First-Order and Modal Logics for Spatial Reasoning

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For the Cypriot parenthood

(and in particular for Hüseyin & Ümran. . .)

"In the 1960s and 1970s, students frequently asked, 'Which kind of representation is the best?' and I usually replied that we'd need more research... But now I would reply: To solve really hard problems, we'll have to use several different representations. This is because each particular kind of data structure has its own virtues and deficiencies, and none by itself would seem adequate for all the different functions involved with what we call common sense."

– *Marvin Minsky*

Doctoral dissertation oral defense took place on **5 September 2011** with the committee consisting of:

- **Chair**: Professor Christian Schindelhauer, University of Freiburg;
- **Sitting in**: Professor Matthias Teschner, University of Freiburg;
- **Supervisor**: Professor Bernhard Nebel, University of Freiburg;
- **Referee**: Professor Frank Wolter, University of Liverpool.

Dean of the Faculty: Professor Bernd Becker.

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Chapter 1

Introduction

Spatial representation and reasoning is mainly concerned with the reasoning about our surrounding physical space and has found many applications in various fields including geographical information systems (GIS), robotic navigation, high level vision, spatial propositional semantics of natural languages, engineering design, specifying visual language syntax and semantics, commonsense reasoning about physical systems and document-structure recognition [24, 20, 23, 9, 33, 37, 1, 2].

There are already various elegant theories of the space developed within mathematics over the centuries, like (point-set) topology and Euclidean (or non-Euclidean) geometry, which can provide a sophisticated basis for the representation of the space. However, developing and investigating the properties of algorithms for reasoning with these theories has not been a topic of interest until the recent attention from certain fields in computer science, especially from the field of artificial intelligence [23, 16].

The history of formalisms of space can perhaps be dated as back as Whitehead's philosophical perspective on the matter [68]. Many formalisms which build on such mathematical theories, especially topology [63, 57, 14, 41, 3], Euclidean geometry [62, 64, 10, 11, 66], affine geometry [4, 5], metric spaces [46, 72] and semi-metric structures [28] have been developed since Whitehead's work, in order to reason about the space. With the computational analysis of (some of) these formalisms [40, 46], it has become obvious that there is no straightforward way of finding practical algorithms which can perform *general* reasoning about the space (e.g., full-geometric reasoning) in an *efficient* manner. This resulted in the natural outcome of developing different formalisms using less expressive techniques (e.g., relation algebras instead of first-order logics) aimed at solving *particular* reasoning problems about the space. For example, there are exclusive formalisms for reasoning about the size and shape aspects of the objects [45, 74, 16], formalisms for describing positional information [47, 37, 59, 17, 43, 49] or formalisms which deal with the topological relationships between objects [56, 21, 32, 30, 8, 6, 15, 71] and yet some other, which are concerned with metric-like [42, 19, 73] or metric [46, 72] information in the space.

Naturally, the most useful techniques are those which attempt to combine together some of the different aspects of reasoning about the space in a practical and computationally feasible way. For example, there are formalisms which combine distance and orientation information to obtain positional representation [17, 43, 34]. Yet another example is the combination of topological relationships with 'time' in order to achieve spatio-temporal formalisms [38, 12]. From a higher point of view, formalisms which combine qualitative techniques with quantitative ones are likely to address real-life problems better than others. Examples of this include the combination of topology with metric information [72].

Apart from the fragmentation of formalisms based on the facts related to the computational costs of general spatial reasoning algorithms, there are simply many different ways of reasoning about the space depending on the application area of interest as well, fuelling the need for even more fragmentation of spatial formalisms.

Spatial reasoning formalisms appeared in the form of various formal languages including relation algebras, propositional logics, intuitionistic logics, modal logics and first-order logics. As a matter of fact, logics which can be interpretted spatially have been a topic of interest long before the appearance of logical formalisms that are bred within the field of spatial reasoning and representation [48]. These naturally include the influential works of Tarski [62, 63, 64] and Grzegorczyk [40, 41] besides Rescher and Garson's 'topological logic' [57], von Wright's 'logic of place' [67, 44], Sergeberg's 'logic of elsewhere' [60] and Venema's 'compass logic' [66, 50].

This thesis aims to contribute formalisms in the form of modal and first-order logics aiming possible applications in different areas of spatial reasoning where so far no known contributions exist, with the exception of Chapter 2[1]. We investigate important theoretical properties of the introduced logics like the axiomatisability, completeness, finite model property, decidability and computational complexity.

In Chapter 2, we introduce logics of comparative distances. Many formalisms in the literature have been devoted to capture the common-sense relationships between objects by using topology (see Figure 1.1). Although topology is a very attractive model for common-sense reasoning about the space due to its computational feasibility, there are many areas where a purely topological representation remains simply inadequate. For example, when two objects are apart from each other (or when they are 'disconnected' in the appro-

[1] One of the areas that this thesis contributes to, is the distance and metric aspects of the space, for which very serious research results already exist, especially by Wolter and Zakharyaschev [46, 72]. Our investigations take an alternative path compared to their line of work on the matter.

priate topological terminology), there is nothing more that can be said about their relationship to each other: They could be very close to each other or they could be miles apart from each other –we can not tell the difference. Hence, there is a certain interest for formalisms that are more expressive than topology and yet somehow remain less expressive than a fully-metric representation.

Logics of comparative distances aims to improve on this, without introducing a fully-metric representation of the space. The novel side of the formalism is that, instead of quantitatively measuring the distances between points in the space (from which the distance between objects can be obtained), it compares the distances between objects, e.g., 'if my arm can reach to my computer but not to my desk lamp, then the distance between me and the desk lamp is greater than the distance between me and my computer.' We employ modal and first-order logic formalisms to perform this kind of spatial reasoning and investigate their logical and computational properties.

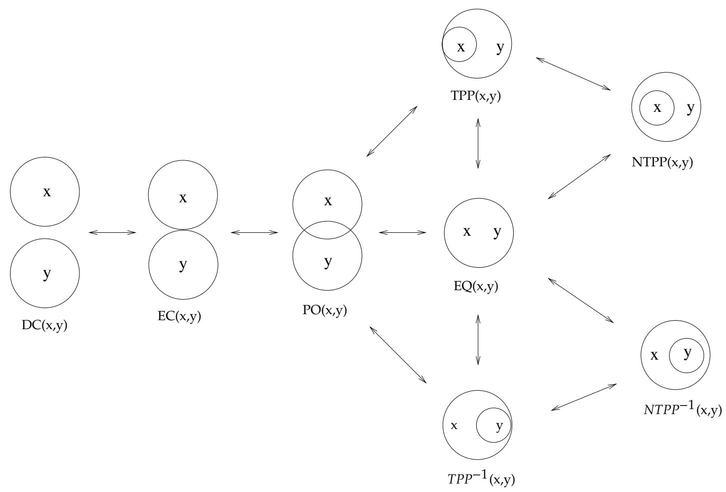

Figure 1.1: A commonly used topological representation of relationships between objects. Eight relations, mostly referred to as the 'RCC-8 relations.'

In Chapter 3, we present a modal logic formalism which can talk about angles. Our underlying goal in constructing this logic is not only to obtain a formalism which can reason about mere angles, but to contribute the development of logical formalisms of trigonometry which will eventually deal with angles.

Trigonometry has an enormous variety of applications. The ones mentioned explicitly in textbooks and courses on trigonometry are its uses in practical endeavors such as navigation, land surveying and building. It is also used extensively in a number of academic fields, primarily mathematics, science and engineering. But perhaps trigonometry is known chiefly for its application to measurement problems. A particular application of trigonometry can be observed at the 'Canadarm2' robotic manipulator on the International Space Station, which is operated by controlling the angles of its joints. Calculating the final position of the tip of the arm requires repeated use of the trigonometric functions of those angles. See Figure 3.3 for an actual picture of the robotic arm.

An example of common-sense reasoning tasks which we would like to tackle by using trigonometric formalisms is as follows (see Figure 1.2 for the illustration): Person A and person B are on the shores of opposite sides of a river, which has a total width of d meters. If B is standing at a degrees of angle with respect to A when A is facing directly the opposite side of the river, then what is the distance between A and B?

Unfortunately, the angular modal logic presented in this thesis deals only with the angle side of the problem. More precisely, our formalism is only able to reason about the angle information within triangles induced by every trio of points in the space and not the distance information between points. Although there is a discussion at the end of Chapter 3 on how angles and distances can be combined in a single formalism, this major but important task is a part of the future research agenda.

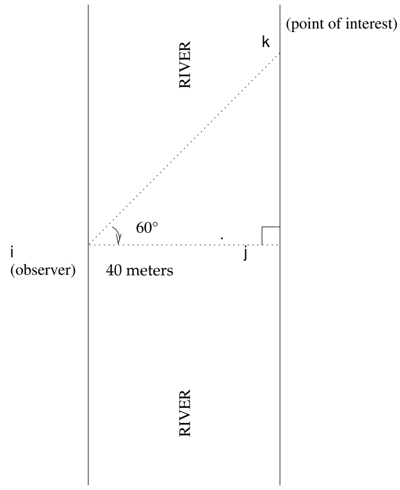

Figure 1.2: A practical use of trigonometry: Calculating the distance between two points on the sides of a river, with the only knowledge of the width of the river and an angle.

For the modal language, we use binary modal operators with the usual Kripkean semantics, e.g., 'φ holds at somewhere and ψ holds at somewhere else, with a degrees of angle in between about here.' As we will show in the chapter, it becomes a trivial task to express the useful qualitative notions of *collinearity* and *betweenness* as well, within this paradigm.

Despite formalisms of distances have been studied extensively (both in qualitative and quantitative settings), formalisms that talk about angles and more importantly, formalisms which can perform reasoning on the combination of distances and angles have not been studied in the setting of formal logic. Given the importance of trigonometric reasoning and its wide areas of possible applications, angular modal logic is an important step in the direction of developing formalisms for trigonometric reasoning.

Chapter 2

Logics for Reasoning with Comparative Distances

2.1 Introduction

In this chapter we deal with the revitalization of Theodore de Laguna's notion of 'can-connect', with the purpose of developing first-order and modal logical formalisms that have the ability to represent and reason with *comparative* distance information. In other words, we are interested in formalisms of qualitative distance information, in contrast to the formalisms of quantitative nature. Laguna's original idea appears in an article [28] which he regards as an appendix to his "revisit to the basic elements of mathematical geometry from the window of actual human experience" (in contrast to the abstract space, which is usually the case with most of the mathematical representations) [26, 27].

At the heart of Laguna's work lies his ontology built on the notion of 'solids.' This is actually very similar to other terms such as 'individuals', 'regions' and 'volumes', that can be easily found throughout the spatial reasoning literature and refer to the very same type of ontological basis as intended by this term. Especially from a philosophical point of view, the discussion regarding the choice of suitable ontologies for spatial formalisms played an important role in the field's research [62, 14, 39, 53, 22, 21]. As Simons says, the problem is that, "nobody has ever perceived a 'point', or ever will do so, whereas people have perceived individuals of finite extent" [61]. So, for the researchers of the field, using an ontological basis like the solids within the formalisms which aim to represent and reason with our physical surrounding space means an attractive harmony between these formalisms and what they claim to be talking about. Unfortunately, choosing such an approach over point-based representations implies abandoning the comfort provided by using well-established mathematical theories.

The intended semantics of the can-connect notion is described as follows by Laguna: A solid a can-connect two other solids, say b and c, whenever a can be moved into simultaneous contact with solids b and c, while all the solids a, b and c remain deformation-free during this process.

By using this notion, very simple but effective distance measurements between solids can be introduced in a very natural way. If a can-connect solids b and c, but it can not d and e, then this implies that solid b is nearer to c, than d is to e. From here, the notion of 'equidistant' can also be trivially defined.

This method of dealing with distances has three main advantages: Firstly, we are able to handle the distance information between solids (instead of points) in a natural way. Secondly, we do not need to incorporate any numeric parameters or values into the formalisms which encompass can-connect, hence allowing mathematically simple and elegant theories to be formed. Thirdly, we are able to compare two distances within the formalisms utilising can-connect, in other words, we can make statements of the form "the distance between a

and b is greater than the distance between c and d". This is important because as the work of Wolter and Zakharyaschev on quantitative distance logics shows, without the use of a notion like can-connect, comparison of distances within formalisms is actually very difficult, if not impossible [72].

From a general perspective, the idea of formalisms designed to talk about distances is certainly not new [57, 67, 60, 44], given the core importance of distance data in many applications, especially the ones dealing with the physical space. This is because distance information allows one of the most basic types (besides topology) of relationships between spatial entities to be established. For example, with detailed distance information, one can represent and reason about the size and shape of the objects [23]. Even non-Euclidean distances are of interest in computer science: In the development of 'logics of similarity' in the field of approximate reasoning [29, 35], similarity measures are used to classify various sets of objects [18] and require reasoning in metric spaces that are non-Euclidean.

The investigation of the theoretical properties of reasoning with distances came with a more recent line of work, which studies knowledge representation formalisms in the form of a combination of modal and description logics and investigates their computational properties [46, 72]. Theoretical and in particular computational properties of distance formalisms have not been addressed throughly in any other work except the aforementioned studies.

We propose logics from two types of languages in which we embed can-connect: First, we embed the ternary can-connect relation into a first-order language and interpret it using standard metric spaces. We provide a finite axiomatisation of the resulting first-order logic. Our axiomatisation provides 'mereology' as a sub theory. This can be compared to the case of spatial theories where the topological 'connection' primitive allows the definition of mereology as a sub theory [3]. Moreover, our first-order logic allows the construction of the new solids from the old ones as well, e.g., given solids a and b, the sum (union) of a and b, the product (intersection) of a and b and their complements can be easily defined inside the theory. This implies that one can make expressions of the form "the sum of solids a and b can-connect the solids c and d".

In the rest of the chapter, we introduce multi-modal languages with the usual Kripkean semantics. These languages mainly consist of a polyadic modality of the form $\langle \mathbf{CC} \rangle(\varphi, \psi)$, with the intended semantics that "here can connect somewhere φ and somewhere ψ." Then, we extend this language by adding more modal operators in order to be able to talk about the lengths of solids. By the length of a solid, we mean the greatest distance between the points of a solid. This new language comes with a parameter set for lengths and nullary modalities of the form $\langle \mathbf{L}_{=x} \rangle$ and $\langle \mathbf{L}_{<x} \rangle$, for each parameter x. The intended meanings of these modal operators are as expected "the length of this solid is equal to x" and "the length of this solid is less than x," respectively. The main results of these investigations are that both modal logics have a satisfiability problem that is NP-complete. Moreover, we show that the first modal logic can be finitely axiomatised.

This chapter is organised as follows: In Section 2.2, we introduce the first-order formalism and provide a semantically complete, finite axiomatisation. In Section 2.3, we concentrate on the modal formalism embracing the notion of can-connect and show that this modal logic is finitely axiomatisable, has the finite model property and that it is decidable. Moreover, we prove that this modal logic has an NP-complete satisfiability problem. In Section 2.4, we extend the modal logic of can-connect and incorporate the notion of lengths for solids. Our results show that this logic enjoys the finite model property, decidability and an NP-complete satisfiability problem as well. We finish with

Section 2.5, where we summarize our achievements and discuss future research topics.

2.2 First-Order Comparative Distance Logic

The goal of this section is to develop a first-order logic which can talk about distance information in a qualitative and cognitively plausible manner, parallel to the main scheme of this chapter. As a part of this section, we will introduce the semantic structure which we will be working with throughout the entire chapter. The structure in question has the novelty of embedding the notion of 'individuals' (in contrast to points) inside the concept of a metric space. Distance information among the individuals will be handled with the help of Laguna's notion of can-connect. Unfortunately, one of the main results of this section will be that despite its simplicity, reasoning about qualitative distances via such structures in a first-order setting is computationally infeasible.

2.2.1 Language and Semantics

We begin by introducing the first-order language $\mathcal{L}^1$, which has the usual properties that can be expected from a first-order language. $\mathcal{L}^1$ contains denumerably many variable symbols, which we generally denote by $x, y, z, \ldots$ etc. and denumerably many constant symbols, which we generally denote by $c_1, c_2, c_3, \ldots$ etc. A term in the language $\mathcal{L}^1$ is either a variable or a constant.

Naturally, $\mathcal{L}^1$ contains the standard basic boolean operators $\vee, \neg$ and the proposition constant of verum $\top$, besides the first-order existential quantifier $\exists$. The operators of $\wedge, \rightarrow, \leftrightarrow, \bot$ and $\forall$ represent the usual duals and shorthands for the aforementioned basic operators. Finally and most importantly, $\mathcal{L}^1$ contains a non-logical, ternary, primitive relation symbol CC.

Atomic formulas of $\mathcal{L}^1$ are expressions in the form of $t_1 = t_2$ or $\mathsf{CC}(t_1, t_2, t_3)$, where t_1, t_2 and t_3 are terms. Arbitrary formulas of $\mathcal{L}^1$ are generated in the usual recursive manner using the basic operators of the language.

The language $\mathcal{L}^1$ is interpreted over models based on structures which consist of a metric space and a set of 'individuals'. Hence, we call them 'metric structures with individuals'. More precisely, we will be dealing with structures in the following form:

$$\mathfrak{F} = \left\langle \mathsf{W}, d, \mathbb{I} \right\rangle,$$

where $\langle \mathsf{W}, d \rangle$ is a metric space and $\mathbb{I}$ is a set, members of which are called as individuals and satisfy the following constraints:

(**CNT**1) $\mathbb{I} \subseteq 2^{\mathsf{W}}$ and $\mathsf{W} \in \mathbb{I}$,

(**CNT**2) $\forall x \in \mathbb{I}[x \neq \emptyset]$,

(**CNT**3) $\forall p \in \mathsf{W}[\{p\} \in \mathbb{I}]$,

(**CNT**4) $\forall x \in \mathbb{I}[x \neq \mathsf{W} \Rightarrow \sim x \in \mathbb{I}]$,

(**CNT**5) $\forall x \in \mathbb{I} \forall y \in \mathbb{I}[x \cap y \neq \emptyset \Rightarrow x \cap y \in \mathbb{I}]$,

(**CNT**6) $\forall x \in \mathbb{I} \forall y \in \mathbb{I}[x \cup y \in \mathbb{I}]$.

Now, our models are pairs in the form

$$\mathfrak{M} = \left\langle \mathfrak{F}, \mathsf{C} \right\rangle,$$

where $\mathfrak{F}$ is a metric structure with individuals and C is a function interpreting the constants symbols of $\mathcal{L}^1$ as individuals from $\mathbb{I}$.

