Existence of Periodic Orbits in Riemannian and Contact Geometry

Dissertation zur Erlangung des Doktorgrades der Fakultät für Mathematik und Physik der Albert-Ludwigs-Universität Freiburg im Breisgau

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September 2014
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Mündliche Prüfung: 19.12.1914

Parts of this work have been published ahead of print in *Inventiones Mathematicae*, [GRZ14].

This document was typeset with \TeX. Figures were created with inkscape.
Acknowledgments

First and foremost I want to thank Professor Victor Bangert for his excellent mentoring. Thanks for the numerous questions you entrusted me with and which guided me into the world of research.

Furthermore I thank all of my colleagues for many discussions, new insights and an open ear to my questions; in particular, I want to mention Anda Degeratu, Patrick Emmerich, Prof. Sebastian Goette, Felix Grimm, Ursula Ludwig, Blaž Mramor, Emanuel Scheidegger, Oliver Straser, Katrin Wendland, Yi-Sheng Wang, and the late Alex Koenen. My thanks also go to the administrative staff which provided a very pleasant work atmosphere.

I would especially like to thank Oliver Straser and Malte Wiemann for proof reading this thesis, and Anda Degeratu for discussing the introduction with me.

Finally, I feel deeply grateful to all my friends, my family, my parents and Malte. Special thanks to all of you for your continuous support over the past years.

I am indebted to the University of Freiburg for funding the work of this thesis. Also I want to thank the German Research Foundation (DFG) for providing generous financial support for travelling to conferences within the framework of the Collaborative Research Center SFB 71 ”Geometric Partial Differential Equations”.

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1. Preface

The study of periodic orbits in riemannian and contact geometry has proven to be very influential in both areas and further afield. In the first part of this thesis we concern ourselves with the geodesic flow and give existence results for closed geodesics on open riemannian manifolds. In the second part we focus on a particular contact manifold and provide a counterexample to a conjecture of Hofer about the existence of periodic orbits for the associated Reeb flow.

If the Weinstein conjecture proves true, each Reeb flow on a closed contact manifold has at least one periodic orbit, [Wei79]. For the geodesic flow this holds by a proof of Lusternik and Fet, [LF51]. On open manifolds, however, closed geodesics exist only under additional conditions. If the open riemannian manifold has convex or concave ends we give existence conditions for closed geodesics. This will be the focus of the first part of this thesis.

In the second part we will attack a problem of a completely different flavor. Considering the standard contact structure on $\mathbb{R}^n$ we answer the following rigidity question: Can we characterize the standard contact form by the fact that its associated Reeb flow has no periodic orbit? By a result of Eliashberg and Hofer, this is true in dimension three if the contact form is standard outside a compact set, [EH94]. By way of examples we show that the answer is “no” in dimension five and higher.

**Statement on prior publication**  The main results of this thesis are contained in chapter 4 and 7. The outcome of chapter 7 has been published in *Inventiones mathematicae*, [GRZ14]. The approach used therein is presented in Section 7.3 but in a reorganized form. Section 7.2 is not published in a peer reviewed journal, but is based on the preprint [Röt13]. The results in chapter 4 have not yet been published.
Part I.

On the existence of closed geodesics on open manifolds
2. Introduction

Closed geodesics can be described as critical points of the energy functional on the (free) loop space of a riemannian manifold. Therefore their existence can be established by variational methods. This is the point of view we take in this thesis.

On a closed riemannian manifold the topology of the loop space modulo point curves is nontrivial and the energy functional satisfies the Palais-Smale-condition. This guarantees the existence of closed geodesics. On an open riemannian manifold both conditions might fail and thus in general one needs additional assumptions.

In two dimensions the situation is fully understood: Thorbergsson [Tho78, Thm. 3.2] proved the existence of infinitely many closed geodesics on every complete, open riemannian surface except for the plane, the cylinder, and the Möbius strip. Furthermore for the Möbius band he proved the existence of at least one geodesic for every complete riemannian metric. Bangert, [Ban80], established the existence of infinitely many closed geodesics on every plane, cylinder, and Möbius strip with a complete riemannian metric of finite area. Hence combined with Thorbergsson’s result, one obtains the existence of infinitely many closed geodesics on every complete riemannian surface of finite area.

Thorbergsson’s result is based on a thorough analysis of fundamental classes which can not be realized outside a certain compact set. This type of argument actually implies the existence of closed geodesics in arbitrary dimensions once the manifold satisfies this topological condition. In his thesis, Thorbergsson illustrated this with two examples: the first is a compact manifold with nontrivial fundamental group from which a totally disconnected, compact set is cut out and the second being the complement of the trefoil knot in \( \mathbb{R}^3 \), cf. [Tho77, 5.2/5.3].

Yet, every product of a compact manifold with \( \mathbb{R} \) is not of this type. Actually the existence of a closed geodesic might not hold on these manifolds, as shown by examples of warped products due to Thorbergsson and Benci-
Giannoni, [Tho77] and [BG92]. However, there are existence results for manifolds of this kind by Tanaka and Secchi: Tanaka, [Tan00], proved the existence of a closed geodesic on $S^n \times \mathbb{R}$ if it is equipped with a riemannian metric that converges uniformly to the standard product metric when $t \in \mathbb{R}$ goes to $\pm \infty$. Working with a perturbative method, Secchi, [Sec01], obtained multiplicity results on $S^n \times \mathbb{R}$ under slightly different conditions; in particular his method demands that the metric is close to the standard metric. His approach applies as well if $S^n$ is substituted by another closed riemannian manifold. He, then, proved the existence of one or even two closed geodesics if there is a sufficiently nice closed geodesic with respect to the riemannian metric on the closed manifold.

The hypotheses of these results are very technical. More geometric conditions are imposed for example by sectional curvature bounds. However, if the open manifold has positive sectional curvature it is known that closed geodesics cannot exist, [GM69a]. Relaxing this condition, Thorbergsson proved the existence of a closed geodesic on every complete, noncontractible riemannian manifold whose sectional curvature is nonnegative outside a compact set, [Tho78, Thm. 4.3]. This is a corollary from the following more general theorem: Every complete riemannian manifold that contains a compact, convex set of nontrivial homotopy type contains a closed geodesic [Tho78, Thm. 4.2] (Gromoll and Meyer announced in [GM69b] the existence of infinitely many geodesics on a simply connected convex set with a loop space having unbounded Betti numbers but the promised article [GM] is unpublished).

Benci and Giannoni, [BG92], obtained an interesting result by use of penalized functionals. They proved the existence of at least one closed geodesic on every $m$-dimensional riemannian manifold $M$ with loop space $\Lambda M$ if 1) $H_q(\Lambda M; F) \neq 0$ for some $q > 2m$ and some field $F$ and 2) $\limsup_{d(x,x_0) \to \infty} K(x) \leq 0$ where $K(x)$ is the maximal sectional curvature at $x$ and $x_0 \in M$ is an arbitrary point. Note that this result does not apply to riemannian manifolds with nonpositive sectional curvature since in this case Hadamard-Cartan’s Theorem implies that the universal cover is diffeomorphic to $\mathbb{R}^m$ and hence every connected component of its loop space is contractible. This situation was analyzed by Eberlein and O’Neil, [EO73] and [Ebe72], if the riemannian manifold has a negative sectional curvature and is of arbitrary dimension.
Results and open problems:

In the first part of this thesis we give existence results for closed geodesics on complete, open manifolds with convex and concave ends. These are based on a Birkhoff type deformation (defined in section 4.2). If all ends are convex, iterations of Birkhoff’s curve shortening process move a closed curve either in one of the ends, to a closed geodesic, or to a point curve. Assuming that there exists no closed geodesic we deduce the existence of a map which is homotopic to the identity and deforms every closed curve into a convex neighborhood of one end or to a point curve. This argument generalizes easily to convex subsets of riemannian manifolds. In particular, it can be adapted to riemannian manifolds whose ends are all convex or concave. As a consequence, closed geodesics arise from topological obstructions to the existence of such a Birkhoff type deformation. We will present sufficient conditions for the existence of closed geodesics: In section 4.4 we deduce from the homology of the loop space strong obstructions for the existence of a riemannian metric without closed geodesics if at least two ends are convex and all the others concave. This induces existence results for several classes of manifolds with at least two convex ends. On riemannian manifolds with exactly one convex end we obtain existence results in section 4.3 from nontrivial relative homotopy groups. If the manifold has exactly one end, these arguments imply the existence of at least one closed geodesic. The method of proof is a relative version of Birkhoff’s minimax method and implies furthermore the existence of a closed geodesic on every riemannian manifold that contains a compact, concave, co-connected subset, i.e. a compact, concave subset whose complement is connected.

We now explain these results more in detail and indicate open questions that arise in their context. The attempt to generalize Bangert’s result, establishing the existence of closed geodesics on surfaces with finite area, to higher dimension uncovers the following problem: In two dimensions finite area implies the existence of arbitrarily small convex neighborhoods of each end. Bangert uses this property in [Ban80] to establish the existence of infinitely many closed geodesics on every plane, Möbius strip and cylinder of finite area using Birkhoff’s minimax method on ”short” homotopies. In higher dimensions finite volume does not imply convex ends as the following example of Bangert shows:

Consider a monotone, smooth, bijective function \( f : \mathbb{R} \to \mathbb{R} \) such that
2. Introduction

$t \mapsto \exp(f(t) + f(-t))$ is integrable. Then $\bar{g} = e^{2f(t)}dx_1^2 \oplus e^{2f(-t)}dx_2^2 \oplus dt^2$ is a complete Riemannian metric on $\mathbb{T}^2 \times \mathbb{R}$ with finite volume. On the other hand a careful consideration of the generators of the fundamental group shows that none of the ends has a convex neighborhood.

This points to two possible generalizations to higher dimension:

**Question 1.** If a complete riemannian manifold has finite volume, does there exist a closed geodesic on it?

**Question 2.** If a complete riemannian manifold has convex ends, does there exist a closed geodesic on it?

See [Ban85, p.59] and [BM13] for short discussions of the first question. In this thesis we focus on the second question. From the variational point of view this means that the ends act as generalized local minimizers of the energy functional. Slightly more generally we give partial results for the existence of closed geodesics on a complete riemannian manifold whose ends are all convex or concave.

Our first result states the following

**Theorem.** If $N$ is a complete riemannian manifold which contains a concave, compact and co-connected subset $K$, then there is a closed geodesic on $N$.

This result generalizes Lusternik and Fet’s theorem since the complement of every sufficiently small ball in a compact manifold yields a compact set as stipulated. The proof is based on a relative version of Birkhoff’s minimax method. We choose a nontrivial homotopy class of $N$ relative to the complement $U$ of $K$. Such a class exists since $K$ is a compact manifold with boundary and we may choose it of minimal degree. This class will give rise to a homotopy class in the loop space $\Lambda N$. Assume for simplicity that this class is a degree 2 homotopy class. We consider the disk, that constitutes the domain of a representative, as a one-parameter-family of concentric circles. In this manner we get a correspondence between relative homotopy classes of the pair $(N, U)$ and homotopy classes of maps from a closed interval into the loop space which map one endpoint to a point curve and the other to a curve in $U$. This type of homotopy classes in the loop space yields exactly the kind of obstructions that we need in order to detect closed geodesics; their higher dimensional analogs are known as
homotopy classes of the triad \((\Lambda N, \Lambda N_0, \Lambda U)\) where \(\Lambda N_0\) denotes the space of point curves in \(N\).

The above theorem implies the existence of a closed geodesic on every manifold with exactly one end, which is convex. The method applies also if we allow additional concave ends. Then we prove the existence of at least one closed geodesic if the manifold has at least three ends such that one end has arbitrarily small convex neighborhoods and the others arbitrarily small concave ones. The most interesting case arises when the manifold has exactly two ends. Then we have the following natural

**Question.** Let \(N\) be a complete riemannian manifold with exactly two ends such that one end has arbitrarily small convex neighborhoods and the other arbitrarily small concave ones. If there is no closed geodesic on \(N\), is then \(N\) a product of a closed manifold with \(\mathbb{R}\)?

We will give further comments on this question in section 4.3 and achieve the following partial result.

**Theorem.** Let \((\bar{N}, N_1, N_2)\) be a cobordism. Assume that the interior \(N\) of \(\bar{N}\) is equipped with a riemannian metric as in the question above where the end with convex neighborhoods corresponds to \(N_1\).

*If there exists no closed geodesic on \(N\), then the inclusion \(\iota : N_1 \hookrightarrow \bar{N}\) is a homotopy equivalence.*

In particular: If \(\bar{N}, N_1,\) and \(N_2\) are simply connected and the dimension of \(N\) is higher than 5, then \(N\) is diffeomorphic to the product \(N_1 \times (0,1)\).

The above method is not effective for complete riemannian manifolds with convex or concave ends such that at least two ends are convex. In this case a relative homotopy class favors by definition the connected component of the neighborhood of the convex ends that contains the base point. Hence the other convex ends remain uncontrolled and the lack of Palais-Smale-condition leads to the problems already discussed. To deal with this issue we choose another approach using homology. Therefor we choose a concave neighborhood of each of the concave ends and denote the complement of their union by \(M\). It still has, say \(n\), remaining ends. Let \(U_1, \ldots, U_n\) be convex neighborhoods of these ends. Then we obtain the following
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Theorem. Let \( k \) be an integer such that

1. \( H_{k-1}(M) = H_{k-2}(M) = 0 \), and
2. \( H_{k}(M, U_i) = H_{k-1}(M, U_i) = 0 \) for \( i = 1, \ldots, n \).

If the homomorphism

\[
\tau_* : \bigoplus_{i=1}^{n} H_{k-1}(\Lambda U_i) \to H_{k-1}(\Lambda M)
\]

that is induced by the inclusions \( \iota_i : \Lambda U_i \to \Lambda M \) via \( \tau_* \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (\iota_i \ast \alpha_i) \) is not an isomorphism, then there exists a closed geodesic on \( M \).

In particular, if for some \( k > \dim M + 2 \)

\[
\tau_* : \bigoplus_{i=1}^{n} H_{k-1}(\Lambda U_i) \to H_{k-1}(\Lambda M)
\]

is not an isomorphism, then there exists a closed geodesic on \( M \).

For an open manifold the obstructions imposed by the above theorem to the existence of a riemannian metric without a closed geodesic seem rather strong. On the other hand, the only examples of complete riemannian manifolds with convex or concave ends and without closed geodesics that are known to us are given by metrics of positive sectional curvature on \( \mathbb{R}^m \) and the warped product examples mentioned before. However, we cannot say if all complete riemannian manifolds with convex or concave ends that do not contain a closed geodesic are of one of these topological types. Here, we will prove the existence of a closed geodesic on every riemannian manifold \( N \) with convex collared ends if \( N \) can be obtained in one of the following two ways:

1. as complement of finitely many disjoint closed balls in a compact manifold
2. as complement of finitely many disjoint tubular neighborhoods of embedded submanifolds in a sphere if at least one of these submanifolds fulfills the Gromoll-Meyer condition.
In these examples the above homomorphism fails to be an isomorphism in infinitely many degrees. Hence it seems reasonable to pose the following question:

**Question.** Do there exist infinitely many closed geodesics on every riemannian manifold with convex collar ends of type 1. or 2.?
3. Preliminaries

In this chapter we introduce notation and recall well-known results which are required for chapter 4.

3.1. Basic definitions and notation from topology

For a topological space $X$ and subsets $A \subseteq B \subseteq X$ we denote by $\text{int}_B A$, $\text{clos}_B A$, $\partial_B A$ and $\text{compl}_B A$ the interior, closure, boundary and complement of $A$ in $B$ respectively. If $\text{compl}_B A$ is connected, we call $A \subseteq B$ co-connected. We omit the subscript where it is clear from context which set we refer to.

For positive $\varepsilon$ we denote the $\varepsilon$-neighborhood of the set $A$ by $B_\varepsilon(A)$ if $X$ is equipped with a metric.

If $(X,x_0)$ is a pointed topological space we denote for each $k \geq 0$ by $\pi_k(X,x_0)$ the $k$-th homotopy group of $(X,x_0)$. The pointed space $(X,x_0)$ is $k$-connected if its homotopy groups of order $\leq k$ vanish. Analogously we use the standard notation $\pi_k(X,A,x_0)$ for the $k$-th relative homotopy group of a pair $(X,A)$ of topological spaces with base point $x_0 \in A$. We emphasize that, in the lowest degree, these sets are not equipped with a group structure but provide only a natural choice of zero element. However, we will use the term ‘isomorphism’ in this context as well, meaning a bijective map that preserves the neutral element, and we adapt ‘exactness’ similarly. We call a pair $(X,A)$ $k$-connected if $\pi_i(X,A,x_0)$ is trivial for all $i \leq k$. Since we will work on path-connected spaces, we will usually omit the base point. If $f$ represents an element of a (relative) homotopy group, we denote the associated homotopy class by $[f]$. For a continuous (base point-preserving) function $g$ we denote the induced map on the associated homotopy groups by $g\#$.

Furthermore we denote by $H_k(X;G)$ and $H_k(X,A;G)$ the singular homology group with coefficients in an abelian group $G$ of a topological space.
3. Preliminaries

... and of a pair of topological spaces \((X, A)\) respectively. If \(G\) is given by the integers we abbreviate with \(H_k(X)\) and \(H_k(X, A)\). Given a continuous map the induced map on homology is indicated by the index \(*\).

3.2. Fibrations and topological properties of loop spaces

A continuous map \(p : E \to B\) between topological spaces is called a Serre fibration if \(p\) has the homotopy lifting property with respect to the collection of cubes \([0, 1]^n\), \(n \geq 0\), i.e. for every homotopy \(H : I^n \times [0, 1] \to B\) and every map \(f : I^n \to E\) such that \(H(\cdot, 0) = p \circ f\) there exists a homotopy \(\tilde{H} : I^n \times [0, 1] \to E\) such that \(p \circ \tilde{H} = H\). Then \(B\) is called the base space and \(E\) the total space of the fibration. For a point \(b \in B\) the fiber over \(b\) is given by \(F := F_b = p^{-1}(b)\).

Given a Serre fibration \(p : E \to B\) and an element \(e\) in the fiber \(F\) over \(b\), then the continuous map \(p\) induce bijections

\[ p_\#: \pi_n(E, F, e) \simeq \pi_n(B, b) \quad n \geq 1. \]

Furthermore if \(\bar{\partial} = \partial \circ p^{-1}_\#\) is the concatenation of the inverse of \(p_\#\) with the boundary operator of the homotopy sequence of the pair \((E, F)\) and \(\iota_\#\) is induced by the inclusion \(\iota : F \hookrightarrow E\), then the following sequence

\[ \ldots \to \pi_n(E, e) \xrightarrow{p_\#} \pi_n(B, b) \xrightarrow{\bar{\partial}} \pi_{n-1}(F, e) \xrightarrow{\iota_\#} \ldots \xrightarrow{p_\#} \pi_0(B, b) \]

is exact. This and further results on Serre fibrations can—for example—be found in Spanier’s book [Spa66, 7.2.9/7.2.10].

