constructed in several stages (changing only the weak equivalences and the model structure, not the category itself):

- The global model structure is defined as injective functor model category, i.e. cofibrations and weak equivalences are defined on sections of simplicial presheaves, as in the Kan model structure on simplicial sets (where cofibrations are just monomorphisms and weak equivalences are maps that induce isomorphisms on the homotopy groups of a geometric realization). Fibrations in the global model structure are defined by the right lifting property along acyclic cofibations (in contrast to pointwise fibrations).
- The local model structure incorporates the structure of the site by a homotopy localization (left Bousfield localization of the model structure) at hypercovers.
- The $\mathbb{A}^1$-local model structure incorporates the interval object $\mathbb{A}^1$ by a homotopy localization (left Bousfield localization) at projection maps $X \times \mathbb{A}^1 \to X$.

The general approach was first described by Morel and Voevodsky [MV99], building on work of Jardine. It is well described by Dugger [Dug01] and Dugger–Hollander–Isaksen [DHI04].

The most important source of $\mathbb{A}^1$-weak equivalences are the projections from the total space of an affine bundle to the base:

Definition 2.1.2. A locally trivial (with respect to some Grothendieck topology τ) fiber bundle $p\colon E \twoheadrightarrow B$ with fibers $p^{-1}(b)$ isomorphic to $\mathbb{A}^n$ is called an *affine bundle* on B (in the τ topology).

Lemma 2.1.3. *Let $p\colon E \twoheadrightarrow B$ be an affine bundle of rank n in the Zariski topology with B smooth. Then p is an $\mathbb{A}^1$-weak equivalence.*

Proof. By definition of a bundle, B admits a Zariski cover $\mathcal{U} = \{U_i \hookrightarrow B\}_{i\in I}$ and there are isomorphisms φ_i over U_i from $U_i \times \mathbb{A}^n$ to $p^{-1} * U_i$. Choose (for convenience of stating the proof) a well-order on I. For every word $\alpha = (i_1, \dots, i_k)$ of length k over I we let φ_α be the restriction of φ_i to $U_\alpha := U_{i_1} \times_B \cdots \times_B U_{i_k}$ for $i = \min(\alpha)$. As the Čech nerve of a cover is defined as $\check{C}^k(\mathcal{U}) = \coprod_{|\alpha|=k} U_\alpha$, we get an isomorphism between $p^*\check{C}^k(\mathcal{U})$ and $\check{C}^k(\mathcal{U}) \times \mathbb{A}^n$ over $\check{C}^k(\mathcal{U})$, by disjoint union of the φ_α.

By definition of the $\mathbb{A}^1$-weak equivalences, the projections $\check{C}^k(\mathcal{U}) \times \mathbb{A}^n \twoheadrightarrow \check{C}^k(\mathcal{U})$ are $\mathbb{A}^1$-weak equivalences, and so is the morphism $p^*\check{C}^k(\mathcal{U}) \twoheadrightarrow \check{C}^k(\mathcal{U})$. As any degree-wise weak equivalence of simplicial objects is a weak equivalence, $p^*\check{C}^\bullet(\mathcal{U}) \twoheadrightarrow \check{C}^\bullet(\mathcal{U})$ is an $\mathbb{A}^1$-weak equivalence. By definition, $p^*\check{C}^\bullet(\mathcal{U}) = \check{C}^\bullet(p^*\mathcal{U})$, where $p^*\mathcal{U} := \{p^*U_i \hookrightarrow E\}_{i\in I}$ is the induced Zariski cover of E. As the homotopy colimit of a Čech nerve is the space covered [DHI04, Theorem 1.2], we get a commutative diagram in which we know that all morphisms except possibly p are $\mathbb{A}^1$-weak equivalences:

$$\begin{array}{ccc} \mathrm{hocolim}(p^*\check{C}^\bullet(\mathcal{U})) & \longrightarrow & \mathrm{hocolim}(\check{C}^\bullet(\mathcal{U})) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad p \quad} & B \end{array}$$

From the diagram we see that p is also an $\mathbb{A}^1$-weak equivalence. □

Definition 2.1.4. A space $X \in \mathrm{Spc}(S)$ such that the structure map $X \to S$ is an $\mathbb{A}^1$-equivalence is called $\mathbb{A}^1$-*contractible*.

Corollary 2.1.5. *Let $p\colon E \to B$ be an affine bundle with E an $\mathbb{A}^1$-contractible space (e.g. $\mathbb{A}^n$). Then B is $\mathbb{A}^1$-contractible.*

Remark 2.1.6. Homotopy theory comes in unpointed and pointed versions. While the definition of motivic spaces we mentioned earlier did not include basepoints, we will always use basepoints in the following. In particular, by $\mathrm{Spc}(S)$ we intend to mean the pointed version from now on. In pointed motivic spaces, there are the operations *wedge sum* $\vee$ and *smash product* $\wedge$ which distribute over each other.

One way to go from the unpointed setting to the pointed setting is to adjoin an extra basepoint, which is written $X \mapsto X_+ := X \sqcup \{*\}$ and provides a left adjoint to the forgetful functor from pointed to unpointed. If the space X in question was already pointed, $X_+ = X \vee \mathrm{S}^0$.

The spaces appearing in this work will be mostly naturally pointed: any group is pointed by its identity element, and so is any homogeneous space.

Remark 2.1.7. There are also stable versions of motivic homotopy theory, i.e. categories of *motivic spectra*. We work with $\mathbb{P}^1$-spectra, which are constructed by homotopy localization of the functor $\mathbb{P}^1 \wedge -$ on pointed spaces. Spaces are mapped to spectra by the construction of a *suspension spectrum*, $X \mapsto \Sigma^\infty X$. If a space is not already pointed, it is often written $X \mapsto X_+ \mapsto \Sigma^\infty X_+ = \Sigma^\infty_+ X$. For more details we refer to Jardine [Jar00].

The relation between classical homotopy theory and motivic homotopy theory over the base scheme $S = \mathrm{Spec}(\mathbb{C})$ is given mostly by two functors (and versions thereof): the interpretation of a simplicial set as constant simplicial presheaf

$$c\colon \mathrm{sSet} \to \mathrm{Spc}(S)$$

and the complex geometric realization functor [MV99, page 120],[DI04]

$$|\cdot|_{\mathbb{C}}\colon \mathrm{Spc}(\mathbb{C}) \to \mathrm{sSet}$$

which takes an algebraic variety to a simplicial approximation of its analytification. For any simplicial set M we have $|c(M)|_{\mathbb{C}} = M$ and geometric realization is a left Quillen functor.

One can extend these functors to stable homotopy theory, i.e.

$$c\colon \mathrm{Spectra} \to \mathrm{Spectra}(S),$$

$$|\cdot|_{\mathbb{C}}\colon \mathrm{Spectra}(\mathbb{C}) \to \mathrm{Spectra}\,.$$

This can be used to relate the stable homotopy groups of the sphere spectrum to the motivic stable homotopy groups of a motivic spectrum called the motivic sphere spectrum (the unit for the monoidal structure on motivic spectra), as done by Levine [Lev14].

Definition 2.1.8. For $p, q \in \mathbb{Z}$ with $p \geq q$, the motivic space

$$\mathrm{S}^{p,q} := \left(\mathbb{G}_{\mathrm{m},S}\right)^{\wedge q} \wedge c\left(\mathrm{S}^{p-q}\right) \in \mathrm{Spc}(S)$$

is called a *motivic sphere*. Here $\mathrm{S}^{p-q} := \left(\mathrm{S}^1\right)^{\wedge p-q} \in \mathrm{sSet}$ is a simplicial sphere, where $\mathrm{S}^1 := \Delta^1/\partial\Delta^1$ (pointed by $\partial\Delta^1$) and $\mathbb{G}_{\mathrm{m},S} := \mathbb{G}_{\mathrm{m}} \times_{\mathbb{Z}} S$ is the multiplicative group scheme over S (pointed by the unit), where $\mathbb{G}_{\mathrm{m}}(R) = R^\times$ for any ring R.

Example 2.1.9. There is an $\mathbb{A}^1$-homotopy equivalence

$$(\mathbb{P}^1, 0) \xrightarrow{\sim} \mathrm{S}^{2,1},$$

that is, an isomorphism in the homotopy category of $\mathrm{Spc}(S)$ [MV99, Example 3.2.18]. One can see this directly by writing $\mathbb{P}^1 = X \cup Y$ with $X = \mathbb{P}^1 \setminus \{0\}$ and $Y = \mathbb{P}^1 \setminus \{\infty\} \xrightarrow{\sim} X$, so that $X \times_{\mathbb{P}^1} Y = \mathbb{A}^1 \setminus \{0\} \simeq \mathrm{S}^{1,1}$ and X and Y are both $\mathbb{A}^1$-contractible (isomorphic to 0 in the homotopy category of $\mathrm{Spc}(S)$). This shows that $(\mathbb{P}^1, 0)$ is the suspension of $\mathrm{S}^{1,1}$.

Example 2.1.10. Affine space with removed origin is a motivic sphere [MV99, Example 3.2.20]:

$$\mathbb{A}^n \setminus \{0\} \simeq \mathrm{S}^{2n-1,n}$$

For odd-dimensional split quadrics, there is a well-known elementary argument to see that they are motivic spheres:

Lemma 2.1.11. *Let $AQ_{2n-1} := \{(x,y) \in \mathbb{A}^n \times \mathbb{A}^n \mid \sum_{i=1}^n x_i y_i = 1\}$ (considered as affine algebraic variety over $\mathbb{Z}$), then $\pi\colon AQ_{2n-1} \twoheadrightarrow \mathbb{A}^n \setminus \{0\}$ given by $(x,y) \mapsto y$ is a rank $n-1$ affine bundle and over any base scheme S there is an isomorphism*

$$AQ_{2n-1} \xrightarrow{\sim} \mathrm{S}^{2n-1,n}.$$

Proof. We can cover $\mathbb{A}^n \setminus \{0\}$ by the varieties $U_i := \{y_i \neq 0\}$, over which $\pi^{-1}(U_i) = \{(x,y) \in \mathbb{A}^n \times \mathbb{A}^n \mid y_i \neq 0,\ \sum_{j=1}^n x_j y_j = 1\}$ can be rewritten as

$$\pi^{-1}(U_i) = \left\{ (x,y) \in \mathbb{A}^n \times \mathbb{A}^n \;\middle|\; y_i \neq 0,\ x_i = y_i^{-1}\left(1 - \sum_{j=1,\ j\neq i}^{n} x_j y_j\right) \right\}$$

so there are isomorphisms

$$\pi^{-1}(U_i) \xrightarrow{\sim} \mathbb{A}^{n-1} \times U_i, \qquad (x,y) \mapsto ((x_1, \ldots, \hat{x}_i, \ldots, x_n), y).$$

For fixed $y \in \mathbb{A}^n \setminus \{0\}$, the equation for x_i is linear in x with x_i removed, which guarantees that the inverse map is linear in the $\mathbb{A}^{n-1}$-component.

Now apply Lemma 2.1.3 and Example 2.1.10. □

With a vastly more complicated proof, one can see that even-dimensional split quadrics are also smooth models of motivic spheres [ADF16, Theorem 2.2.5].

Theorem 2.1.12 (Asok–Doran–Fasel, 2015)**.** *For any base scheme S, there is a pointed weak equivalence of the $2n$-dimensional smooth split affine quadric with a motivic sphere:*

$$AQ_{2n} \xrightarrow{\sim} \mathrm{S}^{2n,n}.$$

Example 2.1.13. A space X which is $\mathbb{A}^1$-contractible need not be an affine space itself. An ample supply of quasi-affine non-affine varieties which are $\mathbb{A}^1$-contractible is given by Asok and Doran [AD07], [AD08]. Dubulouz and Fasel gave examples of smooth affine threefolds over fields of characteristic 0 which happen to be $\mathbb{A}^1$-contractible but are not isomorphic to affine spaces [DF15].

We now state an important technical ingredient for using vector bundles in motivic homotopy theory:

Lemma 2.1.14 (Wendt). *Let R be a ring which is smooth over a Dedekind ring with perfect residue fields (e.g. $\mathbb{Z}$). For E and B two R-varieties and $p\colon E \to B$ a rank n vector bundle projection, the underived pullback p^* preserves homotopy colimits (for $\mathbb{A}^1$-local weak equivalences). For a diagram $D \in \mathrm{Spc}(R)/B$ we can compute*

$$\operatorname{hocolim} p^* D \simeq p^* \operatorname{hocolim} D.$$

Proof. First, we can modify the proof of Wendt [Wen11, Theorem 4.6] in the case of $G = \mathrm{GL}_n$ to get rid of the assumption on the base being an infinite field: By [AHW16, Theorem 5.1.3 and the proof of Theorem 5.2.3], under the stated assumptions on R, the space $B\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} \mathrm{GL}_n$ is $\mathbb{A}^1$-local, hence as in the original proof [Wen11, Theorem 4.6], there is an $\mathbb{A}^1$-local fiber sequence $\mathrm{GL}_n \to \mathcal{E} \to B$, with $\mathcal{E}$ the GL_n-principal bundle associated to E (the frame bundle). It is a general theorem of Rezk that maps which induce simplicial fiber sequences are sharp [Rez98, Theorem 4.1., (1) $\Leftrightarrow$ (3)]. In any right proper model category, sharp maps are those whose (underived) pullback preserves homotopy colimits [Rez98, Proposition 2.7]. □

This proof was suggested by Wendt.

2.2. Motives

We quickly explain a definition of mixed motives we will use and point out some references to the literature. This section serves only motivational purpose and may be skipped on first reading.

The conjectural picture of a category of motives over a field k is a Tannakian category $\mathcal{MM}$ whose extension groups are isomorphic to higher Chow groups, also called motivic cohomology. There is a candidate for such a Tannakian category $\mathcal{MM}_{Nori}$ due to sketches of Nori, for which the extensions are not known yet. There is a candidate for the derived category $D(\mathcal{MM})$ due to Voevodsky, called $\mathrm{DM}_{gm}(k)$, which has a very close relation to motivic homotopy theory. Déglise and Cisinski developed the theory of derived motives over a general base scheme S [CD15].

We will not use motives for the main result of this article, but intend to give a supporting argument (over a perfect field k) why the main construction is slightly more complicated than one might first expect. For this purpose, we use the category of geometric motives, localization triangles and the functor which assigns every compact object in motivic spaces its motive. There are various constructions one might use to get such a functor and category, and the reader may choose one she is familiar with (see the work of Déglise [Dég12] for a detailed review on functoriality of localization triangles).

Fact 2.2.1. For k a perfect field, there is a tensor triangulated category of *derived mixed motives* $\mathrm{DM}_{gm}(k)$ and a tensor triangulated functor $M\colon \mathrm{Spectra}(k)^{cpct} \to \mathrm{DM}_{gm}(k)$ (for the smash product on $\mathbb{P}^1$-spectra as monoidal product). The suspension spectrum of a smooth scheme is a compact object in motivic spectra and we write $M(X) := M(\Sigma^\infty_+ X)$ and $\widetilde{M}(Y, y_0) := M(\Sigma^\infty(Y, y_0))$ for a space X and a pointed space (Y, y_0). Motivic spheres are also compact objects in motivic spaces and spectra. There are *localization exact triangles* (also known as Gysin triangles), i.e. for each codimension n closed immersion

$Z \hookleftarrow\!\!\rightarrow X$ of smooth schemes with open complement $U \hookrightarrow X$, there exists a distinguished triangle

$$\cdots \to M(U) \to M(X) \to M(Z)(n)[2n] \to [1]\cdots$$

Example 2.2.2. The simplicial suspension on spectra is usually denoted by $X[n] := \Sigma^n X := \mathrm{S}^{n,0} \wedge X$ and the reduced motive of the spectrum $\mathrm{S}^{0,1} = \Sigma^{-1}\Sigma^{\infty}(\mathbb{G}_{\mathrm{m}}, 1)$ is written as $\mathbb{1}(1) := \widetilde{M}(\mathrm{S}^{0,1})$, called a *Tate twist*. The unreduced motive is

$$M(\mathrm{S}^{0,1}) = \widetilde{M}(\mathrm{S}^{0,1}_+) = \widetilde{M}(\mathrm{S}^{0,1} \vee \mathrm{S}^{0,0}) = \widetilde{M}(\mathrm{S}^{0,1}) \oplus \widetilde{M}(\mathrm{S}^{0,0}) = \mathbb{1}(1) \oplus \mathbb{1}$$

Consequently for motivic spheres

$$\widetilde{M}(\mathrm{S}^{p,q}) = \mathbb{1}(q)[p]$$

and more generally

$$\widetilde{M}(\mathrm{S}^{p,q} \wedge X) = \widetilde{M}(X)(q)[p].$$

Definition 2.2.3. A motive $M \in DM_{gm}(S)$ which is in the triangulated subcategory generated by tensor powers of $\mathbb{1}(1)$ is called a *mixed Tate motive.*

2.2.1. A Localization Triangle from Diagram Folding.

Fact 2.2.4. Given a 2-*orbit completion*, that is a reductive group G, an affine homogeneous G-space G/H, a proper homogeneous G-space G/P, a proper G-space X and an equivariant embedding $G/H \hookrightarrow X$ with closed complement G/P, the space X is itself a homogeneous space under a different group G', so that $X = G'/P'$.

This can be seen from the classification of Ahiezer [Ahi83] in a case-by-case analysis. It is remarkable that this fact is not yet proved without using the classification, so there is no obvious intrinsic reason for this to happen in the 2-orbit case.

Observation 2.2.5. In several cases, one can understand the relation between the reductive groups G' and G in Fact 2.2.4 as *Dynkin diagram folding*, where the group G' and the parabolic P' give rise to G and P by taking invariants under an outer automorphism that is induced by an automorphism of the Dynkin diagram. These cases are the diagram foldings $A_{2n-1} \rightsquigarrow C_n$, $\mathrm{E}_6 \rightsquigarrow \mathrm{F}_4$, $D_n \rightsquigarrow B_{n-1}$, $D_4 \rightsquigarrow G_2$. In other cases, the relation is a *Dynkin diagram inclusion*, such as $B_n \hookrightarrow D_n$. Roughly the same observation was made independently by Landsberg and Manivel [LM01, Section 4.2].

Remark 2.2.6. From the Hasse diagrams described by Semenov [Sem06] one can read off the decomposition of a motive $M(G/P)$ directly from the combinatorics of the Dynkin diagram. If $G/P \hookleftarrow\!\!\rightarrow G'/P'$ is given by a diagram folding or diagram inclusion, one can compute the map of the corresponding Hasse diagrams and therefore on the motives. That means, one can compute the morphism

$$M(G'/P') \to M(G/P)(1)[2]$$

that appears in the localization triangle of $G/P \hookleftarrow\!\!\rightarrow G'/P'$. From this data, one can compute the third vertex of the triangle, that is $M(G/H)$.

Conjecture 1. *Some expected results following from Observation 2.2.5 are*

$$\begin{aligned} M(B_{n+1}/D_n) &= 1 \oplus 1(n)[2n] \\ M(D_n/B_n) &= 1 \oplus 1(n)[2n-1] \\ M(A_{n+1}/A_1 \times A_n) &= 1 \oplus 1(1)[2] \oplus 1(2)[4] \oplus \cdots \oplus 1(n)[2n] \\ M(C_{n+1}/C_1 \times C_n) &= 1 \oplus 1(2)[4] \oplus 1(4)[8] \oplus \cdots \oplus 1(2n)[4n] \\ M(\mathrm{F}_4\,/\,\mathrm{B}_4) &= 1 \oplus 1(4)[8] \oplus 1(8)[16]. \end{aligned}$$

These computations give restrictions on possible motivic cell structures on these spaces, as we discuss in Remark 2.3.11. Furthermore, given a motivic cell structure, one can deduce that the attaching maps become trivial in the category of mixed motives, as these computations show that the motives split.

2.3. Motivic Cell Structures

We introduce the concept of motivic cell structures, which is the $\mathbb{A}^1$-homotopy analogue of CW complexes, originally due to Dugger and Isaksen. We explain the main tool to obtain unstable motivic cell structures, Morel–Voevodsky's homotopy purity theorem, and how we intend to use it.

2.3.1. Cell Structures.

We recall the definition of motivic cell structures.

Definition 2.3.1. Let $\mathcal{M}$ be a pointed model category and $\mathcal{A} \subset Ob\,\mathcal{M}$ a set of objects. The class of *$\mathcal{A}$-cellular objects* in $\mathcal{M}$ is defined as the smallest class of objects containing $\mathcal{A}$ that is closed under weak equivalence and contains all homotopy colimits over diagrams whose objects are all $\mathcal{A}$-cellular.

This was defined by Dugger and Isaksen [DI05, Definition 2.1].

Definition 2.3.2. For the pointed model category of pointed motivic spaces $\mathrm{Spc}(S)$ let $\mathcal{A} := \{\mathrm{S}^{p,q} \mid p, q \in \mathbb{N},\ p \geq q\}$ be the set of motivic spheres. The $\mathcal{A}$-cellular objects in $\mathrm{Spc}(S)$ are called *motivically cellular*. For motivic spectra with $\mathcal{A}^s := \{\mathrm{S}^{p,q} \mid p, q \in \mathbb{Z}\}$, we call $\mathcal{A}^s$-cellular objects *stably motivically cellular*. A motivic space $X \in \mathrm{Spc}(S)$ with $\Sigma^\infty_+ X$ stably motivically cellular is also called stably motivically cellular.

Example 2.3.3. Projective space $\mathbb{P}^n$ carries a motivic cell structure, as there exists a homotopy cofiber sequence (compare [DI05, Proposition 2.13])

$$\mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1} \to \mathbb{P}^n$$

and $\mathbb{A}^n \setminus \{0\}$ is a motivic sphere $\mathrm{S}^{2n-1,n}$ (up to $\mathbb{A}^1$-homotopy equivalence [DI05, Example 2.11] for a proof of this claim first made by Morel and Voevodsky [MV99, Example 3.2.20]). This homotopy cofiber sequence yields a distinguished triangle in the derived category of motives

$$M(\mathbb{P}^{n-1}) \to M(\mathbb{P}^n) \to 1(n)[2n] \to$$

and one can show that the attaching map $\mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is 0 the level of motives,hence there is a decomposition

$$M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} 1(i)[2i].$$

However, even on the level of spectra, in the stable motivic homotopy category, the attaching map is non-trivial and $\bigvee_{i=0}^n \mathrm{S}^{2i,i}$ is a different motivic space with the same motivic decomposition as $\mathbb{P}^n$.

Remark 2.3.4. If a motivic space X admits a stable motivic cell structure, its motive is of mixed Tate type. This follows directly from the fact that motivic spheres are of mixed Tate type and the homotopy colimits defining the cell structure can be written as homotopy coequalizer and homotopy coproduct, which translates directly to distinguished triangles of mixed motives.

Since "few" varieties have a motive of mixed Tate type (e.g. positive-dimensional Abelian varieties are not of that type), only few varieties can admit a motivic cell structure. This is in strong contrast to the topological situation, where every space admits a weakly equivalent cellular approximation.

Definition 2.3.5. Let X be a motivically cellular space, $f\colon \mathrm{S}^{p-1,q} \to X$ a morphism. We say that $Y := \mathrm{hocofib}(f)$ *has a (p,q)-cell with attaching map f.*

Remark 2.3.6. Given a motivic space X with mixed Tate motive, a description of the motive already puts strong restrictions on possible finite cell structures: suppose $M(X) = 1 \oplus 1(1)[2] \oplus 1(2)[4]$, then there can be no (nontrivial) $\mathrm{S}^{3,2}$-cell, as $M(X)$ would then have to be an extension of some motive with $1(2)[3]$.

2.3.2. Homotopy Purity.

The following is the direct analogue of the Pontryagin–Thom construction in algebraic geometry.

Definition 2.3.7. Given a morphism $f\colon X \to Y$ of motivic spaces, we define the *homotopy cofiber* as the homotopy colimit of the solid-lines diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \vdots \\ * & \cdots\rightarrow & hocofib(f) \end{array}$$

Theorem 2.3.8 (Morel and Voevodsky [MV99, Theorem 3.2.23])**.** *For a closed immersion $\iota : Z \hookrightarrow X$ with open complement $U \hookrightarrow X$, there is a natural homotopy cofiber sequence of pointed motivic spaces*

$$U \to X \to \mathrm{Th}(N_\iota)$$

where $\mathrm{Th}(N_\iota)$ *denotes the* Thom space *of the normal bundle N_ι of ι, which is defined using the zero section $Z \hookrightarrow N_\iota$ as* $\mathrm{Th}(N_\iota) := N_\iota/(N_\iota \setminus Z) := \mathrm{hocofib}(N_\iota \setminus Z \hookrightarrow N_\iota)$.