Let α, β be two formulas, t_1, t_2 be two terms and let $\mathfrak{a}$ be an assignment function mapping free occurring variables to the elements of $\mathbb{I}$. The interpretation of arbitrary $\mathcal{L}^1$ formulas is achieved in the usual inductive manner by defining a relation of truth $\models_{\mathfrak{a}}$ as follows:

- $\mathfrak{M} \models_{\mathfrak{a}} \top$,
- $\mathfrak{M} \models_{\mathfrak{a}} \alpha \land \beta$ iff $\mathfrak{M} \models_{\mathfrak{a}} \alpha$ and $\mathfrak{M} \models_{\mathfrak{a}} \beta$,
- $\mathfrak{M} \models_{\mathfrak{a}} \neg\alpha$ iff $\mathfrak{M} \not\models_{\mathfrak{a}} \alpha$,
- $\mathfrak{M} \models_{\mathfrak{a}} t_1 = t_2$ iff $\mathfrak{a}(t_1) = \mathfrak{a}(t_2)$,
- $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{CC}(x, y, z)$ iff

$$\exists p_1 \exists p_2 \exists p_3 \exists p_4 \Big[p_1 \in \mathfrak{a}(y) \land p_2 \in \mathfrak{a}(z) \land p_3 \in \mathfrak{a}(x) \land p_4 \in \mathfrak{a}(x) \land \\ d(p_1, p_2) \leq d(p_3, p_4) \Big],$$

- $\mathfrak{M} \models_{\mathfrak{a}} \exists x\alpha$ iff $\mathfrak{M} \models_{\mathfrak{b}} \alpha$ where $\mathfrak{b}$ is an assignment which differs from $\mathfrak{a}$, if at all, only on x.

The class of all metric models with individuals is denoted by M. As usual, validity (of a formula α) in every metric model with individuals and every assignment is denoted by writing $\mathsf{M} \models \alpha$.

2.2.2 Axiomatisation

Combining the axioms and inference rules for first-order logic with the axioms intended to capture the necessary properties of comparative distance logic, which are given below through **AXM**1 to **AXM**10, results with the formation of a proof system, its theory which we will denote by **Ax**CD$_1$ and denote its 'relation of proof' by $\vdash$. A proof in this proof system is a usual sequence of sentences of $\mathcal{L}^1$ such that each sentence is either an axiom of the system or derivable from the previous elements of the sequence using modus ponens or universal generalisation, in which case we write $\mathbf{Ax}\mathrm{CD}_1 \vdash \alpha$, where α is the formula proved.

(**AXM**1) $\forall x \forall y[\mathsf{CC}(x, y, y)]$,

(**AXM**2) $\forall x \forall y \forall z[\mathsf{CC}(x, y, z) \rightarrow \mathsf{CC}(x, z, y)]$,

(**AXM**3) $\forall x \forall y \forall z \forall p \forall q[\mathsf{CC}(x, y, z) \wedge \neg\mathsf{CC}(x, p, q) \rightarrow \neg\exists r[\mathsf{CC}(r, p, q) \wedge \neg\mathsf{CC}(r, y, z)]]$,

(**DEF**) $\mathsf{I}(x, y) \equiv_{\mathrm{def}} \forall z[\mathsf{CC}(z, x, y)]$,

(**AXM**4) $\forall x \forall y[\forall z[\mathsf{I}(x, z) \leftrightarrow \mathsf{I}(y, z)] \rightarrow x = y]$,

(**AXM**5) $\exists x \forall y[\mathsf{I}(x, y)]$,

(**DEF**) $\mathsf{P}(x, y) \equiv_{\mathrm{def}} \forall z[\mathsf{I}(x, z) \rightarrow \mathsf{I}(y, z)]$,

(**AXM**6) $\forall x \forall y \exists z \forall p[\mathsf{I}(z, p) \leftrightarrow [\mathsf{I}(x, p) \vee \mathsf{I}(y, p)]]$,

(**AXM**7) $\forall x \forall y[\mathsf{I}(x, y) \rightarrow \exists z \forall p[\mathsf{I}(z, p) \leftrightarrow \exists q[\mathsf{P}(q, x) \wedge \mathsf{P}(q, y) \wedge \mathsf{I}(p, q)]]]$,

(**AXM**8) $\forall x[\exists y \neg\mathsf{I}(x, y) \rightarrow \exists z \forall p[\mathsf{I}(z, p) \leftrightarrow \exists q[\neg\mathsf{I}(q, x) \wedge \mathsf{I}(q, p)]]]$,

(**DEF**) $\mathsf{A}(x) \equiv_{\mathrm{def}} \forall y[\mathsf{P}(y, x) \rightarrow x = y]$,

(**AXM**9) $\forall x \exists y[\mathsf{A}(y) \wedge \mathsf{P}(y, x)]$,

(**DEF**) $\mathsf{AP}(x, y) \equiv_{\mathrm{def}} \mathsf{A}(x) \wedge \mathsf{P}(x, y)$,

(**DEF**) $(x, y) \leq (z, p) \equiv_{\mathrm{def}} \forall q[\mathsf{CC}(q, z, p) \rightarrow \mathsf{CC}(q, x, y)]$,

(**DEF**) $(x, y) = (z, p) \equiv_{\mathrm{def}} (x, y) \leq (z, p) \wedge (z, p) \leq (x, y)$,

(**DEF**) $(x, y) < (z, p) \equiv_{\mathrm{def}} (x, y) \leq (z, p) \wedge \neg(x, y) = (z, p)$,

(**AXM**10) $\forall x \forall y \forall z[\mathsf{CC}(x, y, z) \leftrightarrow \exists p \exists q \exists r \exists s[\mathsf{AP}(p, x) \wedge \mathsf{AP}(q, x) \wedge \mathsf{AP}(r, y) \wedge \mathsf{AP}(s, z) \wedge (r, s) \leq (p, q)]]$.

The first three axioms are very intuitive, they capture the essential properties of the can-connect notion. Axiom **AXM**1 states that, any entity can-connect any other entity with itself. With axiom **AXM**2, the symmetric feature of can-connect notion is captured on CC: If an entity can-connect other two entities y and z, then it also can-connect z and y. **AXM**3 is the following property of can-connect: If an entity can-connect a pair of entities and it can-NOT-connect another pair, then there could be no entity which can-connect the latter pair and yet can-NOT-connect the former pair.

Axiom **AXM**4 is the identity axiom. It allows us to determine the identity of entities based on the primitive non-logical notion of can-connect. Axioms **AXM**5, **AXM**6, **AXM**7 and **AXM**8 create new entities from the old ones with the help of the identity axiom. More precisely, **AXM**5 entails the existence of a unique universe which we will denote by **U**. Given two entities x and y; while **AXM**6 entails the existence of a unique entity $x + y$, **AXM**7 entails the existence of a unique entity $x * y$, whenever we have $\mathsf{I}(x, y)$. Finally, **AXM**8 entails the existence of a unique entity $-x$, whenever we have $\exists y \neg \mathsf{I}(x, y)$.

Many similar axiomatisations can be found in the studies of axiomatic spatial logics. For example, Asher and Vieu present a successful axiomatisation of the mereotopology ("geometry of common sense") [3]. However, there are quite a number of problematic first-order axiomatisation attempts with regard to their basic logical properties as well [48]. Such problems are often in the form of inconsistent axiom systems [14] or semantically incomplete systems [**?**]. There are even studies with the pursuit of achieving an absolutely complete (in contrast to semantic completeness) first-order axiom systems [7], which are impossible tasks given the likely undecidability of such logics [40] and the fact that every absolutely complete and recursively enumerable theory must be decidable.

Finally, axiom **AXM**9 states that every entity contains an atomic entity (entities whose only sub-part is itself) and axiom **AXM**10 manifests the interaction between atomic entities and can-connect primitive.

The following lemma will be used in the forthcoming proofs:

Lemma 2.2.1. *The following formulas are theorems of* $\mathbf{Ax}CD_1$*:*

- $\neg[(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) < (c_1, c_2)]$,
- $[(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) = (c_5, c_6)] \rightarrow (c_1, c_2) \leq (c_5, c_6)$.

Proof. In order to see through the first claim, note that $(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) < (c_1, c_2)$ is, by definition, equivalent to $(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) \leq (c_1, c_2) \wedge \neg(c_3, c_4) = (c_1, c_2)$, which is again by definition equivalent to $(c_3, c_4) = (c_1, c_2) \wedge \neg(c_3, c_4) = (c_1, c_2)$, which is a contradiction.

For the second claim, note that by definition $(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) = (c_5, c_6)$ implies that $(c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) \leq (c_5, c_6)$. This obviously implies that $(c_1, c_2) \leq (c_5, c_6)$. □

2.2.3 Soundness and Completeness Theorems

Overview

Establishing a semantical foundation for any kind of spatial logic is essential. The lack of such investigations in the study of several spatial logics has been subject of righteous criticism from within the field [48]. This is because of the fact that any spatial logic study which lacks necessary semantical investigation, bears the risk of not being able to capture the type of reasoning it promises to achieve. In other words, an established semantical foundation guarantees that a logic is able to represent and reason with the structures that are of interest. For example, if a formalism is intended for reasoning with distance information, we should expect a semantical investigation of this formalism based on metric spaces.

In this section, we will establish that the first-order comparative distance logic is sound and complete with respect to the class of all metric structures with individuals M. While the soundness has a completely standard proof, completeness proof employs a Henkin-style argument which consists of more interesting model construction procedures.

For the Henkin style completeness proof, we begin with a set of formulas and aim to build a model using this set of formulas. This technique uses Lindenbaum and Saturation Lemmas [52] in order to obtain sufficient amount of objects to construct the domain of the target model. Namely, with the help of the mentioned lemmata, it generates a set of witnesses and then, the collection of equivalence classes defined over this set becomes the domain of a new model.

The novelty of our proof lies in the construction of a metric function over this domain set. This involves a procedure which inductively assigns a value for a function for all different pairs from the domain. At the final part of the proof, we establish that the resulting structure is a metric structure with individuals.

Theorem 2.2.2 (Soundness)**.** *Let φ be a formula. Then we have that,* $\mathbf{Ax}CD_1 \vdash \varphi \Rightarrow \mathsf{M} \models \varphi$.

Proof. The proof is by induction on the complexity of φ. It is sufficient to establish the base case, which amounts to show that all of the axioms **AXM**1-**AXM**10 are valid on any metric model with individuals. Let,

$$\mathfrak{M} = \left\langle \mathfrak{F}, \mathsf{C} \right\rangle$$

be a model where,

$$\mathfrak{F} = \left\langle \mathsf{W}, d, \mathbb{I} \right\rangle$$

is a metric structure with individuals.

First, let us establish the case of **AXM**1, i.e., that $\mathfrak{M} \models \forall x \forall y[\mathsf{CC}(x, y, y)]$. It is sufficient to show that for every $x, y \in \mathbb{I}$, there are $p_1, p_2 \in x$ and $p_3, p_4 \in y$ such that $d(p_3, p_4) \leq d(p_1, p_2)$. Since $x \neq \emptyset \neq y$ from **CNT**2, we can simply pick some arbitrary $p_5 \in x$ and $p_6 \in y$ and set $p_1 = p_2 = p_5$ and $p_3 = p_4 = p_6$. But then we have that $d(p_1, p_2) = d(p_3, p_4) = 0$, which gives us what we want.

Now we focus on the case of **AXM**2. So we have to show that $\mathfrak{M} \models \forall x \forall y \forall z[\mathsf{CC}(x, y, z) \rightarrow \mathsf{CC}(x, z, y)]$. We will proceed as follows: Suppose that for some $x, y, z \in \mathbb{I}$, there are $p_1, p_2 \in x$, $p_3 \in y$ and $p_4 \in z$ such that $d(p_3, p_4) \leq d(p_1, p_2)$. But since d is symmetric, it follows that $d(p_4, p_3) \leq d(p_1, p_2)$, which gives us what we want.

Let us now consider axiom **AXM**3. This is slightly more complicated than the previous cases. Assume that for some $x, y, z, p, q \in \mathbb{I}$, there are $p_1, p_2 \in x$, $p_3 \in y$ and $p_4 \in z$ such that $d(p_3, p_4) \leq d(p_1, p_2)$ and on the other hand, for every $p'_1, p'_2 \in x$, $p'_3 \in p$ and $p'_4 \in q$ we have that $d(p'_1, p'_2) < d(p'_3, p'_4)$.

Now, for the sake of a contradiction suppose that there are $r \in \mathbb{I}$ such that there are $p_5, p_6 \in r$, $p_7 \in p$ and $p_8 \in q$ such that $d(p_7, p_8) \leq d(p_5, p_6)$ while for every $p'_1, p'_2 \in r$, $p'_3 \in y$ and $p'_4 \in z$, we have that $d(p'_1, p'_2) < d(p'_3, p'_4)$.

Combining the information we have so far, it easily follows that we have $d(p_5, p_6) < d(p_3, p_4)$ and $d(p_1, p_2) < d(p_7, p_8)$. On the other hand, since $d(p_3, p_4) \leq d(p_1, p_2)$, we conclude that $d(p_5, p_6) < d(p_7, p_8)$, which contradicts with the fact that we have $d(p_7, p_8) \leq d(p_5, p_6)$. This ends the case of axiom **AXM**3.

Before we continue any further, let us establish the fact that for any assignment $\mathfrak{a}$, we have that $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{I}(x, y)$ iff $\mathfrak{a}(x) \cap \mathfrak{a}(y) \neq \emptyset$. To see this from left to right, assume that $\mathfrak{M} \models_{\mathfrak{a}} \forall z[\mathsf{CC}(z, x, y)]$. This means that, for every $z \in \mathbb{I}$, there are $p_1, p_2 \in z$, $p_3 \in \mathfrak{a}(x)$ and $p_4 \in \mathfrak{a}(y)$ such that $d(p_3, p_4) \leq d(p_1, p_2)$. Then, suppose that $z = \{p_5\}$ for some $p_5 \in \mathsf{W}$. But this implies that $p_1 = p_2 = p_5$ and moreover that $d(p_1, p_2) = 0$. Hence, we must have $d(p_3, p_4) = 0$. So, $p_3 = p_4$. Thus, $\mathfrak{a}(x) \cap \mathfrak{a}(y) \neq \emptyset$. The opposite direction can be easily established by using a similar argument.

To see the case of axiom **AXM**4, we will show that for every $x, y \in \mathbb{I}$ we have that $\forall z[x \cap z \neq \emptyset \Rightarrow y \cap z \neq \emptyset] \Rightarrow x \subseteq y$. If $y = \mathsf{W}$ then we are through. So assume that $y \neq \mathsf{W}$. Then from **CNT**4, it follows that $\sim y \in \mathbb{I}$. Suppose we have that for every $x, y \in \mathbb{I}$ we have that $\forall z[x \cap z \neq \emptyset \Rightarrow y \cap z \neq \emptyset]$ and for sake of a contradiction, also suppose that $x \nsubseteq y$. From here it follows that $x \cap \sim y \neq \emptyset$. But from the hypothesis, this implies that $y \cap \sim y \neq \emptyset$, which is a contradiction.

The case of **AXM**6 is trivial once we observe that for every $x, y \in \mathbb{I}$ we have that $x \cup y \in \mathbb{I}$ from **CNT**6 and thus, we can always select this as an assignment for $x + y$. Now all that needs to be done is to show that for every $x, y \in \mathbb{I}$ we have that $\forall z[(x \cup y) \cap z \neq \emptyset \Leftrightarrow [(x \cap z) \neq \emptyset \vee (y \cup z) \neq \emptyset]$, which is a well known fact itself. A similar proof for the validity of axioms **AXM**7 and **AXM**8 can be easily generated by using the constraints **CNT**5 and **CNT**4, respectively. The case of axiom **AXM**5 is absolutely trivial.

It is obvious that we have $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{A}(x)$ iff $\mathfrak{a}(x)$ is a singleton. Now, to see the case of axiom **AXM**9, let $x \in \mathbb{I}$. From **CNT**2, it follows that $x \neq \emptyset$. Pick $p_1 \in x$. Now from **CNT**3, it follows that $\{p_1\} \in \mathbb{I}$. Hence, we have found a singleton $\{p_1\}$ such that $\{p_1\} \subseteq x$.

Before we move into the case of **AXM**10, assume that $\mathfrak{M} \models_{\mathfrak{a}} (x, y) \leq (z, p)$. By definition, we get that $\mathfrak{M} \models_{\mathfrak{a}} \forall q[\mathsf{CC}(q, z, p) \rightarrow \mathsf{CC}(q, x, y)]$. We will show that this means $\min\{d(p_1, p_2) \mid p_1 \in \mathfrak{a}(x), p_2 \in \mathfrak{a}(y)\} \leq \min\{d(p_1, p_2) \mid p_1 \in \mathfrak{a}(z), p_2 \in \mathfrak{a}(p)\}$. For the sake of a contradiction, assume not. Then, $\exists p_1 \in \mathfrak{a}(z), \exists p_2 \in \mathfrak{a}(p)$ such that $\forall p_3 \in \mathfrak{a}(x), \forall p_4 \in \mathfrak{a}(y)[d(p_1, p_2) < d(p_3, p_4)]$. Note that $\{p_1, p_2\} \in \mathbb{I}$. From here, it follows that we have $\mathfrak{M} \models_{\mathfrak{b}} \mathsf{CC}(q, z, p)$ and $\mathfrak{M} \not\models_{\mathfrak{b}} \mathsf{CC}(q, x, y)]$ where $\mathfrak{b}$ is an assignment which differs from $\mathfrak{a}$ only on q such that $\mathfrak{b}(q) = \{p_1, p_2\}$. This is a contradiction.

Now, finally to see the case of **AXM**10, assume that for some assignment $\mathfrak{a}$ we have $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{CC}(x, y, z)$. Then there are $p_1, p_2 \in \mathfrak{a}(x)$, $p_3 \in \mathfrak{a}(y)$ and $p_4 \in \mathfrak{a}(z)$ such that $d(p_3, p_4) \leq d(p_1, p_2)$. First note that from **CNT**3, it follows that $\{p_1\}, \{p_2\}, \{p_3\}, \{p_4\} \in \mathbb{I}$. Now, together with the above paragraph it follows that $\mathfrak{M} \models_{\mathfrak{b}} \mathsf{AP}(p, x) \wedge \mathsf{AP}(q, x) \wedge \mathsf{AP}(r, y) \wedge \mathsf{AP}(s, z) \wedge (r, s) \leq (p, q)$ where $\mathfrak{b}$ is an assignment which differs from $\mathfrak{a}$, if at all, on p, q, r and s such that $\mathfrak{b}(p) = \{p_1\}$, $\mathfrak{b}(q) = \{p_2\}$, $\mathfrak{b}(r) = \{p_3\}$ and $\mathfrak{b}(s) = \{p_4\}$.

□

We now turn our attention to the completeness of the axiomatic system **Ax**CD$_1$ with respect to the class of all metric models with individuals. We begin by remembering one of the standard lemmas in the scheme of Henkin-style completeness proofs for first-order logics [3]. This lemma naturally has a standard proof, hence there is no need to provide one here.

Lemma 2.2.3 (Witness or Saturation Lemma)**.** *Every* **Ax***CD$_1$-consistent set of sentences Σ can be extended to a saturated set Σ' in the extension of $\mathcal{L}^1$, $\mathcal{L}^1(c_1, c_2, \dots)$, such that $\Sigma' \vdash \exists x\varphi \to \varphi[c_k/x]$, for every formula with one free variable φ and c_k is a witness for x.*

So, we have finally arrived at the core of our completeness argument, the Henkin Lemma. Henkin Lemma must be provided with a proof. Since the proof is quite long, it is split into multiple shorter lemmata.