The concept of fibration and the above result can be generalized to pairs of topological spaces. For this purpose we recall that a triad is defined as a triple \((A; A_1, A_2)\) of topological spaces such that \(A_1 \cup A_2 \subseteq A\) and \(A_1 \cap A_2\) is nonempty. The definition of homotopy sets for triads and fundamental properties of these are due to Blakers and Massey [BM51]. We summarize those results that are necessary for the further work:

Let \((A; A_1, A_2)\) be a triad and \(x_0 \in A_1 \cap A_2\). For \(n \in \mathbb{N}, n \geq 2\), the homotopy set \(\pi_n(A; A_1, A_2; x_0)\) of \((A; A_1, A_2)\) with base point \(x_0\) is defined by the set of path components of the space of continuous maps from
3.2. Fibrations and topological properties of loop spaces

$(D^n; S^{n-1}_+, S^{n-1}_-; p_0)$ to $(A; A_1, A_2; x_0)$, i.e. of continuous maps from $D^n$ to $A$ such that the restriction to $S^{n-1}_+$ has image in $A_1$, the restriction to $S^{n-1}_-$ has image in $A_2$, and $p_0$ is mapped to $x_0$. Here and in the following we denote

\[
D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \\
S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \\
S^{n-1}_+ = \{ x \in S^{n-1} \mid x_n \geq 0 \} \\
S^{n-1}_- = \{ x \in S^{n-1} \mid x_n \leq 0 \} \\
p_0 = (1, 0, 0, \ldots, 0).
\]

For $n \geq 3$ the set $\pi_n(A; A_1, A_2; x_0)$ has a group structure with an operation similar to the one for the homotopy group of a pair. As for homotopy groups of topological spaces and pairs we will use for sake of convenience the group theoretic terminology also for $n = 2$ with the explained meaning. Blakers and Massey show that these groups are commutative if $n > 3$. Additionally, each path $\gamma : [0, 1] \to A_1 \cap A_2$ induces an isomorphism $\pi_n(A; A_1, A_2; \gamma(0)) \simeq \pi_n(A; A_1, A_2; \gamma(1))$ that is again similar to the isomorphism known from homotopy groups of pairs. Hence, we will omit the base point here as well if $A_1 \cap A_2$ is pathwise connected.

One important question is when a homotopy class of a triad is the zero class. In this regard we add the following

3.1 Lemma ([BM51] Lemma 4.71). Let $\beta$ be an $n$-th homotopy class of the triad $(A; A_1, A_2)$ and $f : (D^n; S^{n-1}_+, S^{n-1}_-) \to (A; A_1, A_2)$ a representative of $\beta$. Assume furthermore that $D^n$ can be decomposed into the union of two closed sets $C_1$ and $C_2$ such that

\[
S^{n-1}_+ \subseteq C_1 \subseteq f^{-1}(A_1) \\
S^{n-1}_- \subseteq C_2 \subseteq f^{-1}(A_2).
\]

Then $\beta$ is the zero class if the map $f' : (C_2, C_1 \cap C_2) \to (A_2, A_1 \cap A_2)$ induced by $f$ is deformable rel $C_1 \cap C_2$, i.e. if $f'$ is homotopic rel $C_1 \cap C_2$ to a map with image in $C_1 \cap C_2$. (If $C_1 \cap C_2$ is a subcomplex of $C_2$, the condition “rel $C_1 \cap C_2$” can be dropped).
3. Preliminaries

Now, we cite the following

3.2 Theorem ([BM51] Thm. 3.5.4). For a triad $(A; A_1, A_2)$ with base point $x_0 \in A_1 \cap A_2$ the following homotopy sequence is exact:

$$
\ldots \to \pi_n(A_1, A_1 \cap A_2, x_0) \xrightarrow{i_\#} \pi_n(A, A_2, x_0) \xrightarrow{j_\#} \\
\pi_n(A; A_1, A_2; x_0) \xrightarrow{\partial} \pi_{n-1}(A_1, A_1 \cap A_2, x_0) \xrightarrow{i_\#} \\
\ldots \to \pi_2(A; A_1, A_2; x_0) \xrightarrow{\partial} \pi_1(A_1, A_1 \cap A_2, x_0) \xrightarrow{i_\#} \pi_1(A, A_2, x_0).
$$

The boundary map $\partial$ is induced by the restriction of a representative defined on $D^n$ to the upper half sphere $S^{n-1}_+$. The other two maps are induced by the inclusions $i: (A_1, A_1 \cap A_2) \to (A, A_2)$ and $j: (A; x_0, A_2) \to (A; A_1, A_2)$ and the fact that $\pi_n(A; x_0, A_2; x_0) \simeq \pi_n(A, A_2, x_0)$.

3.3 Remark. The above result is true if we exchange the role of $A_1$ and $A_2$ and choose the adequate boundary operator. While in the above setting $\partial$ is a homomorphism for $n \geq 3$, in the reversed case the boundary operator is in general an antihomomorphism, i.e. if the boundary operator is applied to a sum it reverses the order of the summands.

The following result on Serre fibrations for pairs of topological spaces is due to Millett, cf. [Mil73, Prop. 2]. There he uses a slightly different terminology. We start with a definition:

For pairs of topological spaces $(E, E')$ and $(B, B')$ a map $p: (E, E') \to (B, B')$ of pairs, i.e. a map $p: E \to B$ such that $p(E') \subseteq B'$, is called a Serre fibration if both $p: E \to B$ and $p|_{E'}: E' \to B'$ are Serre fibrations.

3.4 Proposition. Let $(E, E')$ and $(B, B')$ be pairs of topological spaces and $p: (E, E') \to (B, B')$ a Serre fibration with fibers $\iota: (F, F') \hookrightarrow (E, E')$ over $b \in B'$. Then for $e \in F'$ the following holds:

1. For every $n \geq 2$ the map $p$ induces an isomorphism

$$
p_\#: \pi_n(E; E', F; e) \simeq \pi_n(B, B', b).
$$
2. The homotopy sequence

\[ \cdots \to \pi_n(F, F', e) \xrightarrow{\iota_\#} \pi_n(E, E', e) \xrightarrow{p_\#} \pi_n(B, B', b) \xrightarrow{\partial^*} \pi_{n-1}(F, F', e) \to \cdots \to \pi_1(E, E', e) \]

is exact for \( \partial^* = \partial \circ (p_\#)^{-1} \).

Now, we recall that for a path-connected topological space \( X \) with base point \( x \in X \) the path space \( PX = PX_x = \{ \gamma : [0,1] \to X \mid \gamma \text{ continuous and } \gamma(0) = x \} \) and the loop space \( \Lambda X = \{ \gamma : S^1 = \mathbb{R}/\mathbb{Z} \to X \mid \gamma \text{ continuous } \} \) equipped with the compact open topology induce Serre fibrations via \( \tilde{\pi} : PX \to X, \gamma \mapsto \gamma(1) \), and \( \pi : \Lambda X \to X, \gamma \mapsto \gamma(0) \). Since \( PX \) is contractible, the boundary operator in the homotopy long exact sequence of the fibration \( \tilde{\pi} \) induces isomorphisms \( \pi_{n+1}(X) \simeq \pi_n(\Omega X) \) for every \( n \geq 1 \) between the homotopy groups of the base space and those of the fiber \( \Omega X = \{ \gamma \in PX \mid \gamma(1) = x \} \).

Furthermore the embedding of \( X \) in \( \Lambda X \) as space of point curves \( \Lambda X_0 \) yields a section of \( \pi \). Hence the homotopy long exact sequence of the fibration \( \pi \) becomes

\[ \cdots \to \pi_{n+1}(\Lambda X) \xrightarrow{\pi_\#} \pi_{n+1}(X) \xrightarrow{0} \pi_{n}(\Omega X) \xrightarrow{\iota_\#} \pi_n(\Lambda X) \to \cdots \]

where we consider \( \Lambda X \) and \( \Omega X \) as pointed spaces whose base points are given by the constant curve \( c_x \) with image \( x \). Thus for every \( i \geq 1 \) the isomorphism \( \pi_n(\Omega X) \simeq \pi_{n+1}(X) \) induces an injective homomorphism \( \pi_{n+1}(X) \hookrightarrow \pi_n(\Lambda X) \) and actually an isomorphism for every \( n \leq k \) if \( X \) is \( k \)-connected. Furthermore the homotopy long exact sequence of the pair \( (\Lambda X, \Lambda X_0) \) yields—in this case—an isomorphism \( \pi_{k+1}(X) \simeq \pi_k(\Lambda X, \Lambda X_0) \).

These standard results can be found for example in [Ban85, 3 §1]. We now generalize them to a pair \( (X, A) \) of path-connected topological spaces with base point \( x \in A \). That will eventually cope with some of the non compactness issues we have to face:

\[ \pi_{n+1}(X, A, x) \simeq \pi_n(\Omega X, \Omega A, c_x) \text{ for each } n \geq 1. \]  \hspace{1cm} (3.1)

Now, we consider the free loop space: Again the inclusion of the base space as point curves is a section and therefore, in the long exact sequence of the
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fibration of pairs, \( \pi \# \) is surjective:

\[
\cdots \rightarrow \pi_{n+1}(\Lambda X, \Lambda A, c_x) \xrightarrow{\pi \#} \pi_{n+1}(X, A, x) \xrightarrow{0} \pi_n(\Omega X, \Omega A, c_x) \xrightarrow{i_\#} \cdots
\]

Hence it follows that \( i_\# : \pi_n(\Omega X, \Omega A, c_x) \rightarrow \pi_n(\Lambda X, \Lambda A, c_x) \) is injective and, if \( \pi_n(X, A, x) = 0 \), the inclusion yields actually a bijection \( \pi_n(\Omega X, \Omega A, c_x) \simeq \pi_n(\Lambda X, \Lambda A, c_x) \).

On the other hand, the homotopy sequence of the triad \( \Delta \Lambda = (\Lambda X; \Lambda X_0, \Lambda A; c_x) \)

\[
\cdots \rightarrow \pi_n(X, A, x) \xrightarrow{i_\#} \pi_n(\Lambda X, \Lambda A, c_x) \xrightarrow{j_\#} \pi_n(\Delta \Lambda) \xrightarrow{\partial} \pi_{n-1}(X, A, x) \rightarrow \cdots
\]

yields an isomorphism \( \pi_n(\Lambda X, \Lambda A, c_x) \simeq \pi_n(\Delta \Lambda) \) if \( \pi_n(X, A, x) = \pi_{n-1}(X, A, x) = 0 \).

Hence, when the pair \((X, A)\) is \( k \)-connected for some \( k \geq 2 \), we get an isomorphism \( \pi_{k+1}(X, A, x) \simeq \pi_k(\Delta \Lambda) \) using equation 3.1.

We finish this section by an explicit description of this isomorphism: An element \( \alpha \in \pi_{k+1}(X, A, x) \) can be represented by a map \( f : (D^k \times I, S^{k-1} \times I, D^k \times \{0, 1\} \cup S^{k-1} \times I) \rightarrow (X, A, x) \). Then \( \alpha \) will be mapped under the above isomorphism to the homotopy class \([\bar{f}]\) where

\[
\bar{f} : (D^k, S^{k-1}_+, S^{k-1}_- ; p_0) \rightarrow (\Lambda X; \Lambda X_0, \Lambda A; c_x), \quad \bar{f}(x)(t) = f(x, t).
\]

3.3. Manifolds with ends

An end \( \alpha \) of a manifold \( N \) is a map \( \alpha : \{K \mid K \subseteq N \text{ compact}\} \rightarrow \{U \mid U \subseteq N \text{ open}\} \) such that \( \alpha(K) \) is a connected component of the complement of \( K \) and \( \alpha(K') \subseteq \alpha(K) \) for every compact set \( K' \supseteq K \). A subset of \( N \) is a neighborhood of \( \alpha \) if it contains an open set which is the image of some compact set under \( \alpha \).

In the following we are particularly interested in a class of manifolds with especially nicely behaved ends, namely manifolds that are diffeomorphic to the interior of a smooth, compact manifold with boundary. Then, by Milnor’s collar neighborhood Theorem, [Mil65, Cor.3.5], each end has a collar neighborhood, i.e. a neighborhood which is diffeomorphic to \( B \times (0, 1) \) such that \( B \times \{0\} \) yields the corresponding boundary component.
Furthermore the fundamental group of such an end can be defined as the fundamental group of the manifold \( B \). Accordingly we will call an end \textit{simply connected} if the related manifold is simply connected. One should note at this point that the definition of fundamental group of an end given here coincides with the (intrinsic) definition given by Siebenmann in [Sie65, chap. 3], for a larger class of manifolds, called tame: When defined, it is given as inverse limit of any sequence of fundamental groups

\[ \pi_1(Y_1) \leftarrow \pi_1(Y_2) \leftarrow \ldots \]

that is defined by a sequence of open, connected neighborhoods \( Y_1 \supset Y_2 \supset \ldots \) of the given end such that \( \bigcap_{i=1}^{\infty} Y_i = \emptyset \) together with an arbitrary choice of base points and base paths.

Hence for each choice of a sequence of neighborhoods as above the inverse limit of the associated fundamental groups exists and is isomorphic to the fundamental group of the corresponding boundary component.

### 3.4. Convex sets

A subset \( S \) of a rieannian manifold \( N \) is called strongly convex if for each pair of points \( p, q \in S \) there exists a unique minimal segment in \( N \) connecting \( p \) with \( q \) and if this segment is completely contained in \( S \). Here and in the following a \textit{minimal segment} denotes a geodesic segment whose length coincides with the distance of its endpoints. The \textit{convexity radius} \( \text{conv}(p) \) of a point \( p \in N \) is a continuous function given by the supremum of all radii \( r > 0 \) such that each open ball in \( B_r(p) \) is strongly convex, cf. [GKM68, p.162]. Note that this implies that the \textit{injectivity radius} \( \text{inj}(p) \) at the point \( p \) is an upper bound for the convexity radius at \( p \). A subset \( C \) is called \textit{convex} if for each \( p \in \text{clos} C \) there exists some positive \( \varepsilon < \text{conv}(p) \) such that \( B_\varepsilon(p) \cap C \) is strongly convex. This definition of convexity is due to Cheeger and Gromoll, [CG72, p.417]. We will later need a slightly different kind of convexity radius which we introduce now: For a convex set \( C \) we define the \textit{convexity radius} \( \text{conv}_C(p) \) \textit{relative to} \( C \) for a point \( p \in C \) by \( \sup\{ \varepsilon \in (0, \text{conv}(p)) \mid B_\varepsilon(p) \cap C \text{ is strongly convex} \} \). Then, if the closure of a geodesic ball \( B \) is contained in \( B_{\text{conv}_C(p)}(p) \), the intersection \( B \cap C \) yields a strongly convex set. Hence, as for the “standard” convexity radius,
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cf. [GKM68, p.162], if $\text{conv}_C$ is infinite at one point its everywhere infinite and otherwise $|\text{conv}_C(p) - \text{conv}_C(q)| \leq d(p,q)$. This implies that $\text{conv}_C$ is continuous on $C$.

Now, we focus on the topological properties of convex sets. Cheeger and Gromoll proved that $\text{clos} \ C$ for a convex set $C$ is an embedded submanifold with smooth totally geodesic interior and (possibly nonsmooth) boundary, [CG72, Thm 1.6]. In [Wal74], Walter studied properties of the same kind of sets, calling them locally convex: In particular, he proved that every compact, convex subset $C$ of a smooth, connected Riemannian manifold is a strong deformation retract of $\text{clos} \ B_r(C)$ for every $r > 0$ small enough, [Wal74, Corollary 2]. The constructed deformation retract is actually locally lipschitz. Since his construction is performed in a small neighborhood of $\partial C$, the result is still true if we relax the compactness condition to that effect that only the boundary of $C$ is compact.

This will be of importance in the following because we will study riemannian manifolds whose ends have neighborhoods with the following convexity properties:

3.5 Definition. An end $\alpha$ of a riemannian manifold $N$ is called convex (concave), if $\alpha$ has a convex (concave) neighborhood of $\alpha$. We call the end $\alpha$ convex at infinity ($\infty$-convex) resp. concave at infinity ($\infty$-concave) if there exists a sequence of convex (concave) neighborhoods $U_i, i \in \mathbb{N}$, of $\alpha$ such that $\bigcap_{i=1}^{\infty} U_i = \emptyset$.

Note, that given a manifold $N$ such that each end has a collar $B \times (0,1)$, one can construct complete riemannian metrics with convex ends as follows: Choose $t \in (0,1)$, a small $\varepsilon > 0$ and a, monotonically increasing $C^2$-function $f : (t - \varepsilon, t + \varepsilon) \to \mathbb{R}_{>0}$. Then each riemannian metric on $N$ which coincides on a neighborhood of $B \times \{t\}$ with the warped product metric $f(s) \hat{g} \oplus ds^2$ for an arbitrary riemannian metric $\hat{g}$ on the closed manifold $B$ induces a convex neighborhood of the corresponding end. In particular $B \times (0,t)$ yields such a convex neighborhood as can be seen as follows: Let $p : B \times (0,1) \to (0,1)$ be the projection. Then with respect to the warped product metric its gradient equals the standard coordinate vector $\partial_s$ and has norm one. Hence $p$ is a distance function, cf. [Pet98, p.34], on a neighborhood $O$ of $B \times \{t\}$. Then it follows that on $O$ the gradient of $p$ lies in the kernel of the hessian.
3.5. Loop spaces of riemannian manifolds and closed geodesics

\[ \nabla^2 p \text{ of } p \text{ and tangent to } N, \text{ we have (cf. } [BR12, \text{ sec. } 2]) \]

\[ \nabla^2 p = \frac{1}{2} f' \cdot \tilde{g}. \]

This implies the (local) convexity property of the sublevel set \( B \times (0, t) \) that we need. Similarly one can construct concave ends.

3.5. Loop spaces of riemannian manifolds and closed geodesics

We consider the (free) loop space \( \Lambda N \) of a riemannian manifold \( N \) as a metric space with metric

\[ d(\gamma_1, \gamma_2) = \max_{t \in S^1} d_N(\gamma_1(t), \gamma_2(t)) \]

where \( d_N : N \times N \to \mathbb{R}_{\geq 0} \) is the distance induced by the riemannian metric on \( N \). Note that the metric \( d \) induces the compact-open topology.

We point out that loop spaces of manifolds have the homotopy type of a CW-complex, [Mil59]. Moreover, this is true for loop spaces of spaces that have the homotopy type of a CW-complex. Hence, it holds also for the loop space of every convex subset of a riemannian manifold since Haver proved that every countable union of compact, finite dimensional, locally contractible metric spaces is an absolute neighborhood retract, [Hav73], and hence has the homotopy type of a CW-complex, [Mil59].