Example 2.3.9. We explain how useful such a statement is with a topological analogue: Suppose $\iota\colon N \hookrightarrow M$ is a closed immersion of an n-dimensional manifold into an m-dimensional manifold, and both N and $M \setminus N$ are cellular (CW complexes). Then $M \setminus N \to M \to \mathrm{Th}(N_\iota)$ is a homotopy cofiber sequence. As Thom spaces over cellular spaces (in particular suspensions) are again cellular (in classical topology), the following is a homotopy cofiber sequence

$$\mathrm{Th}(N_\iota) \to \Sigma(M \setminus N) \to \Sigma M$$

which expresses ΣM as homotopy cofiber of cellular spaces. This shows that ΣM is cellular, and that M is stably cellular.

If $M \setminus N$ is contractible, then M is weakly equivalent to $\mathrm{Th}(N_\iota)$.

Remark 2.3.10. It is not necessarily true that a Thom space over a motivically cellular base is again motivically cellular (it is not known to the author whether counterexamples exist or whether we simply lack a proof). For this reason, and also to be able to describe a cell structure explicitly, it is highly desirable to trivialize normal bundles (locally). Thom spaces over trivial bundles are just suspensions ([MV99, Proposition 3.2.17]):

$$\mathrm{Th}(\mathbb{A}^n \times B \to B) = B_+ \wedge \mathrm{S}^{2n,n}\,.$$

Remark 2.3.11. From Fact 2.2.1 on the functor which associates to a motivic space its motive, we can use the previous remark to see that any codimension c closed immersion $B \hookrightarrow Y$ with trivial normal bundle $N \twoheadrightarrow B$ and contractible complement $Y \setminus B$ results in

$$M(Y) = M\left(\mathrm{Th}\left(N \twoheadrightarrow B\right)\right) = M\left(\left(B \wedge \mathrm{S}^{2c,c}\right) \vee \mathrm{S}^{2c,c}\right) = M(B)(c)[2c] \oplus 1(c)[2c].$$

This insight, together with the computations in Remark 2.2.6, gives a guideline on why Section 4.2.1 is necessary.

An important tool to trivialize a bundle is the Quillen–Suslin theorem

Theorem 2.3.12 (Quillen–Suslin)**.** *Let R be a smooth finite type algebra over a Dedekind ring. Then all algebraic vector bundles on $\mathbb{A}^n_R$ are extended from* $\mathrm{Spec}(R)$.

This is beautifully explained in Lam's book [Lam06, Theorem III.1.8].

It has been used to obtain a generalization, which one may see as a corollary to the vector bundle classification of Asok–Hoyois–Wendt:

Lemma 2.3.13. *There are no non-trivial vector bundles on a smooth affine finite type $\mathbb{A}^1$-contractible variety X over a Dedekind ring.*

Proof. Let $f : \mathrm{Spec}(k) \xrightarrow{\sim} X$ be the isomorphism in the homotopy category of motivic spaces $\mathcal{H}o\,\mathrm{Spc}(k)$ given by contractibility. From [AHW16, Theorem 5.2.3], we know

$$\{\text{rank } r \text{ vector bundles on } X\}/_{\simeq} \xrightarrow{\sim} [X, Gr_r]_{\mathbb{A}^1} \xrightarrow[f^*]{\sim} [*, Gr_r]_{\mathbb{A}^1} = 1. \qquad \square$$

This fails already for smooth non-affine quasi-affine varieties that are $\mathbb{A}^1$-contractible, where one can give infinitely many counterexamples [ADF16, Corollary 4.3.9].

Corollary 2.3.14. *If $V \hookrightarrow M$ is a codimension c closed immersion of smooth varieties over a smooth finite type $\mathbb{Z}$-algebra R, and the complement $M \setminus V$ is an $\mathbb{A}^1$-contractible smooth affine R-variety, and V is the total space of a vector bundle $V \twoheadrightarrow M'$ with M' an $\mathbb{A}^1$-contractible R-variety, M is a motivic sphere $\mathrm{S}^{2c,c}$.*

Proof. Assume that M' is smooth affine as well. Using Lemma 2.3.13, the vector bundle $V \twoheadrightarrow M'$ is trivial, so the total space V is also smooth affine $\mathbb{A}^1$-contractible. Using Lemma 2.3.13 again, the normal bundle $N_\iota \twoheadrightarrow V$ is trivial. The Thom space of a trivial bundle of rank r over a base B is $\mathbb{A}^1$-homotopy equivalent to $(\mathbb{P}^1)^{\wedge r} \wedge B_+$. We conclude by using Theorem 2.3.8, which hands us a homotopy cofiber sequence

$$M \setminus V \to M \to \mathrm{Th}(N_\iota).$$

Contractibility of $M \setminus V$ implies that $M \to \mathrm{Th}(N_\iota)$ is a weak equivalence, so $M \simeq (\mathbb{P}^1)^{\wedge c} \wedge \mathrm{S}^0 \simeq \mathrm{S}^{2c,c}$.

Now if M' is not smooth affine, $N_\iota \twoheadrightarrow V$ may be non-trivial. Since $V \twoheadrightarrow M'$ is still a weak equivalence, V is still $\mathbb{A}^1$-contractible, hence $\mathrm{Th}(N_\iota) \simeq (\mathbb{P}^1)^{\wedge c} \wedge \mathrm{S}^0$, as Thom spaces are invariant under $\mathbb{A}^1$-equivalence. $\square$

Fact 2.3.15. Every *spherical variety*, that is a variety with an action of a reductive group G such that a Borel $B \subset G$ has an open orbit, admits stable motivic cell structures. This follows from the fact that spherical varieties are linear varieties as follows: A linear variety in the sense of Totaro [Tot14, page 8, section 3] is a variety that admits a filtration F^i such that the strata $F^i \setminus F^{i-1}$ are finite disjoint unions of products of affine spaces with split tori $\mathbb{A}^n \times {\mathbb{G}_\mathrm{m}}^l$. Joshua gives a slightly more general definition by a 2-out-of-3 property, namely that for every closed immersion $Z \hookrightarrow X$ with open complement $U \hookrightarrow X$, if Z and either X or U are $(n-1)$-linear, then Z, U, X are n-linear. Furthermore, the empty set and every affine space are 0-linear and a variety is *linear* if it is n-linear for some n. The existence of a stable motivic cell structure now follows by an easy 2-out-of-3 argument, using homotopy purity and the Quillen–Suslin theorem as well as Lam's extension of Quillen–Suslin to tori (compare Gubeladze's more general theorem [Lam06, VIII.4, Theorem 4.1]). Carlsson and Joshua give a slightly more general statement in an equivariant setting [CJ11, Proposition 4.7]. Spherical varieties are linear by the fact that they consist of only finitely many B-orbits and a result of Rosenlicht that describes the B-orbits (compare [Tot14, page 8, Addendum]), as was observed by Totaro around 1996, published much later [Tot14, Introduction]. Special cases of spherical varieties are spherical homogeneous spaces such as affine quadrics, flag varieties G/P and the spaces DP^n that are discussed in this article. We also discuss how wonderful completions yield cell structures in Section 2.3.3.

Definition 2.3.16. For a variety N, a Zariski covering $\mathcal{U} = (U_i \hookrightarrow N)_{i\in I}$ (with $N = \bigcup_{i\in I} U_i$) is called *totally cellular* if the Čech nerve $\check{C}^\bullet(\mathcal{U})$ is a simplicial object in cellular varieties. It is called *totally contractible* if the $U_\alpha = \bigcap_{j\in J} U_j$ for each $J \subset I$ are $\mathbb{A}^1$-contractible. It is called *totally affinely contractible* if there are affine bundles $\widetilde{U}_\alpha \to U_\alpha$ with affine total spaces $\widetilde{U}_\alpha \cong \mathbb{A}^{m_\alpha}$, compatible with the simplicial structure on $\check{C}^\bullet(\mathcal{U})$ (assembling to an affine bundle $\check{C}^\bullet(\widetilde{\mathcal{U}}) \to \check{C}^\bullet(\mathcal{U})$).

The definition of total cellularity was made in the stable context by Dugger and Isaksen [DI05, Definition 3.7], see also [DI05, Lemma 3.8].

Remark 2.3.17. While smash products of unstably cellular spaces are again unstably cellular, Dugger and Isaksen already noticed [DI05, Example 3.5] that it is in general hard to show whether a cartesian product of cellular spaces is unstably cellular. Since it is easy to show that cartesian products of stably cellular spaces are stably cellular, they only prove that Thom spaces of bundles over a totally cellular base are stably cellular [DI05, Corollary 3.10]. As we're interested in unstable cell structures on spaces which are iterated Thom spaces, we need a stronger statement in the sequel. While the proof is very similar to the one of Dugger and Isaksen, we chose a different exposition.

Lemma 2.3.18. *Let $p\colon E \twoheadrightarrow B$ be a vector bundle and $B' \to B$ an $\mathbb{A}^1$-weak equivalence. Then there exists a weak equivalence of Thom spaces*

$$\mathrm{Th}(p) \xrightarrow{\sim} \mathrm{Th}(p').$$

Proof. Since vector bundle projections are sharp (Lemma 2.1.14), the morphism $E\times_B B' \to E$ is a weak equivalence. Let s be the zero section of p and s' the zero section of the base change $p'\colon E\times_B B' \to B'$. By construction of s', we get a weak equivalence of $E\times_B B' \setminus s'(B')$ with $E \setminus s(B)$. We proved that the diagrams whose homotopy colimits are $\mathrm{Th}(p)$ respectively $\mathrm{Th}(p')$ are weakly equivalent. □

Corollary 2.3.19. *Let $p\colon E \twoheadrightarrow B$ be a rank n vector bundle and $B' \to B$ an affine bundle with $B' \cong \mathbb{A}^m$ (as varieties). Then* $\mathrm{Th}(p) \xrightarrow{\sim} B_+ \wedge \mathrm{S}^{2n,n} \simeq \mathrm{S}^{2n,n}$.

Proof. We use Lemma 2.1.3 to apply Lemma 2.3.18 and then Remark 2.3.10. □

Theorem 2.3.20. *Let $p\colon E \to B$ be an algebraic vector bundle of rank r and $\mathcal{U} = (U_i \hookrightarrow B)_{i \in I}$ a totally affinely contractible Zariski cover of B, all defined over a ring R which is smooth and finite type over a Dedekind ring (e.g. $\mathbb{Z}$). Then* $\mathrm{Th}(p)$ *is unstably cellular.*

Proof. We use the morphism $l\colon \check{C}^\bullet(\mathcal{U}) \to B$ which induces a weak equivalence on homotopy colimits, i.e. hocolim $\left(\check{C}^\bullet(\mathcal{U})\right) \simeq B$. The bundle $q\colon E \setminus B \to B$ obtained as sub-bundle of p is a fiber bundle with fiber $\mathbb{A}^r \setminus \{0\}$. The following diagram commutes:

$$\begin{array}{ccccc}
q^*\check{C}^\bullet(\mathcal{U}) & \xrightarrow{l^*i} & p^*\check{C}^\bullet(\mathcal{U}) & \longrightarrow & \mathrm{hocofib}(l^*i) \\
\big\downarrow{\scriptstyle q^*l} & & \big\downarrow{\scriptstyle p^*l} & & \big\downarrow \\
E \setminus B & \xrightarrow{i} & E & \longrightarrow & \mathrm{Th}(p)
\end{array}$$

The rows are homotopy cofiber sequences. The middle column is a weak equivalence by Lemma 2.1.14. The left column is also a weak equivalence, as it is the restriction of the middle column and the model structure is proper. (alternatively one could argue that spherical bundle projections are as sharp as vector bundle projections). We inspect the first row more closely. While the bundle E might not trivialize over U_i, its pullback to an affine space $\widetilde{U}_i$ (given by the property of U_i being totally affinely contractible) is trivial, by Theorem 2.3.12. The same holds for each U_α with obvious definition of $\widetilde{U}_\alpha$. By Lemma 2.3.18 the Thom space of $E|_{U_i}$ is weakly equivalent to the Thom space of the pulled back bundles $E_{|\widetilde{U}_i}$. Now we can form a diagram, commutative up to homotopy

$$\begin{array}{ccccc}
q^*U_\alpha & \xrightarrow{i|_{U_\alpha}} & p^*U_\alpha & \longrightarrow & \mathrm{hocofib}(i|_{U_\alpha}) \\
\big\uparrow{\scriptstyle l^*i} & & \big\uparrow{\scriptstyle p^*l} & & \big\uparrow \\
\mathbb{A}^n \setminus \{0\} \times \widetilde{U}_\alpha & \longrightarrow & \mathbb{A}^n \times \widetilde{U}_\alpha & \longrightarrow & \Sigma_s\left(\mathbb{A}^n \setminus \{0\}\right)
\end{array}$$

whose rows are homotopy cofiber sequences and the leftmost two columns are weak equivalences. Consequently, the last column is a weak equivalence. This exhibits both $E \setminus B$ and $\mathrm{Th}(p)$ as homotopy colimit over cellular spaces.

We can view $E \setminus B \to B$ obtained by composing $E \setminus B \to E$ with the bundle projection $p\colon E \to B$ as an explicit gluing map, as its homotopy colimit is again $\mathrm{Th}(p)$. $\square$

Corollary 2.3.21. *Given a sequence M_i of smooth varieties over a ring R which is smooth and finite type over a Dedekind ring (e.g. $\mathbb{Z}$)*

$$M_n \supset M_{n-1} \supset \cdots \supset M_0 = * \supset M_{-1} = \emptyset$$

such that each M_i admits totally affinely contractible covers and rank r_i vector bundles $V_i \twoheadrightarrow M_{i-1}$ together with a closed immersion of the total space $V_i \hookrightarrow M_i$ of codimension c_i, and each complement $X_i := M_i \setminus V_i$ is $\mathbb{A}^1$-contractible, there exists an unstable motivic cell structure on each M_i.

Proof. We use induction on i, with the base case M_0 being trivially cellular. Let $N_i \twoheadrightarrow V_i$ be the normal bundle of the closed immersion $V_i \hookrightarrow M_i$. As the complement X_i is $\mathbb{A}^1$-contractible, by Theorem 2.3.8, applied as in the proof of Corollary 2.3.14, we get a weak equivalence $M_i \to \mathrm{Th}(N_i)$. From our assumptions, $\mathrm{Th}(N_i)$ carries an unstable cell structure. □

Remark 2.3.22. If the ranks r_i in Corollary 2.3.21 are all 0, this resembles Wendt's unstable cell structure on generalized flag varieties using the Bruhat cells [Wen10, Proposition 3.7].

2.3.3. Cell Structures After a Single Suspension.

We now discuss how to obtain cell structures for spherical varieties which admit wonderful completions. For the spaces DP^n discussed in more detail in other chapters, we get an unstable cell structure after a single suspension.

Let G be a split reductive group. We will use the theory of spherical varieties, as detailed in the comprehensive book by Timashev [Tim11]. A spherical variety is a normal algebraic variety with an algebraic G-action such that a Borel $B \subset G$ acts with a dense orbit. As a consequence it has only finitely many B-orbits.

Remark 2.3.23. Let X be a homogeneous spherical G-variety which admits a *wonderful* equivariant completion $\overline{X}$ with boundary Z, i.e. $\overline{X}$ is smooth,

$$G/H = X \hookrightarrow \overline{X} \hookleftarrow Z,$$

the boundary Z has r irreducible components, where r is the rank of X, and there is a unique closed orbit in Z, which is the intersection of all irreducible components of Z. Furthermore, all open orbits of Z are of lower dimension than X [Tim11, Chapter 5, Definition 30.1].

As $\overline{X}$ is a complete G-variety, we can apply the algebraic Morse theory of Białynicky-Birula [BB73], as Wendt proved [Wen10, Corollary 3.5], to obtain a stable motivic cell structure on $\overline{X}$. The same applies to Z, so by a 2-out-of-3-argument, as in Fact 2.3.15, the variety X is stably motivically cellular.

Remark 2.3.24. As already mentioned in Fact 2.2.4, the rank 1 (two-orbit) wonderful completions are classified and have a particularly rich structure:

$$G/H = X \hookrightarrow \overline{X} = G'/P' \hookleftarrow Z = G/P.$$

We apply Wendt's unstable motivic cell structure [Wen10, Theorem 3.6] arising from the Bruhat decomposition of a flag variety to each G'/P' and G/P. Homotopy purity (Theorem 2.3.8) hands us a homotopy cofiber sequence

$$X \to \overline{X} \to \mathrm{Th}\left(N_{Z \hookrightarrow \overline{X}}\right)$$

that we extend to the right by the next term $\to \Sigma X$. As one can locally trivialize the normal bundle N_Z over the Bruhat cells, which are affine spaces, by the Quillen–Suslin theorem, Theorem 2.3.12, the Thom space is motivically cellular. By the arguments of Theorem 2.3.20 the Thom space is even unstably cellular. Putting everything together, the homotopy cofiber sequence

$$\overline{X} \to \mathrm{Th}\left(N_{Z \hookrightarrow \overline{X}}\right) \to \Sigma X$$

endows ΣX with an unstable motivic cell structure.

Example 2.3.25. For split quaternionic projective space HP^n, the completion is a Grassmannian $\mathrm{Gr}(2, 2n+2)$, with complement the symplectic Grassmannian $\mathrm{SpGr}(2, 2n+2)$ classifying symplectic planes in a $2n+2$-dimensional vector space with the standard symplectic form (compare Section 4.4). Proceeding as in Remark 2.3.24, we get an unstable motivic cell structure

$$\mathrm{Gr}(2, 2n+2) \to \mathrm{Th}\left(N_{\mathrm{SpGr}(2,2n+2)\hookrightarrow \mathrm{Gr}(2,2n+2)}\right) \to \Sigma\, \mathrm{HP}^n\,.$$

Example 2.3.26. For the split octonionic projective plane OP^2, the completion is the complex Cayley plane E_6/P_1, with complement an F_4/P_1. Proceeding as in Remark 2.3.24, we get an unstable motivic cell structure

$$\mathrm{E}_6/P_1 \to \mathrm{Th}\left(N_{\mathrm{F}_4/P_1}\right) \to \Sigma\, \mathrm{OP}^2\,.$$

Remark 2.3.27. It would be desirable to have more explicit knowledge on the cells of $\Sigma\,\mathrm{OP}^2$, as one could use that for first applications in obstruction theory of F_4-bundles.

Remark 2.3.28. In Remark 2.3.24, we may use as torus of G one induced from G' under the diagram relation explained in Observation 2.2.5. This tells us that the map $G/P \hookrightarrow G'/P'$ is compatible with the resulting cellular filtrations from these tori. This means that we have $\mathbb{G}_{\mathrm{m}}$-operations on G/P and G'/P' such that the map is equivariant and maps fixed points to fixed points. Consequently, the corresponding Białynicki-Birula filtrations X_n and X'_n are mapped to each other. These stratifications yield unstable motivic cell structures by [Wen10, Proposition 3.7]. We can define a stratification of the complement of $G/P \hookrightarrow G'/P'$ by taking the complements in each filtration step, but the strata are not necessarily affine spaces. The strata are complements of affine spaces in affine spaces, and it is not clear whether the map from one to the other is linear. If the Abhyankar–Sathaye conjecture holds (see Kraft's Bourbaki talk [Kra96] for an overview), or if one can show the linearity by a concrete computation, one can indeed extract an unstable cell structure on the complement of $G/P \hookrightarrow G'/P'$ by this strategy.

It becomes more complicated in the higher rank cases, where one does not have an unstable cell structure for the completion, and also the complement-strata are just complements of hyperplane arrangements, not necessarily linear ones.

It seems therefore unlikely to obtain explicit unstable motivic cell structures by these methods, which was the original goal of the thesis.

Remark 2.3.29. The bulk of this work may be read as improving upon the two examples just given by "removing the suspension".

CHAPTER 3

Split Forms of Division Algebras

The classical projective spaces over real, complex, quaternion and octonion numbers are smooth manifolds, defined in terms of real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. Given that these composition algebras with anisotropic norm forms arise from $\mathbb{R}$ by the Cayley–Dickson doubling process, their natural analogues in algebraic geometry are projective spaces over composition algebras with split norm forms, obtained by the Cayley–Dickson process from a ring R. The split complex numbers C_R, the split quaternion numbers H_R and the split octonion numbers O_R are the only such composition algebras. In this chapter we review this theory and make some preliminary computations involving Lagrangians for the Norm forms to be used in the following chapter.

3.1. Composition Algebras and the Cayley–Dickson Construction

We give a pragmatic definition of composition algebras and recall the Cayley–Dickson process, to put our discussion in context and to fix the notation. We introduce Convention 3.1.12 which is heavily used in Chapter 4.

Definition 3.1.1. A *composition algebra* C over a unital ring R is a unital R-algebra C, free as an R-module, together with a non-singular quadratic form $N\colon C \to R$ called *norm* that satisfies the composition law $N(xy) = N(x)N(y)$ and an involution $x \mapsto x^*$ such that $x^*x = N(x) \cdot 1_C$ and $x + x^* \in R \cdot 1_C$. Here, N non-singular means $\forall a : N(a + x) = N(x) \implies x = 0$.

Example 3.1.2. A field can be seen as composition algebra over itself, with norm form $x \mapsto x^2$ and the identity as involution.

Petersson generalized this definition from rings to ringed spaces [Pet93].

Lemma 3.1.3. *For any composition algebra C over a ring R, and any elements $x, y, z \in C$, with $\langle x,\ y\rangle := x^*y + y^*x$, we have the following rules for computations:*

(1) $N(x) = N(-x)$.
(2) $\langle x,\ 1\rangle = x + x^*$.
(3) $x^2 - \langle x,\ 1\rangle x + N(x) = 0$. *(*minimum equation*)*
(4) $N(x^*) = N(x)$.
(5) $N(xy) = N(yx)$.
(6) $\langle x,\ y\rangle = \langle y,\ x\rangle$.
(7) $\langle x,\ y\rangle^* = \langle x,\ y\rangle$.
(8) $xy + yx - \langle x,\ 1\rangle y - \langle y,\ 1\rangle x + \langle x,\ y\rangle = 0$.
(9) $N(x + y) = N(x) + N(y) + \langle x,\ y\rangle$

(10) $\langle x^*,\ y^*\rangle = \langle x,\ y\rangle$.
(11) $\langle xy,\ z\rangle = \langle y,\ x^*z\rangle$.
(12) $\langle xy,\ z\rangle = \langle x,\ zy^*\rangle$.
(13) $\langle xy,\ z\rangle = \langle yz^*,\ x^*\rangle$.
(14) $\langle xy,\ xz\rangle = N(x)\langle y,\ z\rangle$.
(15) $(\forall z \in C : \langle x,\ z\rangle = 0 \wedge N(x) = 0) \Leftrightarrow x = 0$.
(16) $x(x^*y) = (xx^*)y$.

For $a, b, c, d \in C$, we also have $\langle ac,\ db^*\rangle = \langle d,\ acb\rangle$ as well as $\langle ac, d^*b\rangle = \langle da,\ bc^*\rangle$.

Proof.

(1) $N(x) = x^*x = (-x^*)(-x) = N(-x)$.
(2) $\langle x,\ 1\rangle = x^*1 + 1^*x = x^* + x$.
(3) $x^2 - (x^* + x)x + x^*x = 0$.
(4) $N(x^*) = xx^* = (x + x^*)x^* - (x^*)^2 = N(x)^* = N(x)$.
(5) $N(xy) = N(x)N(y) = N(y)N(x) = N(yx)$.
(6) $x^*y + y^*x = y^*x + x^*y$.
(7) $(x^*y + y^*x)^* = x^*y + y^*x$.
(8) $xy + yx - (x + x^*)y - (y + y^*)x + x^*y + y^*x = 0$.
(9) $N(x + y) = (x + y)^*(x + y) = x^*x + x^*y + y^*x + y^*y$.
(10) Apply the involution $*$ to the previous equation.
(11) $(xy)^*z + z^*xy = y^*x^*z + (x^*z)^*y$.
(12) $\langle x, zy^*\rangle = \langle x^*,\ yz^*\rangle = \langle y^*x^*,\ z^*\rangle = \langle xy,\ z\rangle$.
(13) $\langle yz^*,\ x^*\rangle = \langle zy^*,\ x\rangle = \langle y^*,\ z^*x\rangle = \langle y^*x^*,\ z^*\rangle = \langle xy,\ z\rangle$.
(14) $N(xy + xz) = N(x)N(y + z) = N(x)\left(N(y) + N(z) + \langle y,\ z\rangle\right)$
$= N(xy) + N(xz) + N(x)\langle y,\ z\rangle$.
(15) $N(x + z) - (N(x) + N(z)) = \langle x,\ z\rangle$, so we can use non-degeneracy of N.
(16) $\langle x(x^*y),\ z\rangle = \langle x^*y,\ x^*z\rangle = N(x)\langle y,\ z\rangle = \langle N(x)y,\ z\rangle$.