Lemma 2.2.4 (Henkin Lemma)**.** *Every* **Ax***CD$_1$-consistent, maximal and saturated set of sentences Γ yields a metric model with individuals $\mathfrak{M}$ such that for any formula φ, we have that $\mathfrak{M} \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. Let γ be a set of **Ax**CD$_1$-consistent set of sentences. It follows from the Lindenbaum's Lemma (Lemma 5.0.2) and from the Saturation Lemma (Lemma

2.2.3) that, we can extend γ to a **Ax**CD$_1$-consistent, maximal and saturated set of sentences Γ. Now, we have a collection of constants $\mathfrak{C}$ occurring in Γ. We will utilise equivalence classes to represent the individuals of our model. In order to achieve this, we first define the relation $\equiv$ such that for every $c_1, c_2 \in \mathfrak{C}$ we have that,

$$c_1 \equiv c_2 \Leftrightarrow \Gamma \vdash c_1 = c_2.$$

Clearly, $\equiv$ is an equivalence relation over $\mathfrak{C}$. Let us define the equivalence classes induced by the relation $\equiv$ as follows:

$$|c_1| = \{c_2 \in \mathfrak{C} \mid c_1 \equiv c_2\}.$$

We have now constructed the basic elements of our model. However, in our models individuals are represented as usual sets of points. So far, we have only created the elements to stand for individuals. So, we now need to "fill" these individuals with (appropriate) points.

First, we define the universe -the set of all points- where our individuals will inherit their points from. We will denote the universe by W and define it as follows:

$$\mathsf{W} = \{c \in \mathfrak{C} \mid \Gamma \vdash \mathsf{A}(c)\}.$$

In other words, points are simply derived from the constants which are "syntactically points" according to Γ. Now we have to assign points to the corresponding individuals. We achieve this as follows:

$$\mathcal{P}(|c_1|) = \{c_2 \in \mathsf{W} \mid \Gamma \vdash \mathsf{P}(c_2, c_1)\}.$$

Now, we arrived at the most complicated and important part of the proof: inducing a metric space over W. More specifically, we have to define a (metric) function $d\colon \mathsf{W} \times \mathsf{W} \to \mathbb{R}^+ \cup \{0\}$ such that the existing metric information within Γ is represented via d.

We will devote a construction procedure which will take points from W as the input and at the end of the procedure, it will return a function d satisfying all the constraints we have mentioned in the above paragraph. Before giving the procedure in detail, we will define some shorthands for simplifying the specification of the procedure. First, we set up some relations on $\mathsf{W} \times \mathsf{W}$. Let $c_1, c_2, c_3, c_4 \in \mathsf{W}$. Then,

- $(c_1, c_2) \sqsubseteq (c_3, c_4)$ iff $\Gamma \vdash (c_1, c_2) \leq (c_3, c_4)$,
- $(c_1, c_2) \sqsubset (c_3, c_4)$ iff $\Gamma \vdash (c_1, c_2) \leq (c_3, c_4) \wedge \neg(c_3, c_4) \leq (c_1, c_2)$,
- $(c_1, c_2) \square (c_3, c_4)$ iff $\Gamma \vdash (c_1, c_2) \leq (c_3, c_4) \wedge (c_3, c_4) \leq (c_1, c_2)$.

The procedure given below works by considering *every different triple* of points from W, one triple in each iteration, until all the combinations of all the points from W are handled. Given an arbitrary triple of points, say c_1, c_2 and c_3, there are three values (one for each of the pairs (c_1, c_2), (c_2, c_3) and (c_1, c_3)) to be assigned by the procedure to the function d. In order to keep a track of the value-assigned pairs (note that same pairs will most likely occur within many different triples), they are added into a set as soon as their value is assigned by the procedure. This "tracking set" is denoted by AV_n (Assigned Values), where n is the number of iterations. Therefore, we will know that the procedure will quit at the nth iteration, if $\mathrm{AV}_n = \{\{c_1, c_2\} \mid c_1, c_2 \in \mathsf{W}\}$, i.e., when all possible pairs are value-assigned. Another similar notation which we will use frequently is as follows:

$$d(\mathrm{AV}_n) = \{d(c_1, c_2) \mid \{c_1, c_2\} \in \mathrm{AV}_n\}.$$

Thus, $d(\mathrm{AV}_n)$ denotes the set of all values, which are so far assigned by the nth iteration of the procedure.

The assignment of the three values for each pair is done in a certain order. Namely, before any processing, the procedure orders the pairs based on the constraints inherited from Γ. For example, if we have that $(c_1, c_2) \leq (c_2, c_3) \leq (c_1, c_3)$, then the procedure begins by dealing with the pair $\{c_1, c_2\}$ first, then deals with the pair $\{c_2, c_3\}$ in the second order and finally finishes assigning all three pairs by processing the pair $\{c_1, c_3\}$.

Before we give the procedure in detail, let us finally analyse the underlying strategy used by the procedure in the construction of d. The procedure has to achieve two main goals: First of all, the metric constraints inherited from Γ must be satisfied. Secondly, d must satisfy the necessary constraints in order to qualify as a metric function.

For the first goal, since $\mathbb{R}^+$ is dense, the procedure is always guaranteed to find appropriate values from $\mathbb{R}^+$ to assign for d such that the constrains inherited from Γ are satisfied.

In order to achieve the second goal, it ensures that in each iteration, the value picked for the maximal pair is less than twice the value picked for the minimal pair. This guarantees that the function d we end up with satisfies the triangle inequality and hence, becomes a metric function. To exemplify this strategy, consider a triple of c_1, c_2 and c_3 fed into the procedure. Suppose that $(c_1, c_2) \leq (c_2, c_3) \leq (c_1, c_3)$. According to this strategy, an assignment for d is made such that

$$\frac{d(c_1, c_3)}{2} < d(c_1, c_2) \leq d(c_2, c_3).$$

As we will establish in the proof below, this implies that d satisfies the triangle inequality, which is the most crucial condition that d must satisfy in order to qualify as a metric function.

Technically speaking, the strategy in question is implemented by "tracking" a dedicated set of values which we will denote by MPV_n (Maximal Pair Values), where n is the number of iterations. It works as follows: In each iteration, half of the value assigned for the maximal pair is added into MPV_n. In the iterations that follow, the value to be assigned for the minimal pair is chosen such that it is greater than all of the elements in MPV_n. Now, let us give the construction procedure in detail.

Construction 2.2.1 (Metric Construction)**.** The procedure consists of two main parts: The initial part in step 1 and the inductive step 2.

1. Assume that the first input to the procedure is the triple $c_1, c_2, c_3 \in \mathsf{W}$ such that $(c_1, c_2) \leq (c_2, c_3) \leq (c_1, c_3)$.

   Pick three arbitrary elements $r_1, r_2, r_3 \in \mathbb{R}^+$ such that the appropriate ones of the following constraints are satisfied:

   First pick r_1 and r_2:

   - If $(c_1, c_2) \sqsubset (c_2, c_3)$ then $0 < r_1 < r_2 < 2 \cdot r_1$ or,
   - if $(c_1, c_2) \square (c_2, c_3)$ then $0 < r_1 = r_2$.

   Now pick r_3 (r_2 is already picked above):

   - If $(c_2, c_3) \sqsubset (c_1, c_3)$ then $0 < r_2 < r_3 < 2 \cdot r_1$ or,
   - if $(c_2, c_3) \square (c_1, c_3)$ then $0 < r_2 = r_3$.

   Now make the assignments for the function d as follows:

   - $d(c_1, c_2) = d(c_2, c_1) = r_1$,
   - $d(c_2, c_3) = d(c_3, c_2) = r_2$,
   - $d(c_1, c_3) = d(c_3, c_1) = r_3$.

   Set $\mathrm{AV}_1 = \{\{c_1, c_2\}, \{c_2, c_3\}, \{c_1, c_3\}\}$ and $\mathrm{MPV}_1 = \{\frac{d(c_1,c_3)}{2}\}$.

2. The core of the procedure which takes place after the initial step above, is given in an inductive manner as follows: Assume that $n-1$th iteration has already been executed. If $\{\{c_1, c_2\} \mid c_1, c_2 \in \mathsf{W}\} = \mathrm{AV}_{n-1}$, then quit the procedure. Otherwise, start executing the nth iteration as follows:

   Pick a triple from W, say $c_1, c_2, c_3 \in \mathsf{W}$, such that at least one of the pairs arising from this trio is not an element of AV_{n-1} -otherwise there is nothing to do. Suppose that we have the following order among the pairs: $(c_1, c_2) \leq (c_2, c_3) \leq (c_1, c_3)$.

   (a) *Firstly, assign a value for d on the minimal pair* (c_1, c_2)*:*

   If the pair is already processed by an earlier iteration of the procedure, i.e., if $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$, then skip this step and continue with step 2b. Otherwise,

      i. If $\forall \{x, y\} \in \mathrm{AV}_{n-1}[(c_1, c_2) \sqsubset (x, y)]$ then:

      - pick $r \in \mathbb{R}^+$ such that $\max \mathrm{MPV}_{n-1} < r < \min d(\mathrm{AV}_{n-1})$ and assign $d(c_1, c_2) = d(c_2, c_1) = r$ and,
      - set $\mathrm{AV}_n = \mathrm{AV}_{n-1} \cup \{\{c_1, c_2\}\}$.

      ii. If $\forall \{x, y\} \in \mathrm{AV}_{n-1}[(x, y) \sqsubset (c_1, c_2)]$ then:

      - pick $r \in \mathbb{R}^+$ such that $\max d(\mathrm{AV}_{n-1}) < r < 2 \cdot \min d(\mathrm{AV}_{n-1})$ and assign $d(c_1, c_2) = d(c_2, c_1) = r$ and,
      - set $\mathrm{AV}_n = \mathrm{AV}_{n-1} \cup \{\{c_1, c_2\}\}$.

      iii. If $\exists \{x, y\} \in \mathrm{AV}_{n-1}[(c_1, c_2) \Box (x, y)]$ then:

      - assign $d(c_1, c_2) = d(c_2, c_1) = d(x, y)$ and,
      - set $\mathrm{AV}_n = \mathrm{AV}_{n-1} \cup \{\{c_1, c_2\}\}$.

      iv. If none of the above is the case then:

      - pick $r \in \mathbb{R}^+$ such that $\max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_{n-1} \wedge (x, y) \sqsubset (c_1, c_2)\} < r < \min\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_{n-1} \wedge (c_1, c_2) \sqsubset (x, y)\}$

and assign $d(c_1, c_2) = d(c_2, c_1) = r$ and,

- set $\mathrm{AV}_n = \mathrm{AV}_{n-1} \cup \{\{c_1, c_2\}\}$.

(b) *Secondly, assign a value for d on the maximal pair* (c_1, c_3)*:*

If the pair is already processed by an earlier iteration of the procedure, i.e., if $\{c_1, c_3\} \in \mathrm{AV}_{n-1}$, then skip this step and continue with step 2c. Otherwise,

i. If $\forall(x, y) \in \mathrm{AV}_n[(x, y) \sqsubset (c_1, c_3)]$ then:

- pick $r \in \mathbb{R}^+$ such that $\max d(\mathrm{AV}_n) < r <$ $\min\{2 \cdot \min d(\mathrm{AV}_n), 2 \cdot d(c_1, c_2)\}$ and assign $d(c_1, c_3) = d(c_3, c_1) = r$ and,
- set $\mathrm{AV}_n = \mathrm{AV}_n \cup \{\{c_1, c_3\}\}$ and $\mathrm{MPV}_n = \mathrm{MPV}_{n-1} \cup \{\frac{d(c_1,c_3)}{2}\}$.

ii. If $\exists(x, y) \in \mathrm{AV}_n[(c_1, c_3) \sqsupset (x, y)]$ then:

- assign $d(c_1, c_3) = d(c_3, c_1) = d(x, y)$ and,
- set $\mathrm{AV}_n = \mathrm{AV}_n \cup \{\{c_1, c_3\}\}$ and $\mathrm{MPV}_n = \mathrm{MPV}_{n-1} \cup \{\frac{d(c_1,c_3)}{2}\}$.

iii. If none of the above is the case then:

- pick $r \in \mathbb{R}^+$ such that $\max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_n \wedge (x, y) \sqsubset (c_1, c_3)\} < r < \min\{\min\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_n \wedge (c_1, c_3) \sqsubset (x, y)\}, 2 \cdot d(c_1, c_2)\}$ and assign $d(c_1, c_3) = d(c_3, c_1) = r$ and,
- set $\mathrm{AV}_n = \mathrm{AV}_n \cup \{\{c_1, c_3\}\}$ and $\mathrm{MPV}_n = \mathrm{MPV}_{n-1} \cup \{\frac{d(c_1,c_3)}{2}\}$.

(c) *Thirdly, assign a value for d on the final pair* (c_2, c_3)*:*

If the pair is already processed by an earlier iteration of the procedure, i.e., if $\{c_2, c_3\} \in \mathrm{AV}_{n-1}$, then skip this step and end the current iteration. Otherwise,

i. If $\exists\{x, y\} \in \mathrm{AV}_n[(c_2, c_3) \sqsupset (x, y)]$ then:

- assign $d(c_2, c_3) = d(c_2, c_3) = d(x, y)$ and,
- set $\mathrm{AV}_n = \mathrm{AV}_n \cup \{\{c_2, c_3\}\}$.

ii. If 2(c)i is not the case then:

- pick $r \in \mathbb{R}^+$ such that $\max\{d(x,y) \mid \{x,y\} \in \mathrm{AV}_n \wedge (x,y) \sqsubset (c_2,c_3)\} < r < \min\{d(x,y) \mid \{x,y\} \in \mathrm{AV}_n \wedge (c_2,c_3) \sqsubset (x,y)\}$ and assign $d(c_2,c_3) = d(c_2,c_3) = r$ and,
- set $\mathrm{AV}_n = \mathrm{AV}_n \cup \{\{c_2,c_3\}\}$.

Now it only remains to make the finishing touch: For every $c \in \mathsf{W}$ set,

$$d(c,c) = 0.$$

This ends the construction procedure.

Now, we have to establish that the function d constructed by the procedure given above is a metric function. First of all, note that from Construction 2.2.1 it is obvious that d satisfies the following two constraints: $\forall c_1, c_2 \in \mathsf{W}$ and $r \in \mathbb{R}^+ \cup \{0\}$,

- $d(c_1,c_2) = 0$ iff $c_1 = c_2$ and,
- $d(c_1,c_2) = r$ iff $d(c_2,c_1) = r$.

In other words, it only remains to establish that d satisfies the triangle inequality. But for this, first we need the following two lemmata:

Lemma 2.2.5. *For every $c_1,c_2,c_3,c_4 \in \mathsf{W}$, we have that $(c_1,c_2) \sqsubseteq (c_3,c_4)$ iff $d(c_1,c_2) \le d(c_3,c_4)$.*

Proof. Let $c_1,c_2,c_3,c_4 \in \mathsf{W}$ and consider the procedure of Construction 2.2.1 by which d is defined. Clearly, we have that $\{c_1,c_2\},\{c_3,c_4\} \in \mathrm{AV}_m$ for some m. So, if we could show that for any n and any $\{c_1,c_2\},\{c_3,c_4\} \in \mathrm{AV}_n$ we have $(c_1,c_2) \sqsubseteq (c_3,c_4) \Leftrightarrow d(c_1,c_2) \le d(c_3,c_4)$ (i.e., at any step of the construction procedure, the claim holds for all the pairs processed so far by the construction procedure), then we will have the proof we are looking for. The proof of this claim is by induction on n.

The base case for $n = 1$ is immediate from step 1 of Construction 2.2.1. For the inductive step, assume that for any $\{c_1, c_2\}, \{c_3, c_4\} \in \mathrm{AV}_{n-1}$ we have $(c_1, c_2) \sqsubseteq (c_3, c_4) \Leftrightarrow d(c_1, c_2) \leq d(c_3, c_4)$. Now pick $\{c_1, c_2\}, \{c_3, c_4\} \in \mathrm{AV}_n$.

If $\{c_1, c_2\}, \{c_3, c_4\} \in \mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$, then we are immediately through by the induction hypothesis. So, suppose that we have $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$ and $\{c_3, c_4\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$. The proof of the alternate case when $\{c_3, c_4\} \in \mathrm{AV}_{n-1}$ and $\{c_1, c_2\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$ is very similar.

Firstly, suppose that the pair $\{c_3, c_4\}$ is added into AV_n via step 2a of the procedure. In this step, there are four sub-cases based on which a value for $d(c_3, c_4)$ is assigned:

In the case of 2(a)i, in order to see through the claim from left to right direction assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. However, sub-case 2(a)i (when $\forall \{x, y\} \in \mathrm{AV}_{n-1}[(c_3, c_4) \sqsubset (x, y)]$) does not apply here, since we have $(c_1, c_2) \sqsubseteq (c_3, c_4)$ and $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$ by the assumption. Conversely, assume that $d(c_1, c_2) \leq d(c_3, c_4)$. Then since $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$ it follows that $d(c_3, c_4) < d(c_1, c_2)$, which is a contradiction. So, it is impossible that a value for $d(c_3, c_4)$ is assigned in step 2(a)i.

In the case of 2(a)ii, in order to see through the claim from left to right direction assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. Then $d(c_3, c_4)$ is assigned a value r such that $\max d(\mathrm{AV}_{n-1}) < r$. Therefore, we have $d(c_1, c_2) < d(c_3, c_4)$ since $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$. Conversely, assume that $d(c_1, c_2) \leq d(c_3, c_4)$. Then obviously we have that $(c_1, c_2) \sqsubset (c_3, c_4)$, which implies that $(c_1, c_2) \sqsubseteq (c_3, c_4)$.

In the case of 2(a)iii, in order to see through the claim from left to right direction assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. Then we have that $(c_3, c_4) \square (c_5, c_6)$ for some $\{c_5, c_6\} \in \mathrm{AV}_{n-1}$ and by the construction such that $d(c_5, c_6) = d(c_3, c_4)$. On the other hand, since we have $(c_1, c_2) \sqsubseteq (c_3, c_4)$, it follows from Lemma 2.2.1 that we also have $(c_1, c_2) \sqsubseteq (c_5, c_6)$. Using the induction hypothesis, it follows that $d(c_1, c_2) \leq d(c_5, c_6) = d(c_3, c_4)$. The opposite direction follows easily using the induction hypothesis.

In the case of 2(a)iv, in order to see through the claim from left to right direction assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. Then $d(c_3, c_4)$ is assigned a value r such that $\max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_{n-1} \wedge (x, y) \sqsubset (c_3, c_4)\} < r$. Since we have that $(c_1, c_2) \sqsubseteq (c_3, c_4)$ and $\neg(c_1, c_2) \Box (c_3, c_4)$ (otherwise step 2a would have been finalised by the case 2(a)iii), this means that we also have $(c_1, c_2) \sqsubset (c_3, c_4)$. Since $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$, it follows that $d(c_1, c_2) < d(c_3, c_4)$. The opposite direction is obvious.