In the following, we are mainly interested in special elements of the loop space, namely closed geodesics. We emphasize that by a closed geodesic we mean a periodic geodesic of positive length respectively energy. Hence they are the nontrivial critical points of the energy functional. On the subspace \( \Lambda N^\infty \subseteq \Lambda N \) of piecewise smooth curves the length \( L : \Lambda N^\infty \to \mathbb{R} \) and energy functional \( E : \Lambda N^\infty \to \mathbb{R} \) are given as follows

\[ L(\gamma) = \int_0^1 |\dot{\gamma}| \, dt, \quad E(\gamma) = \int_0^1 |\dot{\gamma}|^2 \, dt. \]

Observe that we use an energy functional without the constant factor 1/2.
Hence by Cauchy-Schwarz we obtain for $\gamma \in \Lambda N^{\infty}$ the following inequality
\[ L(\gamma)^2 \leq E(\gamma). \tag{3.2} \]

We introduce on $\Lambda N^{\infty}$ the metric
\[ d^{\infty}(\gamma, \tilde{\gamma}) = d(\gamma, \tilde{\gamma}) + \sqrt{\int_0^1 (|\dot{\gamma}| - |\dot{\tilde{\gamma}}|)^2 \, dt}. \]

Then, with respect to the induced topology the length and energy functional become continuous. Thus from the point of view of calculus of variation we want to restrict to this space on which the energy functional is well-behaved (even differentiable but we won’t use this explicitly) when we study the existence of closed geodesics. On the other hand—by methods from algebraic topology—one can relate the topology of $\Lambda N$ to that of $N$ as seen in section 3.2. The following theorem closes the resulting gap.

**3.6 Theorem.** The inclusion $\iota : (\Lambda N^{\infty}, d^{\infty}) \to (\Lambda N, d)$ induces a homotopy equivalence.

The homotopy inverse and the associated homotopies can be chosen stationary on $\Lambda N_0$. Furthermore if $U$ is a convex subset of $M$, there is a homotopy inverse of $\iota$ that leaves $\Lambda U$ invariant, i.e. which makes the restriction $\iota|_{\Lambda U^{\infty}} : (\Lambda U^{\infty}, d^{\infty}) \to (\Lambda U, d)$ a homotopy equivalence.

The main statement is proven by Milnor for the space of curves with fixed endpoints, [Mil63, 17.1]. The proof that we give follows the lines of Milnor’s argument with the exception that we get an additional condition on $U$ involving the convexity radius:

The first step is the construction of a continuous function $p : \Lambda N \to (0, 1]$ such that each segment of $\gamma \in \Lambda N$ whose domain has length smaller than $p(\gamma)$ (Here, Milnor considers $2p(\gamma)$ instead which seems unnecessary for our purpose) can be joined by a unique minimal segment which lies entirely in $U$ if its endpoints do:

Choose a continuous function $f : N \to (0, \infty)$ that is bounded from above by the injectivity radius and fulfills $f < \text{conv}_U$ on $U$. Then the induced map
\[ g : \Lambda N \to (0, \infty), \quad \gamma \mapsto \min_{t \in S^1} f(\gamma(t)) \]
3.6. The Birkhoff curve shortening process on a compact subset

is continuous and any pair of points of $\gamma(S^1)$ with distance $\leq g(\gamma)$ can be joined by a unique minimal segment that lies furthermore in $U$ if this is true for its endpoints.

Now, the important step follows which produces a condition on the domain that controls the distance in the image. For this purpose Milnor defines a continuous function as follows

$$F : \Lambda N \times [0, 1] \to \mathbb{R}, \quad F(\gamma, s) = (s - 1)g(\gamma) + \max_{|t-t'|<s} d_N(\gamma(t), \gamma(t'))$$

where $|t-t'| < s$ must be interpreted in our setting as the minimal distance of $t, t'$ in the sphere $S^1$ of length 1.

Since $s \mapsto F(\gamma, s)$ is strictly monotonically increasing for each $\gamma$ and $F(\gamma, 0) < 0 \leq F(\gamma, 1)$, the equation

$$F(\gamma, p(\gamma)) = 0 \quad (3.3)$$

defines a function $p$ as claimed above:

First we prove continuity of $p$. For a converging sequence $\gamma_i \to \gamma$ each limit point of the sequence $p(\gamma_i)$ must, by continuity of $F$, coincide with a solution of the equation (3.3) for $\gamma$. By uniqueness this shows that $(p(\gamma_i))$ converges and that its limit is $p(\gamma)$.

Furthermore, for a segment of $\gamma$ whose domain has length $s < p(\gamma)$ the distance of its endpoints can be estimated by $(1 - s)g(\gamma) < g(\gamma)$ which proves the second property of $p$.

Thus, a homotopy inverse $r$ of $\iota$ can be defined as follows:

For each $\gamma$ choose $k$ as the largest integer such that $k \cdot p(\gamma) < 1$ and $r(\gamma)$ as the broken geodesic loop which coincides at $j \cdot p(\gamma) \in \mathbb{R}/\mathbb{Z}$ for $j \in \{0, \ldots, k\}$ with $\gamma$ and has no further corners.

Then a homotopy to the identity can be defined similarly to the construction at the beginning of the following section. This proves the claim.

3.6. The Birkhoff curve shortening process on a compact subset

The Birkhoff curve shortening process partitions curves in small segments and substitutes these by minimal segments. On a closed manifold the
division can be chosen uniformly for all curves with common finite energy bound, cf. [Bir17]. This yields a continuous energy-non-increasing deformation and an analog construction works in the non compact case if we consider only curves that are contained in a compact subset; refer to Thorbergsson [Tho77] for a similar construction:

Therefore we denote by $M$ a closed, convex subset of a riemannian manifold with non-empty interior and by $K_1 \subseteq \text{int}(K_2) \subseteq K_2 \subseteq M$ two compact sets. We consider the convex subset $M$ instead of the whole manifold in order to make the following construction applicable to riemannian manifolds with concave ends later on. We choose $\varepsilon \in \mathbb{R}$ positive, denote $\ell = \sqrt{\varepsilon}$, $\Lambda M = \{ \gamma \in \Lambda M^\infty | E(\gamma) < \varepsilon \}$ and extend this notation in the obvious way. Hence by equation (3.2) each curve in $\Lambda M^\varepsilon$ has length less that $\ell$. Let $\delta > 0$ fulfill $\delta < \frac{1}{2} \min \{ \text{conv}_M(K_2), d(K_1, \partial K_2) \}$. We will subdivide $S^1$ in subintervals of length $\frac{2}{k}$ where $k > \frac{2\varepsilon}{\delta^2}$ is a fixed even natural number. Then the choice of $k$ implies for each segment $\gamma_{|[t_1,t_2]} \subseteq K_2$ with $\gamma \in \Lambda M^\varepsilon$ and $|t_1 - t_2| \leq \frac{2}{k}$ the following estimate:

$$d(\gamma(t_1), \gamma(t_2)) \leq L(\gamma_{|[t_1,t_2]}) \leq \sqrt{E(\gamma_{|[t_1,t_2]})} \cdot |t_2 - t_1| < \sqrt{\frac{2\varepsilon}{k}} \leq \delta \quad (3.4)$$

Hence there exists up to reparametrization a unique minimal segment connecting $\gamma(t_1)$ with $\gamma(t_2)$ which lies in $M$. In particular, if $\gamma \in \Lambda K_1^\delta$ this curve is completely contained in $B_\delta(K_1) \subseteq K_2$. We will denote this minimal segment by $c_{\gamma|_{[t_1,t_2]}} : [t_1, t_2] \to M$ when the domain coincides with the original one of $\gamma$.

The following exposition follows Klingenberg [Kli95, section 3.7] with the difference that we consider $\Lambda M^\infty$ with the topology induced by $d^\infty$ as defined in section 3.5.

Let $D_0$ be defined by $\text{id} |_{\Lambda M^\varepsilon}$. For every $j \in \{0, 2, \ldots, k-2\}$ and $\sigma \in (0, \frac{2}{k}]$ we denote by $I_{j, \sigma}$ the projection of $[\frac{j}{k}, \frac{j}{k} + \sigma]$ onto $S^1$. Then for $\gamma \in \Lambda K_2^\delta$ we define

\[ D_{\frac{j}{k} + \sigma} \gamma = \begin{cases} \gamma & \text{on } S^1 \setminus I_{j, \sigma} \\ c_{\gamma|_{I_{j, \sigma}}} & \text{on } I_{j, \sigma} \end{cases} \]
3.6. The Birkhoff curve shortening process on a compact subset

and

\[ D_{1+\ell+\sigma}^+ = \begin{cases} 
\gamma & \text{on } S^1 \setminus I_{j+1,\sigma} \\
\sigma_{I_{j+1,\sigma}} & \text{on } I_{j+1,\sigma}. 
\end{cases} \]

For \( \gamma \in \Lambda K_1^* \) we denote by \( D(s, \cdot), s \in [0, 2] \), the composition \( D_{s} \circ D_{2} \circ \ldots \circ D_{2} \) where \( l \in \mathbb{N} \) is the maximal natural number with \( 2l \leq k \) and we abbreviate \( D(2, \cdot) \) with \( D(\cdot) \). These deformations are well-defined since by equation 3.4 the vertices of each geodesic segment lie in \( K_2 \).

We start with some properties of \( D \):

3.7 Lemma. For every \( \gamma \in \Lambda K_1^* \) the map \( v_\gamma : [0, 2] \to \mathbb{R}, t \mapsto E \circ D(t, \gamma) \), is monotonically decreasing. In addition, for \( \gamma \in \Lambda K_1^* \) it yields, that \( v_\gamma \) is constant if and only if it is a closed geodesic or a point curve.

The proof is literally the same as for cf. [Kli95, 3.7.4].

3.8 Lemma. The map \( D : [0, 2] \times \Lambda K_1^* \to \Lambda M^\epsilon \) is continuous.

Proof. From Lemma 3.7.5 in [Kli95] it follows that \( D \) is continuous with respect to the topology induced by \( d \). As Klingenberg we make use of the fact that the involved spaces are metric spaces. Hence it suffices to prove that converging sequences are mapped to converging sequences. Therefore we choose \( \gamma_m \to \gamma \) in \( \Lambda K_1^* \) and \( \sigma_m \to \sigma \) in \([0, 1]\) and denote \( c_m = D(\sigma_m, \gamma_m) \) and \( c = D(\sigma, \gamma) \). It remains to show that the second term of the \( d^\infty \)-metric

\[ \int_0^1 (|\dot{c}_m| - |\dot{c}|)^2 \, dt \]

converges to zero when \( m \) goes to infinity. First, we obtain from Minkowski’s inequality for \( \tilde{c}_m = D(\sigma_m, \gamma) \):

\[ \int_0^1 (|\dot{c}_m| - |\dot{c}|)^2 \, dt \leq \int_0^1 (|\dot{c}_m| - |\tilde{c}_m|)^2 \, dt + \int_0^1 (|\tilde{c}_m| - |\dot{c}|)^2 \, dt. \]

Then it is easy to see that both summands converge to 0 for \( m \to \infty \). □

3.9 Lemma. For every sequence \( (\gamma_m) \) of curves in \( \Lambda K_1^* \) such that the sequences \( (E(\gamma_m)) \) and \( (E(D\gamma_m)) \) converge both and to the same limit \( \kappa > 0 \) there is a subsequence that converges to a closed geodesic \( \gamma \) with energy \( E(\gamma) = \kappa \).
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The proof is literally the same as for Theorem 3.7.6 in [Kli95] since for \( \gamma \in \Lambda K_1^\epsilon \) the curve \( D\gamma \) lies in a fixed compact neighborhood of \( K_1 \). We add, deviating from Klingenberg’s exposition, the following basic corollary, which is implicitly contained in the proof of Theorem 3.2 in [Tho77].

3.10 Corollary. Let \( K \subseteq M \) be compact and \( \beta \in \pi_1(M) \) not trivial such that each loop \( \gamma \) with \( \gamma \in \beta \) has nontrivial intersection with \( K \). Then there exists a closed geodesic in \( M \).

Proof. Let \( \gamma \in \Lambda M^\epsilon \) be a representative of \( \beta \) of length \( l \) and \( D \) a deformation as defined above for \( K_1 = B_l(K) \). Then \( (D^n\gamma) \) is a well-defined sequence of representatives of \( \beta \), since \( D \) does not change the homotopy class of a curve and hence the intersection condition assures that \( D^n\gamma \subseteq K_1 \) for every \( n \). Since \( D \) is energy non-increasing, we find a subsequence such that \( \lim_{n \to \infty} E(D^n\gamma) = \lim_{n \to \infty} E \circ D(D^n\gamma) = \kappa \). Here it yields \( \kappa > 0 \) since \( \beta \) is nontrivial. By Lemma 3.9 this implies the existence of a closed geodesic. \( \square \)

We finish this section with the following lemma. It shows that the deformation \( D \) decreases energy uniformly in the absence of closed geodesics:

3.11 Lemma. Let \( \kappa \in (0, \epsilon] \) and \( V \) be an open neighborhood of \( C = \{ \gamma \in \Lambda M^\epsilon \mid \gamma \text{ closed geodesic with } E(\gamma) = \kappa \} \) that contains \( \{ \gamma \in \Lambda M^\epsilon \mid \gamma \nsubseteq K_1 \} \). Then there exists an \( \delta = \delta(V, \kappa) > 0 \) such that

\[
D(\Lambda K_1^{\kappa+\delta} \cap \Lambda M^\epsilon) \subseteq V \cup \Lambda K_1^{\kappa-\delta}.
\]

Proof. Since \( D|_C = \text{id}_C \) there is an open neighborhood \( V' \subseteq V \) of \( C \) such that \( D(V') \subseteq V \). If we assume that no such \( \epsilon \) exists, we can find a sequence \( \gamma_m \) in \( \Lambda K_1^\epsilon \setminus V' \) such that

\[
\kappa + \frac{1}{m} \geq E(\gamma_m) \geq E(D(\gamma_m)) > \kappa - \frac{1}{m}.
\]

Now, it follows from Lemma 3.9 that \( (\gamma_m) \) has a convergent subsequence whose limit is contained in \( C \). This yields a contradiction. \( \square \)
4. Closed geodesics on open manifolds with convex and concave ends

4.1. Notation

Throughout the following chapter we will denote by \((N, g)\) a complete, connected, smooth Riemannian manifold of dimension \(m \geq 3\) such that \(N\) is diffeomorphic to the interior of a compact manifold \(\bar{N}\) with boundary \(\partial \bar{N} \neq \emptyset\). We denote the connected components of \(\partial \bar{N}\) by \(N_1, \ldots, N_n\). Then \(N\) has \(n\) ends \(\alpha_1, \ldots, \alpha_n\) which are numbered in order to match under the given diffeomorphism the index of the corresponding boundary component. By \(M\) we will denote a closed, connected, convex subset of \(N\) with nonempty interior and compact boundary \(\partial M = \partial_N M\). Later on, it will be given by the complement of a concave neighborhood of the concave ends of \(N\).

4.2. A Birkhoff type deformation

We assume throughout the following section that there exists a compact, concave set \(K\) in \(M\) but no closed geodesic. We denote by \(U\) the closure of \(\text{compl}_M K\).

Our goal is to define, for arbitrary \(\epsilon > 0\), a continuous map \(D : \Lambda M^\epsilon \to \Lambda M^\epsilon\) that is homotopic to the identity and maps every curve to a point curve or to a curve outside \(K\).

The construction is as follows: We start with a map constructed as in section 3.6 defined on curves that lie completely in a compact neighborhood of \(K\) and extend it to a deformation on \(\Lambda M^\epsilon\) by a cut off function. In the next step we show that we can reduce uniformly the energy of curves that
4. Closed geodesics on open manifolds with convex and concave ends

do not leave $K$ in finitely many steps to an arbitrarily small amount. Then we deform these short curves to point curves.

Therefore we choose $\epsilon > 0$, denote $\ell = \sqrt{\epsilon}$ and $K_\nu = \text{clos} B_{\nu \ell}(K)$, $\nu = 1, 2$. Let $D : [0, 2] \times \Lambda K_2 \to \Lambda M^\epsilon$ denote a map defined as in section 3.6 for $K_1 = K_2$, $K_2$ a compact neighborhood of $K_2$ and $\delta > 0$ so small that $\text{conv}_U > \delta$ on $K_2 \cap U$. Using an Urysohn’s function we get a continuous, energy-nonincreasing map $\bar{D} : [0, 2] \times \Lambda M^\epsilon \to \Lambda M^\epsilon$ which coincides with $D$ on $[0, 2] \times \Lambda K_1$ and with the identity where $D$ was not defined. We denote $\bar{D}(2, \cdot)$ by $\bar{D}$ and deduce from Lemma 3.11 the following

4.1 Corollary. Let $\kappa \in (0, \epsilon]$ and $V$ be an open subset of $\Lambda M^\epsilon$ containing $\{\gamma \in \Lambda M^\epsilon | \gamma \not\subseteq K_1\}$. Then there exists a $\mu = \mu(V, \kappa) > 0$ such that

$$\bar{D}(\Lambda K_1^{\kappa + \mu} \cap \Lambda M^\epsilon) \subseteq V \cup \Lambda K_1^{\kappa - \mu}.$$ 

Since $K$ is concave, $\Lambda U^\epsilon$ is invariant under the map $\bar{D}$. Thus it follows:

4.2 Proposition. For every $\eta > 0$ there exists an $\chi = \chi(\eta) \in \mathbb{N}$ such that

$$\bar{D}^\chi(\Lambda M^\epsilon) \subseteq \Lambda M^\eta \cup \Lambda U^\epsilon.$$ 

Proof. We denote by $U$ the open set $\{\gamma \in \Lambda M^\epsilon | \gamma \cap K = \emptyset\} \subseteq \Lambda U^\epsilon$. Then $U \cup \Lambda K_1^\epsilon = \Lambda M^\epsilon$ by the choice of $\ell$ and furthermore $U$ is invariant under $\bar{D}$. Hence it suffices to find for every $\eta > 0$ an $\chi \in \mathbb{N}$ such that $\bar{D}^\chi(\Lambda K_1^\kappa) \subseteq \Lambda M^\eta \cup U$. For this purpose we choose

$$\kappa := \inf\{\eta > 0 | \forall \eta' > \eta \exists \chi \in \mathbb{N} : \bar{D}^\chi(\Lambda K_1^\kappa) \subseteq \Lambda M^{\eta'} \cup U\}.$$ 

Since $U$ is open it follows from Corollary 4.1 that $\kappa < \epsilon$ and that, if $\kappa > 0$, there is some $\mu > 0$ such that

$$\bar{D}(\Lambda K_1^{\kappa + \mu} \cap \Lambda M^\epsilon) \subseteq \Lambda K_1^{\kappa - \mu} \cup U.$$ 

But this implies for $\chi = \chi(\kappa + \mu)$ and every $\eta' > \kappa - \mu$:

$$\bar{D}^{\chi + 1}(\Lambda K_1^\kappa) \subseteq \Lambda M^{\eta'} \cup U$$

in contradiction to the definition of $\kappa$. \hfill \square

Now we define a map that deforms sufficiently short curves to point
4.2. A Birkhoff type deformation

curves: Choose a continuous function $f$ on $M$ that is bounded from above by half the injectivity radius and additionally on $U$ by $\frac{1}{2} \text{conv}_U$. Then $\mathcal{B} = \{ \gamma \in \Lambda M^\epsilon \mid L(\gamma) < f(\gamma(0)) \}$ is an open subset that deformation retracts to $\Lambda M_0$ via

$$d : [0, 1] \times \mathcal{B} \to \mathcal{B}, \quad d(s, \gamma) = \begin{cases} \gamma_{[0,s]} & \text{on } [0, s] \\ \gamma_{[s,1]} & \text{on } [s, 1] \end{cases}$$

(4.1)

Note that $d(s, \Lambda \mathcal{U}_0 \cap \mathcal{B}) \subseteq \Lambda \mathcal{U}_0 \cap \mathcal{B}$ for every $s \in [0, 1]$. Hence using an Urysohn's function we find a homotopy $\bar{d} : [0, 1] \times \Lambda M^\epsilon \to \Lambda M^\epsilon$ which is stationary on point curves and has the following properties:

1. $\bar{d}(0, \cdot) = \text{id}_{\Lambda M^\epsilon}$,
2. $\bar{d}(s, \Lambda \bar{U}^\epsilon) \subseteq \Lambda \bar{U}^\epsilon$ for each connected component $\bar{U}$ of $U$ and $s \in [0, 1]$
3. $\bar{d}(1, \bar{\mathcal{B}}) \subseteq \Lambda M_0$ for $\bar{\mathcal{B}} = \{ \gamma \in \Lambda M^\epsilon \mid L(\gamma) < \frac{1}{2}f(\gamma(0)) \}$

Then, by Proposition 4.2 there exists $\chi \in \mathbb{N}$ such that $\bar{D}^\chi(\Lambda M^\epsilon) \subseteq \bar{\mathcal{B}} \cup \Lambda \mathcal{U}^\epsilon$ and we define $D : \Lambda M^\epsilon \to \Lambda M^\epsilon$ by $D(\gamma) = \bar{d}(1, \bar{D}^\chi(\gamma))$.