For the last statement, compute

$$\langle ac, d^*b\rangle = \langle a, d^*bc^*\rangle = \langle da, bc^*\rangle. \qquad \square$$

We will use these computations without mention.

Lemma 3.1.4. *For any composition algebra C over a ring R, any $x \in C$ is invertible if and only if $N(x) \in R^\times$. In that case, it has the two-sided inverse $N(x)^{-1}x^*$*

Proof. This is a computation:

$$xN(x)^{-1}x^* = N(x)^{-1}(xx^*) = N(x)^{-1}N(x) = 1$$

$$N(x)^{-1}x^*x = N(x)^{-1}N(x) = 1.$$

From the composition property, if y is a right-inverse, $1 = N(1) = N(xy) = N(x)N(y)$, so $N(y) = N(x)^{-1}$. $\square$

Definition 3.1.5. Given an associative composition algebra C over a ring R, and a unit $\gamma \in R^\times$, we define on the R-module $D := C \oplus C$ a ring structure by

$$(a, b)(a', b') := (aa' + \gamma b'b^*,\ a^*b' + a'b)$$

so that the identity is $(1,0)$, and we define a quadratic form $N\colon D \to R$ by

$$N(a,b) := N(a) - \gamma N(b)$$

and an involution by

$$(a,b)^* := (a^*, -b).$$

The resulting composition algebra D is called the *Cayley–Dickson algebra* over C with parameter γ.

The properties $N(a,b) = (a,b)^*(a,b)$ and the composition property for N are straightforward computations:

$$(a,b)^*(a,b) = (a^*,-b)(a,b) = (a^*a - \gamma bb^*,\ ab - ab) = (N(a) - \gamma N(b), 0)$$

$$\begin{aligned}
N\left((a,b)(a',b')\right) =& N\left((aa' + \gamma b'b^*,\ a^*b' + a'b)\right) \\
=& N(aa' + \gamma b'b^*) - \gamma N(a^*b' + a'b) \\
=& N(aa') + \gamma^2 N(bb') - \gamma N(ab') - \gamma N(a'b) \\
& + \gamma (aa')^*(b'b^*) + \gamma (b'b^*)^*(aa') - \gamma (a^*b')^*(a'b) - \gamma (a'b)^*(a^*b') \\
=& N(a,b)\ N(a',b') \\
& + \gamma \left(\left((aa')^*(b'b^*) + (b'b^*)^*(aa')^*\right) - \left(b'^*(aa')b + b^*(aa')^*b'\right)\right) \\
=& N(a,b)\ N(a',b') \\
& + \gamma \left(\langle aa',\ b'b^*\rangle - \langle b',\ aa'b\rangle\right) \\
=& N(a,b)\ N(a',b')
\end{aligned}$$

The formula for multiplication that we use is taken from the book of involutions [KMRT98, 33.C, p.452]. Springer and Veldkamp [SV00, Prop. 1.5.3, p.13], use a slightly different formula. The same computations with $\langle\ ,\ \rangle$ from Lemma 3.1.3 prove the formula of Springer and Veldkamp correct and they define isomorphic composition algebras. We included a proof here because both books make the claim of the composition property of the norm only for R a field and they both do not give a proof.

Example 3.1.6. With parameter $\gamma = -1$, this construction gives for $C = \mathbb{R}$ the complex numbers $D = \mathbb{C}$ as Cayley–Dickson algebra, and for $C = \mathbb{C}$ it constructs the quaternions $\mathbb{H}$, out of which the octonions $\mathbb{O}$ result. As is well known, these are the only division algebras over $\mathbb{R}$ (that follows from the solution of the Hopf invariant one problem [Ada60]).

Because of a condition on the characteristic, we will not use the following:

Lemma 3.1.7. *For $D = C \oplus C$ a Cayley-Dickson construction over a composition algebra C, over a ring R in which 2 is invertible, the set of $*$-invariant elements in D can be identified with those of C:*

$$\{x \in D \mid x^* = x\} \to \{y \in C \mid y^* = y\}$$

where the map is projection to the first C-component, with the inclusion as inverse.

Proof. We can write $x = (a,b)$ with $a, b \in C$. By definition, $x^* = (a^*, -b)$, so $x^* = x$ implies $a = a^*$ and $2b = 0$. □

We will from now on use the symbol C only for one particular composition algebra (defined below), and denote by D_R any composition algebra that arises by the Cayley–Dickson construction with parameter $\gamma = 1$, i.e. a Cayley–Dickson construction with split norm form. This means we will study exclusively $D_R = \mathrm{C}_R,\ \mathrm{H}_R,\ \mathrm{O}_R$. Note that on R, $N(x) = x^2$ is not a split form.

Definition 3.1.8. Let $\mathrm{C}_R := R \oplus R$ with component-wise multiplication

$$(x_1, x_2)(y_1, y_2) := (x_1 y_1, x_2 y_2),$$

transposition involution $(x_1, x_2)^* := (x_2, x_1)$ and norm form $N(x_1, x_2) := x_1 x_2$. We call C_R the *split complex numbers* over R.

Definition 3.1.9. Let $\mathrm{H}_R := \mathrm{Mat}^{2\times 2}(R)$ with matrix-multiplication, involution given by the adjugate

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and norm form the determinant $N := \det$. We call H_R the *split quaternions* over R.

Definition 3.1.10. Let $\mathrm{O}_R := \mathrm{H}_R \oplus \mathrm{H}_R$ be the Cayley–Dickson construction over H_R with parameter $\gamma := 1$, with norm form $N(h_1, h_2) = N(h_1) - N(h_2)$. We call O_R the *split octonions.*

The split complex numbers C_R are isomorphic to the Cayley–Dickson construction over R with parameter $\gamma = 1$ by the map $(x, y) \mapsto (x + y, x - y)$.

The split quaternions H_R are isomorphic to the Cayley–Dickson construction over C_R with parameter $\gamma = 1$ by the map $((a, d), (b, c)) \mapsto \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$.

It is well known that C_R is commutative, while H_R and O_R are not, C_R and H_R are associative, while O_R is not. The only quality of O_R which is worth mentioning at this point is that it is alternative, which implies that every sub-algebra generated by 2 elements is associative.

A convenient way to write down octonionic computations are *Zorn vector matrices*, where $\mathrm{O}_R \ni x = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}$ with $\alpha, \beta \in R$ and $a, b \in R^3$, multiplication given by the usual rule of matrix multiplication, where one employs the euclidean scalar product in R^3 when appropriate:

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + a \cdot b' & \alpha a' + \beta' a \\ \alpha' b + \beta b' & b \cdot a' + \beta\beta' \end{pmatrix}$$

with norm being the generalized determinant $\alpha\beta - a \cdot b$.

For concrete computations, one can always use Zorn vector matrices, which reduces to ordinary matrices for the quaternions embedded in the octonions, and it further reduces to pairs with component-wise multiplication for the complex numbers embedded in the quaternions (as diagonal matrices).

Definition 3.1.11. An n-fold Cayley–Dickson construction D_R over a ring R consists of R^{2^n} with extra structure (product, involution, norm), so we call $d := 2^n$ the *dimension* of D_R (observing that $d \in \{2, 4, 8\}$) and $e := d/2 - 1$ the *excess dimension.* We note $e(\mathrm{C}_R) = 0$, $e(\mathrm{H}_R) = 1$ and $e(\mathrm{O}_R) = 3$.

Convention 3.1.12. Using Zorn vector matrix notation, for $x \in D_R$ with

$$x = \begin{pmatrix} x^{11} & x^{21} \\ -x^{22} & x^{12} \end{pmatrix}$$

the entries x^{21} and x^{22} are elements of R^e (for $D = \mathrm{C}$, we have $e = 0$ and $x^{21} = 0 = x^{22}$). The sign at the entry x^{22} is a convention we will use throughout, so we will have

$$N(x) = x^{11}x^{12} + x^{21} \cdot x^{22}.$$

Lemma 3.1.13. *Octonions with entries on the diagonal associate with all other entries, i.e. for any $\alpha, \beta \in R$, for*

$$\chi := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \mathrm{O}_R$$

we have $\forall x, y \in \mathrm{O}_R$:

(1) $$(xy)\,\chi = x\,(y\chi)\,.$$

(2) $$(x\chi)\,y = x\,(\chi y)\,.$$

(3) $$(x\chi)\,(y\chi)^* = (xy)\,N(\chi).$$

(4)

Proof. Right-multiplication with the diagonal matrix multiplies the first column by α and the second column by β. Left-multiplication with an element x operates on y column-wise. This (or a direct computation) shows the first equation. The second equation is also just a computation. The third equation follows from the second (or by another direct computation). □

3.2. Computations with a Lagrangian

In this section, we make some computations that will be used in the following chapter.

We remind the reader that we denote by D_R any of the three split composition algebras C_R, H_R, O_R.

Definition 3.2.1. Define $\mathrm{half}_R \colon R^{2n} \to R^n$ to be the projection to the first n coordinates, and $\mathrm{ohalf}_R \colon R^{2n} \to R^n$ the projection to the last n coordinates, such that $\mathrm{half}_R \oplus \mathrm{ohalf}_R = \mathrm{id}_{R^{2n}}$.

For $x \in D_R$ we order the coordinates of $D_R = R^d$ as tuple $(x^{11}) \oplus x^{21} \oplus (x^{12}) \oplus x^{22}$ to define $\mathrm{half}_D \colon D_R \to R^{d/2}$ and $\mathrm{ohalf}_D \colon D_R \to R^{d/2}$. Note that we did not put a minus sign in front of x^{22}, but we still have that $\mathrm{half}_D \oplus \mathrm{ohalf}_D = \mathrm{id}_{D_R}$ with the ordering we chose, because of our choice of sign convention. We will always write $\mathrm{half} := \mathrm{half}_D$ and $\mathrm{ohalf} := \mathrm{ohalf}_D$. We define the map $\mathrm{half} \colon D_R^n \to R^{nd/2}$ as the n-fold direct sum of half on D_R (and do the same for ohalf).

Remark 3.2.2. Our definition of half and ohalf on D_R ensures

$$N(x) = \mathrm{half}(x) \cdot \mathrm{ohalf}(x).$$

One could have chosen a different set of coordinates for $\mathrm{half} \colon D_R \to R^{d/2}$, as long as $\mathrm{half}(x) = 0$ implies $N(x) = 0$. In other words, we made a choice of Lagrangian for the norm form, given by $\{\mathrm{half} = 0\}$.

Lemma 3.2.3. *For any* $x, y \in D_R$

$$\mathrm{half}(x) = 0 \implies \mathrm{half}(xy) = 0$$

and therefore $\mathrm{half}(xy) \neq 0 \implies \mathrm{half}(x) \neq 0$.

Proof. We compute:

$$\mathrm{half}(x) = 0 \Leftrightarrow x = \begin{pmatrix} 0 & 0 \\ -x^{22} & x^{12} \end{pmatrix}, \text{ so}$$

$$xy = \begin{pmatrix} 0 & 0 \\ -(y^{11}x^{22} + x^{12}y^{22}) & x^{12}y^{12} - x^{22} \cdot y^{21} \end{pmatrix}.$$ □

Example 3.2.4. Let $y \in D$ with $N(y) \in R^\times$, then there exists $x \in D$ with $N(x) = 1$ such that $\mathrm{half}(yx) = e_1$. If $N(y)$ is invertible then y is invertible with two-sided inverse $y^* N(y)^{-1}$, since $yy^* = y^*y = N(y)$. With $x := y^*N(y)^{-1}$, we compute $\mathrm{half}(yx) = \mathrm{half}(1) = e_1$. There could, however, be more ways to achieve the goal $\mathrm{half}(yx) = e_1$, i.e. the set $\{x \in D \mid N(x) = 1,\ \mathrm{half}(yx) = e_1\}$ might be larger than just $\{y^*N(y)^{-1}\}$.

For any such x we can compute

$$yx = \begin{pmatrix} 1 & 0 \\ * & N(y) \end{pmatrix}$$

and therefore

$$y = y(xx^*) = (yx)x^* = \begin{pmatrix} x^{12} & -x^{21} \\ * & * \end{pmatrix}$$

consequently

$$x = \begin{pmatrix} * & -y^{21} \\ * & y^{11} \end{pmatrix}$$

Definition 3.2.5. We call $y \in D$ *half-invertible* if there exists an $x \in D$ of unit norm $N(x) = 1$ with $\mathrm{half}(yx) = e_1$. We call $E_y := \{x \in D \mid N(x) = 1,\ \mathrm{half}(yx) = e_1\}$ the set of *half-inverters* of y.

Lemma 3.2.6. *Any* $y \in D$ *with the property* $\exists i : \mathrm{half}(y)_i \in R^\times$ *is half-invertible.*

More generally, if $\gamma\colon Z \to D$ *is any morphism of* R*-varieties with* $\mathrm{half}(\gamma(z))_i \in R^\times$ *for fixed* i*, then there exists a morphism* $\chi_i\colon Z \times \mathbb{G}_\mathrm{a}^{\times \mathrm{e}} \to D$ *such that* $\chi_i(z)$ *is a half-inverter of* $\gamma(z)$*. Here, we write* $\mathbb{G}_\mathrm{a}^{\times \mathrm{e}} = \mathbb{G}_\mathrm{a} \times \cdots \times \mathbb{G}_\mathrm{a}$ *for the* e*-fold product of additive groups.*

Proof. We construct explicit half-inverters x, for a given free parameter $\kappa \in R^e$. We distinguish two cases:

(1) Assume that $\mathrm{half}(y)_0 = y^{11} \in R^\times$, define

$$x_0(\kappa) := \begin{pmatrix} (y^{11})^{-1} - y^{21} \cdot \kappa & -y^{21} \\ y^{11}\kappa & y^{11} \end{pmatrix}$$

(2) Assume that $\mathrm{half}(y)_i = y^{21}_i \in R^\times$ (where we index $\mathrm{half}(y)$ from 0 on and y^{21} from 1 on), define

$$x_i(\kappa) := \begin{pmatrix} -y^{21} \cdot \kappa & -y^{21} \\ y^{11}\kappa + (y^{21}_i)^{-1} e_i & y^{11} \end{pmatrix}$$

The properties $N(x_i(\kappa)) = 1$ and $\mathrm{half}(yx_i(\kappa)) = e_1$ are fulfilled by construction. □

Lemma 3.2.7. *For any $y \in D$ the set of half-inverters E_y admits a nontrivial $\mathbb{G}_a^{\times e}$-action.*

Proof. The $\mathbb{G}_a^{\times e}$-action is given by right multiplication in D along

$$\mathbb{G}_a^{\times e} \hookrightarrow D,\ \alpha \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

and we check for $y \in D$ and x with $\mathrm{half}(yx) = e_1$ that for all $\alpha, \beta \in \mathbb{G}_a^{\times e}$

$$\mathrm{half}(y(x.\alpha)) = e_1,\ \mathrm{half}(y((x.\alpha).\beta)) = \mathrm{half}(y(x.(\alpha+\beta))) = e_1$$

which is a computation that can be done economically by two observations:

$$x.\alpha = \begin{pmatrix} * & -y^{21} \\ * & y^{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} * & -y^{21} \\ * & y^{11} \end{pmatrix}.$$

This already tells us that $(x.\alpha).\beta$ and $x.(\alpha+\beta)$ are also of this shape.

$$y(x.\alpha) = y \begin{pmatrix} * & -y^{21} \\ * & y^{11} \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & N(y) \end{pmatrix}.$$

By $N(y(x.\alpha)) = N(y)N(x.\alpha) = N(y)$ we now see that $\mathrm{half}(y(x.\alpha)) = e_1$. Note also that $(x.\alpha).\beta = x.(\alpha+\beta)$ even if D is non-associative. □

Remark 3.2.8. On the x_i from Lemma 3.2.6 the $\mathbb{G}_a^{\times e}$-action is easy to compute:

$$x_i(\kappa).\alpha = x_i(\kappa+\alpha).$$

Lemma 3.2.9. *If D is associative, i.e. $D = \mathrm{C}$ and $e = 0$ or $D = \mathrm{H}$ and $e = 1$, then for any $y \in D$, the $\mathbb{G}_a^{\times e}$-action on E_y is free and transitive.*

If D is non-associative, i.e. $D = \mathrm{O}$ and $e = 3$, then for $y \in D$ with $y^{11} = 0$, the $\mathbb{G}_a^{\times e}$-action on E_y is not transitive. For $y \in D$ with y^{11} invertible, the $\mathbb{G}_a^{\times e}$-action on E_y is free and transitive.

Proof. First, let $D = \mathrm{H}$. We show that every $x \in E_y$ is of the form $x_i(\kappa)$ for some i and κ (and of course we could have proved Lemma 3.2.7 that way, too). If there exists $x \in E_y$, then we see that $\exists i : \mathrm{half}(y)_i \in R^\times$. We distinguish two cases:

(1) Assume that $\mathrm{half}(y)_0 = y^{11} \in R^\times$, then $x^{11} = (1 + y^{21}x^{22})(y^{11})^{-1}$ and with $\kappa := -x^{22}(y^{11})^{-1}$ we get $x = x_0(\kappa)$.
(2) Assume that $\mathrm{half}(y)_1 = y^{21} \in R^\times$, then $x^{22} = (y^{11}x^{11} - 1)(y^{21})^{-1}$ and with $\kappa := -x^{11}(y^{21})^{-1}$ we get $x = x_1(\kappa)$.

We can extract a proof that for $D = \mathrm{C}$ the ${\mathbb{G}_a}^0$-action on E_y is free and transitive, i.e. E_y consists of a single element, by putting $\mathrm{C} \hookrightarrow \mathrm{H}$ as diagonal, as usual.

Now for $D = \mathrm{O}$ and $y^{11} = 0$, but $y_1^{21}, y_2^{21} \in R^\times$, let us assume there is $\alpha \in {\mathbb{G}_a}^3$ with $x_1(0).\alpha = x_2(0)$:

$$x_1(0).\alpha = \begin{pmatrix} -y^{21} \cdot \alpha & -y^{21} \\ (y_1^{21})^{-1} e_1 & \end{pmatrix} = \begin{pmatrix} 0 & -y^{21} \\ (y_2^{21})^{-1} e_2 & 0 \end{pmatrix}$$

As e_1 and e_2 are linearly independent, we get a contradiction.

For $y^{11} \in R^\times$, we can for $i, j > 0$ always find a unique α such that $x_i(\kappa).\alpha = x_j(\kappa')$, by the expression $\alpha := -\kappa + (y^{11})^{-1} \left((y_j^{21})^{-1} e_j - (y_i^{21})^{-1} e_i \right) + \kappa'$.

Furthermore, $x_0(\kappa).\alpha = x_i(\kappa')$ is uniquely solved by $\alpha := \kappa + (y^{11} y_i^{21})^{-1} e_i + \kappa'$. □

CHAPTER 4

Projective Spaces over Split Composition Algebras

In this section we study the geometry of projective spaces with coordinates in a split composition algebra, defined via rank 1 projectors in the Jordan algebras of hermitian matrices over the composition algebra. This provides the geometric input to motivic homotopy considerations. We also prove surjectivity of the map from the sphere to projective space in this setup, which might be of independent interest.

4.1. Algebraic Geometry of D-Projective Spaces

We define D-projective space over a split composition algebra D over a ring (in any characteristic) via hermitian matrices.

Remark 4.1.1. We apologize to the reader for not using group-theoretic methods more heavily. The reasons are twofold: the octonions do not admit a group structure on the norm 1 elements because groups are usually required to be associative. On the other hand, for general affine quadrics group-theoretic methods seem to be more difficult in characteristic 2, and we wanted to pursue an approach that would work for many D-geometries in any characteristics.

Convention 4.1.2. For the remainder of this chapter, let D be a split composition algebra of dimension d over a commutative unital ring R, i.e. D is either C_R, H_R or O_R. Let $n \in \mathbb{N}$ be a number.

4.1.1. Jordan Algebras.

Good notions of Jordan algebras which behave correctly in any characteristic are the concepts of quadratic Jordan algebra or J-algebra, which need not be algebras at all. We decided against the description of arbitrary Jordan algebras here, and focus instead on the family of Jordan algebras which is relevant to this work: hermitian matrices over a split composition algebra.

Definition 4.1.3. We define $\dagger\colon \mathrm{Mat}^{m\times n}(D) \to \mathrm{Mat}^{n\times m}(D)$ by $\dagger(A)_{ij} = A^*_{ji}$. We will write $A^\dagger := \dagger(A)$.

The special case of $\dagger\colon D^n \to \mathrm{Mat}^{1\times n}(D)$ which converts column vectors to row vectors while applying the involution $*$ to components is particularly important.

Definition 4.1.4. We call

$$J_n(D) := \{A \in \mathrm{Mat}^{n\times n}(D) \mid A = A^\dagger,\ \forall i : A_{ii} \in R\}$$

the set of *hermitian matrices.*

The diagonal entries A_{ii} of a hermitian matrix A satisfy $A_{ii} = A_{ii}^*$, so if 2 is invertible in R, we automatically have $A_{ii} \in R$, due to Lemma 3.1.7.

We remark that $J_n(D)$ is not a Jordan algebra for D of dimension 8 over R and $n > 3$. We will exclude these pathological cases from our study from now on (compare Section 1.3 for more reasons).

Lemma 4.1.5. *For D of dimension less than 8 or $n \leq 2$, the set $J_n(D)$ with matrix multiplication and addition forms a ring.*

Proof. If D is of dimension less than 8, it is associative, hence $J_n(D)$ is. If $n \leq 2$, each matrix multiplication in $J_2(D)$ involves only a subalgebra of D generated by 2 elements, which is always associative. □

A proof of the Jordan algebra property of $J_n(D)$ can be found in the book of Springer and Veldkamp [SV00, Chapter 5, p.117] and the references provided therein for the case of characteristics 2 and 3. For characteristic 2, one has to use the concept of quadratic Jordan algebras (or J-algebras) instead of Jordan algebras, if one wants to consider anything beyond hermitian matrices. We will not use the Jordan algebra property in the sequel and we will not have to use the theory in this generality.

For a more Jordan-theoretic approach towards algebraic geometry over split composition algebras, we refer to Chaput's work [Cha06].

Remark 4.1.6. In the classification theory of Jordan algebras one can see the spin factors, and in fact $J_2(D)$ is a spin factor Jordan algebra. Given the connection between $J_2(D)$ and affine quadrics to be explored in Lemma 4.1.12, it seems promising to relate these spin factors to higher dimensional quadrics. While we do not pursue these ideas in this work, it is likely that some techniques carry over.

4.1.2. Geometric Constructions: Projective Spaces.

We define projective space over a split composition algebra D over a unital ring R by a naive generalization of a classical definition of projective space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.1.7. Inside the hermitian matrices, we define

$$\mathrm{DP}^n := \{A \in J_{n+1}(D) \mid A^2 = A,\ \mathrm{tr}(A) = 1\}$$

to be the *n-dimensional projective space over D*. The square A^2 denotes the usual matrix product in $\mathrm{Mat}^{n+1}(D)$, the trace $\mathrm{tr}(A)$ the usual matrix trace given by the sum of diagonal entries.

Remark 4.1.8. Given the *projector condition* $A^2 = A$ (which we could as well call idempotent condition), the rank (which we define as dimension of the image) is given by the trace: $\mathrm{rk}(A) = \mathrm{tr}(A)$, hence the *trace condition* is geometrically reasonable in the division algebra case. To see this, one has to show that any projector can be transformed by an appropriate Spin group, without changing rank or trace, to a diagonal matrix with entries 0 or 1.