Secondly, suppose that the pair $\{c_3, c_4\}$ is added into AV_n via step 2b of the procedure. In this step, there are three sub-cases based on which a value for $d(c_3, c_4)$ is assigned:

In the case of 2(b)i, in order to see through the claim from left to right direction assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. Then $d(c_3, c_4)$ is assigned a value r such that $\max d(\mathrm{AV}_n) < r$. Therefore, we have $d(c_1, c_2) < d(c_3, c_4)$ since $\{c_1, c_2\} \in \mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$. The opposite direction is obvious.

Alternatively, to see the case of 2(b)ii, first assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. Then we have that $(c_3, c_4) \Box (c_5, c_6)$ for some $\{c_5, c_6\} \in \mathrm{AV}_n$ and $d(c_3, c_4) = d(c_5, c_6)$. Since $\mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$, we have two possibilities: Either $\{c_5, c_6\} \in \mathrm{AV}_{n-1}$ or $\{c_5, c_6\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$. In the former case, since we clearly also have that $(c_1, c_2) \sqsubseteq (c_5, c_6)$ from Lemma 2.2.1, it follows from the induction hypothesis that $d(c_1, c_2) \leq d(c_5, c_6) = d(c_3, c_4)$. In the latter case, first notice that $\{c_5, c_6\}$ must have been added to AV_n via step 2a of the current (nth) iteration of the procedure. But it is already established in the previous paragraphs of this very proof that we have $d(c_1, c_2) \leq d(c_5, c_6)$ in such a case, which gives us what we want, i.e.,

$d(c_1, c_2) \leq d(c_3, c_4)$.

Conversely, assume that $d(c_1, c_2) \leq d(c_3, c_4)$. Then we have that $(c_3, c_4) \square (c_5, c_6)$ for some $\{c_5, c_6\} \in \mathrm{AV}_n$ and $d(c_3, c_4) = d(c_5, c_6)$. Therefore, $d(c_1, c_2) \leq d(c_5, c_6)$. Since $\mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$, we have two possibilities: Either $\{c_5, c_6\} \in \mathrm{AV}_{n-1}$ or $\{c_5, c_6\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$. In the former case, it follows from the induction hypothesis that we have $(c_1, c_2) \sqsubseteq (c_5, c_6)$. Then from Lemma 2.2.1 we have that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. In the latter case, we note that $\{c_5, c_6\}$ must have been added into AV_n via step 2a of the current (nth) iteration of the procedure. But it is already established in the above paragraphs that we have $(c_1, c_2) \sqsubseteq (c_5, c_6)$ under our assumptions. Thus, from Lemma 2.2.1 we get what we are looking for.

In case 2(b)iii $d(c_3, c_4)$ is assigned a value r such that $\max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_n \wedge (x, y) \sqsubset (c_3, c_4)\} < r$. Since we have that $(c_1, c_2) \sqsubseteq (c_3, c_4)$ and $\neg(c_1, c_2) \square (c_3, c_4)$ (otherwise step 2b would have been finalised by the case 2(b)ii), this means that we also have $(c_1, c_2) \sqsubset (c_3, c_4)$. Since $\{c_1, c_2\} \in \mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$, it follows that $d(c_1, c_2) < d(c_3, c_4)$. The opposite direction is obvious.

Thirdly and finally, suppose that the pair $\{c_3, c_4\}$ is added into AV_n via step 2c of the procedure. In this step, there are only two sub-cases based on which a value for $d(c_3, c_4)$ is assigned:

In order to see the cases of 2(c)i and 2(c)ii, first assume that $(c_1, c_2) \sqsubseteq (c_3, c_4)$. In case 2(c)i we have that $(c_3, c_4) \square (c_5, c_6)$ for some $\{c_5, c_6\} \in \mathrm{AV}_n$ and $d(c_3, c_4) = d(c_5, c_6)$. Since $\mathrm{AV}_{n-1} \subseteq \mathrm{AV}_n$, we have two possibilities: Either $\{c_5, c_6\} \in \mathrm{AV}_{n-1}$ or $\{c_5, c_6\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$. In the former case, since we clearly also have that $(c_1, c_2) \sqsubseteq (c_5, c_6)$ from Lemma 2.2.1, it follows from the induction hypothesis that $d(c_1, c_2) \leq d(c_5, c_6) = d(c_3, c_4)$, so we get what we want. In the latter case, we first notice that $\{c_5, c_6\}$ must have been added to AV_n either via step 2a or step 2b of the current (nth) iteration of the procedure. However, we have already established in the previous paragraphs that we will have $d(c_1, c_2) \leq d(c_5, c_6)$ in either situation, which implies that we have what we want. The proof of case 2(c)ii and the proofs of the both cases in the opposite direction have very similar proofs to the corresponding cases from step 2b above. This completes the proof of the lemma. $\square$

Lemma 2.2.6. *For every $n \in \mathbb{N}$, for every $x \in MPV_n$ and for every $\{c_1, c_2\} \in AV_n$, we have that $x < d(c_1, c_2)$.*

Proof. The proof is by induction on n. The base case for $n = 1$ is obvious from step 1 of Construction 2.2.1. For the inductive step, assume that for every $x \in \mathrm{MPV}_{n-1}$ and for every $\{c_1, c_2\} \in \mathrm{AV}_{n-1}$, we have that $x < d(c_1, c_2)$. Let $x \in \mathrm{MPV}_n$ and $\{c_1, c_2\} \in \mathrm{AV}_n$. Suppose that $\{c_1, c_2\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$.

Firstly, suppose that the pair $\{c_1, c_2\}$ is added into AV_n via step 2a of the procedure. In this step, there are four sub-cases based on which a value for $d(c_1, c_2)$ is assigned:

In case 2(a)i, $d(c_1, c_2)$ is assigned a value r such that $\max \mathrm{MPV}_{n-1} < r$. So, if $x \in \mathrm{MPV}_{n-1}$, then we are easily through.

Alternatively, suppose that $x \in \mathrm{MPV}_n - \mathrm{MPV}_{n-1}$. First, note that set MPV_n is only extended in one of the three sub-cases of step 2b. If x is added into MPV_n either in case 2(b)i or in case 2(b)iii, then we obviously have that $x < d(c_1, c_2)$ as desired. Now suppose that x is added into MPV_n in case 2(b)ii. Then, we have that $x = \frac{d(c_3, c_4)}{2}$ for some $\{c_3, c_4\}$ such that either $\{c_3, c_4\} \in \mathrm{AV}_{n-1}$ or $\{c_3, c_4\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$. The former case ($\{c_3, c_4\} \in \mathrm{AV}_{n-1}$) implies that we have $\max \mathrm{MPV}_n = \max \mathrm{MPV}_{n-1}$. Using the induction hypothesis, we get that $x \leq \max \mathrm{MPV}_n = \max \mathrm{MPV}_{n-1} < \max d(\mathrm{AV}_{n-1}) < r$, which is what we want. The latter case ($\{c_3, c_4\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$) implies that $\{c_3, c_4\}$ is added into AV_n in one of the cases of step 2a. But this means that the pair $\{c_3, c_4\}$ is actually pair $\{c_1, c_2\}$. In other words, we have that $x = \frac{d(c_1, c_2)}{2} < d(c_1, c_2)$.

In case 2(a)ii, $d(c_1, c_2)$ is assigned a value r such that $\max d(\mathrm{AV}_{n-1}) < r$. Now, if $x \in \mathrm{MPV}_{n-1}$, then from the induction hypothesis it follows that $x < \max d(\mathrm{AV}_{n-1}) < r$. The case of $x \in \mathrm{MPV}_n - \mathrm{MPV}_{n-1}$ has an almost identical proof to the corresponding part of case 2(a)i in the above paragraph.

In case 2(a)iii, we have that $d(c_1, c_2) = d(c_3, c_4)$ for some $\{c_3, c_4\} \in \mathrm{AV}_{n-1}$. If $x \in \mathrm{MPV}_{n-1}$, then from the induction hypothesis it follows that $x < d(c_3, c_4) = d(c_1, c_2)$. We again leave the proof of case $x \in \mathrm{MPV}_n - \mathrm{MPV}_{n-1}$ since it can be easily derived from the case of 2(a)i.

Finally, in case 2(a)iv, $d(c_1, c_2)$ is assigned a value r such that $\max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_{n-1} \wedge (x, y) \sqsubset (c_1, c_2)\} < r$. If $x \in \mathrm{MPV}_{n-1}$, then from the induction hypothesis it follows that $x < \max\{d(x, y) \mid \{x, y\} \in \mathrm{AV}_{n-1} \wedge (x, y) \sqsubset (c_1, c_2)\} < r$ and we are through. Case $x \in \mathrm{MPV}_n - \mathrm{MPV}_{n-1}$ can be derived from the above corresponding case of 2(a)i.

If the pair $\{c_1, c_2\}$ is added into AV_n either via step 2b or step 2c, then it suffices to notice that there is a pair $\{c_3, c_4\} \in \mathrm{AV}_n - \mathrm{AV}_{n-1}$ added in step 2a such that $(c_3, c_4) \sqsubseteq (c_1, c_2)$ and $x < d(c_3, c_4)$ as can be derived from the above paragraphs. Using Lemma 2.2.5, it follows that $x < d(c_1, c_2)$ as desired. This completes the proof. □

So now, let us show that d satisfies the triangle inequality. Let $c_1, c_2, c_3 \in \mathsf{W}$. It is sufficient to establish that we have $d(c_1, c_3) \leq d(c_1, c_2) + d(c_2, c_3)$.

Consider the ordering among the pairs $\{c_1, c_3\}$, $\{c_1, c_2\}$ and $\{c_2, c_3\}$. We consider two possibilities: Firstly, suppose that the pair $\{c_1, c_3\}$ is not the maximal pair. This means that we have either $(c_1, c_3) \sqsubseteq (c_1, c_2)$ or $(c_1, c_3) \sqsubseteq (c_2, c_3)$. But then from Lemma 2.2.5, it follows that we have either $d(c_1, c_3) \leq d(c_1, c_2)$ or $d(c_1, c_3) \leq d(c_2, c_3)$. In either case, we get that $d(c_1, c_3) \leq d(c_1, c_2) + d(c_2, c_3)$ as desired.

Secondly, suppose that (c_1, c_3) is the maximal pair. Now, it is sufficient to show that $\frac{d(c_1,c_3)}{2} \leq d(c_1, c_2)$ and $\frac{d(c_1,c_3)}{2} \leq d(c_2, c_3)$. From Construction 2.2.1, since (c_1, c_3) is the maximal pair, it follows from step 2b that for some n we have that $\frac{d(c_1,c_3)}{2} \in \mathrm{MPV}_n$ and $\{\{c_1, c_2\}, \{c_2, c_3\}\} \subseteq \mathrm{AV}_n$. Now from Lemma 2.2.6, the desired result follows immediately. This shows that d satisfies the triangle equality. As we have already mentioned preceding Lemma 2.2.5, d has all the other necessary properties and we conclude that the pair $\langle \mathsf{W}, d \rangle$ is a metric space.

Finally, we are ready to put together our "Henkin model" except that we need to define the set of individuals. But this can be done easily by setting:

$$\mathbb{I} = \{\mathcal{P}(|c|) \mid c \in \mathfrak{C}\}.$$

We first set

$$\mathfrak{F} = \left\langle \mathsf{W}, d, \mathbb{I} \right\rangle$$

and now we give our constructed model as follows:

$$\mathfrak{M} = \left\langle \mathfrak{F}, \mathsf{C} \right\rangle,$$

where C is a function interpreting the constant symbols such that for every $c \in \mathfrak{C}$, $\mathsf{C}(c) = \mathcal{P}(|c|)$.

In order to complete the proof of the Henkin Lemma, we provide the following two lemmata, where $\mathfrak{F}$ and $\mathfrak{M}$ refers to the structure and model constructed in the above.

Lemma 2.2.7. *Let φ be a formula. Then we have that $\mathfrak{M} \models \varphi$ iff $\varphi \in \Gamma$.*

Proof. The proof is by induction on the complexity of φ. It is sufficient to establish the base case alone, since the rest of the inductive cases are highly routine. This amounts to prove that we have $\mathfrak{M} \models \mathsf{CC}(c_1, c_2, c_3)$ iff $\mathsf{CC}(c_1, c_2, c_3) \in \Gamma$.

In order to prove the claim in the direction from right to left, assume that $\mathsf{CC}(c_1, c_2, c_3) \in \Gamma$. It follows from axiom **AXM**10 that $\exists c_1' \exists c_1'' \exists c_2' \exists c_3' [\mathsf{AP}(c_1', c_1) \wedge \mathsf{AP}(c_1'', c_1) \wedge \mathsf{AP}(c_2', c_2) \wedge \mathsf{AP}(c_3', c_3) \wedge (c_2', c_3') \leq (c_1', c_1'')]$. Now, by the construction it can be easily shown that $c_1', c_1'' \in \mathcal{P}(|c_1|)$, $c_2' \in \mathcal{P}(|c_2|)$ and $c_3' \in \mathcal{P}(|c_3|)$. Moreover, from Lemma 2.2.5 we get that $d(c_2', c_3') \leq d(c_1', c_1'')$. In other words, we have $\mathfrak{M} \models \mathsf{CC}(c_1, c_2, c_3)$ as desired.

Conversely, suppose that $\mathfrak{M} \models \mathsf{CC}(c_1, c_2, c_3)$. Then $\exists c_1' \in \mathcal{P}(|c_1|)$, $\exists c_1'' \in \mathcal{P}(|c_1|)$, $\exists c_2' \in \mathcal{P}(|c_2|)$ and $\exists c_3' \in \mathcal{P}(|c_3|)$ such that, $d(c_2', c_3') \leq d(c_1', c_1'')$. On the other hand, it follows from the construction that $\mathsf{AP}(c_2', c_2)$, $\mathsf{AP}(c_1', c_1)$, $\mathsf{AP}(c_1'', c_1)$ and $\mathsf{AP}(c_3', c_3)$. Moreover, from Lemma 2.2.5, we get that $(c_2', c_3') \sqsubseteq (c_1', c_1'')$. Now it follows from axiom **AXM**10 and the maximal consistency of Γ that we have $\mathsf{CC}(c_1, c_2, c_3) \in \Gamma$ as desired. This completes the proof. $\square$

Lemma 2.2.8. *$\mathfrak{F}$ satisfies all constraints **CNT**1- **CNT**6, i.e., $\mathfrak{F}$ is a metric structure with individuals.*

Proof. Let us begin by establishing that **CNT**1 is satisfied over $\mathfrak{F}$. By definition, we have that $\mathcal{P}(|c|) \subseteq \mathsf{W}$ for every $c \in \mathfrak{C}$. So, we clearly have that $\mathbb{I} \subseteq 2^{\mathsf{W}}$. On the other hand, from axiom **AXM**5 it follows that $\mathbf{U} \in \mathfrak{C}$ and $\forall c \in \mathfrak{C}$ we have that $\Gamma \vdash \mathsf{P}(c, \mathbf{U})$. By definition, this entails that $\mathcal{P}(|\mathbf{U}|) = \mathsf{W}$. Hence, $\mathsf{W} \in \mathbb{I}$ as desired.

Now lets show the case of **CNT**2. Let some arbitrary $c_1 \in \mathfrak{C}$. We will show that $\mathcal{P}(|c_1|) \neq \emptyset$. However, from axiom **AXM**9 we immediately get that $\exists c_2[\mathsf{A}(c_2) \wedge \mathsf{P}(c_2, c_1)]$. So it follows that $c_2 \in \mathcal{P}(|c_1|)$. This proves **CNT**2.

To see the case of **CNT**3, let $c_1 \in \mathsf{W}$. Then by definition we have that $\Gamma \vdash \mathsf{A}(c_1)$. In other words, $\Gamma \vdash \forall x[\mathsf{P}(x, c_1) \rightarrow c_1 = x]$. However, this means that $\mathcal{P}(|c_1|) = \{c_1\}$. Since $\mathcal{P}(|c_1|) \in \mathbb{I}$, the desired result follows.

We will finally establish that **CNT**6 is satisfied by $\mathfrak{F}$. The cases of **CNT**4 and **CNT**5 follow in a very similar way. Let $x, y \in \mathbb{I}$. By definition, it follows that

$\exists c_1 \in \mathfrak{C}$, $\exists c_2 \in \mathfrak{C}$ such that $x = \mathcal{P}(|c_1|)$ and $y = \mathcal{P}(|c_2|)$. On the other hand, from axiom **AXM**6 we derive that $\exists c_3 \in \mathfrak{C}$ such that $\forall p[\mathsf{I}(c_3, p) \leftrightarrow [\mathsf{I}(c_1, p) \vee \mathsf{I}(c_2, p)]]$.

Let $z \in \mathcal{P}(|c_1|) \cup \mathcal{P}(|c_2|)$. We will show that $z \in \mathcal{P}(|c_3|)$. First assume that $z \in \mathcal{P}(|c_1|)$ (the case of $z \in \mathcal{P}(|c_2|)$ can be established in a similar way). Then by definition we get that $\mathsf{AP}(z, c_1)$. This implies that $\mathsf{I}(c_1, z)$ from the definition of predicate P. So it follows that $\mathsf{I}(c_3, z)$. However, since $\mathsf{A}(z)$, it follows from axiom **AXM**7 and the definition of predicate A that $\mathsf{P}(z, c_3)$. So we finally get that $z \in \mathcal{P}(|c_3|)$ as desired.

Conversely assume that $z \in \mathcal{P}(|c_3|)$. So we have that $\mathsf{AP}(z, c_3)$ and from here that $\mathsf{I}(z, c_3)$. Therefore, $\mathsf{I}(c_1, z) \vee \mathsf{I}(c_2, z)$ from axiom **AXM**7. Since $\mathsf{A}(z)$, it follows that $\mathsf{P}(z, c_1) \vee \mathsf{P}(z, c_2)$. In other words, we have either $z \in \mathcal{P}(|c_1|)$ or $z \in \mathcal{P}(|c_2|)$, i.e., $z \in \mathcal{P}(|c_1|) \cup \mathcal{P}(|c_2|)$. □

With Lemma 2.2.8, we also complete the proof of the Henkin Lemma. □

Now we finally conclude that,

Theorem 2.2.9 (Completeness)**.** *Let φ be a formula. Then we have that* $\mathsf{M} \models \varphi \Rightarrow$ $\mathbf{Ax}CD_1 \vdash \varphi$.

Proof. Follows directly from Lindenbaum's Lemma (Lemma 5.0.2), Saturation Lemma (Lemma 2.2.3) and the Henkin Lemma (Lemma 2.2.4). □

2.3 Modal Comparative Distance Logic

This section is dedicated to the development of a modal logic formalism which can talk about distance information in a comparative and qualitative manner, as we have done with the first-order logic in the previous section. The biggest difference between this section and the previous one is naturally the use of a much less expressive -hence, computationally much more feasible- language to talk about essentially identical semantic structures, i.e., metric structures with individuals.