We summarize the properties of $D$ in the following

4.3 Theorem. Let $M$ contain a compact, concave subset $K$ but no closed geodesic. We denote the closure of $M \setminus K$ by $U$. Then, for every $\epsilon > 0$, there exists a continuous map $D = D(\epsilon) : \Lambda M^\epsilon \to \Lambda M^\epsilon$ with the following properties:

1. $D(\Lambda M^\epsilon) \subseteq \Lambda M_0 \cup \Lambda \mathcal{U}^\epsilon$.
2. $D(\Lambda \bar{U}^\epsilon) \subseteq \Lambda \bar{U}^\epsilon$ for each connected component $\bar{U}$ of $U$.
3. $D$ is homotopic to the identity $\text{id}_{\Lambda M^\epsilon}$ by a homotopy $H : [0, 1] \times \Lambda M^\epsilon \to \Lambda M^\epsilon$ such that it yields $H(t, \gamma) = \gamma$ if $\gamma \in \Lambda M_0$ and $H(t, \gamma) \in \Lambda \mathcal{U}^\epsilon$ if $\gamma \in \Lambda \mathcal{U}^\epsilon$.
4. $D^{-1}(\Lambda M_0)$ is a neighborhood of $D^{-1}(\Lambda K_0)$.

Proof. We choose $D$ as constructed above and use the same notation. Then properties 1., 2., and 3. follow easily from the construction. For the proof of the last property we observe that $(\bar{d}|_{\bar{\mathcal{B}} \cup \Lambda \mathcal{U}^\epsilon})^{-1}(\Lambda K_0) \subseteq \bar{\mathcal{B}} \subseteq (\bar{d}|_{\bar{\mathcal{B}} \cup \Lambda \mathcal{U}^\epsilon})^{-1}(\Lambda M_0)$. Hence $O = (\bar{D}^\chi)^{-1}(\bar{\mathcal{B}})$ is an open neighborhood of $D^{-1}(\Lambda K_0)$ contained in $D^{-1}(\Lambda M_0)$. \qed
4. Closed geodesics on open manifolds with convex and concave ends

4.4 Remark. For the construction of $D$ it suffices that $K_1 = B_\ell(K)$ contains no closed geodesic. Since in the following application we do not have a natural upper bound for $\ell = \sqrt{\epsilon}$, this will not strengthen our statement.

4.3. Concave, compact, co-connected subsets imply the existence of closed geodesics

Here, we study the existence of closed geodesics on non compact riemannian manifolds with a compact, concave, co-connected subset. The method we use is a relative version of Lusternik and Fet’s proof of the existence of closed geodesics on every compact Riemannian manifold, cf. [LF51]. In this way we obtain the following

4.5 Theorem. If $N$ contains a concave, compact and co-connected subset $K$, then there is a closed geodesic on $N$.

4.6 Remark. Montezuma recently proved that the existence of a compact, concave subset $K$ implies the existence of a minimal hypersurface that intersects $K$, [Mon14]. If the manifold is not closed he imposes further asymptotical conditions on the geometry but does not need co-connectedness for $K$. This is due to a different approach using sweep-outs of the manifold that are defined by level sets of certain Morse-functions. However, his ideas do not generalize to codimension higher than one.

If we combine the above result with Thorbergsson’s theorem on the existence of closed geodesics on convex subsets of nontrivial homotopy type, [Tho78, 4.2], we obtain the following

4.7 Corollary. Assume that $N$ has exactly one end $\alpha$. Then there is a closed geodesic in $N$ if $\alpha$ is convex or if $N$ is noncontractible and $\alpha$ is $\infty$-concave.
4.8 Remark.

1. Theorem 4.5 and Corollary 4.7 do not rely on the fact that $N$ is diffeomorphic to the interior of a compact manifold but hold without this assumption.

2. The euclidean $\mathbb{R}^n$, $n \geq 2$, together with an exhaustion by compact balls is an obvious example of a contractible Riemannian manifold with exactly one $\infty$-concave end that does not contain a closed geodesic. Furthermore, Gromoll and Meyer showed that every complete open manifold of positive curvature yields a counterexample: In [GM69a] they proved that these manifolds are contractible, actually diffeomorphic to $\mathbb{R}^n$, the exponential map is proper and they construct a filtration $C_i$ of $N$, $C_i \subseteq \text{int} C_{i+1}$, $\bigcup_{i=1}^{\infty} C_i = N$, by compact and (totally) convex sets $C_i$. This shows that the noncontractibility condition for $N$ cannot be omitted in Corollary 4.7 if the only end is $\infty$-concave.

3. If the manifold has more than one end, then there exists a closed geodesic provided all ends are $\infty$-concave. This is due to the fact that contractible manifolds of dimension higher than one have only one end, [Sta62, Prop. 2·3].

Proof of Corollary 4.7. First we assume that $\alpha$ is convex. Hence it has an open convex neighborhood which can be chosen connected. Thus, from Theorem 4.5, it follows that there exists a closed geodesic on $N$. Otherwise $N$ is not contractible and $\alpha$ is $\infty$-concave. As above this yields an exhaustion by convex, connected, compact sets since $N$ has only one end. Furthermore $N$ is noncontractible. Thus there is some homotopy class of $N$ that does not vanish and we can realize this homotopy class in some compact subset of the given exhaustion. Now, the statement follows from [Tho78, 4.2].

The method we present applies also to the case where $K$ is not co-connected but its complement has exactly one convex component and all the others are concave. This implies the following:

4.9 Theorem. Assume that $N$ has exactly one $\infty$-convex end $\alpha_1$ and that its other ends $\alpha_2, \ldots, \alpha_\ell$ are $\infty$-concave.

If there exists no closed geodesic on $N$, then the inclusion $\iota : N_1 \hookrightarrow \bar{N}$ is a homotopy equivalence.
Therefrom we deduce, depending on the number of ends, the following two corollaries:

4.10 Corollary. Suppose $N$ has at least three ends. Assume furthermore that one end is $\infty$-convex and all the others $\infty$-concave. Then there exists a closed geodesic on $N$.

Proof of Corollary 4.10. Arguing by contradiction, we assume that $N$ contains no closed geodesic. Then Theorem 4.9 implies for the $\infty$-convex end, say $\alpha_1$, that the inclusion $\iota : N_1 \hookrightarrow \tilde{N}$ is a homotopy equivalence. Hence, the relative homotopy groups $\pi_k(\tilde{N}, N_1)$, $k \geq 0$, vanish. Now, by the relative Hurewicz Theorem ( [SZ88, 16.8.2]) also the relative homology $H_*(\tilde{N}, N_1)$ is trivial and therefore it holds $H^*(\tilde{N}, \bigcup_{i=2}^n N_i) = 0$ by the Poincaré-Lefschetz-Duality Theorem ( [Ran02, Thm. 4.8]). But then in particular $H_1(\tilde{N}, \bigcup_{i=2}^n N_i) = 0$ which yields a contradiction since $n \geq 2$.

In the case of two ends the situation is different: It is easy to find examples of complete riemannian manifolds with two ends, one $\infty$-convex and the other $\infty$-concave, that contain no closed geodesic, using a warped product construction, see Benci-Giannoni [BG92] and section 3.4. Here, existence results for closed geodesics depend on more subtle topological arguments.

4.11 Corollary. Suppose $N$ has exactly two ends, an $\infty$-convex end $\alpha_1$ and an $\infty$-concave end $\alpha_2$. Additionally, assume that $\tilde{N}, N_1, N_2$ are simply connected and that $\dim \tilde{N} > 5$. If there exists no closed geodesic on $N$, then $(\tilde{N}, N_1, N_2)$ is a trivial $h$-cobordism and hence $\tilde{N}$ diffeomorphic to the product $N_1 \times (0, 1)$.

From Theorem 4.9 it follows that $(\tilde{N}, N_1, N_2)$ is a “semi” $h$-cobordism. Hence, if $N, N_1, N_2$ are simply connected, the $h$-cobordism theorem [Mil65, 9.2] implies that $\tilde{N}$ is diffeomorphic to $N_1 \times [0, 1]$. By the $s$-cobordism Theorem the conclusion of Corollary 4.11 is still valid if we relax the conditions in the following way: Instead of asking $\tilde{N}, N_1,$ and $N_2$ to be simply connected we stipulate that $\iota : N_2 \hookrightarrow \tilde{N}$ is a homotopy equivalence and the Whitehead torsion $\tau(\tilde{N}, N_2) \in \text{Wh}(\pi_1(N_2))$ vanishes, cf. for example to [Ker65]. It is an interesting question if the non existence of a closed geodesic actually implies one of these conditions:

Does it imply—for example—that $\iota : N_1 \to \tilde{N}$ is a simple homotopy equivalence, i.e. that the Whitehead torsion vanishes? By the $s$-cobordism
Theorem this implies that $\tau(\bar{N}, N_2)$ vanishes if $\iota : N_2 \hookrightarrow \bar{N}$ is a homotopy equivalence. The above question is motivated by the following deliberation: One and hence both boundaries embed exactly by simple homotopy equivalences if there exists an adapted Morse function on the cobordism without critical points. Otherwise the h-cobordism Theorem implies, at least in dimension $> 5$, that the manifolds are not simply connected. Hence the loop space has a connected component of non contractible curves. Now our question becomes the following: Can we deduce that the energy functional on this component of the loop space has a critical point, i.e. that there exists a noncontractible closed geodesic, from the fact that every differentiable function on the manifold has a critical point?

The other case seems less promising. If $\iota : N_2 \hookrightarrow \bar{N}$ is not a homotopy equivalence there are examples of so called semi s-cobordisms, i.e. the inclusion $\iota : N_1 \hookrightarrow \bar{N}$ is a simple homotopy equivalence (Note that the inclusion $\iota : N_2 \hookrightarrow \bar{N}$ induces however isomorphisms on homology, cf. [Mil65, Rem. after Thm. 9.1]). An explicit example can be given as follows: Consider a compact, contractible manifold of dimension $m > 5$ whose boundary is a homology sphere not homeomorphic to the standard sphere, [Ker69]. Then, by cutting a small ball from the interior of this manifold, one obtains a manifold whose boundary is the disjoint union of a nonstandard homology sphere and $S^{m-1}$. The result is simply connected by Seifert-van-Kampens Theorem, [SZ88, 5.3.12]. Furthermore, the Whitehead torsion in $\text{Wh}(\pi_1(S^{m-1}))$ vanishes since $S^{m-1}$ is simply connected. In this case it seems more reasonable to ask if there exists riemannian metrics on these manifolds with ends as above that carry no closed geodesic. Now, we prepare the proof of Theorems 4.5 and 4.9 by the following lemmata. We start with the case of a nontrivial relative fundamental group.

4.12 Lemma. Let $K$ be a compact, concave subset of $M$ and let $U$ be the closure of its complement $\text{compl}_M K$. If $U$ is connected and $\pi_1(M,U)$ is not trivial, then there is a closed geodesic on $M$.

Proof. By the exact sequence of homotopy groups

$$\ldots \to \pi_1(U) \to \pi_1(M) \to \pi_1(M,U) \to \pi_0(U)$$

every nontrivial element in $\pi_1(M,U)$ can be represented by a non contractible curve in $M$ that is not homotopic to a curve in $U$. This precludes
4. Closed geodesics on open manifolds with convex and concave ends

the existence of a map $D$ as in Theorem 4.3, as illustrated in Figure 4.1, and hence proves the Lemma.

If $\pi_1(M,U)$ is trivial, we argue as follows: Assuming that there exists no closed geodesic on $M$ we consider representatives of the first nontrivial higher relative homotopy classes. They yield representatives of homotopy classes of the triad $(\Lambda M; \Lambda M_0, \Lambda U)$, see section 3.2. Then, we use Theorem 4.3 and vanishing of the relative homotopy groups in lower degree in order to construct a homotopy of this map to a map with image in $\Lambda U$. This homotopy leaves both $\Lambda U$ and $\Lambda M_0$ invariant; thus maintaining the homotopy class of the triad. This contradicts the nontriviality of the homotopy class we started with. In the following we will call a homotopy of maps with image in $\Lambda M$ shape-preserving if it preserves $\Lambda U$ and $\Lambda M_0$. First, we consider the case of nonvanishing relative second homotopy groups since it can not be included in the general setting given that $n$th-homotopy sets of triads are only defined for $n \geq 2$. Besides, this approach provides the geometrical idea more easily, cf. to figure 4.2.

4.13 Lemma. Let $K$ be a compact, concave subset of $M$ and let $U$ be the closure of its complement. If $U$ is connected, $(M,U)$ is 1-connected and $\pi_2(M,U)$ is not trivial, then there exists a closed geodesic on $M$.

Proof. We choose a representation $f : (D^2, S^1) \to (M,U)$ of a nontrivial homotopy class $\beta \in \pi_2(M,U)$ and consider $D^2$ as subset of $C$. Then, $f$ induces a continuous map $F : [0,1] \to \Lambda M$ given by $F(s)(t) = f(se^{2\pi i t})$. 
4.3. Concave, compact, co-connected subsets

By definition, $F(0)$ is a point curve and $F(1) \in \Lambda U$. By Theorem 3.6 we may consider $F$ as a continuous map from $[0,1]$ to $\Lambda M^\infty$ and hence there exists $\epsilon > 0$ such that $F([0,1]) \subseteq \Lambda M^\epsilon$. Now, we assume that $M$ contains no closed geodesic and construct a shape-preserving homotopy of $F$ to a map with image in $\Lambda U^\epsilon$:

By Theorem 4.3 there exists a map $D : \Lambda M^\epsilon \rightarrow \Lambda M^\epsilon$ such that $G = D \circ F$ has the following properties: The image of $G$ lies in $\Lambda U^\epsilon \cup \Lambda M_0$, $G(0)$ is a point curve and $G(1) \in \Lambda U^\epsilon$. Furthermore $G$ is homotopic to $F$ by a homotopy that fixes $\Lambda M_0$ and leaves $\Lambda U^\epsilon$ invariant.

The last property of $D$ in Theorem 4.3 implies furthermore that $G^{-1}(\Lambda M_0)$ is a neighborhood of $J := G^{-1}(\Lambda K_0)$. Hence by compactness of $K$ we can find finitely many disjoint intervals $[a_1, b_1], \ldots, [a_n, b_n]$ such that

$$J \subseteq \bigcup_{i=1}^{n} [a_i, b_i] \subseteq G^{-1}(\Lambda M_0),$$

cf. Figure 4.3. If $G(0) \in K$ we can homotop $G$ to a map with endpoint in $U$ by pulling it back along the curve given by $G|_{[a_i,1]}$. Thus we may assume that $G(0) \in U$ and since $\pi_1(M,U)$ is trivial we can find homotopies with fixed endpoints of the curves given by $c_i : [a_i, b_i] \rightarrow M$, $c_i(s) = F(s)$, to curves that lie completely in $U$. That yields a contradiction to the fact that $\beta$ was chosen nontrivial.
4. Closed geodesics on open manifolds with convex and concave ends

4.14 Lemma. Let $K$ be a compact, concave subset of $M$ and $U$ the closure of its complement $\text{compl}_M K$. If $U$ is connected and if for some $k \geq 3$ the pair $(M, U)$ is $(k - 1)$-connected and $\pi_k(M, U)$ is not trivial, there exists a closed geodesic on $M$.

Proof. Arguing by contradiction we assume that there exists no closed geodesic on $M$. For a nontrivial element $\alpha$ of $\pi_k(M, U)$ we consider a representative $f : (D^{k-1} \times I, S^{k-2} \times I, D^{k-1} \times \{0, 1\} \cup S^{k-2} \times I) \to (M, U, p)$.

As discussed in section 3.2 the isomorphism $\pi_{k-1}(\Lambda M; \Lambda M_0, \Lambda U) \simeq \pi_k(M, U)$ maps $f$ to a map $F : (D^{k-1}, S^{k-2}_+, S^{k-2}_-; p_0) \to (\Lambda M; \Lambda M_0, \Lambda U; c_p)$ where $c_p$ is the point curve with image $p \in U$. By Theorem 3.6 we may assume that there exists an $\epsilon > 0$ such that $F(D^{k-1}) \subseteq \Lambda M^\epsilon$. Then it follows from Theorem 4.3 that there exists a shape-preserving homotopy from $F$ to a map $G$ such that $G(D^{k-1}) \subseteq \Lambda M_0 \cup \Lambda U^\epsilon$. From the last property of $\mathcal{D}$ in Theorem 4.3 it follows that $\tilde{K} = G^{-1}(\Lambda K_0)$ has an open neighborhood $V$ that is contained in $G^{-1}(\Lambda M_0)$.