Convention 4.1.9. We defined an affine algebraic variety over R:

$$\mathrm{DP}^n \hookrightarrow \mathbb{A}_R^{(d(n+1))^2}.$$

It is tempting to call DP^n an affine projective space, but we will instead refer to it as *D-projective space.* We call OP^1 the *split octonionic projective line* or just *octonionic projective line* and OP^2 the *(split) octonionic projective plane* or *split Cayley plane* or *affine Cayley plane.* Some authors call OP^2 the *real Cayley plane* (where "real" stands for the real numbers). We call HP^n *(split) quaternionic projective space* and CP^n *split complex projective space.*

Remark 4.1.10. It is obvious how one can define other Grassmannians using higher rank/trace projectors in hermitian matrices as well. We do not pursue it in this work, but it is likely that large, if not all, parts of this work generalize to these other Grassmannians as well. Of course, some restrictions in working with octonions apply.

Lemma 4.1.11. *The dimension of* DP^n *is* dn*.*

Proof. Over a field, this is an easy computation from the definition of DP^n. We can also use the homogeneous space structure of CP^n, HP^n and OP^1, OP^2, which allows for a different and more convenient way of computation. To get the dimension over arbitrary rings, it suffices to see that DP^n over $\mathbb{Z}$ is flat. Torsion-freeness of $\mathcal{O}(\mathrm{DP}^n)$ can be seen from looking at the quadratic terms in the ideal defining DP^n as subset of affine space. □

Lemma 4.1.12. *The spaces* CP^1, HP^1 *and* OP^1 *are affine split quadrics of even dimensions. More precisely, writing*

$$AQ_{2n} := \{(x,y,z) \in \mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^1 \mid \sum_{i=1}^{n} x_i y_i = z(1-z)\}$$

for an even-dimensional split quadric (smooth over $\mathrm{Spec}(\mathbb{Z})$*),*

$$\mathrm{CP}^1 \xrightarrow{\sim} AQ_2 \tag{5}$$

$$\mathrm{HP}^1 \xrightarrow{\sim} AQ_4 \tag{6}$$

$$\mathrm{OP}^1 \xrightarrow{\sim} AQ_8 \tag{7}$$

Proof. By definition,

$$\mathrm{DP}^1 = \{A \in \mathrm{Mat}^{2\times 2}(D) \mid A^\dagger = A,\ A^2 = A,\ \mathrm{tr}(A) = 1\}.$$

We will write

$$A = \begin{pmatrix} z & \psi^* \\ \psi & 1-z \end{pmatrix}$$

then $A^2 = A$ is equivalent to $z^2 + \psi^*\psi = z$. Using our knowledge of $\psi^*\psi = N(\psi)$ being a split quadratic form, we conclude

$$\mathrm{DP}^1 = \left\{ (z,\psi) \in R \oplus R^d \;\middle|\; \sum_{i=1}^{d/2} \psi_i \psi_{d/2+i} = z(1-z) \right\}.$$ □

Remark 4.1.13. While Asok–Doran–Fasel proved that even-dimensional split affine quadrics are motivic spheres with an inductive argument, they provide more elementary proofs for AQ_2 and AQ_4 [ADF16, Subsection 2.1]. We also work out the theory developed here for these examples more explicitly, see Example 4.3.2 and Example 4.4.5. The reader is advised to take a look at Section 4.3 and Section 4.4 for more background information on CP^n and HP^n.

4.1.3. Geometric Constructions: Spheres.

We define a map from a D-sphere to D-projective space, and prove that this map is surjective on R-points.

Definition 4.1.14. We define the *n-th D-sphere* to be

$$\mathrm{DS}^n := \left\{ v \in D^{n+1} \;\middle|\; N(v) := v^\dagger v = 1 \right\}$$

For $D = O$ we also define the *associative second D-sphere*

$$\mathrm{OS}^{2,ass} := \left\{ v \in \mathrm{OS}^2 \;\middle|\; \mathrm{assoc}(v) = 0 \right\}$$

where the associator of $v \in \mathrm{O}^3$ is defined as

$$\mathrm{assoc}(v) := \begin{pmatrix} \{v_2, v_0^*, v_0 v_1^*\} \\ \{v_0, v_1^*, v_1 v_2^*\} \\ \{v_1, v_2^*, v_2 v_0^*\} \end{pmatrix} = \begin{pmatrix} N(v_0) v_2 v_1^* - (v_2 v_0^*)(v_0 v_1^*) \\ N(v_1) v_0 v_2^* - (v_0 v_1^*)(v_1 v_2^*) \\ N(v_2) v_1 v_0^* - (v_1 v_2^*)(v_2 v_0^*) \end{pmatrix} \in \mathrm{O}^3 .$$

In all other cases let $\mathrm{DS}^{n,ass} := \mathrm{DS}^n$.

Example 4.1.15. As an algebraic variety, $\mathrm{DS}^n \simeq AQ_{d(n+1)-1}$, an *odd*-dimensional split affine quadric. We know that DS^n is $d(n+1)-1$-dimensional and in particular DS^0 is $d-1$-dimensional. We remark that $\mathrm{CS}^0 \simeq AQ_1 = \mathbb{G}_\mathrm{m}$ and $\mathrm{HS}^0 \simeq AQ_3 \simeq \mathrm{SL}_2$ have a group structure, while $\mathrm{OS}^0 \simeq AQ_7$ does not inherit a group structure from O, since groups are required to be associative. For $D = \mathrm{O}$ and $n = 1$, it makes sense to define $\mathrm{OS}^{1,ass} := \mathrm{OS}^1$ since every subalgebra of O generated by two elements is associative.

Definition 4.1.16. There is a *canonical projection* map

$$p\colon \mathrm{DS}^{n,ass} \to \mathrm{DP}^n$$

obtained as restriction of the map

$$p\colon D^{n+1} \to J_{n+1}(D),\ v \mapsto vv^\dagger.$$

The dimension drops by $d-1 = \dim \mathrm{DS}^0$ except for $\mathrm{OS}^{2,ass}$. The map p on O^3 restricted to OS^2 does not map to OP^2, as $A := p(v) = vv^\dagger$ does not satisfy $A^2 = A$ in general. It is precisely at this point where one needs the associator condition.

Lemma 4.1.17. *Over a local ring R, the subspace $\mathrm{OS}^{2,ass} \hookrightarrow \mathrm{OS}^2$ is of codimension at most 8.*

Proof. Every element $v \in \mathrm{OS}^2$ satisfies $N(v_i) \in R^\times$ for some $i \in \{0,1,2\}$, as $1 = N(v) = N(v_0) + N(v_1) + N(v_2)$. Without loss of generality, let $i = 0$ be this index. We apply the involution to $\mathrm{assoc}(v)_0 = 0$ and get

$$v_1 v_2^* = N(v_0)^{-1} (v_0 v_1^*)^* (v_2 v_0^*)^*.$$

Plugging this equation into either $\mathrm{assoc}(v)_1$ or $\mathrm{assoc}(v)_2$ shows that they vanish. As $\mathrm{assoc}(v)_i = 0$ is an equation of elements in $\mathrm{O} = R^8$, it cuts out a codimension at most 8 subspace. □

Convention 4.1.18. We will write an element of $J_{n+1}(D)$ as a 2×2-matrix with upper left corner an element of R, lower right corner a matrix in $Mat^{n\times n}(D)$, upper right corner a tuple in D^n (written horizontally) and lower left corner a vector in D^n (written vertically):

$$J_{n+1}(D) \ni A = \begin{pmatrix} z & \psi^\dagger \\ \psi & a \end{pmatrix} \qquad \text{with } z \in R,\ \psi \in D^n,\ a \in J_n(D)$$

The projector condition $A^2 = A$ becomes $a^2 + \psi\psi^\dagger = a$, $a\psi = (1-z)\psi$ and $\psi^\dagger\psi = z(1-z)$. The trace condition implies $\operatorname{tr}(a) = 1 - z$. In this notation, for $v = v_0 \oplus \varphi \in \mathrm{DS}^n$:

$$p(v) = p(v_0 \oplus \varphi) = \begin{pmatrix} v_0 v_0^* & v_0\varphi^\dagger \\ \varphi v_0^* & \varphi\varphi^\dagger \end{pmatrix}.$$

Definition 4.1.19. We introduce open subvarieties of DP^n and $\mathrm{DS}^{n,ass}$:

$Z_i := \{A \in \mathrm{DP}^n \mid A_{ii} \neq 0\} \subset \mathrm{DP}^n$,
$Z_i^\circ := \{A \in \mathrm{DP}^n \mid A_{ii} \text{ invertible}\} \subset Z_i$,
$\widetilde{Z}_i := \{v \in \mathrm{DS}^{n,ass} \mid N(v_i) \neq 0\} \subset \mathrm{DS}^n$, $\widetilde{Z}_i^\circ := \{v \in \mathrm{DS}^{n,ass} \mid N(v_i) \text{ invertible}\} \subset \widetilde{Z}_i$.

Theorem 4.1.20. *The morphism p is dominant and surjective on R-points.*

The open subvarieties Z_i (respectively for R a local ring Z_i°) cover DP^n and the projection p restricted to $\widetilde{Z}_i$ (respectively for R a local ring $\widetilde{Z}_i^\circ$) has image in Z_i (respectively Z_i°).

For R a local ring, there are sections $s_i\colon Z_i^\circ \to \widetilde{Z}_i^\circ$ to p with the property $s_j = \widetilde{\tau}_{ij} \circ s_i \circ \tau_{ji}$ for $\tau_{ji}\colon Z_i \to Z_j$ conjugation with a permutation matrix switching rows and columns (i,j) and $\widetilde{\tau}_{ij}\colon \widetilde{Z}_j \to \widetilde{Z}_i$ a transposition switching the ith and jth entry.

For R a field, $Z_i = Z_i^\circ$ and $\widetilde{Z}_i = \widetilde{Z}_i^\circ$.

As warm-up, we collect some innocent observations about rank 1 projector matrices A before we give the proof. These can be subsumed under "$za = \psi\psi^\dagger$".

Proposition 4.1.21. *If $z = 1$, then $a = \psi\psi^\dagger$.*

Proof. We show that $a - \psi\psi^\dagger$ is a projector of rank 0:

$$\begin{aligned}
(a - \psi\psi^\dagger)^2 &= a^2 - a\psi\psi^\dagger - \psi\psi^\dagger a + (\psi\psi^\dagger)(\psi\psi^\dagger) \\
&= a - \psi\psi^\dagger - 2(1-z)\psi\psi^\dagger + (\psi^\dagger\psi)\psi\psi^\dagger \\
&= a - \psi\psi^\dagger \\
\operatorname{tr}(a - \psi\psi^\dagger) &= \operatorname{tr}(a) - \psi^\dagger\psi = 0 - 0 = 0.
\end{aligned}$$

□

Proposition 4.1.22. *If $z = 0$, then $a + \psi\psi^\dagger$ is a rank 1 projector.*

Proof. We compute:

$$\begin{aligned}
(a + \psi\psi^\dagger)^2 &= a^2 + a\psi\psi^\dagger + \psi\psi^\dagger a + (\psi\psi^\dagger)(\psi\psi^\dagger) \\
&= a - \psi\psi^\dagger + 2(1-z)\psi\psi^\dagger + (\psi^\dagger\psi)\psi\psi^\dagger \\
&= a + \psi\psi^\dagger \\
\operatorname{tr}(a + \psi\psi^\dagger) &= \operatorname{tr}(a) + \psi^\dagger\psi = 1 + 0 = 1.
\end{aligned}$$

□

Proposition 4.1.23. *If $z = 0$ and there is $\varphi \in D^n$ such that $\varphi^\dagger\varphi = 1$ and $\varphi\varphi^\dagger = a + \psi\psi^\dagger$, then $\varphi\varphi^\dagger\psi = \psi$, $\psi\psi^\dagger = 0$ and $a = \varphi\varphi^\dagger$.*

Proof. We first compute

$$\varphi\varphi^\dagger\psi = a\psi + \psi\psi^\dagger\psi = 1\psi + 0\psi = \psi$$

so with $v_0 := \psi^\dagger\varphi$ we obtain $\psi = \varphi v_0^*$. As we already know $\psi^\dagger\psi = z(1-z) = 0$,

$$0 = \psi^\dagger\psi = (v_0\varphi^\dagger)(\varphi v_0^*) = (\varphi^\dagger\varphi)v_0 v_0^* = v_0 v_0^*.$$

This shows $v_0^* v_0 = 0$ as well, so we also have

$$\psi\psi^\dagger = \varphi v_0^* v_0 \varphi^\dagger = 0.$$ □

Proposition 4.1.24. *If $z \neq 0, 1$, then $\mathrm{rk}(a) = 1$ and $za = \psi\psi^\dagger$.*

Proof. As a is a submatrix of A, we have $\mathrm{rk}(a) \leq \mathrm{rk}(A) = 1$. Note that $z \neq 0, 1$ implies $\psi^\dagger\psi \neq 0$. From $a\psi = (1-z)\psi$ and $(\psi\psi^\dagger)\psi = z(1-z)\psi$, we get that the images of a and $\psi\psi^\dagger$ are of dimension at least 1, so $\mathrm{rk}(a) = 1$. On the image of a, both za and $\psi\psi^\dagger$ act as multiplication with the scalar $z(1-z)$, so they are equal on this subspace:

$$(za - \psi\psi^\dagger)a = 0.$$

Using $a = a^2 + \psi\psi^\dagger$ and $\psi^\dagger a = (1-z)\psi^\dagger$ we get

$$0 = za^2 - \psi\psi^\dagger a = za - z\psi\psi^\dagger - \psi\psi^\dagger + z\psi\psi^\dagger = za - \psi\psi^\dagger.$$ □

Remark 4.1.25. In the special cases $z = 0$ and $z = 1$, we can now prove that A has a preimage under p, assuming we proved the general theorem in one dimension lower already:

If $z = 1$, we can pick $v_0 := 1$ and $\varphi := \psi$, then by Proposition 4.1.21 we have $a = \varphi\varphi^\dagger$, and with $v := v_0 \oplus \varphi$ we get $p(v) = vv^\dagger = A$.

If $z = 0$, we know that $a + \psi\psi^\dagger$ is a rank 1 projector by Proposition 4.1.22 so by induction on the size of the matrix A (i.e. on n) there is φ such that $\varphi\varphi^\dagger = a + \psi\psi^\dagger$. By Proposition 4.1.23, we can conclude that $\psi\psi^\dagger = 0$. Define $v_0 := \psi^\dagger\varphi$ and $v := v_0 \oplus \varphi$, then $p(v) = vv^\dagger = A$.

Proof of Theorem 4.1.20. Assume that z is a unit, for then we can pick any v_0 with $v_0^* v_0 = z$, and there is $w_0 := v_0 z^{-1}$ with the property

$$v_0^* w_0 = 1.$$

Let $\varphi := \psi w_0$, then $v := v_0 \oplus \varphi$ satisfies $p(v) = A$.

This shows that there is a section to p on the Zariski open where z is a unit. To fix an explicit section, we pick

$$v_0 := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix},$$

which is a morphism $Z_i \to D$. This shows p is dominant.

If R is a local ring, the sum of diagonal elements is a unit, hence one of its terms is a unit. Without loss of generality, this diagonal entry is $z = A_{00}$, which we use to define s_0. The other s_i are defined via τ_{ij} and $\widetilde{\tau}_{ij}$ as in the statement of the theorem.

The equation $1 = \operatorname{tr} A = \sum_i A_{ii}$ yields a Zariski cover of R by local rings, which, pulled back to DP^n, gives a Zariski cover, such that over each open set, p is surjective on R-points. □

One can also prove the surjectivity on R-points in the split octonion case over a field by a reduction to the quaternion case, but both are hard to find in the literature. Harvey gave a proof for the division algebra octonions [Har90], together with a claim that only minor modifications give a proof for the split case.

4.2. Stratification

We study the geometry of embeddings $\mathrm{DP}^{n-1} \hookrightarrow \mathrm{DP}^n$, and construct a vector bundle V_n of rank $d/2$ over DP^{n-1} with total space inside DP^n. We study the complement X_n of V_n, which is our candidate for the *cell* of a cell structure.

Definition 4.2.1. We choose to embed $J_n(D) \hookrightarrow J_{n+1}(D)$ with i, given by

$$i\colon J_n(D) \ni a \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in J_{n+1}(D)$$

and observe that there is a retraction r to the embedding i, given by

$$r\colon J_{n+1}(D) \ni \begin{pmatrix} z & \psi^\dagger \\ \psi & a \end{pmatrix} \mapsto a \in J_n(D).$$

The embedding i restricts to a closed immersion $i\colon \mathrm{DP}^{n-1} \hookleftarrow\!\!\!\rightarrow \mathrm{DP}^n$.

The retraction r does not map DP^n to DP^{n-1}, as already the trace condition shows.

Lemma 4.2.2. *There is a retraction to i over an open subvariety of* DP^n*:*

$$t^\downarrow\colon Y_n := \{A \in \mathrm{DP}^n \mid 1 - z \text{ invertible}\} \twoheadrightarrow \mathrm{DP}^{n-1},\ A \mapsto (1-z)^{-1}a.$$

Proof. The map is well-defined as a is hermitian and we use $za = \psi\psi^\dagger$ (established before the proof of Theorem 4.1.20) to compute

$$\left((1-z)^{-1}a\right)^2 = (1-z)^{-2}\left(a - \psi\psi^\dagger\right) = (1-z)^{-2}(a - za) = (1-z)^{-1}a.$$

We see immediately that $t^\downarrow \circ i = \mathrm{id}_{\mathrm{DP}^{n-1}}$. □

4.2.1. Correcting the Codimension.

Motivated by the considerations of Remark 2.3.11, we construct a vector bundle $V \twoheadrightarrow \mathrm{DP}^{n-1}$ with total space $V \hookleftarrow\!\!\!\rightarrow \mathrm{DP}^n$ of "correct" codimension $d/2$.

We remind the reader that $\operatorname{codim}\left(\mathrm{DP}^{n-1} \hookleftarrow\!\!\!\rightarrow \mathrm{DP}^n\right) = d$.

Definition 4.2.3. We write $W_n := r^{-1}(\mathrm{DP}^{n-1}) \cap \mathrm{DP}^n \subset J_{n+1}(D)$ and obtain $r\colon W_n \twoheadrightarrow \mathrm{DP}^{n-1}$.

For an element $A \in W_n$ by definition $a \in \mathrm{DP}^{n-1}$, so by the trace condition $z = 0$. As $W_n = \{A \in \mathrm{DP}^n \mid z = 0\}$, it is a codimension 1 subspace. The condition $A^2 = A$ is equivalent to $z = \psi^\dagger\psi = \operatorname{tr}(\psi\psi^\dagger)$, $0 = \psi\psi^\dagger$ and $a\psi = \psi$, so that we can write

$$W_n = \left\{ \begin{pmatrix} 0 & \psi^\dagger \\ \psi & a \end{pmatrix} \in J_{n+1}(D) \,\middle|\, a \in \mathrm{DP}^{n-1},\ \psi \in \operatorname{Ker}(a - \mathrm{Id}),\ \psi\psi^\dagger = 0 \right\}$$

We lift this space along the projection map p to $\mathrm{DS}^{n,ass}$.

Proposition 4.2.4. *The preimage $p^{-1}(W_n)$ can be described as*

$$\widetilde{W}_n := \left\{ v_0 \oplus \varphi \in D \oplus D^n \;\middle|\; v_0^* v_0 = 0,\ \varphi^\dagger \varphi = 1 \right\} \cap \mathrm{DS}^{n,ass} = p^{-1}(W_n).$$

Proof. We show that $v = v_0 \oplus \varphi \in \widetilde{W}_n$ satisfies $p(v) \in W_n$, as $v \notin \widetilde{W}_n$ implies $p(v) \notin W_n$. The conditions $z = v_0^* v_0 = 0$ and $\mathrm{tr}(a) = \mathrm{tr}(\varphi\varphi^\dagger) = \varphi^\dagger\varphi = 1$ are immediate. To see that $\psi = \varphi v_0^*$ satisfies $a\psi = \psi$, just compute $a\psi = (\varphi\varphi^\dagger)(\varphi v_0^*) = \varphi(\varphi^\dagger\varphi)v_0^* = \varphi v_0^* = \psi$, using associativity in any subalgebra of D generated by two elements and the associator condition for $\mathrm{OS}^{2,ass}$. □

The notation $v = v_0 \oplus \varphi$ for an element $v \in D^{n+1} = D \oplus D^n$ will be used from now on without further explanation.

We now want to choose a useful subspace of W_n.

Definition 4.2.5. Using the map half introduced in Definition 3.2.1, let

$$\widetilde{V}_n := \left\{ v_0 \oplus \varphi \in \widetilde{W}_n \;\middle|\; \mathrm{half}(v_0) = 0 \right\}$$

$$V_n := \left\{ A \in \mathrm{DP}^n \;\middle|\; a \in \mathrm{DP}^{n-1},\ z = 0,\ \mathrm{half}(\psi^\dagger) = 0 \right\}$$

It is by definition that $V_n \subset W_n$. The reader may now safely forget the affine variety W_n, as we will only use V_n in the remainder. While W_n appears naturally, V_n depends on the chosen Lagrangian. We also want to remark that for $D = \mathrm{O}$, in $\widetilde{V}_2$ we still need the associator condition, which boils down to the equations

$$\left(v_1^{21} \cdot v_2^{22}\right)\left(v_2^{21} \cdot v_0^{22}\right) = \left(v_2^{21} \cdot v_2^{22}\right)\left(v_1^{21} \cdot v_0^{22}\right),$$

$$\begin{aligned} 0 \;=\; & \left(v_0^{22} \cdot v_2^{21}\right)\left(v_1^{12} v_2^{22} - v_2^{12} v_1^{22}\right) \\ & + \left(v_1^{22} \cdot v_2^{21}\right)\left(v_2^{12} v_0^{22} - v_0^{12} v_2^{22}\right) \\ & + \left(v_2^{22} \cdot v_2^{21}\right)\left(v_0^{12} v_1^{22} - v_1^{12} v_0^{22}\right). \end{aligned}$$

Proposition 4.2.6. *In fact, $V_n = \{A \in \mathrm{DP}^n \mid z = 0,\ \mathrm{half}(\psi^\dagger) = 0\}$.*

Proof. We know by Theorem 4.1.20 that there exists $v_0 \oplus \varphi \in \mathrm{DS}^n$ such that $v_0 v_0^* = z$, $\varphi v_0^* = \psi$ and $\varphi\varphi^\dagger = a$. We have $\psi\psi^\dagger = \varphi v_0^* v_0 \varphi^\dagger = z\varphi\varphi^\dagger$, so that $z = 0$ implies $\psi\psi^\dagger = 0$. Caution: This does not imply $\psi = 0$, as the case $n = 1$ already shows.

The projector condition $A = A^2$ implies $a - a^2 = \psi\psi^\dagger$, so that $\psi\psi^\dagger = 0$ implies $a = a^2$. As $1 = \mathrm{tr}(A) = z + \mathrm{tr}(a)$, $z = 0$ implies $\mathrm{tr}(a) = 1$. We have seen that $z = 0$ implies $a \in \mathrm{DP}^{n-1}$. This shows $\{A \in \mathrm{DP}^n \mid z = 0,\ \mathrm{half}(\psi^\dagger) = 0\} \subset V_n$, and the other inclusion is given by definition. □

Definition 4.2.7. Denote $\widetilde{X}_n := \mathrm{DS}^{n,ass} \setminus \widetilde{V}_n$ and $X_n := \mathrm{DP}^n \setminus V_n$ the complements.