2.3.1 Language and Semantics

We will use a modal language containing denumerably many proposition letters, the set of which will be denoted by $\mathscr{P}$ and its elements by $p, q, r, \ldots$ and the usual basic boolean operators $\vee$ and $\neg$, together with the standard proposition constants $\top$ and $\bot$. The main component of the language is the polyadic modal operator $\langle \mathbf{CC} \rangle(\alpha, \beta)$ ('here can-connect somewhere which α and somewhere which β'). We denote this language by $\mathcal{L}[\langle \mathbf{CC} \rangle]$. As we will demonstrate shortly, the 'global modality' or the standard **S5** modal operator $\exists$ can be easily defined in the modal comparative distance logic. The duals of the modalities $\langle \mathbf{CC} \rangle$ and $\exists$ are denoted by $[\mathbf{CC}]$ and $\forall$, respectively.

Despite of the fact that the modal operator $\exists$ can be obtained from the language outlined above, in some cases we will need a language which explicitly contains this **S5** operator. This language, extending the language $\mathcal{L}[\langle \mathbf{CC} \rangle]$ merely with the **S5** operator $\exists$, will be denoted by $\mathcal{L}[\langle \mathbf{CC} \rangle, \exists]$.

Formulas of modal comparative distance logic are interpreted using a 'comparative distance frame,' which can be given by a pair:

$$\mathfrak{F} = \left\langle \mathsf{W}, \mathsf{CC} \right\rangle$$

where W is the domain set the elements of which ('states') represent 'individuals' and CC is a ternary accessibility relation over $\mathsf{W} \times \mathsf{W} \times \mathsf{W}$ which will be used to interpret the binary $\langle \mathbf{CC} \rangle$ modality. A comparative distance frame $\mathfrak{F}$ satisfies the following constraints:

(**CNT**1) $\forall w \forall u [\mathsf{CC}(w, u, u)]$,

($\mathbf{CNT}2$) $\forall w \forall u \forall v[\mathsf{CC}(w,u,v) \Rightarrow \mathsf{CC}(w,v,u)]$,

($\mathbf{CNT}3$) $\forall w \forall u \forall v \forall y \forall z[\mathsf{CC}(w,u,v) \wedge \neg\mathsf{CC}(w,y,z) \Rightarrow$
$\neg\exists t[\mathsf{CC}(t,y,z) \wedge \neg\mathsf{CC}(t,u,v)]]$.

Therefore, a comparative distance model based on a comparative distance frame $\mathfrak{F}$ is as usual a pair:

$$\mathfrak{M} = \left\langle \mathfrak{F}, \mathsf{V} \right\rangle$$

where V is a valuation function such that $\mathsf{V}\colon \mathscr{P} \to 2^{\mathsf{W}}$, mapping proposition letters to sets of states. Now, we are finally ready to give the interpretation of $\mathcal{L}[\langle\mathbf{CC}\rangle, \exists]$ formulas by defining a relation of truth in the usual inductive way. For all formulas α, β, every $w \in \mathsf{W}$ and $p \in \mathscr{P}$,

- $\mathfrak{M}, w \models p$ iff $w \in \mathsf{V}(p)$,
- $\mathfrak{M}, w \models \alpha \wedge \beta$ iff $\mathfrak{M}, w \models \alpha$ and $\mathfrak{M}, w \models \beta$,
- $\mathfrak{M}, w \models \neg\alpha$ iff $\mathfrak{M}, w \not\models \alpha$,
- $\mathfrak{M}, w \models \exists\alpha$ iff $\exists u[\mathfrak{M}, u \models \alpha]$,
- $\mathfrak{M}, w \models \langle\mathbf{CC}\rangle(\alpha, \beta)$ iff $\exists u \exists v[\mathsf{CC}(w,u,v)$ and $\mathfrak{M}, u \models \alpha$ and $\mathfrak{M}, v \models \beta]$.

We denote the class of all comparative distance frames by F and the class of all comparative distance models by M. We will write $\mathsf{M} \models \varphi$, to denote the validity of the formula φ over every comparative distance model. 'Modal comparative distance logic' is the set of formulas of the language $\mathcal{L}[\langle\mathbf{CC}\rangle]$ valid on every comparative distance model. Similarly, 'modal comparative distance logic with global modality' is the set of formulas of the language $\mathcal{L}[\langle\mathbf{CC}\rangle, \exists]$ valid on every comparative distance model.

Note that, under the given semantics and in particular the constraint $\mathbf{CNT}1$, we can always define the 'global modality' as follows:

$$\exists\varphi := \langle \mathbf{CC} \rangle(\varphi, \varphi).$$

2.3.2 Finite Model Property and Decidability

Overview

In this section, we will show that the modal logic introduced in the previous section enjoys 'strong finite model property' with respect to the class of all comparative distance models. In Section 2.3.1, we defined two modal languages, $\mathcal{L}[\langle \mathbf{CC} \rangle]$ and $\mathcal{L}[\langle \mathbf{CC} \rangle, \exists]$, the latter of which is a simple extension of the former with the 'global modality'. It is a well known fact that some properties of modal logics like the finite model property, are commonly shared with the logics which extend them by the global modality (Blackburn et al. [13], Theorem 7.8, pg. 422). Since $\mathcal{L}[\langle \mathbf{CC} \rangle, \exists]$ is merely an extension of $\mathcal{L}[\langle \mathbf{CC} \rangle]$ by the 'global modality' $\exists$, it would be sufficient to establish that the modal comparative distance logic has the strong finite model property in order to conclude that the modal comparative distance logic with global modality also has the strong finite model property.

The proof that modal comparative distance logic has the strong finite model property is a standard one based on the filtration technique. For this, we will provide a procedure which constructs a finite model $\mathfrak{M}^{\mathrm{Fin}}$ for any given model $\mathfrak{M}$ and formula φ such that, φ is satisfied in $\mathfrak{M}$ iff φ is satisfied in $\mathfrak{M}^{\mathrm{Fin}}$. Let us now start giving the details of this construction procedure.

Construction 2.3.1. We say that a set of formulas Σ is 'symmetry-closed' iff we have that,

$$\langle \mathbf{CC} \rangle(\alpha, \beta) \in \Sigma \text{ iff } \langle \mathbf{CC} \rangle(\beta, \alpha) \in \Sigma.$$

Let Σ be a finite, symmetry and subformula closed set of formulas and $\mathfrak{M} = \langle \mathsf{W}, \mathsf{CC}, \mathsf{V} \rangle$ be a comparative distance model. We begin by defining a relation over $\mathsf{W} \times \mathsf{W}$, which we will denote by $\equiv_\Sigma$. For every $w, u \in \mathsf{W}$, set:

$$w \equiv_\Sigma u \text{ iff } \forall \varphi \in \Sigma[\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{M}, u \models \varphi].$$

In plain words, $\equiv_\Sigma$ is the modal equivalence relation with respect to the set of modal formulas Σ. It is obvious that $\equiv_\Sigma$ is an equivalence relation. We denote the equivalence class of a $w \in \mathsf{W}$ induced by this relation with $|w|$. We will now define the model $\mathfrak{M}^{\mathrm{Fin}}$ bye the 'filtration of $\mathfrak{M}$ through Σ'.

First of all, set the following:

- $\mathsf{W}^{\mathrm{Fin}} = \{|w| \mid w \in \mathsf{W}\}$;
- $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|)$ iff for every$\langle \mathbf{CC} \rangle(\varphi, \psi) \in \Sigma$

$$\Big[[\mathfrak{M}, u \models \varphi \text{ and } \mathfrak{M}, v \models \psi] \Rightarrow \mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\varphi, \psi)\Big];$$

- For every $p \in \mathscr{P}$ such that $p \in \Sigma$, $\mathsf{V}^{\mathrm{Fin}}(p) = \{|w| \mid \mathfrak{M}, w \models p\}$.

Now finally set,

$$\mathfrak{M}^{\mathrm{Fin}} = \Big\langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}}, \mathsf{V}^{\mathrm{Fin}} \Big\rangle,$$

as the filtration of $\mathfrak{M}$ through Σ. It is a trivial task to establish that the conditions of Definition 5.0.5 are satisfied.

Let Σ be a finite, symmetry and subformula closed set of formulas and let $\mathfrak{M} = \langle \mathsf{W}, \mathsf{CC}, \mathsf{V} \rangle$ be a comparative distance model. If $\mathfrak{M}^{\mathrm{Fin}} = \langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}}, \mathsf{V}^{\mathrm{Fin}} \rangle$ is the filtration of $\mathfrak{M}$ through Σ, then we have the following three lemmata:

Lemma 2.3.1. *For every formula $\varphi \in \Sigma$ and every state $w \in \mathsf{W}$, we have that $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^{Fin}, |w| \models \varphi$.*

Proof. The proof is by induction on the complexity of φ. The base case is trivial from Construction 2.3.1 and the boolean cases are straightforward. So, it only remains to establish the modal case when $\varphi = \langle \mathbf{CC} \rangle(\alpha, \beta)$.

To see it from left to right, assume that $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. Then we have that, $\exists u \exists v[\mathsf{CC}(w, u, v) \wedge u \models \alpha$ and $v \models \beta]$. From here and from the basic properties of filtrations (see Definition 5.0.5), it follows that we have $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|)$. Moreover, from the induction hypothesis it follows that $\mathfrak{M}^{\mathrm{Fin}}, |u| \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, |v| \models \beta$. Hence, we get that $\mathfrak{M}^{\mathrm{Fin}}, |w| \models \langle \mathbf{CC} \rangle(\alpha, \beta)$ as desired.

Now in order to see it in the opposite direction, assume that we have $\mathfrak{M}^{\mathrm{Fin}}, |w| \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. So, $\exists |u| \exists |v|[\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|) \wedge \mathfrak{M}^{\mathrm{Fin}}, |u| \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, |v| \models \beta]$. Note that we have $\langle \mathbf{CC} \rangle(\alpha, \beta) \in \Sigma$. Moreover, from the induction hypothesis we have that $\mathfrak{M}, u \models \alpha$ and $\mathfrak{M}, v \models \beta$. So, it follows from Construction 2.3.1 that $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. This completes the proof of the lemma. □

Lemma 2.3.2. *$\mathfrak{M}^{Fin}$ is a comparative distance model.*

Proof. It is sufficient to establish that $\mathfrak{F}^{\mathrm{Fin}} = \langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}} \rangle$ is a comparative distance frame, which amounts to show that the frame constraints **CNT**1-**CNT**3 hold over $\mathfrak{F}^{\mathrm{Fin}}$.

Let us first establish that **CNT**1 is satisfied over $\mathfrak{F}^{\mathrm{Fin}}$. Let $|w|, |u| \in \mathsf{W}^{\mathrm{Fin}}$ and pick some $\langle \mathbf{CC} \rangle(\alpha, \beta) \in \Sigma$. Suppose that $\mathfrak{M}, u \models \alpha$ and $\mathfrak{M}, u \models \beta$. Since $\mathfrak{F}$ satisfies the frame constraint **CNT**1, it follows that we have $\mathsf{CC}(w, u, u)$. Hence, $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. From Construction 2.3.1, we derive that $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |u|)$ as desired.

Let us now consider the case of **CNT**2. Let $|w|, |u|, |v| \in \mathsf{W}^{\mathrm{Fin}}$ and suppose that $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|)$. In order to see that we have $\mathsf{CC}^{\mathrm{Fin}}(|w|, |v|, |u|)$, pick some $\langle \mathbf{CC} \rangle(\alpha, \beta) \in \Sigma$ and assume that $\mathfrak{M}, v \models \alpha$ and $\mathfrak{M}, u \models \beta$. Since Σ is symmetry-closed, we have that $\langle \mathbf{CC} \rangle(\beta, \alpha) \in \Sigma$. From the hypothesis and Construction 2.3.1, it follows that we have $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\beta, \alpha)$. Since $\mathfrak{F}$ satisfies **CNT**2, it is easy to see that this implies $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. Hence, from Construction 2.3.1, we get that $\mathsf{CC}^{\mathrm{Fin}}(|w|, |v|, |u|)$.

Finally we address the case of **CNT**3. Let $|w|, |u|, |v|, |y|, |z| \in \mathsf{W}^{\mathrm{Fin}}$ and suppose that we have $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|) \wedge \neg\mathsf{CC}^{\mathrm{Fin}}(|w|, |y|, |z|)$. For sake of a contradiction, suppose that $\exists |t| \in \mathsf{W}^{\mathrm{Fin}}$ such that $\neg\mathsf{CC}^{\mathrm{Fin}}(|t|, |u|, |v|) \wedge \mathsf{CC}^{\mathrm{Fin}}(|t|, |y|, |z|)$.

From here, it follows that there is a $\langle \mathbf{CC} \rangle(\alpha, \beta) \in \Sigma$ such that $\mathfrak{M}, y \models \alpha$ and $\mathfrak{M}, z \models \beta$ and $\mathfrak{M}, w \not\models \langle \mathbf{CC} \rangle(\alpha, \beta)$. Since we also have that $\mathsf{CC}^{\mathrm{Fin}}(|t|, |y|, |z|)$ from the hypothesis, it follows that $\mathfrak{M}, t \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. In the very same way, there is a formula $\langle \mathbf{CC} \rangle(\gamma, \delta) \in \Sigma$ such that $\mathfrak{M}, u \models \gamma$ and $\mathfrak{M}, v \models \delta$ and $\mathfrak{M}, t \not\models \langle \mathbf{CC} \rangle(\gamma, \delta)$. On the other hand, from the hypothesis it follows that $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\gamma, \delta)$.

To summarise, we have that $\mathfrak{M}, t \models \langle \mathbf{CC} \rangle(\alpha, \beta) \wedge \neg\langle \mathbf{CC} \rangle(\gamma, \delta)$ and $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\gamma, \delta) \wedge \neg\langle \mathbf{CC} \rangle(\alpha, \beta)$. Now it is easy to see that this contradicts with the fact that $\mathfrak{F}$ satisfies **CNT**3. This completes the proof of the lemma. $\square$

Lemma 2.3.3. *The size of $\mathfrak{M}^{Fin}$ is exponential in the size of Σ, i.e., $|\mathsf{W}^{Fin}| \leq 2^{|\Sigma|}$.*

Proof. Define a function $f \colon \mathsf{W}^{\mathrm{Fin}} \to 2^{\Sigma}$ such that for every $|w| \in \mathsf{W}^{\mathrm{Fin}}$ we have,

$$f(|w|) = \{\varphi \in \Sigma \mid \mathfrak{M}^{\mathrm{Fin}}, w \models \varphi\}.$$

It is sufficient to show that f is a well-defined and injective function. To see that f is well-defined, let $|w|, |u| \in \mathsf{W}^{\mathrm{Fin}}$ and suppose that $|w| = |u|$. By definition, this means that w and u are modally equivalent with respect to Σ. From here it immediately follows that $f(|w|) = f(|u|)$.

To see that f is also injective, suppose that $f(|w|) = f(|u|)$ for some $|w|, |u| \in \mathsf{W}^{\mathrm{Fin}}$. By the definition of f, this means that w and u are modally equivalent with respect to Σ. In other words, $w \equiv_{\Sigma} u$. Hence, $|w| = |u|$ as desired. $\square$

Now, it only remains to put the pieces together, which gives us:

Theorem 2.3.4 (Strong Finite Model Property)**.** *Let φ be a formula. If φ is satisfiable over a comparative distance model, then it is satisfiable over a finite comparative distance model of size at most $2^{|\varphi|}$. In other words, modal comparative distance logic has the strong finite model property with respect to* M*, the class of all comparative distance models.*

Corollary 2.3.5 (Strong Finite Model Property)**.** *Modal comparative distance logic with global modality has the strong finite model property with respect to* M*.*

Finally, we present our main result which follows directly from Theorem 2.3.4 and Corollary 2.3.5:

Theorem 2.3.6. *Modal comparative distance logic and modal comparative distance logic with global modality have decidable satisfiability problems.*

2.3.3 Computational Complexity

Overview

In this section we show that the modal comparative distance logic has an NP-complete satisfiability problem. We adapt a proof method which relies on the fact that the logic has the finite model property. In fact, the presented proof will establish that the modal comparative distance logic has the polysize model property (see Definition 5.0.17).

The core part of the proof consists of Construction 2.3.2 below. Given a formula φ and a finite model $\mathfrak{M}^{\mathrm{Fin}}$, construction procedure generates a new model $\mathfrak{M}^{\varphi}$ by appropriately selecting states from $\mathfrak{M}^{\mathrm{Fin}}$ such that the size of $\mathfrak{M}^{\varphi}$ is only polynomial in the size of φ (in contrast to the exponential model generated in the finite model property proof above) and in which the satisfiability of φ can be preserved. Now let us continue with the details.

Construction 2.3.2. Let φ be a formula and,

$$\mathfrak{M}^{\mathrm{Fin}} = \left\langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}}, \mathsf{V}^{\mathrm{Fin}} \right\rangle$$

be a *finite* comparative distance model, such that $\mathfrak{M}^{\mathrm{Fin}}, W \models \varphi$ for some $W \in \mathsf{W}^{\mathrm{Fin}}$. We will select suitable states from $\mathfrak{M}^{\mathrm{Fin}}$ to construct a new model $\mathfrak{M}^{\varphi}$, such that the size of $\mathfrak{M}^{\varphi}$ is polynomial in the size of φ.

First, let $\langle \mathbf{CC} \rangle(\alpha_1, \beta_1), \ldots, \langle \mathbf{CC} \rangle(\alpha_n, \beta_n)$ be an enumeration of all of the subformulas of φ in the form of $\langle \mathbf{CC} \rangle(\cdot, \cdot)$ and which are are satisfiable in $\mathfrak{M}^{\mathrm{Fin}}$. For each pair of formulas α_k and β_k where $1 \leq k \leq n$, choose a pair of states w_k and u_k from $\mathsf{W}^{\mathrm{Fin}}$ such that w_k and u_k is a pair with minimal distance in between satisfying the formulas α_k and β_k, respectively. More precisely, we choose a pair of states w_k and u_k from $\mathsf{W}^{\mathrm{Fin}}$ such that:

$$\forall v \Big[\forall y \forall z [\mathsf{CC}^{\mathrm{Fin}}(v, y, z) \text{ and } \mathfrak{M}^{\mathrm{Fin}}, y \models \alpha_k \text{ and } \mathfrak{M}^{\mathrm{Fin}}, z \models \beta_k] \Rightarrow \mathsf{CC}^{\mathrm{Fin}}(v, w_k, u_k) \Big]. \tag{2.1}$$

Now, the critical question is whether such a pair of states can always be found. However, since every formula $\langle \mathbf{CC} \rangle(\alpha_k, \beta_k)$ is satisfied in $\mathfrak{M}^{\mathrm{Fin}}$ by the assumption and $\mathfrak{M}^{\mathrm{Fin}}$ is a finite model, it is easy to see that such a pair of states w_k and u_k always exists. Now set,

- $\mathsf{W}^{\varphi} = \{W\} \cup \bigcup_{k=1}^{n} \{w_k, u_k\}$,
- $\mathsf{CC}^{\varphi} = \mathsf{CC}^{\mathrm{Fin}} \upharpoonright \mathsf{W}^{\varphi}$,
- $\mathsf{V}^{\varphi} = \mathsf{V}^{\mathrm{Fin}} \upharpoonright \mathsf{W}^{\varphi}$.