Then $(\tilde{K}^c, V)$ yields an open covering for $D^{k-1}$ and we can choose a triangulation of $D^{k-1}$ that is finer than this covering ([Spa66, 3.3.14]).
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We consider the sub complex $X$ of the triangulation of $D^{k-1}$ which contains all simplices that are contained in $V$. Then the associated polyhedron $A \subseteq D^{k-1}$ with boundary $B$ in $D^{k-1}$ yields a relative CW-complex $(A, B)$ of dimension $(k - 1)$ with $G(A) \subseteq \Lambda M_0$ and $G(B) \subseteq G(K^c \cap V) \subseteq \Lambda U_0$. Since $(M, U)$ is $(k - 1)$-connected the map $G|_{(A, B)} : (A, B) \to (M, U)$ is homotopic relative to $B$ to a map with image in $\Lambda U$. Hence there is a shape preserving homotopy of $F$ to a map whose restriction to $S^{k-2}$ has image in $\Lambda U^*$. This is a contradiction to the fact that $\beta$ and therefore its image in $\pi_{k-1}(\Lambda M; \Lambda M_0, \Lambda U)$ is nontrivial as Blakers and Massey showed, see Lemma 3.1.

Now, we prove Theorem 4.5:

Proof. We denote by $U$ the convex, connected set given by the closure of $\text{compl}_N K$. Then by Lemmata 4.12, 4.13, and 4.14 – applied in the case where $M := N$ has no boundary – there exists a closed geodesic if for some $k \geq 1$ the relative homotopy group $\pi_k(N, U)$ is not trivial.

If instead all relative homotopy groups of the pair $(N, U)$ vanish and $U$ is connected, it follows from the relative Hurewicz Theorem ([SZ88, 16.8.2]) that $H_k(N, U) = 0$ for all $k$. From the existence of a collar of $U$ in $N$ (refer to section 3.4) we deduce that $H_k(K, \partial K) \cong H_k(N, U) = 0$ ([SZ88, 9.3.5/9.3.6]). Then by the universal coefficient theorem ([Gre67, 29.12]) the $m$-th relative homology group with coefficients in $\mathbb{Z}_2$ is trivial. Since $K$ is concave, the interior of $K$ is nonempty and by section 3.4 its boundary $\partial K = \partial U$ is a compact topological submanifold. Hence there exists a $\mathbb{Z}_2$-fundamental class of $(K, \partial K)$ ([SZ88, 11.3.4]) which yields a contradiction.

We finish this section with the following

Proof of Theorem 4.9. The important step will be to show that the relative homotopy groups $\pi_k(\bar{N}, N_1)$ are trivial for every $k \geq 1$ if there exist no closed geodesic. This will imply that the inclusion $\iota : N_1 \hookrightarrow \bar{N}$ induces isomorphisms on the homotopy groups and hence is a homotopy equivalence since $N_1$ and $\bar{N}$ are CW complexes, cf. [SZ88, 16.7.10]:

Arguing by contradiction we assume there is a nonzero homotopy group but no closed geodesic. The following argument is technical due to the fact that we allow arbitrary convex neighborhoods rather than convex collar
neighborhoods, see Figure 4.3 for a clarifying sketch. Now, let \( k \) be the smallest natural number such that \( \pi_k(\tilde{N}, N_1) \) is not trivial and \( \beta \) a nonzero \( k \)-th homotopy class with representative \( f : (D^k, S^{k-1}) \to (\tilde{N}, N_1) \). Then there exists a closed convex neighborhood \( U \) of \( \alpha_1 \) that is contained in the collar neighborhood of \( \alpha_1 \). Furthermore by concavity of \( \alpha_2 \) we find a larger closed, convex neighborhood \( \bar{M} \) of \( \alpha_1 \) such that \( \bar{M} = \text{clos}_S M \) contains the image of \( f \). Note that \( f \) induces a non vanishing homotopy class in \( \pi_k(\bar{M}, N_1) \) and therefore can not be realized in the collar neighborhood of \( \alpha_1 \). Furthermore the collar structure implies for every \( \varepsilon \in (0, 1) \) with \( N_1 \times [0, 1 - \varepsilon) \subseteq M \) isomorphisms

\[
\pi_k(\bar{M}, N_1) \simeq \pi_k(\bar{M}, N_1 \times [0, 1 - \varepsilon)) \simeq \pi_k(M, N_1 \times (0, 1 - \varepsilon))
\]

Hence for \( \varepsilon \) so small that \( U \subseteq N_1 \times (0, 1 - \varepsilon) \) we can push \( f \) slightly off \( N_1 \times \{0\} \) such that the image of \( S^{k-1} \) is mapped into \( U \) and this new map \( \tilde{f} \) represents a nontrivial element of \( \pi_k(M, N_1 \times (0, 1 - \varepsilon)) \).

Then, Lemmata 4.12, 4.13, and 4.14 imply the existence of a homotopy of \( \tilde{f} \) relative to \( U \) to a map whose image lies completely in \( U \subseteq N_1 \times (0, 1 - \varepsilon) \) yielding a contradiction. \( \square \)
4.4. Closed geodesics on manifolds with at least two convex ends

In this section we assume that every end of \((N, g)\) is convex or concave in the sense that there exists a compact set \(K\) such that \(K = \bigcup_{i=1}^{n} \alpha_i(K)\) and \(\alpha_1(K), \ldots, \alpha_n(K)\) are disjoint sets and each of them is convex or concave. By \(M\) we will denote a closed, connected, convex subset of \(N\) with open interior and compact boundary \(\partial M\) such that \(M\) and each end of \(M\) is convex.

Now, we generalize the following simple idea: Assume that \(N\) is a cylinder \(S^1 \times (0, 1)\) with two convex ends. Choose convex neighborhoods \(U_1\) and \(U_2\) of each end. Then there exists a homotopy between a curve representing a nontrivial homotopy class of \(N\) in \(U_1\) to a curve representing this class in \(U_2\). This contradicts the existence of a map \(D\) as in Theorem 4.3 and hence shows the existence of a closed geodesic.

The above argument implies the following generalization:

4.15 Proposition. If there exists a nontrivial class \(\beta\) in \(\pi_1(M)\) that can be represented in two ends, i.e. if there exist disjoint, convex neighborhoods \(U_1\) and \(U_2\) of these ends and a homotopy from a representative of \(\beta\) in \(U_1\) to a representative in \(U_2\), then there exists a closed geodesic on \(M\). \(\square\)

In this section we elaborate on methods to detect closed geodesics if at least two ends are convex. Before we do so, we want to indicate the main problem that arises from the existence of more than one convex end: Under the assumption that there exists no closed geodesic, a closed curve is deformed by the Birkhoff process to a point curve when it does not leave a given compact, concave set. However, the same is not true when the Birkhoff process is applied to a \(S^k\)-family of closed curves if this family is induced by a \((k+1)\)-homotopy class that cannot leave the compact, concave set. In this case some curves move to point curves in the compact, concave set and others in the convex ends where they are usually not reduced to a point. From the homological point of view this phenomenon can be described as ”splitting” of the associated homology class. Hence the idea is to use homology classes that do not “split” to find closed geodesics.
We start with the main result of this section:

4.16 Theorem. Let $M$ contain a compact concave set $K$ and denote by $U_1, \ldots, U_n$ the closures of the connected components of $M \setminus K$. Furthermore let $k$ be an integer such that

1. $H_{k-1}(M) = H_{k-2}(M) = 0$, and
2. $H_k(M, U_i) = H_{k-1}(M, U_i) = 0$ for $i = 1, \ldots, n$.

If the homomorphism

$$\iota_* : \bigoplus_{i=1}^n H_{k-1}(\Lambda U_i) \to H_{k-1}(\Lambda M)$$

that is induced by the inclusions $\iota_i : \Lambda U_i \hookrightarrow \Lambda M$ via $\iota_* \sum_{i=1}^n \alpha_i = \sum_{i=1}^n (\iota_i)_* \alpha_i$ is not an isomorphism, then there exists a closed geodesic on $M$.

In particular, if for some $k > \dim M + 2$

$$\iota_* : \bigoplus_{i=1}^n H_{k-1}(\Lambda U_i) \to H_{k-1}(\Lambda M)$$

is not an isomorphism, then there exists a closed geodesic on $M$.

From the above theorem we deduce that, in analogy to the two dimensional case, Riemannian manifolds with convex spherical ends contain a closed geodesic:

4.17 Corollary. Let $N$ be the complement of finitely many disjoint closed balls in a closed manifold. If $g$ is a complete Riemannian metric on $N$ which induces a convex collar neighborhood around each end, then there exists a closed geodesic on $N$ with respect to $g$.

Proof. Recall that the dimension of $N$ is higher than two. Hence each end of $N$ is simply connected in the described situation. If $N$ itself is not simply connected, then it is easy to see that there exists a noncontractible curve that cannot be deformed continuously to a curve whose image is contained in a collar neighborhood of one end. Thus the existence of a closed geodesic
4.4. Closed geodesics on manifolds with at least two convex ends

follows in this case from Corollary 3.10 and is true without the assumption of convex ends. In fact, this is a simple case of example 5.2 in [Tho78].

If $N$ is simply connected, we denote by $X$ a closed manifold as in the corollary. We denote the balls cut out of $X$ together with their convex collar neighborhoods $U_1, \ldots, U_n$ by $B_1, \ldots, B_n$. Then we apply the Mayer-Vietoris’ sequence on the associated loop spaces and obtain the following long exact sequence

$$
\ldots \to H_{k+1}(\Lambda X) \xrightarrow{\Delta} H_k\left(\bigcup_{i=1}^n \Lambda U_i\right) \xrightarrow{j_*} H_k\left(\bigcup_{i=1}^n \Lambda B_i\right) \oplus H_k(\Lambda N) \to \ldots
$$

Since the sets $B_1, \ldots, B_n$ are disjoint and contractible, the first summand in the above direct sum vanishes for $k > 0$. Thus $j_*$ coincides with $\iota_*$ as defined in Theorem 4.16. On the other hand, by a Theorem of Sullivan [Sul75, p. 46], the loop space of a closed, simply connected manifold, has infinitely many nonzero Betti numbers. Therefore, using exactness of the above sequence, $\iota_*$ is not an isomorphism for some $k > m + 1$, actually for infinitely many of those $k$, and hence by Theorem 4.16 there exists a closed geodesic on $N$.

The following corollary gives another class of open manifolds with convex ends that contain at least one closed geodesic:

4.18 Corollary. Let $P_1, \ldots, P_n$ be closed submanifolds of $S^m$ with pairwise disjoint, closed tubular neighborhoods $T_1, \ldots, T_n$. Let $N$ be the complement of $\bigcup_{i=1}^n T_i$ in $S^m$ and let $g$ be a complete riemannian metric on $N$ such that each end has a convex collar neighborhood. Then there exists a closed geodesic on $N$ if at least one of the submanifolds $P_1, \ldots, P_n$ is simply connected and its cohomology has more than one generator.

Proof. Arguing as in the second part of Corollary 4.17 the Mayer-Vietoris sequence implies in this case a long exact sequence as follows:

$$
\ldots \to H_{k+1}(\Lambda S^m) \xrightarrow{\Delta} H_k\left(\bigcup_{i=1}^n \Lambda U_i\right) \xrightarrow{j_*} H_k\left(\bigcup_{i=1}^n \Lambda T_i\right) \oplus H_k(\Lambda N) \to \ldots
$$

where $U_i$ denote again the stipulated convex collar neighborhoods. Since $T_i$ deformation retracts to $P_i$ the Theorem of Vigué-Poirrier and Sullivan,
see [VPS76], implies that the Betti numbers of \( \bigcup_{i=1}^{n} \Lambda T_i \) are unbounded while the Betti numbers of \( \Lambda S^m \) are bounded. Hence Theorem 4.16 implies the existence of a closed geodesic. \( \square \)

In order to prove Theorem 4.16, we establish the following proposition which has an interest of its own as it does not require condition 2.

**4.19 Proposition.** Let \( M \) contain a compact concave subset \( K \) and denote by \( U_1, \ldots, U_n \) the closures of the connected components of \( M \setminus K \). If \( M \) contains no closed geodesic, then for every \( k \in \mathbb{Z} \) such that \( H_{k-1}(M) = H_{k-2}(M) = 0 \) and for every \( \epsilon > 0 \), the inclusions \( \iota_i : \Lambda U_i^\epsilon \cup \Lambda M_0 \hookrightarrow \Lambda M^\epsilon \) with \( i \in \{1, \ldots, n\} \) induce an isomorphism

\[
\iota_* : \bigoplus_{i=1}^{n} H_{k-1}(\Lambda U_i^\epsilon \cup \Lambda M_0) \xrightarrow{\simeq} H_{k-1}(\Lambda M^\epsilon)
\]

via \( \iota_* \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} (\iota_i)_* \alpha_i \).

**4.20 Remark.** If, in the above proposition, we omit the condition \( H_{k-2}(M) = 0 \), then \( \iota_* \) is at least injective. Similarly, if we omit the condition \( H_{k-1}(M) = 0 \), then \( \iota_* \) is at least surjective.

The following corollary gives an example where we establish the existence of a closed geodesic from Proposition 4.19 while condition 2 in Theorem 4.16 is too restrictive.

**4.21 Corollary.** Let \( N_1 \) be a closed manifold. If \( g \) is a complete riemannian metric on \( N \cong N_1 \times (0,1) \) with two convex ends, then there exists a closed geodesic on \( N \) with respect to \( g \).

Before we give the proof we explain on homotopy level where the nonsplitting homology class comes from in the non simply connected case: The idea is virtually the same as in Proposition 4.15. Hence one considers a nontrivial homotopy class \( \beta \in \pi_k(N_1) \) for minimal \( k \). This is representable in both ends and we can consider the associated representations in the loop space. Then by the Birkhoff deformation we defined, the nonexistence of closed geodesics implies the existence of a homotopy \( S^{k-1} \times [-1,1] \to \Lambda N \) such that the boundary maps \( S^{k-1} \times \{\pm1\} \to \Lambda N \) yield the nontrivial homotopy class we started with. However, there exists a subset \( T \subseteq S^{k-1} \times [-1,1] \)
of its domain which separates $S^{k-1} \times \{ -1 \}$ from $S^{k-1} \times \{ 1 \}$ and which is mapped to point curves. If $T$ contains $S^{k-1} \times \{ t_0 \}$ for some $t_0 \in (0, 1)$ we immediately see the contradiction. In general it will follow more easily by consideration of the associated homology classes as we see in an instant.

**Proof of Corollary 4.21.** If $N_1$ is not simply connected, the existence of a closed geodesic follows from Proposition 4.15. Otherwise we assume that $N$ is simply connected and does not contain any closed geodesic. Then we choose convex neighborhoods $U_1$ and $U_2$ around each end and $k > 1$ such that $N_1$ is $(k-1)$-connected and $\pi_k(N_1) \cong H_k(N_1) \cong H_{k-1}(\Lambda N_1)$ is not trivial. Then Proposition 4.19 applies in the weaker sense exposed in Remark 4.20.

Now, we construct a homology class of $\Lambda N$ that contradicts the injectivity. Therefor we choose a nontrivial $\beta \in H_{k-1}(\Lambda N_1)$ and denote the images under the natural inclusions by $\beta_i \in H_{k-1}(\Lambda U_1 \cup \Lambda N_0)$, $i = 1, 2$. Then for a large enough $\epsilon > 0$ we get $0 \neq \beta_1 - \beta_2 \in H_{k-1}(\Lambda U_1 \cup \Lambda N_0) \oplus H_{k-1}(\Lambda U_1 \cup \Lambda N_0)$ but $\iota_*(\beta_1 - \beta_2) = 0$. This finishes the proof.

Now, we present the following

**Proof of Proposition 4.19.** Denote for arbitrary $\epsilon > 0$ and $i \in \{1, \ldots, n\}$:

$$
U_i = \{ \gamma \in \Lambda M^\epsilon \mid D(\gamma) \subseteq \Lambda U_i \cup \Lambda M_0 \} \\
\mathcal{M} = \{ \gamma \in \Lambda M^\epsilon \mid D(\gamma) \subseteq \Lambda M_0 \}
$$

where $D = D(\epsilon)$ is a map as in Theorem 4.3.

Then we will show in a first step that the inclusions $\iota_i : U_i \hookrightarrow \Lambda M^\epsilon$ induce an isomorphism

$$
\iota_* : \bigoplus_{i=1}^n H_{k-1}(U_i) \xrightarrow{\sim} H_{k-1}(\Lambda M^\epsilon). \tag{4.2}
$$

via $\iota_* \sum_{i=1}^n \beta_i = \sum_{i=1}^n (\iota_i)_* \beta_i$. We assume that $n > 1$ since otherwise by Theorem 4.3 1) $U_1 = \Lambda M^\epsilon$. For $V_1 = U_2 \cup \ldots \cup U_n$ it follows from Theorem 4.3 that $\text{int} U_1 \cup \text{int} V_1 = \Lambda M^\epsilon$, i.e. the tuple $(U_1, V_1)$ is an excisive couple. Hence the Mayer-Vietoris sequence for excisive couples, cf. [Spa66, p.189],
yields the following exact sequence

\[
\ldots \rightarrow H_{k-1}(\mathcal{M}) \xrightarrow{\Delta} H_{k-1}(\mathcal{U}_1) \oplus H_{k-1}(\mathcal{V}_1) \xrightarrow{(j_1)_*} H_{k-1}(\Lambda M^\epsilon) \rightarrow \ldots
\]

where \((j_1)_*(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2\) if we identify \(\alpha_i, i = 1, 2\), with its image under the inclusion. Since \(H_l(\mathcal{M}) \simeq H_l(M) = 0\) for \(l = k-1, k-2\), the map \((j_1)_*(\alpha_1, \alpha_2)\) yields an isomorphism

\[
H_{k-1}(\mathcal{U}_1) \oplus H_{k-1}(\mathcal{V}_1) \xrightarrow{\simeq} H_{k-1}(\Lambda M^\epsilon).
\]

If \(n > 2\) we consider \(\mathcal{V}_2 = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n\). Now, Theorem 4.3 implies that \(\mathcal{V}_1 = \text{int}_{\mathcal{V}_1} \mathcal{U}_2 \cup \text{int}_{\mathcal{V}_1} \mathcal{V}_2\) and on the other hand it yields \(\mathcal{U}_2 \cap \mathcal{V}_2 = \mathcal{M}\). By application of the Mayer-Vietoris sequence we deduce that \(H_{k-1}(\mathcal{V}_1)\) splits as a direct sum of \(H_{k-1}(\mathcal{U}_2)\) and \(H_{k-1}(\mathcal{V}_2)\). Thus equation 4.2 follows after finite iteration of this argument. This implies the proposition since the embeddings \(\Lambda U_i \cup \Lambda M_0 \hookrightarrow U_i\) for \(i \in \{1, \ldots, n\}\) are homotopy equivalences by Theorem 4.3 3).