Lemma 4.2.8. *The restriction of p to $\widetilde{V}_n$ has image in V_n and is surjective on R-points:*

$$p|_{\widetilde{V}_n} : \widetilde{V}_n \twoheadrightarrow V_n.$$

Proof. For $v \oplus \varphi \in \widetilde{V}_n$, with $\mathrm{half}(v_0) = 0$, also $\mathrm{half}(v_0\varphi^\dagger) = 0$, so $p(\widetilde{V}_n) \subset V_n$. Given any $A \in V_n$, we find a preimage $v_0 \oplus \varphi \in p^{-1}(\{A\})$. While not necessarily $\mathrm{half}(v_0) = 0$, we can still define $\widetilde{v_0}$ with $\mathrm{half}(\widetilde{v_0}) := 0$ and $\mathrm{ohalf}(\widetilde{v_0}) := \mathrm{ohalf}(v_0)$. Now $\widetilde{v_0}^*\widetilde{v_0} = 0 = z = v_0^* v_0$ and $\widetilde{v_0}\varphi^\dagger = v_0\varphi^\dagger$, since the modification $\widetilde{v_0}$ of v_0 only affects $\mathrm{half}(v_0\varphi^\dagger)$, which vanishes anyway. □

Corollary 4.2.9. *We have $p^{-1}(X_n) \subset \widetilde{X}_n$.*

Proof. For any $A \in \mathrm{DP}^n$, any $v = v_0 \oplus \varphi \in p^{-1}(A)$ with $\mathrm{half}(v_0) = 0$ also satisfies $v_0 v_0^* = 0$ and $\mathrm{half}(v_0\varphi^\dagger) = 0$, so $A \notin X_n$. □

Lemma 4.2.10. *We can describe $\widetilde{X}_n$ as*

$$\widetilde{X}_n = \{v_0 \oplus \varphi \in \mathrm{DS}^{n,ass} \mid \mathrm{half}(v_0) \neq 0\}$$

Proof. By definition, $\widetilde{X}_n$ is the union of the right hand side of the equation and the space $\{N(\varphi) \neq 1\}$. If we have $N(\varphi) \neq 1$ and $1 = N(v_0 \oplus \varphi) = N(v_0) + N(\varphi)$, we also have $N(v_0) \neq 0$, hence $\mathrm{half}(v_0) \neq 0$. □

Lemma 4.2.11. *The restriction of the map r to V_n is a rank $d/2$ vector bundle*

$$r \colon V_n \twoheadrightarrow \mathrm{DP}^{n-1}, \ A \mapsto a$$

The codimension of the closed immersion $V_n \hookrightarrow \mathrm{DP}^n$ is $d/2$.

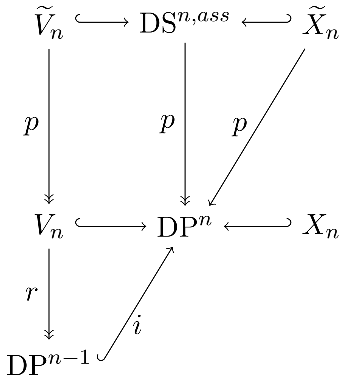
Proof. The fiber of r over a point $a \in \mathrm{DP}^{n-1}$ consists of matrices

$$A = \begin{pmatrix} 0 & \psi^\dagger \\ \psi & a \end{pmatrix} \in \mathrm{DP}^n$$

with the only conditions on ψ that $\mathrm{half}(\psi^\dagger) = 0$ and $a\psi = \psi$, which are linear conditions on an nd-dimensional vector space, that cut out a locus of dimension $d/2$. □

For $n = 2$ we would like to call the bundle $V_2 \twoheadrightarrow \mathrm{DP}^1$ a *Hopf bundle*.
We can put these objects in a diagram as in Fig. 1.

FIGURE 1. Stratification, thickening V and lift to the sphere.



We generalize the definition of V_n slightly:

Definition 4.2.12. For k, m non-negative integers, let $V_{k,m}$ be spaces given by $V_{n-1,n} := V_n \hookleftarrow \mathrm{DP}^n$ as defined before and

$$V_{n-k,n} := V_{n-k-1,n-1} \times_{\mathrm{DP}^{n-1}} V_{n-k,n}.$$

We think of $V_{k,n} \hookrightarrow \mathrm{DP}^n$ as an avatar of $\mathrm{DP}^k \hookleftarrow \mathrm{DP}^n$.

Lemma 4.2.13. *Each $V_{k,n} \hookrightarrow \mathrm{DP}^n$ is a closed immersion and the induced maps $V_{k,n} \to \mathrm{DP}^k$ have a vector bundle structure. In particular, $V_{0,n}$ is isomorphic to an affine space $\mathbb{A}^{d/2}$.*

Proof. Closed immersions and vector bundles are stable under base change. The pullback of a vector bundle along a closed immersion is merely a restriction, and $V_{0,n}$ is the restriction of $V_{n-1,n} \twoheadrightarrow \mathrm{DP}^{n-1}$ to a point DP^0, hence a single fiber of the vector bundle. The dimension follows from the rank in Lemma 4.2.11. □

The notation $V_{k,n}$ is easily confused for Stiefel varieties, so we will stop this digression.

4.2.2. Fibration from an Affine Space to the Cell.

An appropriate restriction of the map $p\colon \mathrm{DS}^{n,ass} \to \mathrm{DP}^n$ will be exhibited as a map from an affine space to the "cell" X_n, the complement of V_n.

Definition 4.2.14. Define a map $\mathrm{DS}^n \to \mathbb{A}^{d/2}$ by $v = v_0 \oplus \varphi \mapsto \mathrm{half}(v_0)$. The restriction of this map to $\widetilde{X}_n^{na} := \{v_0 \oplus \varphi \in \mathrm{DS}^n \mid \mathrm{half}(v_0) \neq 0\}$ has, by definition, image in $\mathbb{A}^{d/2} \setminus \{0\}$, and we denote this map by

$$\pi\colon \widetilde{X}_n^{na} \twoheadrightarrow \mathbb{A}^{d/2} \setminus \{0\}.$$

The pullback of π along the inclusion of the point $e_1 := (1, 0, \dots, 0) \in \mathbb{A}^{d/2}$ into $\mathbb{A}^{d/2} \setminus \{0\}$ shall be named

$$\pi'\colon \widetilde{X}_n^{1,na} \twoheadrightarrow \{e_1\}.$$

For the associative version, we write

$$\widetilde{X}'_n := \widetilde{X}_n^{1,na} \cap \widetilde{X}_n.$$

Proposition 4.2.15. *The map π is a rank $dn + e$ affine bundle (Zariski bundle with affine space fibers) and $\widetilde{X}_n^{1,na}$ is isomorphic to affine space $\mathbb{A}^{dn+e}$. For $n \neq 2$ or $D \neq \mathbb{O}$, $\widetilde{X}'_n = \widetilde{X}_n^{1,na} \xrightarrow{\sim} \mathbb{A}^{dn+e}$.*

Proof. The dimension of DS^n is $d(n+1) - 1$, so the fiber of any affine bundle to a base of dimension $d/2$ is an affine space of dimension $dn + e$. If π is an affine bundle, $\widetilde{X}_n^{1,na}$ is such a fiber.

We know that $\mathrm{DS}^n \xrightarrow{\sim} AQ_{d(n+1)-1}$ and we use the commutative diagram

$$\begin{array}{ccc}
\mathrm{DS}^n & \xrightarrow{\quad\sim\quad} & AQ_{d(n+1)-1} = \left\{ (x,y) \in \left(\mathbb{A}^{(n+1)d/2}\right)^2 \;\middle|\; \sum_{i=0}^{nd/2} x_i y_i = 1 \right\} \\
\uparrow & & \uparrow \\
\widetilde{X}_n^{1,na} & \xrightarrow{\sim} & \left\{ (x,y) \in AQ_{d(n+1)-1} \;\middle|\; y_0 = \dots = y_{d/2} = 0 \right\}.
\end{array}$$

Now $\pi\colon \widetilde{X}_n^{na} \to \mathbb{A}^{d/2+1}\setminus\{0\}$ is isomorphic to the map $AQ_{d(n+1)-1} \to \mathbb{A}^{d/2+1}\setminus\{0\}$ given by $(x,y)\mapsto(y_0,\dots,y_{d/2})$. After covering $\mathbb{A}^{d/2+1}\setminus\{0\}$ by charts of the form $\{y_i\neq 0\}$, we see that the preimage of a chart is just an affine space of dimension $d(n+1)-1$, and compatibly so, i.e., π is indeed an affine bundle.

The associator condition only makes a difference between OS^2 and $\mathrm{OS}^{2,ass}$. □

Remark 4.2.16. For $D=\mathrm{O}$ we can describe $\widetilde{X}_2'$ as subspace of $\widetilde{X}_2^{1,na}$ explicitly as zero locus of the four functions

$$\begin{gathered}
\left(v_1^{21}\cdot v_2^{22}\right)v_2^{21}-\left(v_2^{21}\cdot v_2^{22}\right)v_1^{21},\\
\left(v_2^{21}\cdot v_1^{22}\right)v_1^{21}-\left(v_1^{21}\cdot v_1^{22}\right)v_2^{21},\\
\begin{aligned}
&\left(v_0^{22}\cdot v_2^{21}\right)\left(v_1^{12}v_2^{22}-v_2^{12}v_1^{22}\right)\\
+&\left(v_1^{22}\cdot v_2^{21}\right)\left(v_2^{12}v_0^{22}-v_0^{12}v_2^{22}\right)\\
+&\left(v_2^{22}\cdot v_2^{21}\right)\left(v_0^{12}v_1^{22}-v_1^{12}v_0^{22}\right),\\
&\left(v_0^{22}\cdot v_1^{21}\right)\left(v_2^{12}v_1^{22}-v_1^{12}v_2^{22}\right)\\
+&\left(v_2^{22}\cdot v_1^{21}\right)\left(v_1^{12}v_0^{22}-v_0^{12}v_1^{22}\right)\\
+&\left(v_1^{22}\cdot v_1^{21}\right)\left(v_0^{12}v_2^{22}-v_2^{12}v_0^{22}\right).
\end{aligned}
\end{gathered}$$

Proposition 4.2.17. *We have $p(\widetilde{X}_n')\subset X_n$.*

Proof. Given $v_0\oplus\varphi$ with $1=N(v_0\oplus\varphi)$ and $\mathrm{half}(v_0)=e_1$, we show that either $v_0v_0^*\neq 0$ or $\mathrm{half}(v_0\varphi^\dagger)\neq 0$. First, $\mathrm{half}(v_0\varphi^\dagger)=\mathrm{half}(\varphi^\dagger)$ because of $\mathrm{half}(v_0)=e_1$. Suppose $v_0v_0^*=0$, then $1=N(v_0\oplus\varphi)=v_0^*v_0+\varphi^\dagger\varphi$ implies $1=\varphi^\dagger\varphi$, so that $0\neq\mathrm{half}(\varphi^\dagger)=\mathrm{half}(v_0\varphi^\dagger)$ which proves the claim. □

Remark 4.2.18. Our strategy to prove that X_n is $\mathbb{A}^1$-contractible is to exhibit the structure of an affine bundle on $p|_{\widetilde{X}_n'}\colon \widetilde{X}_n'\to X_n$ by explicitly constructing local sections and fiberwise free and transitive $\mathbb{G}_a^{\times e}$-actions. As affine bundles are $\mathbb{A}^1$-weak equivalences and affine spaces are $\mathbb{A}^1$-contractible, this suffices as soon as $\widetilde{X}_n'$ is isomorphic to affine space. One way to prove the affine bundle structure is to use the local sections to p we already have from Theorem 4.1.20 and modify them (e.g. by right multiplication with a norm 1 element), for then right multiplication with suitable norm 1 elements would already give the transitive $\mathbb{G}_a^{\times e}$-action. This is what we will do for split complex numbers and split quaternions.

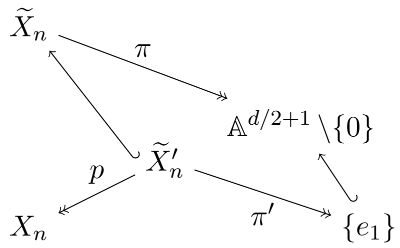
For the octonions, this strategy does not seem to work: both $\widetilde{X}_2'$ fails to be an affine space, and the right multiplication with norm 1 elements does not preserve p-fibers. Luckily, OP^1 requires so little associativity that the strategy still works, as $\widetilde{X}_1'$ is an affine space and we can prove the affine bundle structure.

In Fig. 2 we show a diagram including the constructions just made.

Before we treat the different composition algebras separately, we explain how to cover DP^n with subvarieties isomorphic to X_n. We use the morphisms τ_{ij} and $\widetilde{\tau}_{ij}$ from Theorem 4.1.20.

Lemma 4.2.19. *Let $X_n^{(j)} := \tau_{0j}X_n$ and $\widetilde{X}_n^{(j)} := \widetilde{\tau}_{0j}\widetilde{X}_n^{1,na}$. Then*

FIGURE 2. The desired section s to the restriction of p.



(1) *This is a Zariski covering:* $\mathrm{DP}^n = \bigcup_{j=0}^n X_n^{(j)}$.

(2) *For any subset* $J \subset \{0,\dots,n\}$*, the intersections* $\widetilde{X}_n^J := \bigcap_{j\in J} \widetilde{X}_n^{(j)}$ *are affine spaces.*

(3) *The morphism* $p|_{\widetilde{X}_n^J}$ *is an affine bundle if and only if* $p|_{\widetilde{X}_n^{1,na}}$ *is one.*

Proof. The complement of the union of the $X_n^{(j)}$ is the intersection of the complements, hence contained in $\{A \in \mathrm{DP}^n \mid \forall i = 0,\dots,n : A_{ii} = 0\} = \emptyset$, which proves the first claim. The intersections of the $\widetilde{X}_n^{(j)}$ can be computed explicitly,

$$\widetilde{X}_n^{(\{0,1\})} = \widetilde{X}_n^{(0)} \cap \widetilde{X}_n^{(1)} = \{v \in \mathrm{DS}^n \mid \mathrm{half}(v_0) = e_1,\ \mathrm{half}(v_1) = e_1\}\,.$$

The statement on p is true by construction. □

This already hands us our proof scheme:

Proposition 4.2.20. *If* $p|_{\widetilde{X}'_n}$ *is an affine bundle, then* DP^n *carries an unstable motivic cell structure, except for the case of* OP^2*, which is not settled by this method.*

Proof. By Lemma 4.2.19, the assumptions of Corollary 2.3.21 are fulfilled. □

4.3. Split Complex Projective Spaces

We apply the previous constructions to $D := \mathrm{C}$, the split complex numbers, where $d = 2$, hence $e = 0$. This section finishes the proof that CP^n has a motivic cell structure.

As defined in Definition 3.1.8, we write $x \in C$ as a pair $x = (x^1, x^2)$ such that $x^* = (x^2, x^1)$, hence $N(x) = x^1x^2$ and $\mathrm{half}(x) = x^1$.

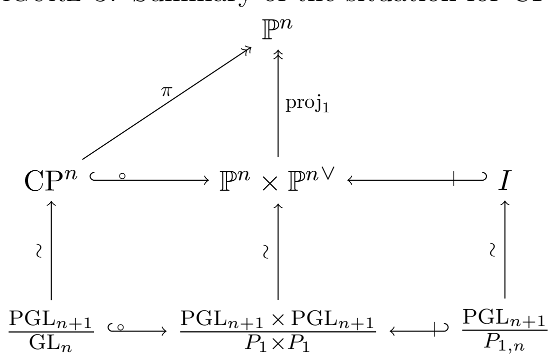
Lemma 4.3.1. *We collect several well-known facts on* CP^n*:*

(1) *Split complex projective space* CP^n *can be identified with the affine variety* U_n *of rank* 1 *idempotent matrices in* $\mathrm{Mat}^{n+1}(R)$.
(2) *The map* $\pi\colon U_n \to \mathbb{P}^n$ *that maps a rank* 1 *idempotent matrix to its image is an affine bundle. The bundle* π *is usually called* Jouanolou torsor.
(3) *The map* π *factors as* $\iota\colon U_n \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{n\vee}$ *followed by the projection onto the first factor, with* ι *given by* $A \mapsto (\mathrm{Im}(A), \mathrm{Ker}(A^t))$.
(4) *The complement to* ι *in* $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ *is the* incidence variety $I := \{(\ell, H) \mid \ell \subset H\}$.
(5) *There is an involution* σ *on* $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ *which leaves* I *invariant.*
(6) *Both* CP^n *and* I *are* PGL_{n+1}*-homogeneous spaces, and* $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ *is a complete* $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$*-homogeneous space.*
(7) $\mathrm{CP}^n \overset{\circ}{\hookrightarrow} \mathbb{P}^n \times \mathbb{P}^{n\vee}$ *is a* PGL_{n+1}*-equivariant open embedding (with* PGL_{n+1} *acting on* $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ *diagonally).*
(8) *The involution* σ *is induced by an involution on* $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$*, given by switching the factors and inverse-transpose. The* σ*-fixed points of the parabolic* $P_1 \times P_1$ *that stabilizes a point of* $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ *form the parabolic* $P_{1,n}$ *that stabilizes a point of* I*.*
(9) *The involution* σ *is given by a diagram automorphism of the Dynkin diagram* $A_n \sqcup A_n$*.*
(10) *The motive of* CP^n *is*

$$M(\mathrm{CP}^n) = \bigoplus_{i=0}^{n} 1(i)[2i].$$

We summarize the situation with a diagram in Fig. 3

FIGURE 3. Summary of the situation for CP^n.



Proof. (1) We decompose $A \in J_{n+1}(C)$ as $A = (A_1, A_2)$ with $A_i \in \mathrm{Mat}^{n+1}(R)$ along $\mathrm{C} = R \oplus R$ with component-wise multiplication. With this definition, $A_2 = A_1^t$ (from $A^\dagger = A$) and $\mathrm{tr}(A_i) = 1$ as well as $A_i^2 = A_i$. The map $A \mapsto A_1$ is therefore an isomorphism $\mathrm{CP}^n \xrightarrow{\sim} U_n$.

(2) For A a rank 1 projector, also A^t is a rank 1 projector, as $(A^t)^2 = (A^2)^t$ and the diagonal elements are invariant under transposition. The datum of a projector $A \in U_n$ induces a choice of direct sum complement $\mathrm{Ker}(A)$ to $\mathrm{Im}(A)$. Conversely, a direct sum decomposition $\ell \oplus H = R^{n+1}$ induces projection operators $A, 1 - A$ onto ℓ resp. H. Now $\pi^{-1}(\ell)$ can be identified with the set of hyperplanes complementing ℓ, that is the set of hyperplanes not containing ℓ, which is an affine space.

Furthermore, let $X_i := \{\ell = [x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_i \neq 0\}$, then $\pi^{-1}(X_i)$ can be identified with the set of hyperplanes not containing any vector x with $x_i = 0$, i.e. the hyperplanes H which are not orthogonal to the standard basis vector $e_i \in R^{n+1}$. We can identify $\varphi_i \colon \pi^{-1}(X_i) \xrightarrow{\sim} X_i \times \mathbb{A}^n$ via $A \mapsto (\pi(A), \mathrm{Ker}(A^t))$, where we consider $\mathrm{Ker}(A^t) \in \mathbb{P}^{n\vee}$ and notice that it is in the complement of $\mathbb{P}^{n-1\vee} \subset \mathbb{P}^{n\vee}$ given by orthogonality to e_i. An inverse map is given by reconstructing A from $\pi(A) \oplus \mathrm{Ker}(A^t) = R^{n+1}$. The map $\varphi_i \circ \varphi_j^{-1}$ is the identity in the first component, and a change of charts of $\mathbb{P}^{n\vee}$ in the second component, hence π is an affine bundle.

(3) We only have to check that ι is well-defined: The kernel of A^t is of rank $n - 1$, so it defines a hyperplane.

(4) For any pair (ℓ, H) with $\ell \oplus H = R^{n+1}$ we may find a preimage A under ι, and every pair (ℓ, H) in the image of ι satisfies $\ell \oplus H = R^{n+1}$. The latter condition is equivalent to $\neg(\ell \subset H)$.

(5) The involution is given by $\sigma(\ell, H) := (H^\perp, \ell^\perp)$. For any pair $(\ell, H) \in I$, we have $\ell \subset H$, hence $H^\perp \subset \ell^\perp$, so that I is a σ-invariant subset.

(6-8) The PGL_{n+1}-action on CP^n is given in terms of an action on U_n, by conjugation (change of basis). The PGL_{n+1}-action on I is given from the diagonal action on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$, where $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$ acts with the product action of the standard action of PGL_{n+1} on $\mathbb{P}^n$ with stabilizer a parabolic P_1. The stabilizer of $\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}$ acting on $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ is the parabolic $P_1 \times P_1$. The stabilizer of I is given by the diagonal $P_{1,n} \subset (P_1 \times P_1) \times_{\mathrm{PGL}_{n+1} \times \mathrm{PGL}_{n+1}} \mathrm{PGL}_{n+1}$, which is invariant under σ again.

(9) Numbering the simple roots in A_n by $\{1, \ldots, n\}$ and furthermore writing $\{\alpha_i\} \cup \{\beta_i\}$ for the simple roots in $A_n \sqcup A_n$, σ is given by $\alpha_i \mapsto \beta_{n-i}$ and $\beta_i \mapsto \alpha_{n-i}$.

(10) This was explained in Remark 2.2.6. Alternatively, one may use $M(\mathrm{CP}^n) = M(\mathbb{P}^n)$.

□

We now look at the special case of CP^1 over a field, where we want to describe everything explicitly in coordinates.

Example 4.3.2. Let R be a field. We show that the morphism $p|_{\widetilde{X}_1'}$ introduced in Definition 4.1.16 and Definition 4.2.14 is an isomorphism.

We already know that $\widetilde{X}_1' \xrightarrow{\sim} \mathbb{A}^2$, explicitly

$$\widetilde{X}_1' = \left\{ \begin{pmatrix} (1, v_0^2) \\ (v_1^1, v_1^2) \end{pmatrix} \;\middle|\; v_0^2 = 1 - v_1^1 v_1^2 \right\}$$

It is also easy to see that the base X_1 is 2-dimensional:

$$X_1 = \left\{ \begin{pmatrix} z & (\psi^2, \psi^1) \\ (\psi^1, \psi^2) & 1 - z \end{pmatrix} \;\middle|\; \psi^1\psi^2 = z(1-z),\ (z \neq 0 \vee \psi^2 \neq 0) \right\}$$

With the definition

$$Z_{11} := \{A \in \mathrm{CP}^1 \mid z \neq 1, \psi^2 \neq 0\}$$

we see that $X_1 = Z_0 \cup Z_{11}$, using the varieties Z_i, $\widetilde{Z}_i$ from Theorem 4.1.20:

$$Z_i = \left\{ A = \begin{pmatrix} z & \psi^* \\ \psi & 1 - z \end{pmatrix} \in \mathrm{CP}^1 \;\middle|\; z \neq i \right\} \text{ for } i \in \{0, 1\}$$

$$\widetilde{Z}_i = \left\{ v = v_0 \oplus v_1 \in \mathrm{CS}^1 \mid N(v_i) \neq 0 \right\} \text{ for } i \in \{0, 1\}$$

Now we define

$$\widetilde{Z}_{11} = \{v \in \mathrm{CS}^1 \mid v_1^1 = 1,\ v_1^2 \neq 0,\ v_0^1 \neq 0\} \subset \widetilde{Z}_1$$

$$\widetilde{Z}_{11}' = \{v \in \mathrm{CS}^1 \mid v_0^1 = 1,\ v_1^2 \neq 0,\ v_1^1 \neq 0\} \subset \widetilde{Z}_1 \cap \widetilde{X}_1'$$

and the next step is to define a morphism $\lambda_{11} \colon \widetilde{Z}_{11} \to \widetilde{Z}_{11}'$ such that the morphisms $s_0 \colon Z_0 \to \widetilde{Z}_0$ and $s_{11}' := \lambda_{11} \circ s_1 \colon Z_{11} \to \widetilde{Z}_{11}'$ are sections to $p|_{\widetilde{X}_1'}$. Note that $\widetilde{X}_1' = \widetilde{Z}_0 \cup \widetilde{Z}_{11}'$.

The morphism λ_{11} can be defined by right multiplication as $v \mapsto vx(v)$, with $x(v) := (v_0^1, (v_0^1)^{-1}) \in \mathrm{CS}^0$. As right multiplication with a norm 1 element does not change the image under p and s_1 was already a p-section, s_{11}' is still a p-section. We write down the sections s_0 and s_{11}' explicitly:

$$s_0 \colon Z_0 \to \widetilde{Z}_0, \quad A \mapsto \begin{pmatrix} (1, z) \\ (\psi^1 z^{-1}, \psi^2) \end{pmatrix}$$

$$s_{11}' \colon Z_{11} \to \widetilde{Z}_{11}' \quad A \mapsto \begin{pmatrix} (1, \psi^1\psi^2(1-z)^{-1}) \\ ((\psi^2)^{-1}(1-z), \psi^2) \end{pmatrix}$$

Over $Z_0 \cap Z_{11}$, where we have $\psi^1 z^{-1} = (\psi^2)^{-1}(1-z)$ and $\psi^2\psi^2(1-z)^{-1} = z$, the sections s_0 and s_{11}' coincide. This shows that they patch together to a section $s \colon X_1 \to \widetilde{X}_1'$ to p. One may now compute $s(p(v)) = v$ or, what amounts to the same, observe that every fiber

$$p^{-1}(A) \cap \widetilde{X}_1' = \left\{ \begin{pmatrix} (1, z) \\ (v_1^1, \psi^2) \end{pmatrix} \;\middle|\; v_1^1\psi^2 = 1 - z,\ v_1^1 z = \psi^1 \right\}$$

consists of a single element (the image of s). This concludes the (admittedly lengthy) proof that $p|_{\widetilde{X}_1'}$ is an isomorphism, and we conclude that X_1 itself is isomorphic to $\mathbb{A}^2$.