And finally set,

$$\mathfrak{M}^{\varphi} = \left\langle \mathsf{W}^{\varphi}, \mathsf{CC}^{\varphi}, \mathsf{V}^{\varphi} \right\rangle.$$

Lemma 2.3.7. *$\mathfrak{M}^{\varphi}$ is a comparative distance model.*

Proof. Since $\mathfrak{M}^{\varphi}$ is a restriction of $\mathfrak{M}^{\mathrm{Fin}}$ and $\mathfrak{M}^{\mathrm{Fin}}$ is a comparative distance model, it follows straightforwardly that $\mathfrak{M}^{\varphi}$ satisfies constraints **CNT**1-**CNT**3. $\square$

Lemma 2.3.8. *For every subformula ψ of φ and every state $w \in \mathsf{W}^{\varphi}$, we have that $\mathfrak{M}^{Fin}, w \models \psi$ iff $\mathfrak{M}^{\varphi}, w \models \psi$.*

Proof. Let ψ be a subformula of φ. The proof is naturally by induction on the complexity of ψ. Let $w \in \mathsf{W}^{\varphi}$. Since $\mathfrak{M}^{\varphi}$ is simply a restriction of $\mathfrak{M}^{\mathrm{Fin}}$, base case follows trivially.

Now suppose $\psi = \neg\alpha$. Then we have that $\mathfrak{M}^{\mathrm{Fin}}, w \models \neg\alpha$ iff $\mathfrak{M}^{\mathrm{Fin}}, w \not\models \alpha$ iff (by the induction hypothesis) $\mathfrak{M}^{\varphi}, w \not\models \alpha$ iff $\mathfrak{M}^{\varphi}, w \models \neg\alpha$.

Alternatively suppose that $\psi = \alpha \wedge \beta$. Then, $\mathfrak{M}^{\mathrm{Fin}}, w \models \alpha \wedge \beta$ iff $\mathfrak{M}^{\mathrm{Fin}}, w \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, w \models \beta$ iff (by the induction hypothesis) $\mathfrak{M}^{\varphi}, w \models \alpha$ and $\mathfrak{M}^{\varphi}, w \models \beta$ iff $\mathfrak{M}^{\varphi}, w \models \alpha \wedge \beta$.

Now, we address the case of $\psi = \langle \mathbf{CC} \rangle(\alpha, \beta)$. To see through the claim in the direction from left to right, suppose that we have $\mathfrak{M}^{\mathrm{Fin}}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. Then, $\exists u \exists v[\mathsf{CC}^{\mathrm{Fin}}(w, u, v) \wedge \mathfrak{M}^{\mathrm{Fin}}, u \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, v \models \beta]$. On the other hand, by Construction 2.3.2, there is a pair of states u_{α} and v_{β} in W^{φ} with minimal distance in between such that the formulas α and β are satisfied, respectively. In other words, $\mathfrak{M}^{\mathrm{Fin}}, u_{\alpha} \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, v_{\beta} \models \beta$. From the induction hypothesis, it follows that $\mathfrak{M}^{\varphi}, u_{\alpha} \models \alpha$ and $\mathfrak{M}^{\varphi}, v_{\beta} \models \beta$. Moreover, it follows as a consequence of (2.1) that we must have $\mathsf{CC}^{\mathrm{Fin}}(w, u_{\alpha}, v_{\beta})$ and thus, $\mathsf{CC}^{\varphi}(w, u_{\alpha}, v_{\beta})$ by the construction. This gives the desired result.

In the opposite direction, suppose that $\mathfrak{M}^{\varphi}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$. Then we have that $\exists u \exists v[\mathsf{CC}^{\varphi}(w, u, v) \wedge \mathfrak{M}^{\varphi}, u \models \alpha$ and $\mathfrak{M}^{\varphi}, v \models \beta]$. Since $\mathfrak{M}^{\varphi}$ is a restriction of $\mathfrak{M}^{\mathrm{Fin}}$, it follows from here that $\mathsf{CC}^{\mathrm{Fin}}(w, u, v)$. On the other hand, from the induction hypothesis we get that $\mathfrak{M}^{\mathrm{Fin}}, u \models \alpha$ and $\mathfrak{M}^{\mathrm{Fin}}, v \models \beta$. This obviously implies that $\mathfrak{M}^{\mathrm{Fin}}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$ as desired.

$\square$

Lemma 2.3.9. *Modal comparative distance logic has the polysize model property.*

Proof. A quick examination of Construction 2.3.2 reveals that the size of $\mathfrak{M}^{\varphi}$ is only polynomial in the size of input formula φ. More precisely, the size of $\mathfrak{M}^{\varphi}$ is equal to twice the number of modalities in the input formula plus 1 at the maximum. Thus, from Lemmas 2.3.8 and 2.3.7, we conclude that the modal comparative distance logic has the polysize model property. □

Theorem 2.3.10. *The satisfiability problem of modal comparative distance logic is NP-complete.*

Proof. It follows from the fact that the class of comparative distance frames can be defined by a first-order sentence (see Lemma 5.0.5) and from lemmas 2.3.9 and 5.0.4 that the satisfiability problem of modal comparative distance logic is NP-complete. □

2.3.4 Soundness and Completeness Theorems

Overview

In this section, we provide an axiomatic system for syntactic reasoning about comparative distances. We will introduce an axiomatic system and it will be shown in this section that the introduced system is sound and complete with respect to the class of all comparative distance frames. This means that, reasoning with comparative distance frames using the modal language $\mathcal{L}[\langle \mathbf{CC} \rangle, \exists]$ can be performed equally by the introduced axiomatic system alone. Both the soundness and the completeness proofs follow a standard methodology. For the completeness proof of the axiomatic system, we use a simple canonical model argumentation.

Axiomatic System

We begin by constructing an axiomatic system which we will call $\mathbf{Ax}\mathrm{CD}_\diamond$. Naturally, $\mathbf{Ax}\mathrm{CD}_\diamond$ consists of axioms for propositional logic, the standard axioms of minimal modal logic **K** for each modal operator we use and the axioms which capture the essential nature of comparative distance reasoning. In addition, it contains the standard inference rules of uniform substitution, generalization and of course, modus ponens. This results with the following axiom schemata for $\mathbf{Ax}\mathrm{CD}_\diamond$:

(**AXM**1) $[\mathbf{CC}](p \to q, r) \to [[\mathbf{CC}](p, r) \to [\mathbf{CC}](q, r)]$,

(**AXM**2) $[\mathbf{CC}](p, q \to r) \to [[\mathbf{CC}](p, q) \to [\mathbf{CC}](p, r)]$,

(**AXM**3) $\forall(p \to q) \to [\forall p \to \forall q]$,

(**AXM**4) $\exists\exists p \to \exists p$,

(**AXM**5) $p \to \exists p$,

(**AXM**6) $p \to \forall\exists p$,

(**AXM**7) $\langle\mathbf{CC}\rangle(p, q) \to \exists p \wedge \exists q$,

(**AXM**8) $\exists(p \wedge q) \to \langle\mathbf{CC}\rangle(p, q)$,

(**AXM**9) $\langle\mathbf{CC}\rangle(p, q) \to \langle\mathbf{CC}\rangle(q, p)$,

(**AXM**10) $[\langle\mathbf{CC}\rangle(p, q) \wedge \neg\langle\mathbf{CC}\rangle(r, s)] \to \forall[\langle\mathbf{CC}\rangle(r, s) \to \langle\mathbf{CC}\rangle(p, q)]$.

Axioms **AXM**1, **AXM**2 and **AXM**3 are those corresponding to the axioms of the minimal modal logic **K** (**AXM**1 and **AXM**2 in the polyadic form) making the logic generated by $\mathbf{Ax}\mathrm{CD}_\diamond$ a 'normal modal logic.' This is a property which will be necessary in the application of some of the theorems that are fundamental to our argumentation.

Axioms **AXM**4, **AXM**5 and **AXM**6 are the axioms more commonly known by the names **4** (of transitive frames), **T** (of reflexive frames) and **B** (of symmetric frames), respectively. They constitute (together with the **K** axioms) the axiom system of the modal logic **S5** and in the current context, they define the behaviour of our $\exists$ operator, which is obviously intended as an **S5** modality. Axiom **AXM**7 is called the inclusion axiom and it defines the interaction between the modal operators $\langle \mathbf{CC} \rangle$ and $\exists$.

Finally, axioms **AXM**8, **AXM**9 and **AXM**10 aim to syntactically capture the nature of the comparative distance frame conditions **CNT**1, **CNT**2 and **CNT**3, respectively.

We will denote deduction in **Ax**CD$_\diamond$ by using the notation $\vdash_{\mathbf{Ax}\mathrm{CD}_\diamond}$. So, for any formula φ which is deductible in **Ax**CD$_\diamond$ we write $\vdash_{\mathbf{Ax}\mathrm{CD}_\diamond} \varphi$ to denote that φ is a theorem of the logic arising from system **Ax**CD$_\diamond$. The following theorem establishes that all theorems of the axiomatic system **Ax**CD$_\diamond$ are tautologies for the class of all comparative distance frames F.

Theorem 2.3.11 (Soundness)**.** *For every formula φ, we have that $\vdash_{\mathbf{Ax}CD_\diamond} \varphi \Rightarrow \mathsf{F} \models \varphi$.*

Proof. It is sufficient to establish the validity of axioms **AXM**1 - **AXM**10 over arbitrary frames from F. So, let $\mathfrak{F} = \langle \mathsf{W}, \mathsf{CC} \rangle \in \mathsf{F}$ and set $\mathfrak{M} = \langle \mathfrak{F}, \mathsf{V} \rangle$ for some arbitrary valuation V.

While axioms **AXM**1, **AXM**2 and **AXM**3 are the axioms of minimal modal logic **K**, axioms **AXM**4, **AXM**5, **AXM**6 and **AXM**7 are the well-known axioms of **S5**. So, we will skip the well-known proofs for the soundness of these axioms, which are obvious to the mind of the experienced reader. Now, let us focus on the axioms **AXM**8, **AXM**9 and **AXM**10.

Let $w \in \mathsf{W}$. In order to establish the soundness of **AXM**8, assume that $\mathfrak{M}, w \models \exists(p \wedge q)$. So, there is a $u \in \mathsf{W}$ such that $\mathfrak{M}, u \models p \wedge q$. On the other hand, since $\mathfrak{F}$ is a comparative distance frame, it satisfies frame condition **CNT**1.

Hence, we derive that $\mathsf{CC}(w,u,u)$. Thus, we get $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(p,q)$, which is what we want.

For the case of **AXM**9, assume that we have $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(p,q)$. We will show that this implies $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(q,p)$. From the hypothesis, it follows that $\exists u \exists v[\mathsf{CC}(w,u,v) \wedge \mathfrak{M}, u \models p$ and $\mathfrak{M}, v \models q]$. Since $\mathfrak{F}$ satisfies **CNT**2, we have that $\mathsf{CC}(w,v,u)$. Hence, $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(q,p)$ as desired.

Finally, to address the case of axiom **AXM**10, first suppose that we have $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(p,q) \wedge \neg\langle \mathbf{CC} \rangle(r,s)$. From here, it follows that $\exists u \exists v[\mathsf{CC}(w,u,v) \wedge \mathfrak{M}, u \models p$ and $\mathfrak{M}, v \models q]$.

For sake of a contradiction, assume that we have $\mathfrak{M}, w \models \exists \neg[\langle \mathbf{CC} \rangle(r,s) \rightarrow \langle \mathbf{CC} \rangle(p,q)]$. This means that $\exists y[\mathfrak{M}, y \models \langle \mathbf{CC} \rangle(r,s)$ and $\mathfrak{M}, y \models \neg\langle \mathbf{CC} \rangle(p,q)]$. So, it follows that $\exists z \exists t[\mathsf{CC}(y,z,t) \wedge \mathfrak{M}, z \models r$ and $\mathfrak{M}, t \models s]$.

Now let us put the pieces together: Since we have that $\mathfrak{M}, w \models \neg\langle \mathbf{CC} \rangle(r,s)$, it follows that $\neg\mathsf{CC}(w,z,t)$. Similarly, since $\mathfrak{M}, y \models \neg\langle \mathbf{CC} \rangle(p,q)$, we conclude that $\neg\mathsf{CC}(y,u,v)$. But then we have that $\mathsf{CC}(w,u,v) \wedge \neg\mathsf{CC}(w,z,t)$ while on the other hand that $\mathsf{CC}(y,z,t) \wedge \neg\mathsf{CC}(y,u,v)$. This clearly violates the frame condition **CNT**3 and it contradicts with the fact that $\mathfrak{F}$ is a comparative distance frame. □

We now turn our attention to the semantic completeness of axiomatic system **Ax**CD$_\diamond$ with respect to the class of all comparative distance frames. Completeness proof exploits the canonical model method, based on maximal consistent sets of the logic. For more on maximal consistent sets and their properties, which is highly advised for the inexperienced reader, see Definition 5.0.10 and Lemma 5.0.1 first.

We begin with the construction of the canonical model. Note that, since **Ax**CD$_\diamond$ is a normal modal logic, it must be strongly complete with respect to its canonical model (see Theorem 5.0.3), which is defined by the following construction.

Construction 2.3.3. This construction simply builds a model by using the set of all maximal consistent sets in the following well-known way:

- $\mathsf{W} = \{w \mid w \text{ is a maximal } \mathbf{Ax}\mathrm{CD}_\diamond\text{-consistent set}\}$;
- For every $w, u, v \in \mathsf{W}$ set:

$$\mathsf{CC}(w, u, v) \text{ iff } \forall\alpha\forall\beta\Big[[\alpha \in u \text{ and } \beta \in v] \Rightarrow \langle\mathbf{CC}\rangle(\alpha, \beta) \in w\Big];$$

- For every $p \in \mathscr{P}$ set:

$$\mathsf{V}(p) = \{w \in \mathsf{W} \mid p \in w\}.$$

Finally, canonical frame and canonical model are set as follows:

$$\mathfrak{F} = \Big\langle \mathsf{W}, \mathsf{CC} \Big\rangle \quad \text{and} \quad \mathfrak{M} = \Big\langle \mathfrak{F}, \mathsf{V} \Big\rangle.$$

Now, we note the following lemma which will act as the cornerstone of our completeness proof:

Lemma 2.3.12 (Truth Lemma)**.** *For every formula φ and state $w \in \mathsf{W}$, we have that $\mathfrak{M}, w \models \varphi \Leftrightarrow \varphi \in w$.*

Proof. See Blackburn et al., Lemma 4.21, pg. 199 [13]. □

Combined with the Lindenbaum Lemma (see Lemma 5.0.2), Lemma 2.3.12 immediately gives the following result:

Theorem 2.3.13 (Canonical Model Theorem)**.** $\mathbf{Ax}CD_\diamond$ *is strongly complete with respect to $\mathfrak{M}$.*

In order to establish strong completeness of $\mathbf{Ax}\mathrm{CD}_\diamond$ with respect to the class of all comparative distance frames, all that is needed to be demonstrated is that the canonical frame $\mathfrak{F}$ satisfies the frame conditions **CNT**1 - **CNT**3. The following lemma deals with this issue.

Lemma 2.3.14. *$\mathfrak{F}$ is a comparative distance frame, i.e., it satisfies frame conditions **CNT**1 - **CNT**3.*

Proof. We begin by showing that constraint **CNT**1 is satisfied by the canonical frame $\mathfrak{F}$. Let $w, u \in \mathsf{W}$ and also let φ and ψ be any two formulas such that $\varphi, \psi \in u$. From Lemma 2.3.12, it follows that $\mathfrak{M}, u \models \varphi \wedge \psi$. Hence, we have that $\mathfrak{M}, w \models \exists(\varphi \wedge \psi)$. So, again from Lemma 2.3.12, it follows that $\exists(\varphi \wedge \psi) \in w$. On the other hand, since w is a maximal consistent set, by Lemma 5.0.1, it must contain the formula $\exists(\varphi \wedge \psi) \rightarrow \langle \mathbf{CC} \rangle(\varphi, \psi)$, which is an instance of **AXM**8. Since w is closed under modus ponens by Lemma 5.0.1, it follows that $\langle \mathbf{CC} \rangle(\varphi, \psi) \in w$. From Construction 2.3.3, we conclude that $\mathsf{CC}(w, u, u)$ as desired.

Now, let us establish that constraint **CNT**2 is satisfied by $\mathfrak{F}$. Let $w, u, v \in \mathsf{W}$ and assume that $\mathsf{CC}(w, u, v)$. This means that, for all formulas φ' and ψ', we have $[\varphi' \in u$ and $\psi' \in v] \Rightarrow \langle \mathbf{CC} \rangle(\varphi', \psi') \in w$. Now let $\varphi \in v$ and $\psi \in u$. From the hypothesis, it follows that $\langle \mathbf{CC} \rangle(\psi, \varphi) \in w$. As the formula $\langle \mathbf{CC} \rangle(\psi, \varphi) \rightarrow \langle \mathbf{CC} \rangle(\varphi, \psi)$ is an instance of **AXM**9, it must be contained in w. Using modus ponens, it follows that $\langle \mathbf{CC} \rangle(\varphi, \psi) \in w$. Thus, from Construction 2.3.3, we get that $\mathsf{CC}(w, v, u)$.

Finally, we address the more interesting case of **CNT**3. Let $w, u, v, y, z \in \mathsf{W}$ and assume that we have $\mathsf{CC}(w, u, v) \wedge \neg\mathsf{CC}(w, y, z)$. From here and from Construction 2.3.3, it follows that for all formulas φ and ψ, we have that $[\varphi \in u$ and $\psi \in v] \Rightarrow \langle \mathbf{CC} \rangle(\varphi, \psi) \in w$. On the other hand, we derive that there are formulas $\alpha \in y$ and $\beta \in z$ such that $\langle \mathbf{CC} \rangle(\alpha, \beta) \notin w$ or equivalently, that $\neg\langle \mathbf{CC} \rangle(\alpha, \beta) \in w$ from Lemma 5.0.1.

For sake of a contradiction, assume that there exists a $t \in \mathsf{W}$ such that $\neg\mathsf{CC}(t, u, v) \wedge \mathsf{CC}(t, y, z)$. So, we have that for all formulas φ and ψ, $[\varphi \in$

y and $\psi \in z] \Rightarrow \langle \mathbf{CC} \rangle(\varphi, \psi) \in t$. Moreover, it follows that there are formulas $\gamma \in u$ and $\delta \in v$ such that $\neg\langle \mathbf{CC} \rangle(\gamma, \delta) \in t$.