In order to prove Theorem 4.16 we start with a Lemma that will form the passage from the setting with finite energy to the whole loop space:

**4.22 Lemma.** Let \(V\) be a closed subset of \(M\) and assume that for some \(l \in \mathbb{Z}\) and every \(\epsilon > 0\) the inclusion \(\iota^\epsilon_V : \Lambda^{V^\epsilon} \rightarrow \Lambda M^\epsilon\) induces an isomorphism

\[
(i_V^\epsilon)_* : H_l(\Lambda^{V^\epsilon}) \xrightarrow{\simeq} H_l(\Lambda M^\epsilon).
\]

Then the inclusion \(i_V : \Lambda V \hookrightarrow \Lambda M\) induces an isomorphism

\[
(i_V)_* : H_l(\Lambda V) \xrightarrow{\simeq} H_l(\Lambda M).
\]

**Proof.** Since the energy functional is continuous on \(\Lambda M^\infty\) each compact set in \(\Lambda V^\infty\) is contained in some subset \(\Lambda V^\epsilon\) and hence it follows from Proposition 3.33 in [Hat02] that the direct limit \(\lim_{\longrightarrow} H_l(\Lambda V^\epsilon)\) is isomorphic to \(H_l(\Lambda V^\infty)\). Then, by the universal property of the direct limit the inclusion \(i_V^\infty : \Lambda V^\infty \rightarrow \Lambda M^\infty\) induces an isomorphism on the \(l\)-th homology groups if this holds for every \(i_V^\epsilon\). Now, the statement follows from Theorem 3.6. \(\square\)
4.4. Closed geodesics on manifolds with at least two convex ends

We now give the Proof of Theorem 4.16:

**Proof.** By Lemma 4.22, applied for $V = \bigcup_{i=1}^{n} U_i$, it suffices to show for arbitrary $\epsilon > 0$ that the inclusions $\iota_{i}^{\epsilon} : \Lambda U_{i}^{\epsilon} \hookrightarrow \Lambda M^{\epsilon}$ induce an isomorphism

$$\iota_{i}^{\epsilon} : \bigoplus_{i=1}^{n} H_{k-1}(\Lambda U_{i}^{\epsilon}) \xrightarrow{\cong} H_{k-1}(\Lambda M^{\epsilon}) \quad (4.3)$$

via $\iota_{i}^{\epsilon} \sum_{i=1}^{n} \alpha_{i} = \sum_{i=1}^{n} (\iota_{i}^{\epsilon})_{*} \alpha_{i}$. For this purpose we show for every $i \in \{1, \ldots, n\}$ that the inclusion $j_{i}^{\epsilon} : \Lambda U_{i}^{\epsilon} \hookrightarrow \Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}$ induces an isomorphism on the $(k-1)$-th homology of these sets. Then equation (4.3) follows from Proposition 4.19.

By the long exact sequence of the pair $(\Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}, \Lambda U_{i}^{\epsilon})$ it is enough to show that $H_{l}(\Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}, \Lambda U_{i}^{\epsilon})$ is trivial for $l = k, k - 1$.

For this purpose we denote $B = \{ \gamma \in \Lambda U_{i}^{\epsilon} \cup \Lambda M_{0} \mid L(\gamma) < f(\gamma(0)) \}$ where $f$ is a continuous function that is bounded from above by half the injectivity radius and on $U$ additionally by $\frac{1}{2} \text{conv} U$. Then $B$ is an open neighborhood of $\Lambda M_{0}$ in $\Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}$. Since $\Lambda U_{i}^{\epsilon}$ and $B$ are open in $\Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}$ the following version of excision, cf. [Spa66, 4.6.3, 4.6.4], applies for every $l$:

$$H_{l}(\Lambda U_{i}^{\epsilon} \cup \Lambda M_{0}, \Lambda U_{i}^{\epsilon}) \simeq H_{l}(B, B \cap \Lambda U_{i}^{\epsilon}).$$

But $(\Lambda M_{0}, (\Lambda U_{i})_{0})$ is a deformation retract of $(B, B \cap \Lambda U_{i})$ by equation 4.1. Hence

$$H_{l}(B, B \cap \Lambda U_{i}) \simeq H_{l}(\Lambda M_{0}, (\Lambda U_{i})_{0}) \simeq H_{l}(M, U_{i}) = 0$$

for $l \in \{k, k - 1\}$. 

\[\Box\]
Part II.

On Reeb flows in dimension five and higher
5. Introduction

On $\mathbb{R}^3$ there exists, up to coordinate transformation, exactly one contact form which coincides with the standard contact form $\alpha_{st} = dz + xdy$ on the complement of a compact set and has no periodic orbit. This is the standard contact form itself, [EH94]. Hence the Reeb flow of a compactly perturbed standard contact form has a periodic orbit exactly if it has a trapped orbit, i.e. an orbit $\gamma(t) = (q(t), z(t)) \in \mathbb{R}^2 \times \mathbb{R}$ that fulfills

$$\lim_{t \to -\infty} z(t) = -\infty, \quad \limsup_{t \to \infty} z(t) < \infty.$$ 

That is a striking property of the Reeb flow. In 2012, Hofer conjectured at the conference on 'Recent Progress in Lagrangian and Hamiltonian Dynamics' in Lyon that a similar result holds for higher dimension:

**Conjecture.** Let $\alpha$ be a contact form on $\mathbb{R}^{2n+1}$, $n > 1$, that induces the standard contact structure and equals the standard contact form outside a compact set. Then the existence of a trapped orbit of the induced Reeb vector field implies the existence of a periodic Reeb orbit.

We disprove this conjecture by showing the following

**Theorem.** There exists a contact form $\alpha$ on $\mathbb{R}^{2n+1}$, $n > 1$, inducing the standard contact structure, with the following properties:

1. The contact form $\alpha$ equals the standard contact form outside a compact set.

2. The induced Reeb flow has a nonempty, compact invariant set.

3. At least one orbit of the induced Reeb flow is trapped.

4. There are no periodic Reeb orbits.
In the proof we present explicit examples for $n > 1$. In the described setting the associated Reeb vector fields have trapped orbits but their only compact invariant set is a torus. This demonstrates that Reeb dynamics in higher dimensions can differ fundamentally from those in dimension three. According to Hofer this could be very interesting in the context of symplectic invariants in higher dimensions, which go beyond symplectic field theory, [Hof].

Another reason for the significance of this result becomes apparent if we recall Wilson’s plug construction for general vector fields, [Wil66]: He considers a flow box $B \times [-1,1]$ whose base is a ball $B$. After a smart deformation of the vector field in the interior of the flow box, the limit sets of the new flow are all given by invariant tori. Additionally, the flow has the mirror-symmetry-property, i.e. is invariant under reflection along $B \times \{0\}$. The first property yields orbits trapped in forward and backward time while the mirror-symmetry-property implies that orbits which traverse the flow box are in fact perturbed only locally.

Wilson uses these plugs in order to produce examples of non zero vector fields without periodic orbits on every compact manifold of dimension higher than 3 with vanishing Euler characteristic; more explicitly all limit sets of these vector fields are contained in a finite set of invariant tori. Then Schweitzer and K. Kuperberg refined Wilson’s idea in order to construct counterexamples to the Seifert conjecture which asserts that every non zero vector field on $S^3$ has a periodic orbit, [Sch74], [Kup94].

Our examples have the first property of Wilson’s plugs but lack the mirror-symmetry-property and hence can be considered as semi contact plugs. Thus insertion of such a plug in a Reeb flow on a compact manifold can open one periodic orbit but it creates in general new ones. Therefore such a construction does not produce examples of Reeb flows without periodic orbits, i.e. counterexamples to the Weinstein conjecture, refer to [Wei79] for the original version. Since the Weinstein conjecture has been proven in certain cases, for instance for PS-overtwisted contact structures in arbitrary odd dimension, [AH09], a (mirror-symmetric) contact plug cannot exist. Doubling of the plug could still be possible in a slightly more general class of vector fields providing simple constructions of plugs in these classes. This could be of interest for example in the Hamiltonian setting.

The remarks above show that the construction is delicate and relies on a good choice of a trap that can actually be realized as invariant set of
a Reeb flow. In our example it is an invariant torus of a well-known Reeb flow on the contact submanifold $S^{2n-1} \subseteq \{0\} \times \mathbb{R}^{2n}$. This promising idea is motivated by a construction of Bangert and the author in [BR12] for the geodesic flow. There, we constructed a Riemannian $n$-ball for $n \geq 4$ with strictly convex boundary that carries a complete geodesic but not a closed one. Although the restriction of a geodesic flow to the unit tangent bundle is a Reeb flow, this riemannian example does however not yield counterexamples to Hofer’s conjecture since the boundary conditions are different.
6. Preliminaries II

Here we briefly present notions and techniques from contact geometry that we need in the second part. This chapter does not include any new results but recalls basic facts and fixes notation. The general reference is Geiges’ book [Gei08].

6.1. Basic definitions and notation

A pair \((M,\xi)\) of a \((2n+1)\)-dimensional smooth manifold and a codimension 1 distribution \(\xi \subseteq TM\) is a contact manifold and \(\xi\) is called a contact structure on \(M\) if there exists an open cover \(\{U_i\}_{i \in I}\) of \(M\) and for each \(i \in I\) a 1-form \(\alpha_i\) on \(U_i\) such that \(\xi|_{U_i} = \ker(\alpha_i)\) and

\[
\alpha_i \wedge (d\alpha_i)^n \quad \text{is a volume form.} \quad (6.1)
\]

We call a 1-form that fulfills condition (6.1) a contact form. Note that property (6.1) implies that \(d\alpha|_{\xi}\) is nondegenerate. In the following we will assume that \(\xi\) is coorientable, i.e. there exists a globally defined contact form \(\alpha\) inducing \(\xi\), i.e. \(\xi = \ker(\alpha)\). In this case every such contact form is given by \(\tau\alpha\) for a smooth function \(\tau : M \to \mathbb{R} \setminus \{0\}\).

Given a contact form \(\alpha\) we can associate a uniquely defined vector field \(R\), called the Reeb vector field, by the following equations:

\[
\iota_R d\alpha = 0 \quad \text{and} \quad \alpha(R) = 1 \quad (6.2)
\]

where \(\iota_R d\alpha\) denotes the contraction of \(d\alpha\) with \(R\).

Two contact manifolds \((M_i, \xi_i = \ker(\alpha_i))\), \(i = 1, 2\), are contactomorphic if there exists a diffeomorphism \(f : M_1 \to M_2\) such that the differential \(Df\) of \(f\) maps \(\xi_1\) onto \(\xi_2\). We call \(f\) a contactomorphism in this case.

Since condition (6.1) is invariant under pullback by a diffeomorphism, it follows that \(f^*\alpha_2\) is a contact form inducing \(\xi_1\). Thus there exists a nowhere
vanishing smooth function $\tau$ on $M_1$ such that

$$f^*\alpha_2 = \tau \cdot \alpha_1.$$  

If $\tau = 1$ we call $f$ a strict contactomorphism and $(M_1, \xi_1)$ and $(M_2, \xi_2)$ strictly contactomorphic. A submanifold $N$ of a contact manifold $(M, \xi_M)$ equipped with a contact structure $\xi_N$ is a contact submanifold if $\xi_N = \xi_M \cap TN$.

In the following we will be interested in the case of $M = \mathbb{R}^{2n+1}$ equipped with the standard contact structure $\xi_{st} = \text{ker} \alpha_{st}$ given by the standard contact form

$$\alpha_{st} = dz + \sum_{i=1}^{n} x_i dy_i - y_i dx_i.$$  

By Darboux’ Theorem this is a local model for every contact manifold, cf. for example to [Gei08, 2.5.1]. Note that after coordinate transformation we get the more common form $\Phi^*\alpha_{st} = dz + \sum_{i=1}^{n} y_i dx_i$ where the map $\Phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is given by $\Phi(x_1, y_1, \ldots, x_n, y_n, z) = ((x_1 + y_1)/2, (x_1 - y_1)/2, \ldots, (x_n + y_n)/2, (x_n - y_n)/2, z + \frac{1}{2} \sum_{i=1}^{n} x_i y_i)$.

### 6.2. Gray stability

Gray’s stability Theorem, cf. [Gra59], states that smooth deformations $(\xi_t)_{t \in [0,1]}$ of contact structures on closed manifolds do not change the contact structure up to contactomorphism. The result stays true if we consider compact, smooth deformations on arbitrary manifolds, i.e. smooth deformations that are constant outside a compact set. Since the theorem is quoted usually for the closed case we will repeat here the proof using Moser’s trick as presented in Geiges’ book, [Gei08, Thm. 2.2.2], to give evidence to this assertion:

We choose a smooth family $(\alpha_t)_{t \in [0,1]}$ of contact forms on a manifold $M$ inducing the contact structures $(\xi_t)_{t \in [0,1]}$. By assumption there exists a compact set $K$ such that $\alpha_t|_{M \setminus K} \equiv \alpha_0|_{M \setminus K}$. Now, we have to prove the existence of a diffeomorphism $f : M \to M$ and a smooth map $g : M \to (0, \infty)$ such that

$$f^*\alpha_1 = g\alpha_0.$$  

(6.3)
The main idea of Moser’s trick is to find $f$ as a time-1 map of the flow $\Phi_t$ of some time dependent vector field $X_t$. Hence we make the following ansatz:

$$\Phi_t^* \alpha_t = g_t \alpha_0 \quad (6.4)$$

where $g_t : M \to (0, \infty), t \in [0, 1]$, is a smooth family of maps.

Differentiation of this equation yields on the right side

$$\dot{g}_t \alpha_0 = \frac{\dot{g}_t}{g_t} \Phi_t^* \alpha_t = \Phi_t^* (\mu_t \alpha_t)$$

with $\mu_t = \frac{d}{dt} \ln(g_t) \circ \Phi_t^{-1}$ and on the left side

$$\frac{d}{dt} \Phi_t^* \alpha_t = \lim_{h \to 0} \frac{\Phi_{t+h}^* (\alpha_{t+h}) - \Phi_t^* (\alpha_t)}{h}$$

$$= \lim_{h \to 0} \frac{\Phi_{t+h}^* (\alpha_{t+h}) - \Phi_{t+h}^* (\alpha_t) + \Phi_{t+h}^* (\alpha_t) - \Phi_t^* (\alpha_t)}{h}$$

$$= \Phi_t^* \left( \dot{\alpha}_t + \lim_{h \to 0} \frac{(\Phi_{t+h} \circ \Phi_t^{-1})^* (\alpha_t) - (\alpha_t)}{h} \right)$$

$$= \Phi_t^* (\dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t)$$

where in the last equation the definition of the Lie derivative $\mathcal{L}_{X_t}$ for a time-dependent vector field is applied. Since $\Phi_t$ is a diffeomorphism and Cartan’s formula holds as well for time-dependent vector fields, cf. [Gei08, B.2], this equation is equivalent to the following

$$\mu_t \alpha_t = \dot{\alpha}_t$$

If we assume $X_t \in \xi_t$, the above condition simplifies to

$$\iota_{X_t} d\alpha_t = \mu_t \alpha_t - \dot{\alpha}_t.$$  

(6.5)

By inserting the Reeb vector field $R_t$ associated to $\alpha_t$ into this equation we find that $\mu_t = \dot{\alpha}_t (R_t)$. Hence the right hand side yields a one form that vanishes on the Reeb vector field $R_t$. Therefore condition (6.5) defines a unique vector in $\xi_t$ at every point of $M$ and thus a smooth vector field since $d\alpha_t|_{\xi_t}$ is a symplectic form and $TM$ splits as $\xi_t \oplus \mathbb{R} R_t$. In particular
it follows from $\dot{\alpha}_t = 0$ on $M \setminus K$, that $X_t$ has compact support on $M$. This assures that the flow $\Phi_t$ is defined for every $t \in \mathbb{R}$. Now, we integrate
\[
\frac{d}{dt} \ln(g_t) = \dot{\alpha}_t(R_t) \circ \Phi_t^{-1}
\]
and find in this way a smooth family $g_t$ of positive functions such that equation (6.4) is fulfilled for $\Phi_t$ and $g_t$ as constructed. This finishes the proof.

6.3. The standard contact structure on $S^3$ and associated Reeb dynamics

The standard contact structure $\xi_0$ on $S^3 \subseteq \mathbb{C}^2$ is given by the complex lines tangent to $S^3$. It is easy to verify that $\xi_0 = \ker \alpha_0$ with
\[
\alpha_0 = \left( \sum_{i=1}^{2} x_idy_i - y_idx_i \right) \bigg|_{S^3}
\]
and this one form yields actually a contact form on $S^3$, [Gei08, Example 1.4.8]. If we consider $(S^3, \xi_0)$ as a subset of the hyperplane $\{z = 0\}$ in $(\mathbb{R}^5, \xi_{st})$ it yields a contact submanifold as follows easily from a comparison of $\alpha_0$ with the standard contact form on $\mathbb{R}^5$. For $\varepsilon > 0$ the smooth family of one forms $\alpha_\varepsilon = x_1dy_1 - y_1dx_1 + (1 + \varepsilon) (x_2dy_2 - y_2dx_2)$ yields a deformation of $\alpha_0$ by contact forms as in the Gray stability Theorem. Hence it preserves the contact structure up to contactomorphism. On the other hand the associated Reeb vector fields $R_\varepsilon = R(\alpha_\varepsilon)$ are given by
\[
R_\varepsilon = x_1\partial_{y_1} - y_1\partial_{x_1} + \frac{1}{1+\varepsilon} (x_2\partial_{y_2} - y_2\partial_{x_2})
\]
Therefore the flow lines of $R_\varepsilon$ are all closed if $\varepsilon \in \mathbb{Q}$ and in particular yield the foliation known from the Hopf fibration in the case $\varepsilon = 0$. But for $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$ there exists only two periodic orbits, each being a circle in the $(x_i, y_i)$-plane for $i \in \{1, 2\}$. All remaining flow lines correspond, when considered in proper coordinates, to an irrational linear foliation on the so called Hopf tori $\{x_1^2 + y_1^2 = t^2, \ x_2^2 + y_2^2 = 1 - t^2\}, t \in (0, 1)$. 

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6.4. Contact vector fields

We fix a contact form $\alpha$ for the contact manifold $(M, \xi)$. A vector field $X$ is a contact vector field if its flow preserves the contact structure or equivalently if

$$\mathcal{L}_X \alpha = \tau \alpha$$

for a smooth function $\tau : M \to \mathbb{R}$. By inserting the Reeb vector field $R = R(\alpha)$ into this equation it follows using Cartan’s formula that $\tau$ is given by

$$\tau = dH(R)$$  \hspace{1cm} (6.6)

for $H = \alpha(X)$. In the following we are interested in transverse contact vector fields $X$, i.e. we assume that $H$ is a nowhere vanishing function. This class of vector fields is given exactly by the set of Reeb vector fields associated to contact forms inducing $\xi$ as the following argument shows:

Starting with a transverse contact vector field $X$ we consider the contact form $\tilde{\alpha} = \frac{1}{H} \alpha$. Then $\tilde{H} := \tilde{\alpha}(X) \equiv 1$ and since $d\tilde{H} = 0$ it follows from equation (6.6) that

$$\iota_X d\tilde{\alpha} = \mathcal{L}_X \tilde{\alpha} = 0.$$  

Conversely it follows easily from Cartan’s formula that every Reeb vector field is a contact vector field.