Recall from Theorem 4.1.20 that we have a Zariski cover of CP^n

$$\mathrm{CP}^n = \bigcup_{i=0}^{n} Z_i^\circ, \qquad Z_i^\circ = \{A \in \mathrm{CP}^n \mid z_i \text{ a unit}\},$$

where we write $z_i := A_{ii}$ as shorthand for the i-th diagonal element.

Lemma 4.3.3. *For $A \in \mathrm{CP}^n$, for any two $v, w \in p^{-1}(A)$, there exists a unique $x = x(v, w) \in \mathrm{CS}^0$ with $v = wx$.*

Proof. For some i, we have $A \in Z_i^\circ$, so that $v_i v_i^* = z_i = w_i w_i^*$ is invertible. From $p(v) = p(w)$ we get by looking at the i-th column that $\forall j : v_j v_i^* = w_j w_i^*$. With $1 = v_i^* v_i z_i^{-1}$ this implies

$$\forall j : v_j = v_j v_i^* v_i z_i^{-1} = w_j w_i^* v_i z_i^{-1}.$$

The x in question is $x(v, w) := w_i^* v_i z_i^{-1}$, which is of norm 1 and easily seen to not depend on i. □

Remark 4.3.4. Identifying the spaces CS^n and CP^n as homogeneous varieties by isomorphisms $\mathrm{PGL}_{n+1} / \mathrm{PGL}_n \xrightarrow{\sim} \mathrm{CS}^n$ and $\mathrm{PGL}_{n+1} / \mathrm{GL}_n \xrightarrow{\sim} \mathrm{CP}^n$, the map $p\colon \mathrm{CS}^n \twoheadrightarrow \mathrm{CP}^n$ is the $\mathbb{G}_\mathrm{m}$-principal bundle $\mathrm{PGL}_{n+1} / \mathrm{PGL}_n \twoheadrightarrow \mathrm{PGL}_{n+1} / \mathrm{GL}_n$. One can use Anthony Bak's theory of quadratic modules [Bak69] to define $\mathrm{U}_n(\mathrm{C})$ in general, of which $\mathrm{U}_1(\mathrm{C}) = \mathrm{CS}^0 = \mathbb{G}_\mathrm{m}$ is a special case.

Proposition 4.3.5. *The morphism $p|_{\widetilde{X}'_n} : \widetilde{X}'_n \to X_n$ is an isomorphism for R a local ring.*

Proof. We define a Zariski cover $X_n = Z_0^\circ \cup \bigcup_{i=1}^n \bigcup_{j=1}^n Z_{ij}^\circ$ by

$$Z_{ij}^\circ := \left\{ A \in \mathrm{CP}^n \;\middle|\; z_i \text{ unit, } \operatorname{half}(\psi_j^*) \text{ unit} \right\} \subset Z_i^\circ$$

Over Z_0°, the section s_0 from Theorem 4.1.20 has image in $\widetilde{Z}_0^\circ \subset \widetilde{X}'_n$ already. We now define $s'_0 := s_0$ and for $i > 0$

$$s'_{ij} \colon Z_{ij}^\circ \to \widetilde{Z}'_{ij} := \{ v \in \widetilde{Z}_i \cap \widetilde{X}'_n \mid \operatorname{half}(v_i) \text{ unit} \}$$

by right multiplication of s_i with $x_{ij} \colon Z_{ij}^\circ \to \mathrm{CS}^0$ defined by Lemma 3.2.6. From the previous Lemma 4.3.3 we can conclude for any $v \in \widetilde{X}'_n$ with $A := p(v) \in Z_\tau$, where $\tau = ij$ or $\tau = 0$, that there exists a unique x such that $vx = s'_\tau(v)$. As already $v_0 = (1, z_0)$ and $(s'_\tau(v))_0 = (1, z_0)$, we see that $x = 1$. This shows at once that the s'_τ glue to a section of p over X_n and that this section is an inverse to $p|_{\widetilde{X}'_n}$. □

As a corollary, we obtain now

Theorem 4.3.6. *Over any field, split complex projective space CP^n carries an unstable motivic cell structure obtained from gluing cells to CP^{n-1}.*

Proof. By Proposition 4.3.5, the assumptions of Proposition 4.2.20 are fulfilled. □

This gives another (much more involved) proof of the commonly used unstable motivic cell structure of $\mathbb{P}^n$.

4.4. Split Quaternionic Projective Spaces

We apply the previous constructions to $D := \mathrm{H}$, the split quaternions, where $d = 4$. This section finishes the proof that HP^n has a motivic cell structure.

As in Definition 3.1.9, we write $x \in \mathrm{H}$ as a matrix $x = \begin{pmatrix} x^{11} & x^{21} \\ -x^{22} & x^{12} \end{pmatrix}$ such that $x^* = \begin{pmatrix} x^{12} & -x^{21} \\ x^{22} & x^{11} \end{pmatrix}$ and $N(x) = x^*x = \det(x) = x^{11}x^{12} + x^{21}x^{22}$ as well as

$$\mathrm{half}(x) = (x^{11}, x^{21}).$$

Lemma 4.4.1. *We collect well-known facts on* HP^n*:*

(1) *Split quaternionic projective space* HP^n *embeds as open subvariety of* $\mathrm{Gr}(2n+2, 2)$*, the Grassmannian of* 2*-dimensional linear subspaces of a* $2n+2$*-dimensional vector space. The closed complement of* $\mathrm{HP}^n \overset{\circ}{\hookrightarrow} \mathrm{Gr}(2n+2, 2)$ *is the* symplectic Grassmannian $\mathrm{SpGr}(2n+2, 2)$ *of* 2*-dimensional symplectic subspaces of a* $2n+2$*-dimensional vector space. Equivalently,* HP^n *is the space of* 2*-dimensional isotropic subspaces of a* $(2n+2)$*-dimensional vector space.*
(2) *There is an involution* σ *on* $\mathrm{Gr}(2n+2, 2)$ *which leaves both* HP^n *and* $\mathrm{SpGr}(2n+2, 2)$ *invariant.*
(3) *Both* HP^n *and* $\mathrm{SpGr}(2n+2, 2)$ *are homogeneous under* Sp_{2n+2}*. The variety* $\mathrm{Gr}(2n+2, 2)$ *is a complete homogeneous space under* SL_{2n+2} *with stabilizer a parabolic* P_2*.*
(4) $\mathrm{HP}^n \overset{\circ}{\hookrightarrow} \mathrm{Gr}(2n+2, 2)$ *is an* Sp_{2n+2}*-equivariant embedding, with* Sp_{2n+2} *acting via* $\mathrm{Sp}_{2n+2} \hookrightarrow \mathrm{SL}_{2n+2}$ *on* $\mathrm{Gr}(2n+2, 2)$*.*
(5) *The involution* σ *is induced from an involution on* SL_{2n+2} *with invariants* Sp_{2n+2}*, namely the inverse-transpose of a matrix.*
(6) *The involution* σ *is given by a diagram automorphism of the Dynkin diagram* A_{2n+2}*.*
(7) *The motive of* HP^n *is*

$$M(\mathrm{HP}^n) = \bigoplus_{i=0}^{n} 1(2i)[4i].$$

Summarizing, we have the diagram in Fig. 4

FIGURE 4. Summary of the situation for HP^n.

$$\begin{array}{ccccc}
\mathrm{HP}^n & \overset{\circ}{\hookrightarrow} & \mathrm{Gr}(2n+2,2) & \hookleftarrow & \mathrm{SpGr}(2n+2,2) \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr \\
\frac{\mathrm{Sp}_{2n+2}}{\mathrm{Sp}_2 \times \mathrm{Sp}_{2n}} & \overset{\circ}{\hookrightarrow} & \frac{\mathrm{SL}_{2n+2}}{P_2} & \hookleftarrow & \frac{\mathrm{Sp}_{2n+2}}{P_2}
\end{array}$$

Proof. First we fix a $2n+2$-dimensional free R-module V and a symplectic form on it, as well as a Lagrangian in V. For our purposes it is fine to think of R^{2n+2} with the standard symplectic form and the standard symplectic basis. Of course, one may easily write up these arguments coordinate-free.

(1) Given a point $A \in \mathrm{HP}^n$, we may consider the H-right line $\mathrm{Im}(A) = \{Av \mid v \in \mathrm{H}^{n+1}\}$ and project onto the Lagrangian $R^{2n+2} \subset \mathrm{H}^{n+1}$ by the map half. Then $\mathrm{half}(\mathrm{Im}(A)) \subset R^{2n+2}$ is an isotropic 2-dimensional linear subspace. The complement in $\mathrm{Gr}(2n+2,2)$ is given by non-isotropic, that is symplectic 2-dimensional subspaces, which is a closed condition on the set of all 2-dimensional linear subspaces.
(2) We may decompose V into a sum of two Lagrangians, and let σ the symplectomorphism exchanging the two. To be more precise, in a symplectic basis x_i, y_i on R^{2n+2} we let $\sigma(x_i) := y_i$ and $\sigma(y_i) := x_i$. This gives rise to σ on $\mathrm{Gr}(2n+2,2)$. Since σ is a symplectomorphism, it preserves symplectic as well as isotropic subspaces.
(3) As Sp_{2n+2} acts on V via symplectomorphisms, it acts on the space of isotropic 2-planes as well as on the space of symplectic 2-planes in V. Transitivity follows as well. The SL_{2n+2}-action on V also descends to an action on the space of 2-planes, and the parabolic which stabilizes a given 2-plane is the maximal parabolic P_2, as one may check by computation on R^{2n+2} with the standard basis for example.
(4) This is clear from the construction of the embedding.
(5) An element on SL_{2n+2} is invariant under inverse-transpose iff it preserves the symplectic form. The inverse-transpose induces on $\mathrm{Gr}(2n+2,2)$ the involution which maps a 2-dimensional symplectic linear subspace to itself and a 2-dimensional isotropic linear subspace to another 2-dimensional isotropic subspace in the complement with the property that the spanned 4-dimensional linear subspace is symplectic again.
(6) Labeling simple roots of A_{2n+1} with α_i with $i \in \{1,\ldots,2n+1\}$, the involution σ is given by $\alpha_i \mapsto \alpha_{2n+2-i}$ for $i \leq n+1$ and $\alpha_{2n-i} \mapsto \alpha_i$ for $i \leq n+1$.
(7) This was explained in Remark 2.2.6. □

We study the projection p introduced in Definition 4.1.16 in the quaternionic case.

Lemma 4.4.2. *The norm 1 elements $\mathrm{HS}^0 \subset \mathrm{H}$ act on HS^n freely by right multiplication. This action preserves p-fibers and the restriction to any p-fiber is transitive.*

Proof. The action $\mathrm{HS}^0 \times \mathrm{HS}^n \to \mathrm{HS}^n$, $(x,v) \mapsto vx$ is p-fiberwise, as

$$p(vx) = (vx)(vx)^\dagger = (vx)(x^*v^\dagger) = v(xx^*)v^\dagger = vv^\dagger = p(v).$$

For each $A \in \mathrm{HP}^n$ and each two $v, w \in p^{-1}(A)$, we have $A \in Z_i^\circ$ for some i. From $p(v) = A = p(w)$ follows (in the ith row and column) that $N(v_i) = z_i = N(w_i)$ is invertible, and from the i-th column we also obtain $\forall j : v_j v_i^* = w_j w_i^*$ so that from $1 = v_i^* v_i z_i^{-1}$ we get

$$\forall j : v_j = v_j v_i^* v_i z_i^{-1} = w_j w_i^* v_i z_i^{-1}.$$

Now we let $x_i(v,w) := w_i^* v_i z_i^{-1}$ and observe that this is unique (in particular it does not depend on i): for $x \in \mathrm{HS}^0$ another element with $v = wx$, we have

$$1 = z_i^{-1} v_i^* v_i = z_i^{-1} v_i^* w_i x = x_i(w,v)x$$

but $x_i(w,v)^{-1} = x_i(v,w)$ by construction. □

Remark 4.4.3. Under the isomorphisms

$$\mathrm{Sp}_{2n+2} / \mathrm{Sp}_{2n} \xrightarrow{\sim} \mathrm{HS}^n \text{ and}$$
$$\mathrm{Sp}_{2n+2} / \mathrm{Sp}_{2n} \times \mathrm{Sp}_2 \xrightarrow{\sim} \mathrm{HP}^n,$$

the map $p\colon \mathrm{HS}^n \twoheadrightarrow \mathrm{HP}^n$ is the Sp_2-principal bundle $\mathrm{Sp}_{2n+2} / \mathrm{Sp}_{2n} \twoheadrightarrow \mathrm{Sp}_{2n+2} / \mathrm{Sp}_{2n} \times \mathrm{Sp}_2$. One may also use Anthony Bak's theory of quadratic modules [Bak69] to define $\mathrm{U}_n(\mathrm{H})$ in general, of which $\mathrm{U}_1(\mathrm{H}) = \mathrm{HS}^0 = \mathrm{Sp}_2$ is a special case.

Lemma 4.4.4. *Via the right multiplication action of* HS^0 *on* HS^n *and the inclusion*

$$\mathbb{G}_\mathrm{a} \hookrightarrow \mathrm{HS}^0,\ y \mapsto x(y) := \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$$

the additive group $\mathbb{G}_\mathrm{a}$ *acts p-fiberwise on* HS^n *and restricts to a free, fiberwise transitive, action on the subspace* $\widetilde{X}'_n$ *introduced in Definition 4.2.14.*

Proof. We see that $\mathbb{G}_\mathrm{a}$ stabilizes $\widetilde{X}'_n$ by computing

$$\mathrm{half}(v_0.y) = \mathrm{half}(v_0 x(y)) = (v_0^{11}, v_0^{21}) = \mathrm{half}(v_0) = e_1.$$

It remains to see that this action is still transitive on each fiber, i.e. that for $A \in Z_i$ and $v, w \in p^{-1}(A) \cap \widetilde{X}'_n$ the unique element $x(v,w) \in \mathrm{HS}^0$ with $v = wx$ satisfies $\mathrm{half}(x) = e_1$, which is again a direct computation from $\mathrm{half}(v) = \mathrm{half}(wx)$, compare Lemma 3.2.9. □

To illustrate the differences between C and H we give the example of HP^1 now (compare Example 4.3.2).

Example 4.4.5. Let R be a field. We cover X_1 by the subvarieties

$$Z_0 = \left\{ A = \begin{pmatrix} z & \psi^* \\ \psi & 1-z \end{pmatrix} \;\middle|\; z \neq 0 \right\}$$
$$Z_{11} = \left\{ A = \begin{pmatrix} z & \psi^* \\ \psi & 1-z \end{pmatrix} \;\middle|\; z \neq 1,\ \mathrm{half}(\psi^*)_1 \neq 0 \right\} \subset Z_1$$
$$Z_{12} = \left\{ A = \begin{pmatrix} z & \psi^* \\ \psi & 1-z \end{pmatrix} \;\middle|\; z \neq 1,\ \mathrm{half}(\psi^*)_2 \neq 0 \right\} \subset Z_1$$

so that $X_1 = Z_0 \cup Z_{11} \cup Z_{12}$. Note that $\mathrm{half}(\psi^*) = (\psi^{12}, -\psi^{21})$. We now proceed to construct sections s'_τ to $p|_{\widetilde{X}'_1}$ over each over these Z_τ, by modifying the sections s_i from Theorem 4.1.20 with right multiplication by HS^0:

$$s'_\tau := \mu \circ (s_\tau \times x_\tau) \circ \Delta\colon Z_\tau \to \widetilde{Z}'_\tau,$$

where Δ is the diagonal, μ the pointwise quaternion multiplication and $x_\tau\colon Z_\tau \to \mathrm{HS}^0$ a morphism with the property

$$\mathrm{half}(s_\tau(A)x_\tau(A)) = e_1.$$

Here, we write $\widetilde{Z}'_0 := \widetilde{Z}_0$ and $\widetilde{Z}'_{ij} := \widetilde{Z}_i \cap \widetilde{X}'_1$.

The existence of x_τ is proved in Lemma 3.2.6, but we will construct x_τ explicitly here:

$$x_{11}\colon Z_{11} \to \mathrm{HS}^0,\ A \mapsto \begin{pmatrix} z_1(\psi^{12})^{-1} & \psi^{21} \\ 0 & z_1^{-1}\psi^{12} \end{pmatrix},$$
$$x_{12}\colon Z_{12} \to \mathrm{HS}^0,\ A \mapsto \begin{pmatrix} 0 & \psi^{21} \\ -(\psi^{21})^{-1} & z_1^{-1}\psi^{12} \end{pmatrix}.$$

With our previous definitions, we have

$$s_0\colon Z_0 \to \widetilde{Z}_0,\ A \mapsto v,\ v_0 = \begin{pmatrix} 1 & 0 \\ 0 & z_0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} z_0^{-1}\psi^{11} & \psi^{21} \\ -z_0^{-1}\psi^{22} & \psi^{12} \end{pmatrix},$$
$$s_1\colon Z_1 \to \widetilde{Z}_1,\ A \mapsto v,\ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & z_1 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} z_1^{-1}\psi^{12} & -\psi^{21} \\ z_1^{-1}\psi^{22} & \psi^{11} \end{pmatrix}.$$

The resulting sections are $s_0' := s_0$ and

$$s_{11}'\colon Z_{11} \to \widetilde{Z}_{11}',\ A \mapsto v, \qquad v_0 = \begin{pmatrix} 1 & 0 \\ \psi^{22}(\psi^{12})^{-1} & z_0 \end{pmatrix},$$
$$v_1 = \begin{pmatrix} z_1(\psi^{12})^{-1} & \psi^{21} \\ 0 & \psi^{12} \end{pmatrix},$$
$$s_{12}'\colon Z_{12} \to \widetilde{Z}_{12}',\ A \mapsto v, \qquad v_0 = \begin{pmatrix} 1 & 0 \\ -\psi^{11}(\psi^{21})^{-1} & z_0 \end{pmatrix},$$
$$v_1 = \begin{pmatrix} 0 & \psi^{21} \\ -z_1(\psi^{21})^{-1} & \psi^{12} \end{pmatrix}.$$

On each intersection, each s_{τ_1} and s_{τ_2} are sections to $p|_{\widetilde{X}_1'}$, so by Lemma 4.4.4 there is a unique $\sigma_{\tau_1,\tau_2}\colon Z_{\tau_1} \cap Z_{\tau_2} \to \mathbb{G}_\mathrm{a}$ which gives a cocycle. We can compute it explicitly, e.g. for $\tau_1 = 0$ and $\tau_2 = 11$, we have

$$\sigma_{0,11} := A \mapsto \begin{pmatrix} 1 & 0 \\ z_0^{-1}\psi^{22}(\psi^{12})^{-1} & \end{pmatrix}$$

with the property $s_0(A)\sigma_{0,11}(A) = s_{11}'(A)$ for $A \in Z_0 \cap Z_{11}$.

If we denote the $\mathbb{G}_\mathrm{a}$-action on $\widetilde{X}_1'$ by ρ, we have commutative diagrams

$$\begin{array}{ccc} \widetilde{Z}_\tau & \xleftarrow{\ \rho \circ (s_\tau' \times \mathrm{id})\ } & Z_\tau \times \mathbb{G}_\mathrm{a} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_1} & \\ & Z_\tau & \end{array}$$

that give $p|_{\widetilde{X}_1'}$, together with the cocycle σ, the structure of a $\mathbb{G}_\mathrm{a}$-bundle.

In particular, we have just proved

Theorem 4.4.6. *The split quaternionic projective line* HP^1 *over a field is a motivic sphere* $\mathrm{S}^{4,2}$.

Proof. We just saw that X_1 is $\mathbb{A}^1$-contractible, by applying Corollary 2.1.5, hence HP^1 carries a motivic cell structure by Corollary 2.3.14. The Thom space is $\Sigma_s \mathbb{A}^4 \setminus \{0\}$. □

Theorem 4.4.7. *Let R be a local ring. The morphism $p|_{\widetilde{X}'_n} : \widetilde{X}'_n \to X_n$ is a rank 1 affine bundle.*

This was essentially already proved by Panin and Walter [PW10a].

Proof. We cover X_n by the subvarieties Z_0° and

$$Z_{ijk} = \left\{ A \in Z_i^\circ \;\middle|\; \operatorname{half}(\psi_j^*)_k \text{ invertible} \right\}.$$

With the notation $Z_\tau = Z_0^\circ$ for $\tau = 0$ and then for $\tau = ijk$

$$\widetilde{Z}'_{ijk} := \widetilde{Z}_i^\circ \cap \widetilde{X}'_n.$$

we will construct p-sections

$$s'_\tau := \mu \circ (s_\tau \times x_\tau) \circ \Delta \colon Z_\tau \to \widetilde{Z}'_\tau,$$

where we write $s_{ijk} := s_i$ and x_τ is defined by the requirement

$$\operatorname{half}(s_\tau(A) x_\tau(A)) = e_1.$$

Existence of x_τ is proved in Lemma 3.2.6 (we make some arbitrary choice of x_τ). From Lemma 4.4.4 we see that any such x_τ differ by the $\mathbb{G}_a$-action ρ on $\widetilde{X}'_n$, so that we have an affine bundle trivialization

$$\begin{array}{ccc} \widetilde{Z}_\tau & \xleftarrow{\;\rho \circ (s'_\tau \times \mathrm{id})\;} & Z_\tau \times \mathbb{G}_a \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_1} & \\ & Z_\tau & \end{array}$$

□

As a corollary, we obtain now

Theorem 4.4.8. *Over any field, split quaternionic projective space HP^n carries an unstable motivic cell structure obtained from gluing cells to HP^{n-1}.*

Proof. By Theorem 4.4.7, the assumptions of Proposition 4.2.20 are fulfilled. □

4.5. Split Octonionic Projective Spaces

We apply the previous constructions to $D := \mathbb{O}$, the split octonions, where $d = 8$ and $e = 3$. We only look at the cases $n = 1$ and $n = 2$. In this section, we give a proof that $\mathbb{O}\mathrm{P}^1$ is a motivic sphere. This section does not prove that $\mathbb{O}\mathrm{P}^2$ has a motivic cell structure, but we discuss a motivic cell structure for $\Sigma\,\mathbb{O}\mathrm{P}^2$.

4.5.1. The Octonionic Projective Line.

To use the morphism p introduced in Definition 4.1.16, we need some explicit computations of associators.

Here we use Zorn vector matrix notation as introduced in Convention 3.1.12.