Combining all the information we have gathered so far, on the one hand we have that $\langle \mathbf{CC} \rangle(\gamma, \delta) \in w$ and $\neg\langle \mathbf{CC} \rangle(\alpha, \beta) \in w$, which entails that $\langle \mathbf{CC} \rangle(\gamma, \delta) \wedge \neg\langle \mathbf{CC} \rangle(\alpha, \beta) \in w$ from Lemma 5.0.1. Since the formula $\langle \mathbf{CC} \rangle(\gamma, \delta) \wedge \neg\langle \mathbf{CC} \rangle(\alpha, \beta) \rightarrow \forall[\langle \mathbf{CC} \rangle(\alpha, \beta) \rightarrow \langle \mathbf{CC} \rangle(\gamma, \delta)]$ is an instance of **AXM**10, using modus ponens we derive that $\forall[\langle \mathbf{CC} \rangle(\alpha, \beta) \rightarrow \langle \mathbf{CC} \rangle(\gamma, \delta)] \in w$. Using Lemma 2.3.12, it is easy to see that $\langle \mathbf{CC} \rangle(\alpha, \beta) \rightarrow \langle \mathbf{CC} \rangle(\gamma, \delta) \in t$. Since we also have $\langle \mathbf{CC} \rangle(\alpha, \beta) \in t$ from the above, it follows that $\langle \mathbf{CC} \rangle(\gamma, \delta) \in t$, which is a contradiction since $\neg\langle \mathbf{CC} \rangle(\gamma, \delta) \in t$ and t is consistent. This completes the proof of the lemma. □

We summarize our achievements with the following completeness theorem:

Theorem 2.3.15 (Strong Completeness)**.** $\mathbf{Ax}CD_{\diamond}$ *is strongly complete with respect to the class of all comparative distance frames, i.e., for every formula* φ *we have that* $\mathsf{F} \models \varphi \Rightarrow \vdash_{\mathbf{Ax}CD_{\diamond}} \varphi$.

Proof. Follows directly from Theorem 2.3.13 and Lemma 2.3.14. □

2.4 Modal Logic of Comparative Distances and Lengths

The aim of this section is to try to extend the language of the modal comparative distance logic with the notion of 'length' in such a way that the "feasible" computational properties of the previous section can be preserved. The intended meaning of 'length of an individual' will be rectified below, but for now one can interpret it as the greatest distance between any two points in the set representing an individual.

2.4.1 Language and Semantics

Let $\mathbb{A} \subseteq \mathbb{R}$ be a set of 'parameters'. The language of $\mathcal{L}^l$ extends the language $\mathcal{L}[\langle \mathbf{CC} \rangle, \exists]$ defined in Section 2.3 by the addition of two families of nullary modal operators (i.e., modal constants), which can be given as follows:

$$\left\{ \langle \mathbf{L}_{=x} \rangle \mid x \in \mathbb{A} \right\} \text{ and } \left\{ \langle \mathbf{L}_{<x} \rangle \mid x \in \mathbb{A} \right\}.$$

On the other hand, $\mathcal{L}^l$ is interpreted by an extension of the comparative distance frames called 'comparative distance frames with lengths'. A comparative distance frame with lengths (from $\mathbb{A}$) is a tuple such as,

$$\mathfrak{F}[\mathbb{A}] = \left\langle \mathsf{W}, \mathsf{CC}, \{\mathsf{L}_{=x}\}, \{\mathsf{L}_{<x}\} \right\rangle_{x \in \mathbb{A}}$$

where the roles of W ('individuals') and the ternary relation CC ('can-connect') are the same as in a comparative distance frame. In addition to that, for every $x \in \mathbb{A}$, unary relations of $\mathsf{L}_{=x}$ ('length is equal to x') and $\mathsf{L}_{<x}$ ('length is less than x') will be used to interpret the modalities of $\langle \mathbf{L}_{=x} \rangle$ and $\langle \mathbf{L}_{<x} \rangle$, respectively.

The intended meaning of 'length of an individual' is defined as the greatest distance between the points of a set in a metric space. More precisely, if $S \subseteq \mathsf{W}$ is a set in a metric space $\langle \mathsf{W}, d \rangle$, then the length of the individual S can be defined as follows:

$$\text{length}(S) = \max \{ d(p_1, p_2) \mid p_1, p_2 \in S \}.$$

For every $x, y \in \mathbb{A}$, a comparative distance frame with lengths $\mathfrak{F}[\mathbb{A}]$ must satisfy the following constraints:

(**CNT**1) $\forall w \forall u [\mathsf{CC}(w, u, u)]$,

(**CNT**2) $\forall w \forall u \forall v [\mathsf{CC}(w, u, v) \Rightarrow \mathsf{CC}(w, v, u)]$,

(**CNT**3) $\forall w \forall u \forall v \forall y \forall z [\mathsf{CC}(w, u, v) \wedge \neg \mathsf{CC}(w, y, z) \Rightarrow$

$\neg\exists t[\mathsf{CC}(t,y,z) \wedge \neg\mathsf{CC}(t,u,v)]]$,

(**CNT**4) $\forall w[\mathsf{L}_{=x}(w) \Rightarrow \neg\mathsf{L}_{=y}(w)]$ whenever $x \neq y$,

(**CNT**5) $\forall w[\mathsf{L}_{<x}(w) \Rightarrow \mathsf{L}_{<y}(w)]$ whenever $x \leq y$,

(**CNT**6) $\forall w[\mathsf{L}_{<x}(w) \Rightarrow \neg\mathsf{L}_{=x}(w)]$,

(**CNT**7) $\forall w\forall u\forall v\forall y[[\mathsf{CC}(w,u,v) \wedge \mathsf{L}_{=x}(w) \wedge \mathsf{L}_{=x}(y)] \Rightarrow \mathsf{CC}(y,u,v)]$,

(**CNT**8) $\forall w\forall u\forall v\forall y[[\mathsf{CC}(w,u,v) \wedge \mathsf{L}_{<x}(w) \wedge \neg\mathsf{L}_{<x}(y)] \Rightarrow \mathsf{CC}(y,u,v)]$.

Constraints **CNT**1, **CNT**2 and **CNT**3 are the same constrains that occur in comparative distance frames. On the other hand, constraints **CNT**4, **CNT**5 and **CNT**6 perform the necessary sanity checks on the relations of lengths. Finally, the constraints **CNT**7 and **CNT**8 regulate how the can-connect relation and the relations of lengths interact with each other.

Now, a comparative distance model with lengths is a pair such as:

$$\mathfrak{M}[\mathbb{A}] = \left\langle \mathfrak{F}[\mathbb{A}], \mathsf{V} \right\rangle$$

where $\mathfrak{F}[\mathbb{A}]$ is a comparative distance frame with lengths from $\mathbb{A}$ and V is a valuation function such that $\mathsf{V}\colon \mathscr{P} \to 2^{\mathsf{W}}$. For the sake of simplicity, we frequently write $\mathfrak{M}$ and $\mathfrak{F}$ instead of $\mathfrak{M}[\mathbb{A}]$ and $\mathfrak{F}[\mathbb{A}]$, whenever the parameter set is clear from the context. The interpretation of $\mathcal{L}^l$ formulas is given in the usual inductive way by defining the following relation of truth:

For every $w \in \mathsf{W}$, $p \in \mathscr{P}$ and $x \in \mathbb{A}$,

- $\mathfrak{M}, w \models p$ iff $w \in \mathsf{V}(p)$,
- $\mathfrak{M}, w \models \alpha \wedge \beta$ iff $\mathfrak{M}, w \models \alpha$ and $\mathfrak{M}, w \models \beta$,
- $\mathfrak{M}, w \models \neg\alpha$ iff $\mathfrak{M}, w \not\models \alpha$,

- $\mathfrak{M}, w \models \exists\alpha$ iff $\exists u[\mathfrak{M}, u \models \alpha]$,

- $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\alpha, \beta)$ iff $\exists u \exists v[\mathsf{CC}(w, u, v) \wedge \mathfrak{M}, u \models \alpha \wedge \mathfrak{M}, v \models \beta]$,

- $\mathfrak{M}, w \models \langle \mathbf{L}_{=x} \rangle$ iff $\mathsf{L}_{=x}(w)$,

- $\mathfrak{M}, w \models \langle \mathbf{L}_{<x} \rangle$ iff $\mathsf{L}_{<x}(w)$.

We will denote the class of all comparative distance frames with lengths by F and the class of all comparative distance models with lengths by M. As usual, $\mathsf{M} \models \varphi$ denotes the validity of the formula φ over every comparative distance model with lengths. 'Modal comparative distance logic with lengths' is the set of all formulas of the language $\mathcal{L}^l$ which are true at every state in every comparative distance model with lengths.

2.4.2 Finite Model Property and Decidability

Overview

In this section we will establish that the modal comparative distance logic with lengths has the finite model property. We will use the same filtration argument from Section 2.3.2 only with a slight modification.

Construction 2.4.1. Let $\mathbb{A}^{\mathrm{Fin}} \subseteq \mathbb{A}$ be a finite set of parameters and $\Sigma[\mathbb{A}^{\mathrm{Fin}}]$ be a finite set of formulas such that,

- Σ is subformula closed and;

- For every x, we have that $x \in \mathbb{A}^{\mathrm{Fin}}$ iff ($\langle \mathbf{L}_{=x} \rangle \in \Sigma$ and $\langle \mathbf{L}_{<x} \rangle \in \Sigma$).

As before, we will drop $\mathbb{A}^{\mathrm{Fin}}$ from our notation and write Σ instead of $\Sigma[\mathbb{A}^{\mathrm{Fin}}]$, as long as $\mathbb{A}^{\mathrm{Fin}}$ is clear from the context. Let,

$$\mathfrak{M} = \left\langle \mathsf{W}, \mathsf{CC}, \{\mathsf{L}_{=x}\}, \{\mathsf{L}_{<x}\}, \mathsf{V} \right\rangle_{\{x \in \mathbb{A}\}}$$

be an arbitrary comparative distance model with lengths. Define the relation $\equiv_\Sigma$ over $\mathsf{W} \times \mathsf{W}$ as follows: For every $w, u \in \mathsf{W}$,

$$w \equiv_\Sigma u \Leftrightarrow \forall \varphi \in \Sigma[\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{M}, u \models \varphi].$$

In plain words, $\equiv_\Sigma$ is the modal equivalence relation with respect to the set of modal formulas Σ. It is obvious that $\equiv_\Sigma$ is an equivalence relation. We denote the equivalence class of a $w \in \mathsf{W}$ induced by $\equiv_\Sigma$ with $|w|$.

The 'filtration of $\mathfrak{M}$ through Σ', denoted $\mathfrak{M}^{\mathrm{Fin}}$, is defined as follows:

- $\mathsf{W}^{\mathrm{Fin}} = \{|w| \mid w \in \mathsf{W}\}$;
- $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|)$ iff for every $\langle \mathbf{CC} \rangle(\varphi, \psi) \in \Sigma$,

$$\Big[[\mathfrak{M}, u \models \varphi \wedge \mathfrak{M}, v \models \psi] \Rightarrow \mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\varphi, \psi)\Big];$$

- $\mathsf{L}^{\mathrm{Fin}}_{=x}(|w|)$ iff $\exists u \in |w|[\mathsf{L}_{=x}(u)]$ for every $x \in \mathbb{A}^{\mathrm{Fin}}$;
- $\mathsf{L}^{\mathrm{Fin}}_{<x}(|w|)$ iff $\exists u \in |w|[\mathsf{L}_{<x}(u)]$ for every $x \in \mathbb{A}^{\mathrm{Fin}}$;
- For every $p \in \mathscr{P}$ such that $p \in \Sigma$, set $\mathsf{V}^{\mathrm{Fin}}(p) = \{|w| \mid w \in \mathsf{V}(p)\}$.

Finally we set,

$$\mathfrak{M}^{\mathrm{Fin}}[\mathbb{A}^{\mathrm{Fin}}] = \Big\langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}}, \{\mathsf{L}^{\mathrm{Fin}}_{=x}\}, \{\mathsf{L}^{\mathrm{Fin}}_{<x}\}, \mathsf{V}^{\mathrm{Fin}} \Big\rangle_{x \in \mathbb{A}^{\mathrm{Fin}}}.$$

It is a trivial task to establish that the conditions of Definition 5.0.5 are satisfied.

We have the following three lemmata concerning the properties of the newly constructed, filtrated model $\mathfrak{M}^{\mathrm{Fin}}$:

Lemma 2.4.1. *For every $\varphi \in \Sigma$ and $w \in \mathsf{W}$, we have that $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^{Fin}, |w| \models \varphi$.*

Proof. The proof is by induction on the complexity of φ. The base case is trivial from Construction 2.4.1 and the boolean cases are as usual. A proof for the case $\varphi = \langle \mathbf{CC} \rangle(\alpha, \beta)$ can be found in the proof of Lemma 2.3.1.

Let $x \in \mathbb{A}^{\mathrm{Fin}}$ and suppose that $\varphi = \langle \mathbf{L}_{=x} \rangle$. Then, $\mathfrak{M}, w \models \langle \mathbf{L}_{=x} \rangle$ iff $\mathsf{L}_{=x}(w)$ iff $\exists u \in |w|[\mathsf{L}_{=x}(u)]$ iff $\mathsf{L}^{\mathrm{Fin}}_{=x}(|w|)$ iff $\mathfrak{M}^{\mathrm{Fin}}, |w| \models \langle \mathbf{L}_{=x} \rangle$. Case of $\varphi = \langle \mathbf{L}_{<x} \rangle$ follows in a similar way. □

Lemma 2.4.2. $\mathfrak{M}^{Fin}$ *is a comparative distance model with lengths.*

Proof. It is sufficient to establish that $\mathfrak{F}^{\mathrm{Fin}} = \langle \mathsf{W}^{\mathrm{Fin}}, \mathsf{CC}^{\mathrm{Fin}}, \{\mathsf{L}^{\mathrm{Fin}}_{=x}\}, \{\mathsf{L}^{\mathrm{Fin}}_{<x}\} \rangle_{x \in \mathbb{A}^{\mathrm{Fin}}}$ is a comparative distance frame with lengths, which amounts to show that the frame constraints **CNT**1-**CNT**8 (see Section 2.4.1) hold over $\mathfrak{F}^{\mathrm{Fin}}$. The proof for the cases **CNT**1, **CNT**2 and **CNT**3 can be found in the proof of Lemma 2.3.2.

In order to see that $\mathfrak{M}^{\mathrm{Fin}}$ satisfies **CNT**4, let $|w| \in \mathsf{W}^{\mathrm{Fin}}$ and $x, y \in \mathbb{A}^{\mathrm{Fin}}$ such that $x \neq y$. Suppose $\mathsf{L}^{\mathrm{Fin}}_{=x}(|w|)$. By Construction 2.4.1, it follows that $\exists u \in |w|$ such that $\mathsf{L}_{=x}(u)$. Since by definition we must have $\langle \mathbf{L}_{=x} \rangle \in \Sigma$, this clearly implies that for $\forall v \in |w|$, we have $\mathsf{L}_{=x}(v)$. Since $\mathfrak{M}$ is a comparative distance model with lengths, it obeys **CNT**4. So, we get that $\neg \exists y \in |w|$ such that $\mathsf{L}_{=y}(y)$. By Construction 2.4.1, this means that $\neg \mathsf{L}^{\mathrm{Fin}}_{=y}(|w|)$.

Next, we will establish that $\mathfrak{M}^{\mathrm{Fin}}$ satisfies **CNT**5. Let $|w| \in \mathsf{W}^{\mathrm{Fin}}$ and $x, y \in \mathbb{A}^{\mathrm{Fin}}$ such that $x \leq y$. Suppose $\mathsf{L}^{\mathrm{Fin}}_{<x}(|w|)$. It follows by Construction 2.4.1 that $\exists u \in |w|$ such that $\mathsf{L}_{<x}(u)$. Since $\mathfrak{M}$ is a comparative distance model with lengths, it obeys **CNT**5. Thus, we get $\mathsf{L}_{<y}(u)$. But this implies that $\mathsf{L}^{\mathrm{Fin}}_{<y}(|w|)$ as desired.

Now we will show that $\mathfrak{M}^{\mathrm{Fin}}$ satisfies **CNT**6. Let $|w| \in \mathsf{W}^{\mathrm{Fin}}$ and $x \in \mathbb{A}^{\mathrm{Fin}}$. Suppose that we have $\mathsf{L}^{\mathrm{Fin}}_{<x}(|w|)$. By Construction 2.4.1, it follows that $\exists u \in |w|$ such that $\mathsf{L}_{<x}(u)$. Since by definition we have that $\langle \mathbf{L}_{<x} \rangle \in \Sigma$, it follows that $\forall v \in |w|$ we have $\mathsf{L}_{<x}(v)$. Since $\mathfrak{M}$ is a comparative distance model with lengths, it obeys **CNT**6. Thus, we get $\neg \mathsf{L}_{=x}(v)$ for every $v \in |w|$. Hence, we get $\neg \mathsf{L}^{\mathrm{Fin}}_{=x}(|w|)$ by Construction 2.4.1.

Let us now show that $\mathfrak{M}^{\mathrm{Fin}}$ satisfies **CNT**7. Let $|w|, |u|, |v|, |y| \in \mathsf{W}^{\mathrm{Fin}}$ and $x \in \mathbb{A}^{\mathrm{Fin}}$. Suppose that we have $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|) \wedge \mathsf{L}^{\mathrm{Fin}}_{=x}(|w|) \wedge \mathsf{L}^{\mathrm{Fin}}_{=x}(|y|)$. Now, let $\langle \mathbf{CC} \rangle(\varphi, \psi) \in \Sigma$ and assume that $\mathfrak{M}, u \models \varphi$ and $\mathfrak{M}, v \models \psi$. It will be sufficient to establish that $\mathfrak{M}, y \models \langle \mathbf{CC} \rangle(\varphi, \psi)$. From the assumption, it is easy to see that $\exists w' \in |w|$ such that $\mathsf{L}_{=x}(w')$ and $\exists y' \in |y|$ such that $\mathsf{L}_{=x}(y')$. On the other hand, we also have that $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\varphi, \psi)$. From here, we get that $\exists u' \exists v' [\mathsf{CC}(w, u', v')$ and $\mathfrak{M}, u' \models \varphi$ and $\mathfrak{M}, v' \models \psi]$. Moreover, since by definition we have that $\langle \mathbf{L}_{=x} \rangle \in \Sigma$, it follows that $\mathsf{L}_{=x}(w)$ and $\mathsf{L}_{=x}(y)$. Now, since $\mathfrak{M}$ is a comparative distance model with lengths, it obeys **CNT**7. Therefore, we get that $\mathsf{CC}(y, u', v')$. This entails that $\mathfrak{M}, y \models \langle \mathbf{CC} \rangle(\varphi, \psi)$ as desired.