On the other hand, the map

$$\{X \mid X \text{ contact vector field} \} \rightarrow \{H : M \rightarrow \mathbb{R} \mid H \text{ smooth} \}$$

$$X \mapsto \alpha(X)$$

is a one to one correspondence. Here the inverse image $X$ of a smooth function $H$ is defined by the following equations

$$\alpha(X) = H \quad \text{and} \quad \iota_X d\alpha = dH(R)\alpha - dH.$$  \hspace{1cm} (6.7)

The inverse map is well-defined since the contact condition for $\alpha$ implies
6. Preliminaries II

a splitting \( \mathbb{R}R \oplus \xi \). Hence the first equation determines obviously the first component while \( d\alpha|_{\xi} \) is nondegenerate and therefore specifies the projection to \( \xi \) via the second equation. Furthermore the vector field \( X \) is contact because

\[
\mathcal{L}_X \alpha = \iota_X d\alpha + dH = dH(R)\alpha
\]

by equation (6.7). Combining these two results we get the following fact that will be important in section 7.2.

6.1 Fact. Let \( H : M \to \mathbb{R} \setminus \{0\} \) be a smooth function. The Reeb vector field \( \tilde{R} \) associated to \( \alpha = \frac{1}{R} \alpha \) is given by the following equations:

\[
\alpha(\tilde{R}) = H \quad \text{and} \quad \iota_{\tilde{R}} d\alpha = dH(R)\alpha - dH. \quad (6.8)
\]
7. Trapped Reeb orbits do not imply periodic ones

In this chapter we present two methods for providing examples that prove Theorem 7.1. The construction described in section 7.2 is very similar to the one in [Röt13] and outlines the geometric idea. It yields an example proving Theorem 7.1 in dimension 5. Its main idea gave rise to the joint article with Hansjörg Geiges and Kai Zehmisch, [GRZ14], where another approach to this problem is used. This other method is very elegant and generalizes easily to higher dimension. We will present this shorter proof for arbitrary odd dimension in section 7.3 with only minor modifications.

7.1. The main result

The goal of this chapter is to prove

7.1 Theorem. Assume $n > 1$. There exists a contact form $\alpha$ on $\mathbb{R}^{2n+1}$, inducing the standard contact structure, with the following properties:

1. The contact form $\alpha$ equals the standard contact form outside a compact set.

2. The induced Reeb flow has a nonempty compact invariant set.

3. At least one orbit of the induced Reeb flow is trapped.

4. There are no periodic Reeb orbits.

We start with a qualitative description of the Reeb dynamics induced by examples of the above type that will be constructed in section 7.2 and 7.3. The idea is illustrated in figure 7.1: Outside the compact cylinder
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The Reeb flow is standard, i.e.

\[ t \mapsto (q_0, z_0 + t) \in \mathbb{R}^{2n} \times \mathbb{R} \]

But on the \( n \)-dimensional torus

\[ \mathcal{T} = \{(x_1, y_1, \ldots, x_n, y_n, 0) \mid x_i^2 + y_i^2 = 1, \; i \in \{1, \ldots, n\}\} \]

the Reeb vector field induces a one-dimensional foliation by dense leaves. This yields in particular a compact invariant set without periodic orbits. In addition the \( z \)-component of the Reeb vector field is positive on \( \mathbb{R}^{2n+1} \setminus \mathcal{T} \). For the remainder of this chapter we will refer to the last condition as monotonicity property of the contact form which induces a Reeb vector field with this condition. The above properties immediately imply that there does not exist any periodic Reeb orbit in such an example since every orbit either increases strictly monotonically in \( z \)-direction or lies entirely on \( \mathcal{T} \).

We summarize this in the following proposition

**7.2 Proposition.** There exists a contact form \( \alpha \) on \( \mathbb{R}^{2n+1} \), \( n > 1 \), inducing \( \xi_{st} = \ker \alpha_{st} \), with the following properties:

1. the contact form \( \alpha \) coincides with \( \alpha_{st} \) outside \( Z = B \times [-1, 1] \).

2. the torus \( \mathcal{T} \) is the only compact invariant set of the induced Reeb vector field.
3. the contact form $\alpha$ has the monotonicity property, i.e. it yields $dz(R) > 0$ for the induced Reeb vector field $R$ on $\mathbb{R}^{2n+1} \setminus \mathcal{T}$.

Proof of Theorem 7.1. By the above considerations Theorem 7.1 follows from Proposition 7.2 once we establish the existence of a trapped Reeb orbit. This orbit arises as limit of orbits that pass by the torus in arbitrarily small distance.

First we prove that every orbit apart from those on the torus $\mathcal{T}$ leaves $\mathbb{R}^{2n} \times [-1, 1]$ in forward or backward time: Therefor we recall that $B \times \mathbb{R}$ is an invariant set of the flow $\Phi$ induced by $R$. Hence each orbit through a point $(q_0, z_0) \in B \times (0, 1]$ stays in $B \times [z_0, \infty)$ in positive time. Since $R = \partial_z$ for $z > 1$ the monotonicity property yields a positive lower bound for $dz(R)$ on $B \times [z_0, \infty)$. This implies the existence of a point of intersection with $B \times \{1\}$ after finite time. The analog is true in backward direction for points in $B \times [-1, 0)$. With this, the above claim follows easily.

Now, we choose a sequence of points $(p_n)_{n \in \mathbb{N}}$ with $p_n \in B \times [-1, 0)$ that converges to a point $p \in \mathcal{T}$. By the above argument we find sequences $(q_n)_{n \in \mathbb{N}}$ with $q_n \in B \times \{-1\}$ and $(t_n)_{n \in \mathbb{N}}$ with $t_n > 0$ such that $\Phi(q_n, t_n) = p_n$. Since $B$ is compact we can choose a converging subsequence $q_n \to q$. We want to show that $\lim_{n \to \infty} t_n = \infty$. Otherwise we may assume that $\lim_{n \to \infty} t_n = t \in [0, \infty)$. From continuity of $\Phi$ it would then follow that

$$\Phi(q, t) = \lim_{n \to \infty} \Phi(q_n, t_n) = p \in \mathcal{T}.$$ 

But $\mathcal{T}$ is an invariant set of the Reeb flow yielding a contradiction.

We emphasize at this point that the monotonicity property implies $dz(R) \geq 0$ on $\mathbb{R}^{2n+1}$ since the interior of $\mathcal{T}$ is empty. Thus again by continuity of $\Phi$ we obtain for every $T > 0$

$$\Phi(q, T) = \lim_{n \to \infty} \Phi(q_n, T) \in B \times [-1, 0].$$

Thus we need to establish Proposition 7.2.

7.2. Examples on $\mathbb{R}^5$

In this section we restrict to the case $n = 2$. The following consideration motivates the choice of $\mathcal{T}$ as compact invariant set: We recall that $S^3 \cong \ldots$
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\{ q \in \mathbb{R}^4 \mid |q| = 1 \} \times \{0\} is a contact submanifold of \( \mathbb{R}^5 \) when both spaces are equipped with their standard contact structures. In addition, for positive \( \varepsilon \in \mathbb{R} \setminus \mathbb{Q} \) the contact form \( \alpha_\varepsilon \) on \( S^3 \) introduced in section 6.3 yields a Reeb vector field \( R_\varepsilon \) inducing an irrational linear flow on \( T \) in proper coordinates, refer to section 7.3 for a more explicit description. Hence \( R_\varepsilon \) is a promising choice for the Reeb vector field on \( T \). Note that if \( \tilde{\alpha} \) is a contact form on \( \mathbb{R}^5 \) whose pullback to \( S^3 \) is \( \alpha_\varepsilon \), it follows

\[
\tilde{\alpha}(R_\varepsilon) = 1 \quad \text{and} \quad \iota_{R_\varepsilon} d\tilde{\alpha}|_{TS^3} = 0. \tag{7.1}
\]

Here and in the remainder of this section we identify \( R_\varepsilon \) with its push forward to \( TR^5 \).

At this point we want to give an overview of the following construction: In the first step we fix \( \varepsilon \in \mathbb{R} \setminus \mathbb{Q} \) and deform \( \alpha_{\text{st}} \) on a neighborhood of \( S^3 \) to obtain a contact form \( \tilde{\alpha} \) with \( \ker \tilde{\alpha} = \xi_{\text{st}} \) that fulfills condition (7.1) and the monotonicity property. Without changing these properties in the second step we deform \( \tilde{\alpha} \) in a neighborhood of \( T \) into a contact form \( \alpha \) whose Reeb vector field coincides with \( R_\varepsilon \) on \( T \). Note that the monotonicity property in particular precludes that the Reeb vector field of \( \alpha \) coincides with \( R_\varepsilon \) on the whole sphere \( S^3 \) and in particular that the two periodic orbits of this vector field become orbits of the Reeb vector field induced by \( \alpha \) on \( \mathbb{R}^5 \).

**Step 1:** We consider the one form \( \beta \) on \( \mathbb{R}^5 \) given by \( x_2 dy_2 - y_2 dx_2 \) and a bump function \( h \in C^\infty(\mathbb{R}^5, [0, 1]) \) that is constantly 1 on an open neighborhood \( \mathcal{U} \) of \( S^3 \) and vanishes outside the cylinder \( Z = B \times [-1, 1] \). Now, since the contact condition and the monotonicity property are both open conditions we can find an \( \bar{\varepsilon} > 0 \) such that for every \( 0 < \varepsilon < \bar{\varepsilon} \) the one form

\[
\tilde{\alpha}_\varepsilon = \alpha_{\text{st}} + \varepsilon h \beta
\]

is a contact form and the associated Reeb vector field \( \tilde{R}_\varepsilon \) fulfills \( dz(\tilde{R}_\varepsilon) > 0 \). Additionally, since \( h \) has compact support, Gray’s stability, cf. section 6.2, implies that \( \ker \tilde{\alpha}_\varepsilon \) yields the standard contact structure up to contactomorphism for every \( 0 < \varepsilon < \bar{\varepsilon} \). Now, we fix \( \varepsilon \in (0, \bar{\varepsilon}) \setminus \mathbb{Q} \) and denote \( \tilde{\alpha} = \tilde{\alpha}_\varepsilon \).
Step 2: We introduce the following coordinate system:

\[ F : \mathbb{R} \times (0, \infty) \times (0, \pi/2) \times \mathbb{R}^2 \rightarrow \mathbb{R}^5 \]

\[
(z, r, \psi, \varphi, \theta) \mapsto \\
\begin{pmatrix}
  r \cos \psi & \cos \varphi \\
  r \cos \psi & \sin \varphi \\
  r \sin \psi & \cos \theta \\
  r \sin \psi & \sin \theta \\
  z
\end{pmatrix}.
\]

These coordinates are well-suited for the problem since they yield charts for both the sphere and the Clifford-torus \( T \) given by the set \( \{ F(z, r, \psi, \varphi, \theta) | (z, r, \psi) = (0, 1, \pi/4) \} \). We will denote \( p_T = (0, 1, \pi/4) \) and perturb the contact form in a tubular neighborhood of \( T \) of the form

\[ V := F(D \times \mathbb{R}^2) \subseteq U \]

with \( D = (z^-, z^+) \times (r^-, r^+) \times (\psi^-, \psi^+) \). By step 1, the contact form \( \tilde{\alpha} \) restricted to \( V \) is given in these coordinates by

\[ \tilde{\alpha}|_V = dz + r^2 \cos^2 \psi d\varphi + (1 + \varepsilon) r^2 \sin^2 \psi d\theta. \quad (7.2) \]

Now, we obtain the contact form \( \alpha \) by a deformation of \( \tilde{\alpha} \) on \( V \) using the following ansatz:

\[ \alpha = f dz + g d\varphi + h d\theta \]

where \( f, g, h \in C^\infty(D, \mathbb{R}) \) meet reasonable boundary conditions, see \((F_B), (G_B), (H_B)\) in Lemma 7.4. The contact condition (6.1) restricted to \( V \) gives:

\[ f (g \psi h_r - g_r h_\psi) + f_r (g \psi h_r - h_\psi) + f_\psi (h g_r - h g_\psi) < 0. \quad (C) \]

Here the choice of sign is stipulated by the boundary condition since the above estimate simplifies for \( \tilde{\alpha}|_V \) to the following estimate

\[ -4(1 + \varepsilon) r^3 \cos \psi \sin \psi < 0 \]

and hence by \( \psi \in (0, \pi/2) \) is valid near the boundary \( \partial V \) of \( V \).

Now, our main theorem 7.1 follows by Gray stability from the following

7.3 Proposition. There exists a contact form \( \alpha \) on \( \mathbb{R}^5 \) that coincides with \( \tilde{\alpha} \) on \( \mathbb{R}^5 \setminus V \) and meets the following conditions:

T1. The line segment \( \alpha_t = t \alpha + (1-t)\tilde{\alpha}, t \in [0, 1], \) is a smooth family of
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T2. On the Clifford torus the Reeb vector field of \( \alpha \) is given by \( R = \partial_\varphi + \chi \partial_\theta \) for \( \chi = \frac{1}{1+\varepsilon} \).

T3. \( dz(R) > 0 \) on \( \mathbb{R}^5 \setminus \mathcal{T} \).

We will show that there exists \( f, g, h \in C^\infty(D, \mathbb{R}_{>0}) \) such that \( \alpha \) with \( \alpha|_V = fdz + gd\varphi + hd\theta \) has the required properties. For this purpose we start with the following technical

7.4 Lemma. There exist open neighborhoods \( V \subseteq U \subseteq D \) of \( p_T \) with compact closure \( \bar{V} \subseteq U, \bar{U} \subseteq D \), and functions \( f, g \) and \( h \in C^\infty(D, \mathbb{R}_{>0}) \) with the following properties:

\[(F_B)\] \( f|_{D \setminus U} \equiv 1 \)

\[(F_M)\] \( f_r|_V < 0 \) and \( f_\psi|_V \equiv 0 \)

\[(F_C)\] \( (2f - rf_r)|_{D \setminus V} > 0 \)

\[(G_B)\] \( g|_{D \setminus V} = r^2 \cos^2 \psi \)

\[(G_M)\] \( g_r|_{D \setminus \{p_T\}} > 0 \) and \( g_\psi < 0 \)

\[(G_T)\] \( g(p_T) = \frac{1}{2} \) and \( g_\psi(p_T) = -1 \)

\[(G_T')\] \( g_z(p_T) = 0 \) and \( g_r(p_T) = 0 \)

\[(H_B)\] \( h|_{D \setminus V} = (1 + \varepsilon)r^2 \sin^2 \psi \)

\[(H_M)\] \( h_r|_{D \setminus \{p_T\}} > 0 \) and \( h_\psi > 0 \)

\[(H_T)\] \( h(p_T) = \frac{1 + \varepsilon}{2} \) and \( h_\psi(p_T) = 1 + \varepsilon \)

\[(H_T')\] \( h_z(p_T) = 0 \) and \( h_r(p_T) = 0 \)
These functions can be constructed explicitly. The exact proof will be given in the appendix.

7.5 Remark. The conditions with index $B$ are boundary conditions which make sure that the new contact form on $V$ can be extended by $\tilde{\alpha}$ to a smooth contact form on $\mathbb{R}^5$. Properties $(G_T)$ and $(H_T)$ ensure that $\alpha$ meets condition (7.1) on $T$. From conditions $(G_T')$ and $(H_T')$ it follows then that on $T$ the Reeb vector field of $\alpha$ is actually given by $R_\varepsilon = \partial_\varphi + \chi \partial_\theta$. The monotonicity property of $\alpha$ will be implied by equations $(F_M)$, $(G_M)$, and $(H_M)$.

We finish this section with the

Proof of Theorem 7.3. We define $\alpha$ on $V$ by $fdz + gd\varphi + hd\theta$ where $f$, $g$, $h$ are functions as in Lemma 7.4 and on the complement of $V$ by $\tilde{\alpha}$. This yields a smooth one form, see Remark 7.5. Now, we prove that $\alpha$ meets conditions T1-T3:

T1 is fulfilled:

We denote $\alpha_t =: f_t dz + g_t d\varphi + h_t d\theta$. Then $f_t$, $g_t$ and $h_t$ are convex combinations of $f$, $g$ and $h$ and the coefficient functions of $\tilde{\alpha}$ on $V$. It is easy to see that $f_t$, $g_t$ and $h_t$ meet every condition of Lemma 7.4 except for $(G_T')$ and $(H_T')$ if $t \in (0, 1]$.

Now we show that this implies condition (C) for $t \in (0, 1]$; On $V$ we reduce condition (C) by applying property $(F_M)$ to the following form

$$f_t \left( \frac{(g_t r h_t \cdot r - (g_t r h_t \cdot r)}{<0} \right) + \left( f_t r (g_t r h_t \cdot r - h_t g_t r \cdot r) < 0. \right)$$

On the complement $D \setminus V$ we use equations $(G_B)$ and $(H_B)$. Thus (C) simplifies to

$$-2(1 + \varepsilon)r^3 \cos \psi \sin \psi (2f_t - r(f_t r) < 0.$$

T2 is fulfilled:

Since $g$ and $h$ do not depend on the variables $\varphi$ and $\theta$, properties $(G_T)$ and $(H_T)$ ensure condition (7.1) on $T$ and properties $(G_T')$ and $(H_T')$ make
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\[ R_z = \partial_{\varphi} + \chi \partial_{\psi} \] is a Reeb vector field for \( \alpha \) on \( T \) as can be seen as follows:

\[
\alpha(\partial_{\varphi} + \chi \partial_{\psi}) = g(p_T) + \chi h(p_T) = 1
\]

\[
\iota_{\partial_{\varphi} + \chi \partial_{\psi}} d\alpha = - (g_r + \chi h_r)(p_T) \, dr - (g_\psi + \chi h_\psi)(p_T) \, d\psi -
\]

\[
(g_z + \chi h_z)(p_T) \, dz = 0
\]

\( T^3 \) is fulfilled:

By definition the Reeb vector field fulfills—among others—the following equations:

\[
\alpha(R) = 1
\]

\[
d\alpha(R, \partial_r) = 0
\]

\[
d\alpha(R, \partial_\psi) = 0
\]

We assume that the \( z \)-component of \( R \) vanishes. Then from the above system of linear equations for the coefficients of \( R = R_z \partial_z + R_r \partial_r + R_\psi \partial_\psi + R_\varphi \partial_\varphi + R_\theta \partial_\theta \) it follows:

\[
g_r R_\varphi + h_r R_\theta = 0
\]

\[
g_\psi R_\varphi + h_\psi R_\theta = 0
\]

\[
g R_\varphi + h R_\theta = 1
\]

Then, conditions \((G_M)\) and \((H_M)\), applied to the first two equations, imply \( R_\varphi = R_\theta = 0 \) outside \( T \). This contradicts the last equation. So \( R_z \) vanishes only on \( T \) and is equal to 1 outside of \( U \). Since \( \mathbb{R}^5 \setminus T \) is connected, we conclude that \( R_z \) is positive on this set. \( \square \)

7.3. Examples on \( \mathbb{R}^{2n+1} \) for \( n \geq 2 \)

Here we present the method of [GRZ14] to construct examples of contact forms on \( \mathbb{R}^{2n+1}, n \geq 2 \), that induce Reeb dynamics with the same properties as in section 7.2. This proves Theorem 7.1 in every odd dimension larger than 3. Additionally, this technique allows for better control of the Reeb vector field, refer to property R2 below, yielding an \( n \)-dimensional family of trapped Reeb orbits. Based on this work, Arai, Inaba, and Kano enlarged the class of examples of this type showing that on the torus every flow that
is positively transverse to the contact structure is actually realizable by this
method, cf. [AIK14]. In [GRZ14] the construction is done explicitly for the
case of dimension 5. Here we will work it out in arbitrary odd dimension
and enlarge the class of constructed Reeb flows in [GRZ14] slightly. We do
not, however, achieve the generality of [AIK14].