Computation 4.5.1. If $v \in \mathrm{DS}^n$ and $x \in \mathrm{DS}^0$, then $p(vx) = (vx)(vx)^\dagger$ differs from $v(xx^*)v^\dagger = vv^\dagger = p(v)$ by the associators $\{v_i,\ x,\ (v_jx)^*\}$. If $D = \mathbb{C}$ or $D = \mathbb{H}$, all associators vanish, but for $D = \mathbb{O}$, this is a crucial difficulty. We make some computations:

$$v_ix = \begin{pmatrix} v_i^{11}x^{11} - v_i^{21}\cdot x^{22} & v_i^{11}x^{21} + v_i^{21}x^{12} \\ -(v_i^{22}x^{11} + v_i^{12}x^{22}) & v_i^{12}x^{12} - v_i^{22}\cdot x^{21} \end{pmatrix}$$

$$(v_jx)^* = \begin{pmatrix} v_j^{12}x^{12} - v_j^{22}\cdot x^{21} & -(v_j^{11}x^{21} + v_j^{21}x^{12}) \\ v_j^{22}x^{11} + v_j^{12}x^{22} & v_j^{11}x^{11} - v_j^{21}\cdot x^{22} \end{pmatrix}$$

$$v_iv_j^* = \begin{pmatrix} v_i^{11}v_j^{12} + v_i^{21}\cdot v_j^{22} & v_i^{21}v_j^{11} - v_i^{11}v_j^{21} \\ v_i^{12}v_j^{22} - v_i^{22}v_j^{12} & v_i^{22}\cdot v_j^{21} + v_i^{12}v_j^{11} \end{pmatrix}$$

We will use that $x \in \mathrm{DS}^0$ means $1 = N(x) = x^{11}x^{12} + x^{21}\cdot x^{22}$. Writing

$$\{v_i,\ x,\ (v_jx)^*\} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we can now compute as in Fig. 5 the elements a, b, c, d:

$$\begin{aligned} a &= \quad (v_i^{21}\cdot x^{22})(v_j^{22}\cdot x^{21}) - (v_i^{21}\cdot v_j^{22})(x^{21}\cdot x^{22}), \\ b &= ((v_j^{11}v_i^{21} - v_i^{11}v_j^{21})\cdot x^{22})x^{21} - (v_j^{11}v_i^{21} - v_i^{11}v_j^{21})(x^{22}\cdot x^{21}) \\ &\quad + v_j^{21}(v_i^{21}\cdot x^{12}x^{22}) - v_i^{21}(v_j^{21}\cdot x^{12}x^{22}), \\ c &= ((v_i^{12}v_j^{22} - v_j^{12}v_i^{22})\cdot x^{21})x^{22} - (v_i^{12}v_j^{22} - v_j^{12}v_i^{22})(x^{21}\cdot x^{22}) \\ &\quad + v_i^{22}(v_j^{22}\cdot x^{11}x^{21}) - v_j^{22}(v_i^{22}\cdot x^{11}x^{21}), \\ d &= \quad (v_i^{22}\cdot x^{21})(v_j^{21}\cdot x^{22}) - (v_i^{22}\cdot v_j^{21})(x^{21}\cdot x^{22}). \end{aligned}$$

Lemma 4.5.2. *Let $v \in \mathbb{O}\mathrm{S}^1$ and $x \in \mathbb{O}\mathrm{S}^0$. If* $\mathrm{half}(x) = e_1$ *and either* $\mathrm{half}(v_0) = e_1$ *or* $\mathrm{half}(v_1) = e_1$*, or if we make no restriction on x but insist that* $v_0 = \mathrm{diag}(1, z_0)$ *or* $v_1 = \mathrm{diag}(1, z_1)$ *for some $z_i \in R$, then the associators $\{v_i,\ x,\ (v_jx)^*\}$ vanish for $i, j \in \{0, 1\}$.*

Proof. For the first case, we use Computation 4.5.1. Given $\mathrm{half}(x) = e_1$, from $N(x) = 1$ we conclude $x^{11} = 1$, $x^{21} = 0$ and $x^{12} = 1$. The only non-vanishing part of the associator is

$$b = v_j^{21}(v_i^{21}\cdot x^{22}) - v_i^{21}(v_j^{21}\cdot x^{22}).$$

For $i = j$, the associator vanishes, as we also see directly in this case. The only interesting case up to sign is $(i, j) = (0, 1)$. We assume $\mathrm{half}(v_1) = e_1$ (the case $\mathrm{half}(v_0) = e_1$ is completely analogous). This entails $v_1^{21} = 0$, so that $b = 0$.

For the second case, where no restriction on x is made, we assume $v_1 = \mathrm{diag}(1, z_1)$ (again, the case of $v_0 = \mathrm{diag}(1, z_0)$ is analogous). Instead of a more monstrous computation, we make use of Lemma 3.1.13:

$$(v_0x)(x^*v_1^*) = ((v_0x)x^*)\,\mathrm{diag}(z_1, 1) = (v_0N(x))\,\mathrm{diag}(z_1, 1) = v_0v_1^*. \qquad \square$$

Lemma 4.5.3. *For* OP^1*, one can construct sections* s'_{1j} *to* $p|_{\widetilde{X}'_1}$ *over* Z_{1j} *with* $j \in \{1, 2, 3\}$ *and* $p|_{\widetilde{X}'_1} \colon \widetilde{X}'_1 \twoheadrightarrow X_1$ *is an affine bundle.*

Proof. As for HP^1 in Example 4.4.5 or Theorem 4.4.7, we modify the sections s_τ of Z_τ defined in Theorem 4.1.20 with a right multiplication by an element of norm 1, via Lemma 3.2.6. The resulting maps s'_τ are again p-sections over the same fiber by the vanishing of associators from Lemma 4.5.2, since the sections s_i can be made to satisfy $v_i = \mathrm{diag}(1, z_i)$ by construction in the proof of Theorem 4.1.20. The condition for transitivity studied in Lemma 3.2.9 for the octonion-case is fulfilled on $\widetilde{X}'_n$, as the y in Lemma 3.2.9 is the $v_0 = (s'_\tau(A))_0$ here, which has $v_0^{11} = 1$ by the half-inverting procedure. This tells us that the $\mathbb{G}_a^{\times 3}$-action on each $\widetilde{Z}_\tau = p^*(Z_\tau)$ is free and transitive on the fibers. The transition maps are affine, so we obtain an affine bundle trivialization. $\square$

Theorem 4.5.4. *Over a field k, the octonionic projective line* OP^1 *is a motivic sphere:*

$$\mathrm{OP}^1 \simeq \mathrm{S}^{8,4}\,.$$

Proof. As before, using that X_1 was proved to be $\mathbb{A}^1$-contractible by Corollary 2.1.5 and Lemma 4.5.3 to apply Corollary 2.3.14. $\square$

As $\mathrm{OP}^1 = AQ_8$, this is a new proof of a special case of [ADF16, Theorem 2.2.5].

Remark 4.5.5. By [ADF16, Corollary 3.2.3], in $\mathrm{OP}^1 = AQ_8$, the subset X_1 (called X_8 in the article by Asok–Doran–Fasel) may not be realized as a unipotent quotient. This is the reason why we aimed to construct the X_n as quotients of vector bundle action from the beginning on (although for HP^1 one could define a global $\mathbb{G}_\mathrm{a}$-action on $\widetilde{X}'_1$ which has X_1 as quotient).

FIGURE 5. Details of Computation 4.5.1

$$
\begin{aligned}
a = {} & (v_i^{11}x^{11} - v_i^{21}\cdot x^{22})(v_j^{12}x^{12} - v_j^{22}\cdot x^{21}) \\
& + (v_i^{11}x^{21} + v_i^{21}x^{12})(v_j^{22}x^{11} + v_j^{12}x^{22}) \\
& - (v_i^{11}v_j^{12} + v_i^{21}\cdot v_j^{22}) \\
= {} & v_i^{11}v_j^{12}x^{11}x^{12} - (v_j^{12}v_i^{21})\cdot(x^{12}x^{22}) - (v_i^{11}v_j^{22})\cdot(x^{11}x^{21}) + (v_i^{21}\cdot x^{22})(v_j^{22}\cdot x^{21}) \\
& + (v_i^{11}v_j^{22})\cdot(x^{11}x^{21}) + (v_i^{21}\cdot v_j^{22})x^{11}x^{12} + v_i^{11}v_j^{12}(x^{21}\cdot x^{22}) + (v_j^{12}v_i^{21})\cdot(x^{12}x^{22}) \\
& - v_i^{11}v_j^{12} - v_i^{21}\cdot v_j^{22} \\
= {} & (v_i^{11}v_j^{12})(x^{11}x^{12} + x^{21}\cdot x^{22} - 1) \\
& + (v_i^{21}\cdot x^{22})(v_j^{22}\cdot x^{21}) \\
& + (v_i^{21}\cdot v_j^{22})(x^{11}x^{12} - 1) \\
= {} & (v_i^{21}\cdot x^{22})(v_j^{22}\cdot x^{21}) - (v_i^{21}\cdot v_j^{22})(x^{21}\cdot x^{22}),
\end{aligned}
$$

$$
\begin{aligned}
b = {} & - (v_i^{11}x^{11} - v_i^{21}\cdot x^{22})(v_j^{11}x^{21} + v_j^{21}x^{12}) \\
& + (v_i^{11}x^{21} + v_i^{21}x^{12})(v_j^{11}x^{11} - v_j^{21}\cdot x^{22}) \\
& - (v_i^{21}v_j^{11} - v_i^{11}v_j^{21}) \\
= {} & - v_i^{11}v_j^{11}x^{11}x^{21} + (v_j^{11}v_i^{21}\cdot x^{22})x^{21} - v_i^{11}v_j^{21}x^{11}x^{12} + (v_i^{21}\cdot x^{12}x^{22})v_j^{21} \\
& + v_i^{11}v_j^{11}x^{11}x^{21} + v_i^{21}v_j^{11}x^{11}x^{12} - (v_i^{11}v_j^{21}\cdot x^{22})x^{21} - v_i^{21}(v_j^{21}\cdot x^{12}x^{22}) \\
& - v_i^{21}v_j^{11} + v_i^{11}v_j^{21} \\
= {} & ((v_j^{11}v_i^{21} - v_i^{11}v_j^{21})\cdot x^{22})x^{21} - (v_j^{11}v_i^{21} - v_i^{11}v_j^{21})(x^{22}\cdot x^{21}) \\
& + v_j^{21}(v_i^{21}\cdot x^{12}x^{22}) - v_i^{21}(v_j^{21}\cdot x^{12}x^{22}),
\end{aligned}
$$

$$
\begin{aligned}
c = {} & - (v_i^{22}x^{11} + v_i^{12}x^{22})(v_j^{12}x^{12} - v_j^{22}\cdot x^{21}) \\
& + (v_i^{12}x^{12} - v_i^{22}\cdot x^{21})(v_j^{22}x^{11} + v_j^{12}x^{22}) \\
& - (v_i^{12}v_j^{22} - v_i^{22}v_j^{12}) \\
= {} & - v_j^{12}v_i^{22}x^{11}x^{12} - v_i^{12}v_j^{12}x^{12}x^{22} + v_i^{22}(v_j^{22}\cdot x^{11}x^{21}) + (v_i^{12}v_j^{22}\cdot x^{21})x^{22} \\
& + v_i^{12}v_j^{22}x^{11}x^{12} - v_j^{22}(v_i^{22}\cdot x^{11}x^{21}) + v_i^{12}v_j^{12}x^{12}x^{22} - (v_j^{12}v_i^{22}\cdot x^{21})x^{22} \\
& - v_i^{12}v_j^{22} + v_i^{22}v_j^{12} \\
= {} & ((v_i^{12}v_j^{22} - v_j^{12}v_i^{22})\cdot x^{21})x^{22} - (v_i^{12}v_j^{22} - v_j^{12}v_i^{22})(x^{21}\cdot x^{22}) \\
& + v_i^{22}(v_j^{22}\cdot x^{11}x^{21}) - v_j^{22}(v_i^{22}\cdot x^{11}x^{21}),
\end{aligned}
$$

$$
\begin{aligned}
d = {} & (v_i^{22}x^{11} + v_i^{12}x^{22})(v_j^{11}x^{21} + v_j^{21}x^{12}) \\
& + (v_i^{12}x^{12} - v_i^{22}\cdot x^{21})(v_j^{11}x^{11} - v_j^{21}\cdot x^{22}) \\
& - (v_i^{22}\cdot v_j^{21} + v_i^{12}v_j^{11}) \\
= {} & (v_j^{11}v_i^{22}\cdot x^{11}x^{21}) + v_j^{11}v_i^{12}(x^{22}\cdot x^{21}) + (v_i^{22}\cdot v_j^{21})x^{11}x^{12} + (v_i^{12}v_j^{21}\cdot x^{12}x^{22}) \\
& + v_i^{12}v_j^{11}x^{11}x^{12} - (v_j^{11}v_i^{22}\cdot x^{11}x^{21}) - (v_i^{12}v_j^{21}\cdot x^{12}x^{22}) + (v_i^{22}\cdot x^{21})(v_j^{21}\cdot x^{22}) \\
& - v_i^{22}\cdot v_j^{21} - v_i^{12}v_j^{11} \\
= {} & (v_i^{22}\cdot x^{21})(v_j^{21}\cdot x^{22}) - (v_i^{22}\cdot v_j^{21})(x^{21}\cdot x^{22}).
\end{aligned}
$$

4.5.2. The Cayley Plane.

We discuss all that we can say about motivic cell structures on the Cayley plane OP^2.

Lemma 4.5.6. *We collect some known facts on* OP^2*:*

(1) *The split Cayley plane* OP^2 *embeds as open subvariety of the* complete Cayley plane $\mathrm{E}_6\,/P_1$ *(also known as "complex" Cayley plane* $O_{\mathbb{C}}P^2$*, in analogy to the "real" Cayley plane), with closed complement* $\mathrm{F}_4\,/P_1$ *(a nice discussion of the Hasse diagrams was written up by Iliev and Manivel [IM05]).*

(2) *The Cayley plane* OP^2 *is an* F_4*-homogeneous space and the completion* $\mathrm{OP}^2 \hookrightarrow \mathrm{E}_6\,/P_1$ *is* F_4*-equivariant.*

(3) *There exists an involution on the Dynkin diagram* E_6 *whose invariants on the level of root systems are of type* F_4*; This involution descends to* $\mathrm{E}_6\,/P_1$ *and leaves* $\mathrm{F}_4\,/P_1$ *as well as the split Cayley plane invariant.*

Summarizing, we have Fig. 6.

FIGURE 6. Homogeneous space structure and completion of OP^2

$$\begin{array}{ccccc} \mathrm{OP}^2 & & & & \\ \wr\uparrow & & & & \\ \frac{\mathrm{F}_4}{\mathrm{B}_4} & \overset{\circ}{\hookrightarrow} & \frac{\mathrm{E}_6}{P_1} & \hookleftarrow & \frac{\mathrm{F}_4}{P_1} \end{array}$$

The proof of the homogeneous structure as an algebraic variety is established by Springer and Veldkamp [SV00, Chapter 7] for a field k with $\mathrm{char}(k) \neq 2, 3$. The completion is explained in the division algebra case by Pazourek, Tuek and Franek [PTF11]. A comprehensive guide to octonion planes which also touches the split case was written by Faulkner [Fau70]. □

Remark 4.5.7. The compactification $\mathrm{E}_6\,/P_1$ of OP^2 also shows up in the classification of so-called *Severi varieties* in the work of Zak on a conjecture of Hartshorne about complete intersections. Here, $\mathrm{E}_6\,/P_1$ is the last Severi variety. An introduction to Severi varieties and the results of Zak is given by Lazarsfeld and Van de Ven [LV84]. Most strikingly, the dimension of a Severi variety is in $2, 4, 8, 16$ ([LV84, Thm 3.1]), which has prompted several authors (e.g. Atiyah and Berndt [AB03, Appendix] who also studied OP^2 as algebraic variety) to ask for an analogy or even a precise mathematical relationship between Zak's theorem and the Hopf invariant one theorem. Manivel and his coauthors have studied the Severi varieties in several interesting papers. Landsberg and Manivel [LM01] managed to construct a Freudenthal magic square relating the complexifications of the split composition algebras (or, which amounts to the same, the complexifications of the division algebras) to the Severi varieties. In this work however, we do not study $\mathrm{E}_6\,/P_1$ except to understand the open F_4-orbit OP^2.

Remark 4.5.8. There is a different approach to OP^2 than using the hermitian matrices (in fact, also for the other DP^n), namely pseudo-homogeneous coordinates. These were originally invented by Aslaksen in a way that did not work for $\mathbb{OP}^2$ and then Allcock [All97] gave an improved account and showed that the definition (for division algebras) is equivalent to the hermitian-matrix based one that we use. Held, Stavrov and van Koten have proved that one can modify the pseudo-homogeneous coordinates to an analogous construction for split octonions [HSV09], where they write $\mathbb{O}'\mathbb{P}^2$ for what we denote as OP^2.

Remark 4.5.9. On the topological space $\mathrm{OP}^2(\mathbb{C})^{an} \simeq \mathbb{OP}^2$, there is a cell structure consisting of a 0-cell, an 8-dimensional cell and a 16-dimensional cell. The algebraic K-theory of the Cayley plane OP^2 over a field k is $K_*(\mathrm{OP}^2) = K_*(k) \oplus K_*(k) \oplus K_*(k)$, a computation of Ananyevskiy [Ana12, 9.5.]. This strongly suggests that any motivic cell structure would also consist of three cells, a 0-cell and two cells that realize to S^8 and S^{16}. The twist one would expect for the cells can be read off from the motive, see Remark 2.2.6 where we mention that one can compute $M(\mathrm{OP}^2) = M(\mathrm{F}_4 / \mathrm{B}_4) = 1 \oplus 1(4)[8] \oplus 1(8)[16]$. This shows that the expected motivic cell structure of OP^2 has a 0-cell, an $\mathrm{S}^{8,4}$ and an $\mathrm{S}^{16,8}$.

With similar arguments, also looking at the topological cell structure, the K-theory as computed by Ananyevskiy (or earlier authors) and the computations of Remark 2.2.6 we see that $\mathrm{OP}^1 = AQ_8$ is expected to admit a motivic cell structure with a 0-cell and an $\mathrm{S}^{8,4}$. As mentioned before, Asok, Doran and Fasel proved that AQ_8 is $\mathbb{A}^1$-equivalent to $\mathrm{S}^{8,4}$ [ADF16].

Conjecture 2. *There exists a subspace $A \subset \widetilde{X}'_2$ which is isomorphic to an affine space and the morphism $p|_A \colon A \to X_2 \subset \mathrm{OP}^2$ is an affine bundle.*

By the same proof strategy as for CP^n and HP^n, i.e. an invocation of Proposition 4.2.20, we would get an unstable motivic cell structure for OP^2.

The missing step to prove this conjecture is that we do not have a good candidate for A right now. It is also no longer possible to modify sections with any right multiplication, since the associator condition fails in general. One may not always have $p(vx) = p(v)$ for an x which half-inverts v_0. It is conceivable that a good choice of κ in Lemma 3.2.6 could supply a half-inverter which associates. The sections s_i from Theorem 4.1.20 map to $\mathrm{OS}^{2,ass}$ already (due to the computations in Lemma 3.2.6), and a good half-inverter would stay in the p-fiber, so that the $\mathbb{G}_{\mathrm{a}}^{\times \mathrm{e}}$-orbits of the sections s'_τ would be subsets of $\widetilde{X}'_2 \cap \mathrm{OS}^{2,ass}$. It is not clear whether the union of these $\mathbb{G}_{\mathrm{a}}^{\times \mathrm{e}}$-orbits (= p-fibers) would be an affine space or $\mathbb{A}^1$-contractible for any other reason.

Remark 4.5.10. We conclude this section by a reference to Example 2.3.26 where we constructed an unstable motivic cell structure on $\Sigma\,\mathrm{OP}^2$ by using the F_4-equivariant completion to E_6 / P_1:

$$\mathrm{E}_6 / P_1 \to \mathrm{Th}\left(N_{\mathrm{F}_4 / P_1}\right) \to \Sigma\,\mathrm{OP}^2 .$$

It would be interesting to analyze how the embedding $\mathrm{OP}^1 \to \mathrm{OP}^2$ behaves with respect to this cell structure. For $\mathrm{OP}^1 = AQ_8$ we can use the equivariant completion as well, which is $AQ_8 \hookrightarrow PQ_8 \hookleftarrow PQ_7$, so that we obtain a motivic cell structure:

$$PQ_8 \to \mathrm{Th}\left(N_{PQ_7}\right) \to \Sigma\,\mathrm{OP}^1$$

As homogeneous varieties, $PQ_8 \xleftarrow{\simeq} \mathrm{Spin}(5,5)/P_1$ and $PQ_7 \xleftarrow{\simeq} \mathrm{Spin}(4,5)/P_1$. An embedding $\mathrm{OP}^1 \hookrightarrow \mathrm{OP}^2$ can be identified as an orbit under the group $\mathrm{Spin}(4,5)$ acting on $\mathrm{OP}^2 \xleftarrow{\simeq} \mathrm{F}_4 / \mathrm{Spin}(4,5)$ on the left via $\mathrm{Spin}(4,5) \hookrightarrow \mathrm{F}_4$. The stabilizer of such a $\mathrm{Spin}(4,5)$ is $\mathrm{Spin}(4,4)$, and $\mathrm{OP}^1 \xleftarrow{\simeq} \mathrm{Spin}(4,5)/\,\mathrm{Spin}(4,4)$. It seems likely that the morphism $\mathrm{Spin}(4,5) \hookrightarrow \mathrm{F}_4$ is compatible with a morphism $\mathrm{Spin}(5,5) \hookrightarrow \mathrm{E}_6$, so that we obtain a morphism of equivariant completions extending $\mathrm{OP}^1 \to \mathrm{OP}^2$. In turn, we obtain a morphism of unstable motivic cell structures extending $\Sigma\,\mathrm{OP}^1 \to \Sigma\,\mathrm{OP}^2$. Connectivity of the morphism $\Sigma\,\mathrm{OP}^1 \to \Sigma\,\mathrm{OP}^2$ can be inferred from connectivity of the morphism $PQ_8 \to \mathrm{E}_6\,/P_1$ and the morphism of Thom spaces of the normal bundles of the boundary components.

From the theory of Albert algebras (that is, algebras of hermitian 3×3-matrices over an 8-dimensional composition algebra), one can see the morphism $\mathrm{Spin}(5,5) \hookrightarrow \mathrm{E}_6$ directly: decomposing as before

$$A = \begin{pmatrix} z & \psi^\dagger \\ \psi & a \end{pmatrix} \in J_3(\mathrm{O})$$

we obtain a decomposition of $J_3(\mathrm{O})$ (as a vector space) into $R \oplus \mathrm{O}^2 \oplus J_2(\mathrm{O})$. By the analogous decomposition, the component $J_2(\mathrm{O})$ is a 10-dimensional affine space, equipped with a split form given by the trace. The linear action of $\mathrm{Spin}(5,5)$ can be seen in this decomposition, as it acts on $J_2(\mathrm{O})$ preserving the trace and on O^2 via the 16-dimensional half-spin representation of $\mathrm{Spin}(5,5)$. Since E_6 are the linear automorphisms of $J_2(\mathrm{O})$, this hands us a morphism $\mathrm{Spin}(5,5) \hookrightarrow \mathrm{E}_6$. We can also see that this morphism is compatible with the morphism $\mathrm{Spin}(4,5) \hookrightarrow \mathrm{F}_4$, as F_4 is the automorphism group of the traceless part of $J_3(\mathrm{O})$ and likewise $\mathrm{Spin}(4,5)$ is the automorphism group of the traceless part of $J_2(\mathrm{O})$.

We leave it up to the reader to draw conclusions on the motivic connectivity of the space $\Sigma\,\mathrm{OP}^2$ and its consequences for motivic homotopy sheaves and obstruction theory.

4.6. Explicit Cell Structures

We describe a conjectural method to construct inductive cell structures for the DP^n by attaching a single cell along a map with domain a motivic sphere in each step. This is analogous to the topological construction described in Example 1.5.3. We assume contractibility of X_n and an additional conjecture.

Conjecture 3. *There is a space Y_n which acts as avatar of $\mathrm{DP}^n \setminus \iota_0 \mathrm{DP}^0$ in the sense that $Y_n \hookrightarrow \mathrm{DP}^n \setminus \iota_0 \mathrm{DP}^0$ and $\mathrm{DP}^{n-1} \hookrightarrow Y_n$ is a section to the composition of an $\mathbb{A}^1$-weak equivalence $t^{\downarrow} \colon Y_n \to V_n$ with $V_n \twoheadrightarrow \mathrm{DP}^{n-1}$.*

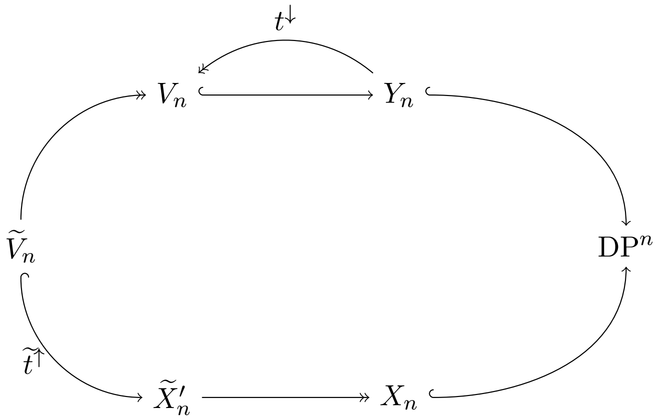
We remind that $\iota_0 \colon \mathrm{DP}^0 \hookleftarrow\!\!\rightarrow \mathrm{DP}^n$ is the embedding as upper left block. A candidate for such a Y_n would be W_n from Definition 4.2.3 or $\{z - 1 \text{ invertible}\}$, compare Lemma 4.2.2. We also conjecture that the obvious truncation map $W_n \twoheadrightarrow V_n$ is a vector bundle, hence $W_n \twoheadrightarrow \mathrm{DP}^{n-1}$ an $\mathbb{A}^1$-weak equivalence.