First, we describe the Reeb flow on \( T \) using polar coordinates \((r_i, \theta_i)\) on
the \((x_i, y_i)\)-planes for \( i \in \{1, \ldots, n\} \). Then on \( T = \{ r_1 = r_2 = \ldots = r_n = 1, z = 0 \} \) the Reeb flow is the linear flow described by

\[
\begin{align*}
\dot{\theta}_i &= \chi_i, \\
\dot{r}_i &= 0, \\
\dot{z} &= 0
\end{align*}
\]

for \( i \in \{1, \ldots, n\} \)

where \( \chi_1, \chi_2, \ldots, \chi_n \in \mathbb{R} \) are positive and rationally independent, i.e. the
only vector \((k_1, \ldots, k_n) \in \mathbb{Z}^n \) with \( \sum_{i=1}^n k_i \chi_i = 0 \) is the zero vector. In this
case it follows that each flow line is dense on the torus and, in particular,
nonperiodic, cf. for example [KH95, Prop. 1.5.1].

The following proposition implies Theorem 7.1:

7.6 Proposition. There exists a positive smooth function \( H : \mathbb{R}^{2n+1} \to (0, \infty), n > 1 \), that equals \( 1 \) outside a compact set such that the Reeb vector
field \( R \) associated to \( \alpha = 1/H \cdot \alpha \) has the following properties:

R1. \( R = \sum_{i=1}^n \chi_i \partial_{\theta_i} \) on \( T \)

R2. The cylinder \( Z_T = \{ r_1 = \ldots = r_n = 1, z \in [-1, 0] \} \) over \( T \) is mapped
to itself under the flow induced by \( R \).

R3. \( dz(R) > 0 \) on \( \mathbb{R}^{2n+1} \setminus T \)

7.7 Remark. Even in the case \( n = 2 \) this proposition is slightly more general
than Proposition 4 in [GRZ14] since we do not claim any upper bounds
on the sum of \( \chi_1, \chi_2, \ldots, \chi_n \). This can be achieved by a small change in
the proof. Actually the function \( H \) constructed on the following pages has
its maximum necessarily above \( \sum_{i=1}^n \chi_i \) while the construction in [GRZ14]
yields always a function with image in \((0, 1]\).
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Proof. We start with a list of sufficient conditions for $H$:

\[
\begin{align*}
H &= \sum_{i=1}^{n} \chi_i \\
H_{x_i} &= 2\chi_i x_i, \quad i \in \{1, \ldots, n\} \\
H_{y_i} &= 2\chi_i y_i, \quad i \in \{1, \ldots, n\} \\
H_z &= 0.
\end{align*}
\]

\[
H = \sum_{i=1}^{n} \chi_i \text{ on } T. \tag{H-i}
\]

\[
H - \frac{1}{2} \sum_{i=1}^{n} (x_i H_{x_i} + y_i H_{y_i}) > 0 \quad \text{on } \mathbb{R}^{2n+1} \setminus T. \tag{H-iii}
\]

\[
H \equiv 1 \text{ outside a compact set.} \tag{H-iv}
\]

To this point we assume the existence of a smooth function $H : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ with these properties and prove that the Reeb vector field $R$ associated to $1/H \cdot \alpha_{st}$ meets conditions (R1)-(R3). The crucial point in the argument relies on the defining equations for the Reeb vector field stated in fact 6.1. Since our reference contact form is $\alpha_{st}$ and its Reeb vector field is given by $\partial_z$, equations (6.8) become:

\[
H = \alpha_{st}(R) \text{ and } \iota_R d\alpha_{st} = H\alpha_{st} - dH.
\]

Using the explicit formula for $\alpha_{st}$ the second equation transforms to

\[
2 \sum_{i=1}^{n} \iota_R dx_i \wedge dy_i = \sum_{i=1}^{n} \left( (H_z x_i - H_{y_i}) dy_i - (H_z y_i + H_{x_i}) dx_i \right).
\]

In the following we write $R = H\partial_z + Y$ where $Y$ is the projection of $R$ to $\xi_{st}$ with respect to the splitting $\mathbb{R}\partial_z \oplus \xi_{st}$. Since the standard contact structure is spanned by the following frame

\[
e_i = \partial_{x_i} + y_i \partial_z, \quad f_i = \partial_{y_i} - x_i \partial_z, \quad i \in \{1, \ldots, n\},
\]
the projection of $R$ on $\xi$ can be made explicit as follows

\[ Y = \frac{1}{2} \sum_{i=1}^{n} ((Hzy_i + Hx_i) f_i + (Hzx_i - Hy_i) e_i). \] (7.3)

With this representation we easily obtain from property (H-i) that, on $T$, the $z$ component of $R$ vanishes and more explicitly:

\[ R = \sum_{i=1}^{n} \chi_i (x_i \partial y_i - y_i \partial x_i). \]

On the other hand, since

\[ dz(Y) = -\frac{1}{2} \sum_{i=1}^{n} (x_i H_{x_i} + y_i H_{y_i}), \]

we deduce from inequality (H-iii) that the Reeb vector field fulfills property (R3).

Hence $T$ is an invariant set and the $z$-component of every orbit disjoint from this set increases strictly monotonically. Therefore it suffices to prove for the remaining constraint that all radial components of $R$ vanish on $Z_T$: At first we note that, on $Z_T$, equation (H-ii) implies $H_z = 0$ and $x_i H_{y_i} + y_i H_{x_i} = H_{\theta_i} = 0$ for $i \in \{1, \ldots, n\}$. Now, it follows easily that for $i \in \{1, \ldots, n\}$

\[ r_i dr_i(R) = \sum_{i=1}^{n} \left( y_i (Hzy_i + Hx_i) + x_i (Hzx_i - Hy_i) \right) \]

\[ = \sum_{i=1}^{n} (y_i H_{x_i} - x_i H_{y_i}) = 0 \text{ on } Z_T. \]

It remains thus to construct a function $H$ with the properties assumed above. We start with the following map

\[ (x_1, y_1, \ldots, x_n, y_n, z) \mapsto \sum_{i=1}^{n} \chi_i (x_i^2 + y_i^2) \]
fulfilling properties (H-i) and (H-ii) and modify it smoothly such that the outcome meets all claimed conditions. Here one should observe that for the above function we have equality in condition (H-iii). Thus in a first step we make its radial growth on the hyperplanes \( \{ z = \text{const} \} \) outside the torus smaller, yielding a strict inequality.

Therefore we choose \( c > \frac{1}{\min\{\chi_i, i \in \{1, \ldots, n\}\}} \sum_{i=1}^{n} \chi_i \) and a smooth family \( f_z : \mathbb{R}_0^+ = \{ x \in \mathbb{R} \mid x \geq 0 \} \rightarrow \mathbb{R}, z \in \mathbb{R}, \) of smooth, monotonically increasing functions with the following properties

1. \( f_z(1) = 0 \)
2. \( tf_z'(t) \leq 1 \) and \( tf_z'(t) = 1 \) exactly for \( z = 0 \) and \( t = 1 \)
3. \( \exists T > 0: f_z(t) > \ln c \) for every \( t > T \) and \( z \in \mathbb{R} \).

We set

\[
H_0 = \sum_{i=1}^{n} \chi_i \exp(f_z(x_i^2 + y_i^2)).
\]

Then the first two properties of \( f_z \) imply (H-i) and (H-ii) for \( H_0 \). Actually, also the third condition is fulfilled but we only need the following estimate for the radial derivatives of \( H_0 \) on \( \mathbb{R}^{2n+1} \setminus \mathcal{T} \):

\[
x_i(H_0)_x + y_i(H_0)_y = 2 \chi_i \exp(f_z(x_i^2 + y_i^2))f_z'(x_i^2 + y_i^2)(x_i^2 + y_i^2) < 2 \chi_i \exp(f_z(x_i^2 + y_i^2))
\]

where \( i \in \{1, \ldots, n\} \).

Here one should note that with the monotonicity condition for \( f_z \), which is added in the proof presented here, the radial derivatives of \( H_0 \) are all non-negative.

Next, we consider a smooth function \( g : \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \} \rightarrow \mathbb{R} \) that meets the following conditions

1. \( g(t) = \ln(t) \) near \( \sum_{i=1}^{n} \chi_i \)
2. \( g(t) = 0 \) if \( t \geq c \cdot \min\{\chi_i \mid i \in \{1, \ldots, n\}\} \)
3. \( g'(t) \leq 1/t \) for \( t \in \mathbb{R}^+ \).
We define
\[ H_1 = \exp(g \circ H_0). \]
By the first property of \( g \) it is obvious that equations (H-i) and (H-ii) are achieved also by \( H_1 \). Additionally \( H_1 \) equals 1 outside a cylinder over a compact set in the hyperplane \( \{ z = 0 \} \). In order to verify condition (H-iii) we compute again the radial derivative and obtain
\[
\begin{align*}
x_i(H_1)_{x_i} + y_i(H_1)_{y_i} &= H_1 \cdot (x_i(H_0)_{x_i} + y_i(H_0)_{y_i}) \cdot g' \circ H_0 \\
&< H_1 \cdot 2 \chi_1 \exp(f_z(x_i^2 + y_i^2)) \cdot 1/H_0
\end{align*}
\]
where we used estimate (7.4) and the non-negativity of the radial derivative of \( H_0 \) in the last step. The above inequality yields easily condition (H-iii).

We finish the construction by applying a cutoff function. This implements condition (H-iv) also for large \( z \). We denote by \( h : \mathbb{R} \to [0, 1] \) a smooth function with compact support and \( h|_{[-1,1]} = 1 \). Then, it is easy to see that the following function has properties (H-i)-(H-iv):
\[
H(x_1, y_1, \ldots, x_n, y_n, z) = 1 - h(z) + h(z)H_1(x_1, y_1, \ldots, x_n, y_n, z).
\]
Appendix

A.1. Proof of Lemma 7.4

We start with some notation: For real numbers $t_1 < t_2 \leq t_3 < t_4$ let
$a = a^{t_1,t_2,t_3,t_4} : \mathbb{R} \to [0,1]$ be a smooth bump function such that

$$a|_{\mathbb{R}\setminus (t_1,t_4)} \equiv 0, \quad a|_{[t_2,t_3]} \equiv 1 \quad \text{and} \quad a'(t) \begin{cases} > 0, & \text{if } t \in (t_1,t_2) \\ < 0, & \text{if } t \in (t_3,t_4). \end{cases} \quad (A.1)$$

Now we choose open neighborhoods of $p_T$

$$U = (z_1^-, z_1^+) \times (r_1^-, r_1^+) \times (\psi_1^-, \psi_1^+),$$
$$V = (z_2^-, z_2^+) \times (r_2^-, r_2^+) \times (\psi_2^-, \psi_2^+),$$
$$V_\delta = (z_2^-, z_2^+) \times (1 - \delta, 1 + \delta) \times (\psi_2^-, \psi_2^+),$$
$$W_\delta = (z_3^-, z_3^+) \times (1 - \delta, 1 + \delta) \times (\psi_3^-, \psi_3^+)$$

such that $\bar{W}_\delta \subseteq V \subseteq \bar{V} \subseteq U \subseteq \bar{U} \subseteq D$.

The construction of $f$:

Set $\bar{z} = z_1^-, z_2^-, z_3^-; z_1^+, z_2^+, z_3^+$ (mind the choice of parameters!),
$a_{\bar{z}} = a^{z_1-, z_2-, z_3-; z_1^+, z_2^+, z_3^+}$ and $C := \sup_{r \in [r_2^+, r_4^+]} -a_{\bar{r}}(r) < \infty$. Then for $0 < \varepsilon < \min\{\frac{1}{2}, \frac{1}{C \cdot r_4^+}\}$ the map

$$f(z, r, \psi) = 1 - \varepsilon \cdot a_{\bar{z}}(z) \cdot a_{\bar{r}}(r) \cdot a_{\bar{\psi}}(\psi)$$

clearly meets conditions $(F_B)$ and $(F_M)$ by the choice of parameters. To verify property $(F_C)$ we consider two cases: By definition of $\varepsilon$ it holds for
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\[ r \in [r_2^+, r_1^+] \]

\[
(2f(z, r, \psi) - r f_r(z, r, \psi)) = \frac{2 - \alpha^2(z) \cdot a^\alpha(\psi) \cdot (2 \varepsilon a^\varepsilon(r) - \varepsilon r a^{r'}(r))}{2 - (1 + 1) = 0}.
\]

On the other hand, this estimate follows from \( f > 0 \) and \( f_r(z, r, \psi) \leq 0 \) if \( r \in \mathbb{R} \setminus [r_2^+, r_1^+] \). So \((F_C)\) is fulfilled.

The construction of \( g \) and \( h \):

We start with the construction of \( h \). Let \( u : D \to \mathbb{R} \) be \( u = \tilde{\alpha}(\partial_\theta) = \chi r^2 \sin^2(\psi) \). Then \( u \) meets all conditions for \( h \) in Lemma 7.4 except for \((H_M)\). We define

\[
b^\delta(z, r, \psi) = a^{z_3}z_3^+, z_2^+(z) - a^{1-61-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\delta}(r) \cdot a^{\psi_1^+, \psi_1^+, \psi_1^+, \psi_1^+(\psi)}.
\]

Then there exists \( C_1 < \infty \) such that for every \( \delta > 0 \) the function \( b^\delta \) has the following properties:

\[
b^\delta|_{\mathbb{R}^3 \setminus V_\delta} \equiv 0, \quad b^\delta|_{W_2} \equiv 1 \quad \text{and} \quad |(b^\delta)| < C_1 \quad (A.2)
\]

and it yields for \( p = (z, r, \psi) \in V \)

\[
(b^\delta)_r(p) = \begin{cases} 
> 0, & \text{if } 1 - \delta < r < 1 - \frac{1}{2}\delta \\
= 0, & \text{if } 1 - \frac{1}{2}\delta \leq r \leq 1 + \frac{1}{2}\delta \\
< 0, & \text{if } 1 + \frac{1}{2}\delta < r < 1 + \delta 
\end{cases} \quad (A.3)
\]

Furthermore, we define for \( \eta \in \mathbb{R} \)

\[
v^\eta(z, r, \psi) = (1 - r^2) \chi \sin^2(\psi) + \eta \left[ \left( z^2 + \left( \psi - \frac{\pi}{4} \right)^2 \right) r + (r - 1)^3 \right].
\]

Note that \( v^\eta(p_T) = 0 \) for every \( \eta \in \mathbb{R} \). Additionally, the partial derivative of \( u \) with respect to \( r \) and the first summand of this map cancel at the point \( p_T \). If \( \eta > 0 \), the second summand will ensure that this partial derivative

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remains positive outside of \( p_T \). Moreover, we see that for \( p = (z, r, \psi) \in V \)
\[
\partial_\psi (u + b^\delta v^0)(p) = 2\chi \sin\psi \cos\psi \cdot r^2 \\
+ \left( (b^\delta)\chi \sin^2\psi 2b^\delta \chi \sin\psi \cos\psi \right) (1 - r^2) \\
\geq C_2 r^2 - C_3 \cdot |1 - r^2|.
\]
with a positive constant \( C_2 := \min_{\psi \in [\psi^-_2, \psi^+_2]} 2\chi \sin\psi \cos\psi \) and a finite constant \( C_3 := \max_{\psi \in [\psi^-_2, \psi^+_2]} \chi (C_1 \sin^2\psi + 2 \sin\psi \cos\psi) \) that is by (A.2) independent of \( \delta \). Hence we can choose \( \tilde{\delta} > 0 \) small enough, so that for \( p \in V_{\tilde{\delta}} \subseteq V \)
\[
\partial_\psi (u + b^\delta v^0)(p) > 0.
\]
By definition it holds
\[
v^0(z, r, \psi) = (1 - r^2)\chi \sin^2\psi \begin{cases} > 0, & \text{if } r < 1 \\ < 0, & \text{if } r < 1. \end{cases}
\]
Using again continuity and the compactness of \( \overline{V}_\delta \) we find an irrational \( \tilde{\eta} > 0 \) such that for \( v = v^\tilde{\eta} \) and every \( p = (z, r, \psi) \in V_{\tilde{\delta}} \)
\[
v(p) \begin{cases} > 0, & \text{if } 1 - \delta < r < 1 - \frac{1}{2}\delta \\ < 0, & \text{if } 1 + \frac{1}{2}\delta < r < 1 + \delta \end{cases} \quad (A.4)
\]
\[
\partial_\psi (u + b^\delta v)(p) > 0.
\]
We define
\[
h = u + b^\delta v.
\]
It is then easy to verify that this function meets conditions \((H_B)\), \((H_T')\), and \((H_T)\). For \((H_M)\) we consider \( h_r \) on \( V_{\tilde{\delta}} \):
\[
h_r = u_r + (b^\delta)_r v + b^\delta v_r
\]
Appendix

From equations (A.2), (A.3), and (A.4) it follows that for all $p \in V_{\delta}$:

$$h_r(p) \geq 2r \chi \sin^2 \psi$$

$$+ b^\delta \left[-2r \chi \sin^2 \psi + \eta \left(z^2 + \left(\psi - \frac{\pi}{4}\right)^2 + 3(r - 1)^2\right)\right]$$

$$= (1 - b^\delta)2r \chi \sin^2 \psi$$

$$+ b^\delta \left[\eta \left(z^2 + \left(\psi - \frac{\pi}{4}\right)^2 + 3(r - 1)^2\right)\right]$$

$$\geq 0.$$ 

Hence $h_r|_{D\setminus\{p\tau\}} > 0$. Moreover $h_\psi > 0$ by construction. This yields $(H_M)$.

The construction of $g$ can be realized in an analog way.
Bibliography


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Bibliography