Lemma 4.6.1. *There is a closed immersion $\widetilde{t}^{\uparrow} \colon \mathrm{DS}^{n-1} \hookleftarrow\!\!\rightarrow \widetilde{X}'_n$ given by*

$$\varphi \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \varphi.$$

Proof. The image is an explicitly described closed subvariety, with fixed v_0. □

Given Conjecture 3, we have a diagram:



analogous to Example 1.5.3. We would get a homotopy cofiber sequence

$$\widetilde{V}_n \twoheadrightarrow V_n \to \mathrm{DP}^n$$

which is weakly equivalent to

$$\mathrm{DS}^{n-1} \twoheadrightarrow \mathrm{DP}^{n-1} \to \mathrm{DP}^n$$

exhibiting DP^n as DP^{n-1} with a cell DS^{n-1} attached along the Hopf map

$$p \colon \mathrm{DS}^{n-1} \twoheadrightarrow \mathrm{DP}^{n-1}.$$

CHAPTER 5

Applications

5.1. Hermitian K-Theory

Panin and Walter constructed a spectrum representing hermitian K-Theory using the quaternionic Grassmannians. We discuss cellularity of this spectrum, prompted by results of Spitzweck and Hornbostel.

As explained by Hornbostel [Hor15], cellularity of the hermitian K-theory spectrum has interesting consequences.

In unpublished work of Spitzweck ([Spi10], private communication) stable motivic cellularity of the hermitian K-theory spectrum is proved, starting from a proof of stable cellularity of HP^n and the quaternionic Grassmannians $\mathrm{HGr}(n, m+n)$ along the lines of Panin–Walter's cohomological cell structure [PW10b].

By repeating the proof of the main theorem for the spaces $\mathrm{HGr}(m, n+m)$ instead of $\mathrm{HP}^n = \mathrm{HGr}(1, n+1)$, working with trace m projectors in hermitian matrices, one may obtain the same results. In particular, we can state the obvious:

Conjecture 4. *Let k be a field, then the motivic spaces $\mathrm{HGr}(n, 2n)$ have a finite motivic cell structure. Furthermore, $\mathrm{HGr} = \mathrm{colim}_n \mathrm{HGr}(n, 2n)$ and $K\,\mathrm{H} := \mathbb{Z} \times \mathrm{HGr}$ have a stable cell structure.*

Panin and Walter showed [PW10a, Theorem 8.2] that the space $K\,\mathrm{H}$ is $\mathbb{A}^1$-weakly equivalent to $K\,\mathrm{Sp}$, the infinite loop space representing Schlichting's hermitian K-theory.

As we mentioned in Fact 2.3.15, stable cellularity of $K\,\mathrm{H}$ is rather easy to prove, so that an easy consequence is the second part of the conjecture just made.

Theorem 5.1.1. *The hermitian K-theory spectrum $K\,\mathrm{Sp}$ is stably motivically cellular (over any field).*

Proof. The variety $\mathrm{HGr}(n, 2n)$ is homogeneous under Sp_{2n}. It is spherical (as it has only finitely many orbits under a Borel subgroup), and the wonderful completion is given by $\mathrm{Gr}(n, 2n)$. We use the direct construction of a stable motivic cell structure from Remark 2.3.23 or just the general theorem of Fact 2.3.15 to see that $\mathrm{HGr}(n, 2n)$ is stably cellular. The colimit HGr is then also stably cellular (as it is a homotopy colimit) and the product with a discrete space $\mathbb{Z}$ is obviously again stably cellular. We then apply [PW10a, Theorem 8.2] for $K\,\mathrm{H} \simeq K\,\mathrm{Sp}$. □

By this argument, one can drop the assumption on the characteristic made by Spitzweck [Spi10] and Hornbostel [Hor15, Proposition 3.3]. This does not improve the main result [Hor15, Theorem 3.2], as it applies only to characteristic 0.

5.2. Algebraic Geometry: Symplectic Bundles

We explain in an example how the connectivity of HP^n given by the motivic cell structure can be used to obtain information on algebraic principal bundles under symplectic groups, analogous to the topological story in Section 1.4. For this, we need to recall some unstable motivic connectivity theory.

Convention 5.2.1. For this section, let k be a field. We work with smooth k-varieties and motivic spaces $Spc(k)$ over k.

Definition 5.2.2. Let X be a motivic space and (p,q) a pair of non-negative integers. We say X is $\mathbb{A}^1$-*connected* if $\pi_0^{\mathbb{A}^1}(X) \xrightarrow{\sim} 1$. We say X is *(p,q)-connected* if $\pi_{p,q}^{\mathbb{A}^1}(X) = [\mathrm{S}^{p,q}, X]_{\mathbb{A}^1} \xrightarrow{\sim} 1$.

For a comparison of different connectedness assumptions including the one made here, see Asok's article [Aso16].

Theorem 5.2.3 (Morel)**.** *Let (n,q') and (m,q) be pairs of non-negative integers. If $n < m$ or both $n = m$ and $q' > 0 = q$, in the pointed $\mathbb{A}^1$-homotopy category*

$$[\mathrm{S}^{n+q',q'}, \mathrm{S}^{m+q,q}]_{\mathbb{A}^1} = 0.$$

For $p = m + q$, the motivic sphere $\mathrm{S}^{p,q}$ is $(p-1,q)$-connected and more generally $(p'-1-q+q',q')$-connected for any $p' \leq p$ and any q'.

This result is part of Morel's $\mathbb{A}^1$-connectivity theorem [Mor12, Corollary 6.43].

Lemma 5.2.4. *If X and Y are motivic spaces with motivic cell structures of the form that there are diagrams $D(X)$ and $D(Y)$ indexed by sets I and J respectively with* $\mathrm{hocolim}(D(X)) = X$ *and* $\mathrm{hocolim}(D(Y))$ *such that for each $i \in I$ and each $j \in J$*

$$D(X)_i \simeq \mathrm{S}^{p_i,q_i} \ \text{ and } D(Y)_j \simeq \mathrm{S}^{p_j,q_j} \ \text{ with } p_i - q_i < p_j - q_j$$

then $[X,Y]_{\mathbb{A}^1} = 0$.

Proof. Any morphism $X \to Y$ can be decomposed into morphisms $D(X)_i \to D(Y)_j$, which are all $\mathbb{A}^1$-nullhomotopic due to Theorem 5.2.3. □

Definition 5.2.5. In the situation of the previous lemma, we say that X *has cells* in bidegrees (p_i, q_i).

Lemma 5.2.6. *If X is a smooth affine variety with motivic cell structure which has cells in bidegrees (p_i,q_i) such that $(p_i,q_i) \leq (4,1)$, then every symplectic bundle which becomes trivial upon adding a trivial line bundle is already trivial.*

Proof. From the assumptions, $[X, \mathrm{HP}^n] = 0$. We remember that

$$\mathrm{HP}^n \xleftarrow{\sim} \mathrm{Sp}_{2n+2} / \mathrm{Sp}_2 \times \mathrm{Sp}_{2n}\,.$$

Apply the covariant functor $[X,-]$ to the fiber sequence

$$\mathrm{Sp}_{2n+2} / \mathrm{Sp}_2 \times \mathrm{Sp}_{2n} \to B\,(\mathrm{Sp}_2 \times \mathrm{Sp}_{2n}) \to B\,\mathrm{Sp}_{2n+2}$$

and then the Sp_{2n}-torsor classification in $\mathbb{A}^1$-homotopy theory [AHW15, Theorem 4.1.2]. □

Remark 5.2.7. A motivic cell structure for $\mathrm{OP}^2 = \mathrm{F}_4 / \mathrm{Spin}_9$ which consists of a cell OP^1 and higher-degree cells would have similar implications for Spin_9 resp. F_4-torsors on smooth affine varieties. Our cell structure for $\Sigma\,\mathrm{OP}^2$ from Example 2.3.26, further discussed in Remark 4.5.10, can be used to get connectivity results on $\Sigma\,\mathrm{OP}^2$ from $\Sigma\,\mathrm{OP}^1 \simeq \mathrm{S}^{9,4}$. This might be useful to obtain results in motivic obstruction theory, compare Remark 1.4.6. The blueprint for such results are the topological computations of Mimura [Mim67], where $\pi_n(\mathbb{OP}^2)$ is computed for small n, which is also used to compute $\pi_n(\mathrm{F}_4)$ for small n.

5.3. Motivic Homotopy Theory: Stable Stems

It is desirable to lift the motivic Hopf elements $\eta, \nu, \sigma \in \pi_*(\mathbb{S})$ to the motivic sphere spectrum. If one had an explicit cell structure as conjectured in Section 4.6, this would follow. We mention the related construction of Dugger and Isaksen.

Remark 5.3.1. Given a homotopy cofiber sequence

$$\mathrm{DS}^{n-1} \to \mathrm{DP}^{n-1} \to \mathrm{DP}^n$$

for $n = 2$ for one D, we have as attaching map (the first map) $f_D\colon \mathrm{DS}^1 \twoheadrightarrow \mathrm{DP}^1$, which is isomorphic to $f_D\colon AQ_{2d-1} \twoheadrightarrow AQ_d$. This is weakly equivalent to $f_D\colon \mathrm{S}^{2d-1,d} \to \mathrm{S}^{d,d/2}$. It sits in a fiber sequence $\mathrm{S}^{d-1,d/2} \to \mathrm{S}^{2d-1,d} \xrightarrow{f_D} \mathrm{S}^{d,d/2}$. We have $[f_D] \in \pi^s_{(d-1,d/2)}(\mathrm{S}^{0,0})$. This means, up to the existence of the conjectured homotopy cofiber sequence from Section 4.6, there are elements

$$\eta = [f_C] \in \pi_{(1,1)}(\mathbb{S}) \tag{8}$$

$$\nu = [f_H] \in \pi_{(3,2)}(\mathbb{S}) \tag{9}$$

$$\sigma = [f_O] \in \pi_{(7,4)}(\mathbb{S}). \tag{10}$$

Remark 5.3.2. The constructions in Chapter 4 are made such that the complex points of the motivic cell structure have the homotopy type (as diagram) of the classical topological cell structure, so that the $[f_D]$ would be motivic lifts of the classical Hopf elements.

Remark 5.3.3. Dugger and Isaksen constructed motivic Hopf elements [DI13] with a homotopical construction (that they call *geometric*) which also lift the classical Hopf elements. It is not yet known whether the elements described here coincide with these, but the author conjectures so (at least up to sign).

Bibliography

[AB03] Atiyah, M.; Berndt, J.: Projective planes, Severi varieties and spheres. MathSciNet:2039984. doi:10.4310/SDG.2003.v8.n1.a1. In: *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*. Int. Press, Somerville, MA, 2003 (Surv. Differ. Geom., VIII), p. 1–27

[AD07] Asok, A.; Doran, B.: On unipotent quotients and some $\mathbb{A}^1$-contractible smooth schemes. *Int. Math. Res. Pap. IMRP* (2007), Nr. 2, p. Art. ID rpm005, 51. MathSciNet:2335246. arXiv:math/0703137

[AD08] Asok, A.; Doran, B.: Vector bundles on contractible smooth schemes. *Duke Math. J.* 143 (2008), Nr. 3, p. 513–530. – MathSciNet:2423761. doi:10.1215/00127094-2008-027

[Ada60] Adams, J. F.: On the non-existence of elements of Hopf invariant one. *Ann. of Math. (2)* 72 (1960), p. 20–104. MathSciNet:0141119. JSTOR:1970147

[ADF16] Asok, A.; Doran, B. ; Fasel, J.: Smooth Models of Motivic Spheres and the Clutching Construction. *International Mathematics Research Notices* (2016). doi:10.1093/imrn/rnw065

[Ahi83] Ahiezer, D.: Equivariant completions of homogeneous algebraic varieties by homogeneous divisors. *Annals of Global Analysis and Geometry* 1 (1983), Nr. 1, p. 49–78. – MathSciNet:739893. doi:10.1007/BF02329739

[AHW15] Asok, A.; Hoyois, M. ; Wendt, M.: Affine representability results in $\mathbb{A}^1$-homotopy theory II: principal bundles and homogeneous spaces. *Preprint* (2015). arXiv:1507.08020

[AHW16] Asok, A.; Hoyois, M. ; Wendt, M.: Affine representability results in $\mathbb{A}^1$-homotopy theory I: vector bundles. *Preprint, to appear in Duke Math. J.* (2016). arXiv:1506.07093

[All97] Allcock, D.: Identifying models of the octave projective plane. *Geom. Dedicata* 65 (1997), Nr. 2, p. 215–217. – MathSciNet:1451975. doi:10.1023/A:1017926505948

[Ana12] Ananyevskiy, A.: On the algebraic K-theory of some homogeneous varieties. *Documenta Mathematica* 17 (2012), p. 167–193. K-Theory Preprint Archive, Nr. 1000

[Aso16] Asok, A.: Stably $\mathbb{A}^1$-connected varieties and universal triviality of CH_0. *Preprint* (2016), January. arXiv:1601.01615

[Bae02] Baez, J. C.: The octonions. *Bull. Amer. Math. Soc. (N.S.)* 39 (2002), Nr. 2, p. 145–205. – MathSciNet:1886087. doi:10.1090/S0273-0979-01-00934-X

[Bak69] Bak, A.: On modules with quadratic forms. (1969), p. 55–66. MathSciNet:0252431

[BB73] Białynicki-Birula, A.: Some Theorems on Actions of Algebraic Groups. *The Annals of Mathematics* 98 (1973), November, Nr. 3, p. 480–497. MathSciNet:0366940. JSTOR:1970915

[Bor50] Borel, A.: Le plan projectif des octaves et les sphères comme espaces homogènes. *C. R. Acad. Sci. Paris* 230 (1950), p. 1378–1380. MathSciNet:0034768

[CD15] Cisinski, D.-C.; Déglise, F.: Integral mixed motives in equal characteristic. *Doc. Math.* (2015), Nr. Extra vol.: Alexander S. Merkurjev's sixtieth birthday, p. 145–194. MathSciNet:3404379

[CF03] Cupit-Foutou, S.: Classification of two-orbit varieties. *Comment. Math. Helv.* 78 (2003), Nr. 2, p. 245–265. – MathSciNet:1988197. doi:10.1007/s000140300011

[Cha06] Chaput, P.-E.: Geometry over composition algebras: projective geometry. *J. Algebra* 298 (2006), Nr. 2, p. 340–362. – MathSciNet:2217619. doi:10.1016/j.jalgebra.2006.02.008

[CJ11] Carlsson, G.; Joshua, R.: Motivic Spanier-Whitehead Duality and Motivic Becker-Gottlieb Transfer. *Preprint (self-hosted)* (2011), September. https://people.math.osu.edu/joshua.1/Mot_Sd10.pdf

[Dég12] Déglise, F.: Around the Gysin triangle I. MathSciNet:2953410. doi:10.1090/conm/571/11323. In: *Regulators* Vol. 571. Amer. Math. Soc., Providence, RI, 2012, p. 77–116

[DF15] Dubouloz, A.; Fasel, J.: Families of $\mathbb{A}^1$-contractible affine threefolds. *Preprint* (2015). arXiv:arXiv:1512.01933

[DHI04] Dugger, D.; Hollander, S. ; Isaksen, D. C.: Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.* 136 (2004), Nr. 1, p. 9–51. – MathSciNet:2034012. doi:10.1017/S0305004103007175

[DI04] Dugger, D.; Isaksen, D. C.: Topological hypercovers and $\mathbb{A}^1$-realizations. *Math. Z.* 246 (2004), Nr. 4, p. 667–689. – MathSciNet:2045835. doi:10.1007/s00209-003-0607-y

[DI05] Dugger, D.; Isaksen, D. C.: Motivic cell structures. *Algebraic & Geometric Topology* 5 (2005), June, p. 615–652 (electronic). – MathSciNet:2153114. doi:10.2140/agt.2005.5.615

[DI13] Dugger, D.; Isaksen, D. C.: Motivic Hopf elements and relations. *New York J. Math* 19 (2013), p. 823–871. MathSciNet:3141814. arXiv:1307.6596v2

[DM15] Dray, T.; Manogue, C. A.: *The geometry of the octonions*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015. – xviii+210 S. MathSciNet:3361898. doi:10.1142/8456

[Dug01] Dugger, D.: Universal homotopy theories. *Adv. Math.* 164 (2001), Nr. 1, p. 144–176. – MathSciNet:1870515. doi:10.1006/aima.2001.2014

[Fau70] Faulkner, J. R.: *Octonion planes defined by quadratic Jordan algebras*. American Mathematical Society, Providence, R.I., 1970 (Memoirs of the American Mathematical Society, No. 104). – ii+71 S. MathSciNet:0271180

[Har90] Harvey, F.: *Spinors and calibrations*. Academic Press, 1990 (Spinors and Calibrations v. 9). MathSciNet:1045637

[Has10] Hasebe, K.: The split algebras and noncompact Hopf maps. *J. Math. Phys.* 51 (2010), Nr. 5, p. 053524, 35. – MathSciNet:2667002. doi:10.1063/1.3372628

[Hat02] Hatcher, A.: *Algebraic topology*. Cambridge University Press, Cambridge, 2002. – xii+544 S. MathSciNet:1867354. www.math.cornell.edu/~hatcher/AT/AT.pdf

[Hor15] Hornbostel, J.: Some comments on motivic nilpotence. *Preprint* (2015), November. arXiv:1511.07292

[HSV09] Held, R.; Stavrov, I. ; VanKoten, B.: (Semi-)Riemannian geometry of (para-)octonionic projective planes. *Differential Geom. Appl.* 27 (2009), Nr. 4, p. 464–481. – MathSciNet:2547826. doi:10.1016/j.difgeo.2009.01.007

[IM05] Iliev, A.; Manivel, L.: The Chow ring of the Cayley plane. *Compos. Math.* 141 (2005), Nr. 1, p. 146–160. – MathSciNet:2099773. doi:10.1112/S0010437X04000788

[Jar00] Jardine, J. F.: Motivic symmetric spectra. *Documenta Mathematica* 5 (2000), p. 445–553 (electronic). MathSciNet:1787949

[Jor49] Jordan, P.: Über eine nicht-desarguessche ebene projektive Geometrie. *Abh. Math. Sem. Univ. Hamburg* 16 (1949), Nr. nos. 1-2, p. 74–76. MathSciNet:0030773

[KMRT98] Knus, M.-A.; Merkurjev, A.; Rost, M. ; Tignol, J.-P.: *American Mathematical Society Colloquium Publications*. Vol. 44: *The book of involutions*. American Mathematical Society, Providence, RI, 1998. – xxii+593 S. MathSciNet:1632779. – With a preface in French by J. Tits

[Kno14] Knop, F.: Spherical roots of spherical varieties. *Ann. Inst. Fourier (Grenoble)* 64 (2014), Nr. 6, p. 2503–2526. – MathSciNet:3331173. doi:10.5802/aif.2919

[Kra96] Kraft, H.: Challenging problems on affine n-space. *Astérisque* (1996), Nr. 237, p. Exp. No. 802, 5, 295–317. MathSciNet:1423629. – Séminaire Bourbaki, Vol. 1994/95

[Lam06] Lam, T.-Y.: *Serre's Problem on Projective Modules*. Springer-Verlag, 2006 MathSciNet:2235330. doi:10.1007/978-3-540-34575-6

[Lev14] Levine, M.: A comparison of motivic and classical stable homotopy theories. *Journal of Topology* 7 (2014), Nr. 2, p. 327–362. – MathSciNet:3217623. doi:10.1112/jtopol/jtt031

[LM01] Landsberg, J. M.; Manivel, L.: The projective geometry of Freudenthal's magic square. *J. Algebra* 239 (2001), Nr. 2, p. 477–512. – MathSciNet:1832903. doi:10.1006/jabr.2000.8697

[LV84] Lazarsfeld, R.; Ven, A. Van d.: Topics in the geometry of projective space. 4 (1984), p. 52. – MathSciNet:808175. doi:10.1007/978-3-0348-9348-0. – Recent work of F. L. Zak, With an addendum by Zak

[Mim67] Mimura, M.: The homotopy groups of Lie groups of low rank. *J. Math. Kyoto Univ.* 6 (1967), p. 131–176. MathSciNet:0206958

[Mor12] Morel, F.: $\mathbb{A}^1$*-algebraic topology over a field*. Vol. 2052. Springer, 2012 MathSciNet:2934577

[MV99] Morel, F.; Voevodsky, V.: $\mathbb{A}^1$-homotopy theory of schemes. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 90 (1999), Nr. 90, p. 45–143. MathSciNet:1813224. Numdam:PMIHES_1999__90__45_0

[Pet93] Petersson, H. P.: Composition algebras over algebraic curves of genus zero. *Trans. Amer. Math. Soc.* 337 (1993), Nr. 1, p. 473–493. – MathSciNet:1108613. doi:10.2307/2154334

[Pez10] Pezzini, G.: Lectures on spherical and wonderful varieties. *Les cours du CIRM* 1 (2010), Nr. 1, p. 33–53

[PTF11] Pazourek, K.; Tuček, V. ; Franek, P.: Hyperplane section $\mathbb{OP}_0^2$ of the complex Cayley plane as the homogeneous space $\mathrm{F}_4/\mathrm{P}_4$. *Comment. Math. Univ. Carolin.* 52 (2011), Nr. 4, p. 535–549. MathSciNet:2863997

[PW10a] Panin, I.; Walter, C.: On the motivic commutative ring spectrum BO. *Preprint* (2010), November. arXiv:1011.0650

[PW10b] Panin, I.; Walter, C.: Quaternionic Grassmannians and Pontryagin classes in algebraic geometry. *Preprint* (2010), November. K-Theory Preprint Archive, Nr. 0977

[Rez98] Rezk, C.: Fibrations and homotopy colimits of simplicial sheaves. *Preprint* (1998), November. arXiv:math/9811038

[Sem06] Semenov, N. S.: Hasse diagrams and motives of homogeneous projective varieties. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 330 (2006), Nr. Vopr. Teor. Predst. Algebr. i Grupp. 13, p. 236–246, 274. – MathSciNet:2253576. doi:10.1007/s10958-007-0013-9

[Spi10] Spitzweck, M.: *Cooperations of hermitian K-theory and E_∞-structures.* November 2010. – Unpublished preprint (private communiation)

[Sta63] Stasheff, J. D.: Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.* 108 (1963), Nr. 2, p. 275–312. MathSciNet:0158400. JSTOR:1993608

[Ste51] Steenrod, N.: *The Topology of Fibre Bundles.* Princeton University Press, Princeton, N. J., 1951 (Princeton Mathematical Series, vol. 14). – viii+224 S. MathSciNet:0039258

[SV00] Springer, T. A.; Veldkamp, F. D.: *Octonions, Jordan algebras and exceptional groups.* Springer-Verlag, Berlin, 2000 (Springer Monographs in Mathematics). – viii+208 S. MathSciNet:1763974. doi:10.1007/978-3-662-12622-6

[Tim11] Timashev, D. A.: *Encyclopaedia of Mathematical Sciences.* Vol. 138: *Homogeneous spaces and equivariant embeddings*. Springer, Heidelberg, 2011. – xxii+253 S. MathSciNet:2797018. doi:10.1007/978-3-642-18399-7. – Invariant Theory and Algebraic Transformation Groups, 8

[Tot14] Totaro, B.: Chow groups, Chow cohomology, and linear varieties. *Forum Math. Sigma* 2 (2014), p. e17, 25. – MathSciNet:3264256. doi:10.1017/fms.2014.15

[Voe03] Voevodsky, V.: Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.* (2003), Nr. 98, p. 1–57. – MathSciNet:2031198. doi:10.1007/s10240-003-0009-z

[Wen10] WENDT, M.: More examples of motivic cell structures. *Preprint* (2010), Dezember. arXiv:1012.0454

[Wen11] WENDT, M.: Rationally trivial torsors in $\mathbb{A}^1$-homotopy theory. *J. K-Theory* 7 (2011), Nr. 3, p. 541–572. – MATHSCINET:2811715. DOI:10.1017/is011004020jkt157