Finally, we show that $\mathfrak{M}^{\mathrm{Fin}}$ satisfies **CNT**8. Let $|w|, |u|, |v|, |y| \in \mathsf{W}^{\mathrm{Fin}}$ and $x \in \mathbb{A}^{\mathrm{Fin}}$. Suppose that we have $\mathsf{CC}^{\mathrm{Fin}}(|w|, |u|, |v|) \wedge \mathsf{L}^{\mathrm{Fin}}_{<x}(|w|) \wedge \neg \mathsf{L}^{\mathrm{Fin}}_{<x}(|y|)$. Now, let $\langle \mathbf{CC} \rangle(\varphi, \psi) \in \Sigma$ and assume that $\mathfrak{M}, u \models \varphi$ and $\mathfrak{M}, v \models \psi$. It will be sufficient to show that $\mathfrak{M}, y \models \langle \mathbf{CC} \rangle(\varphi, \psi)$. From the assumption, it is easy to see that $\exists w' \in |w|$ such that $\mathsf{L}_{<x}(w')$ and $\forall y' \in |y|$ we have $\neg \mathsf{L}_{<x}(y')$. On the other hand, from the above we get that $\mathfrak{M}, w \models \langle \mathbf{CC} \rangle(\varphi, \psi)$. From here, it follows that $\exists u' \exists v' [\mathsf{CC}(w, u', v')$ and $\mathfrak{M}, u' \models \varphi$ and $\mathfrak{M}, v' \models \psi]$. Moreover, since by definition we have that $\langle \mathbf{L}_{<x} \rangle \in \Sigma$, it follows that $\mathsf{L}_{<x}(w)$ and $\neg \mathsf{L}_{<x}(y)$. Now, since $\mathfrak{M}$ is a comparative distance model with lengths, it obeys **CNT**8. Therefore, we get that $\mathsf{CC}(y, u', v')$. Hence, $\mathfrak{M}, y \models \langle \mathbf{CC} \rangle(\varphi, \psi)$ as desired. □

Lemma 2.4.3. *$\mathfrak{M}^{Fin}$ is a finite model. In fact, $|\mathsf{W}^{Fin}| \leq 2^{|\Sigma|}$.*

Proof. Define a function $f \colon \mathsf{W}^{\mathrm{Fin}} \to 2^{\Sigma}$ such that for any $|w| \in \mathsf{W}^{\mathrm{Fin}}$, we have

$$f(|w|) = \{\varphi \in \Sigma \mid \mathfrak{M}^{\mathrm{Fin}}, w \models \varphi\}.$$

To complete the proof, it is sufficient to show that f is a well-defined and injective function. To see that f is well-defined, let $|w|, |u| \in \mathsf{W}^{\mathrm{Fin}}$ and suppose that $|w| = |u|$. By definition, this means that w and u are modally equivalent with respect to Σ. From here it clearly follows that $f(|w|) = f(|u|)$.

To see that f is also injective, suppose that $f(|w|) = f(|u|)$ for some $|w|, |u| \in \mathsf{W}^{\mathrm{Fin}}$. By the definition of f, this means that w and u are modally equivalent with respect to Σ. Hence, $|w| = |u|$ as desired. $\square$

Now, it only remains to put the pieces together, which gives us the following two main results of this section:

Theorem 2.4.4 (Strong Finite Model Property)**.** *Let φ be a formula. If φ is satisfiable over a comparative distance model with lengths, then it is satisfiable over a finite comparative distance model with lengths of size at most $2^{|\varphi|}$. In other words, modal comparative distance logic has the strong finite model property with respect to M.*

Proof. Let φ be a formula and let $\mathbb{A}^{\mathrm{Fin}}$ be the set of parameters that occur in φ. Let $\Sigma[\mathbb{A}^{\mathrm{Fin}}]$ be the closure of $\{\varphi\}$ under subformulas such that for every x we have that: $x \in \mathbb{A}^{\mathrm{Fin}}$ iff $\langle \mathbf{L}_{=x} \rangle \in \Sigma$ and $\langle \mathbf{L}_{<x} \rangle \in \Sigma$. Now suppose that $\mathfrak{M}, w \models \varphi$ for some comparative distance model with lengths $\mathfrak{M}$. Let $\mathfrak{M}^{\mathrm{Fin}}$ be the filtration of $\mathfrak{M}$ through Σ.

From Lemma 2.4.3 and Lemma 2.4.2, it follows that $\mathfrak{M}^{\mathrm{Fin}}$ is a finite comparative distance model with lengths. On the other hand, from Lemma 2.4.1 it follows that $\mathfrak{M}^{\mathrm{Fin}}, |w| \models \varphi$. $\square$

Theorem 2.4.5 (Decidability)**.** *The satisfiability problem of the modal comparative distance logic with lengths is decidable.*

Proof. Observe that the class of all finite comparative distance models with lengths is recursive. Since modal comparative distance logic with lengths has the finite model property by Theorem 2.4.4, it follows that its satisfiability problem is decidable. $\square$

2.4.3 Computational Complexity

Overview

In this section we will show that the satisfiability problem of the modal comparative distance logic with lengths is NP-complete. The main proof argument which we will utilise is identical to the one used in Section 2.3.3. Therefore, reader can refer to that section for more details, some of which are left out from this section.

We begin by giving a procedure for constructing polysize models from finite models:

Construction 2.4.2. Let φ be a formula and,

$$\mathfrak{M}^{\text{Fin}}[\mathbb{A}^{\text{Fin}}] = \left\langle \mathsf{W}^{\text{Fin}}, \mathsf{CC}^{\text{Fin}}, \{\mathsf{L}^{\text{Fin}}_{=x}\}, \{\mathsf{L}^{\text{Fin}}_{<x}\}, \mathsf{V}^{\text{Fin}} \right\rangle_{\{x \in \mathbb{A}^{\text{Fin}}\}}$$

be a *finite* comparative distance model with lengths such that $\mathfrak{M}^{\text{Fin}}, W \models \varphi$ for some $W \in \mathsf{W}^{\text{Fin}}$. We will select suitable states from $\mathfrak{M}^{\text{Fin}}$ in order to construct a model $\mathfrak{M}^{\varphi}$ size of which is only polynomial in $|\varphi|$. Let $\mathbb{A}^{\varphi}$ be the set of parameters that occur in φ.

First, let $\langle \mathbf{CC} \rangle(\alpha_1, \beta_1), \ldots, \langle \mathbf{CC} \rangle(\alpha_n, \beta_n)$ be an enumeration of all of the subformulas of φ in the form of $\langle \mathbf{CC} \rangle(\alpha, \beta)$ and which are satisfiable in $\mathfrak{M}^{\text{Fin}}$. For each pair of formulas α_k and β_k where $1 \leq k \leq n$, choose a pair of states w_k and u_k from W^{Fin} such that w_k and u_k is a pair with minimal distance in between satisfying the formulas α_k and β_k, respectively. Now, for every $x \in \mathbb{A}^{\varphi}$ set,

- $\mathsf{W}^{\varphi} = \{W\} \cup \bigcup_{k=1}^{n} \{w_k, u_k\}$,
- $\mathsf{CC}^{\varphi} = \mathsf{CC}^{\text{Fin}} \restriction \mathsf{W}^{\varphi}$,
- $\mathsf{L}^{\varphi}_{=x} = \mathsf{L}^{\text{Fin}}_{=x} \restriction \mathsf{W}^{\varphi}$,
- $\mathsf{L}^{\varphi}_{<x} = \mathsf{L}^{\text{Fin}}_{<x} \restriction \mathsf{W}^{\varphi}$,

- $\mathsf{V}^{\varphi} = \mathsf{V}^{\mathrm{Fin}} \restriction \mathsf{W}^{\varphi}$.

Finally set,

$$\mathfrak{M}^{(\mathbb{A}^{\varphi})} = \left\langle \mathsf{W}^{\varphi}, \mathsf{CC}^{\varphi}, \{\mathsf{L}^{\varphi}_{=x}\}, \{\mathsf{L}^{\varphi}_{<x}\}, \mathsf{V}^{\varphi} \right\rangle_{\{x \in \mathbb{A}^{\varphi}\}}.$$

Lemma 2.4.6. *$\mathfrak{M}^{\varphi}$ is a comparative distance model with lengths.*

Proof. Since $\mathfrak{M}^{\varphi}$ is a restriction of $\mathfrak{M}^{\mathrm{Fin}}$ and $\mathfrak{M}^{\mathrm{Fin}}$ is a comparative distance model with lengths, the proof that $\mathfrak{M}^{\varphi}$ satisfies constraints **CNT**1-**CNT**8 is trivial. □

Lemma 2.4.7. *For every subformula ψ of φ and every state $w \in \mathsf{W}^{\varphi}$, we have that $\mathfrak{M}^{Fin}, w \models \psi$ iff $\mathfrak{M}^{\varphi}, w \models \psi$.*

Proof. Let ψ be a subformula of φ. The proof is by induction on the complexity of ψ. Let $w \in \mathsf{W}^{\varphi}$. For the proof of the base case, boolean cases and the case for the modality $\langle \mathbf{CC} \rangle$, refer to the proof of Lemma 2.3.8 -the proofs are identical.

Now let $x \in \mathbb{A}^{\varphi}$. Then, $\mathfrak{M}^{\mathrm{Fin}}, w \models \langle \mathbf{L}_{=x} \rangle$ iff $\mathsf{L}^{\mathrm{Fin}}_{=x}(w)$ iff (by Construction 2.4.2) $\mathsf{L}^{\varphi}_{=x}(w)$ iff $\mathfrak{M}^{\varphi}, w \models \langle \mathbf{L}_{=x} \rangle$. The case for $\langle \mathbf{L}_{<x} \rangle$ follows similarly. □

Lemma 2.4.8. *Modal comparative distance logic with lengths has the polysize model property.*

Proof. It easily follows from Construction 2.4.2 that the size of $\mathfrak{M}^{\varphi}$ is only polynomial in the size of input formula. Hence, from Lemmas 2.4.7 and 2.4.6, we conclude that the modal comparative distance logic with lengths has the polysize model property. □

Theorem 2.4.9. *The satisfiability problem of the modal comparative distance logic with lengths is NP-complete.*

Proof. First of all, observe that in order to determine the satisfiability of a formula φ, without the loss of generality, we can restrict ourselves to a finite similarity type by restricting the parameter set to only those which occur in

φ. Call this set of parameters as $\mathbb{A}^{\varphi}$. In order to determine the satisfiability of a formula φ, we need only search though the models with the parameter set equal to $\mathbb{A}^{\varphi}$. From here, it follows that the class of comparative distance frames with lengths from $\mathbb{A}^{\varphi}$ can be defined by a first-order sentence. Thus, from Lemma 5.0.5, it is decidable in polynomial time if a given frame belongs to this class.

To conclude; It follows from lemmas 2.4.8 and 5.0.4 that the satisfiability problem of modal comparative distance logic with lengths is NP-complete. $\square$

2.5 Conclusion & Future Research

2.5.1 Conclusion

We employed one first-order and two modal logical languages (one of which is an extension of the other one) in order to represent and reason with comparative distance and length information of spatial solids, also known as 'individuals' in some part of the literature.

For the first order logic, we used semantics which explicitly utilises metric spaces for the interpretation of the comparative distance relation 'can-connect'. For the modal logical formalisms, we developed a relational representation of comparative distance information in metric spaces.

Laguna's can-connect relation provides a simple way to work with distance information, without the need of using any numerical parameters in our formalisms to represent quantitative distance information. It also provides a natural way of talking about distances between spatial solids, instead of talking about distances between (as generally perceived by some in the field) 'theoretically motivated, but practically faulty' points. Actually, this is a very important debate from a spatial cognitive point of view, since on a day-to-day basis humans mostly refer to distance information using this kind of qualitative expressions, e.g., "Mehmet's arm can not reach the upper shelf" which can be formalised as,

$$\neg \mathsf{CC}(\mathsf{MEHMET\_ARM}, \mathsf{SHELF}, \mathsf{MEHMET\_BODY})$$

and the informal expression that "cinema A is closer to home than cinema B is" can be formalised as,

$$\forall x[\mathsf{CC}(x, \mathsf{CINEMA\_B}, \mathsf{HOME}) \rightarrow \mathsf{CC}(x, \mathsf{CINEMA\_A}, \mathsf{HOME})]$$

within this framework.

In comparison to the work of Kutz et al. [46], where the only decidable logics of metric spaces have NEXPTIME upper bound (the question of lower bound remains an open problem), this work presents much less expressive modal formalisms that are computationally much more feasible. Moreover, as noted in Wolter and Zakharyaschev [72], using that kind of quantitative languages makes it very difficult to have comparative distance expressions. Wolter and Zakharyaschev's solution to this problem is to extend the language with variables for parameters in a rather complicated way.

Our main theoretical investigations have established that, while the first-order comparative distance logic is finitely axiomatisable, modal comparative distance logic is finitely axiomatisable and also enjoys the finite model property. Moreover, we established that this modal logic has a satisfiability problem which is decidable and NP-complete.

We also investigated a very simple extension modal logic, which encompasses quantitative length information besides the qualitative distance information by the use of parameters. Our results show that this logic has the very same properties as its ancestor logic, with a decidable satisfiability problem which is NP-complete.

2.5.2 Future Research

The modal logic of comparative distances can be extended with the use of nominals, to obtain a more expressive hybrid modal logic. Note that, this would enable the definition of many standard set operations quite easily: The following formula,

$$\forall\langle \mathbf{CC}\rangle(\mathsf{i},\mathsf{j})$$

would be true in a model, whenever the solids named by the nominal letters i and j intersect. Using this idea, many useful set operations which we defined with the first-order comparative distance logic can be defined in the modal counterpart as well. It is an interesting question if such an extended modal logic would be decidable and if so, what its computational complexity would be.

Another potential for further research is the combination of the can-connect relation with a topological closure operator. This would provide a formalism which combines comparative distances with another qualitative methodology, the topology. For example, in the case of first-order logic, the interpretation of the can-connect relation can be altered by setting:

$$\begin{aligned}\mathfrak{M} \models_{\mathfrak{a}} \mathsf{CC}(x,y,z) \Leftrightarrow \exists p_1 \exists p_2 \Big[& p_1 \in \mathbf{C}(\mathfrak{a}(y)) \wedge p_2 \in \mathbf{C}(\mathfrak{a}(z)) \wedge \\ & \exists p_3 \exists p_4 [p_3, p_4 \in \mathbf{C}(\mathfrak{a}(x)) \wedge d(p_1,p_2) \leq d(p_3,p_4)] \Big]\end{aligned}$$

where **C** denotes a topological closure operator in the extended model. From here, a definition for the commonly used 'connection' relation can be given with the following formula:

$$\mathsf{C}(x, y) \equiv_{\text{def}} \forall z[\mathsf{CC}(z, x, y)].$$

Chapter 3

Angular Modal Logic of Points (or, Towards Trigonometric Modal Logics)

3.1 Introduction

While the trend of developing spatial logics has gained a considerable speed and maturity over the recent years, there still remains quite a lot to be explored in order to enrich the field to contain sufficient amount of logical formalisms to cover the wide range of different aspects of spatial reasoning.

Until recently, topological and mereological[1] aspects of spatial reasoning

[1] In mathematical logic, mereology is a collection of axiomatic first-order theories dealing with parts and their respective wholes. Mereology is both an application of predicate logic and a branch of formal ontology.

had the most of the attention [68, 14, 31, 32, **?**, 3, 73]. This can be explained by the common perception within the field that qualitative reasoning based on topological information is a sufficient reasoning platform for the most of the applications, while providing computationally feasible algorithms compared to more quantitative approaches, e.g., geometric approaches.

A common slogan in the literature is to say that "a doughnut and a coffee mug are topologically equivalent (i.e., homeomorphic) objects", in order to demonstrate the expressive weakness of topology: Generally speaking, we can stretch, squeeze and reshape objects as much as we like, but as long as the objects are not torn apart into multiple sub-pieces and as long as we do not create any new holes in the objects, topologically speaking, it is impossible to notice the difference. This observation is the source of inspiration for developing more expressive, perhaps metric or semi-metric theories of space, since it is only natural that there are many application areas where a topological representation of space is simply inadequate.

In order to fill the gap, formalisms (not necessarily logical) which can deal with metric information or the combination of metric and topological information [42, 46, 72] and formalisms which can perform reasoning in affine geometry[2] [4, 5] have been developed and their computational and logical properties were investigated. More specific formalisms on orientation [47, 58, 37, 43], shape [45, 19] and size [55, 74] also exist. For example, Jungert [45] uses a technique to qualitatively describe the angles of a polygon for dealing with polygonal shape problems and Liu [49] gives semantics for 'qualitative distance' and 'qualitative angle' and develops a formalism of 'qualitative trigonometry' and provides a 'composition table' for the formalism, which can combine both types

[2]Affine geometry can be shortly described as a generalization of Euclidean geometry, characterized by slant and scale distortions. More precisely, affine geometry is the study of geometric properties which remain unchanged by affine transformations. In general, an affine transformation is composed of linear transformations (rotation, scaling or shear) and a translation (or "shift"). Several linear transformations can be combined into a single one.

of information. Unfortunately, only a small fragment of these studies got the necessary theoretical treatment that they deserve [23].

One path for reasoning about space is via the use of formal logics, which is generally in the form of modal or first-order theories. Bennett [6, 8] uses the topological interpretation of the modal logic **S4**, as originally established by Tarski [63, 65] and shows that propositional, intuitionistic and modal logics can be used for the purpose of spatial reasoning, each with a different level of expressiveness. Other studies use first-order languages to deliver axiomatic theories of topological connection [14, **?**, 3], all of which are known to be, unfortunately but as expected, undecidable [40].

Balbiani and Tinchev [5, 4] developed first-order and modal logics of affine geometry using two-sorted languages with 'points' and 'lines' combined with the relations of 'parallelism' and 'convergence.' Kutz et al. [46] uses a combination of description and classical modal logic languages, which can talk about metric spaces and weaker 'distance spaces.' They use a collection of 'parameterised modalities,' which have the semantics in the form of 'somewhere in the x units of distance from here, φ holds.' Based on the same line of study, Wolter and Zakharyaschev [72] propose combining 'metric modalities' with 'topological modalities,' and underline the importance of formalisms which combine quantitative and qualitative methods together. Venema [66] uses a two-sorted modal logic, the 'compass logic,' to build a formalism of cardinal directions. Unfortunately, reasoning with the compass logic is known to be undecidable [50].

The aim of this chapter is to help bridge the gap arising from the lack of formalisms which can deal with different aspects of spatial reasoning other than topology. A parallel greater goal of this work is to contribute to the development of logics that are capable of performing trigonometric reasoning.

Naturally, in order to perform any kind of trigonometric reasoning, the concept of triangle is *sine qua non*. Henceforth, as a first step in achieving this greater goal, with this study we devote a formalism which basically talks about the interior angles of triangles formed by any three points in space.

We develop a modal logic which can talk about angles in triangles in the setting of relational Kripke frames. For the usual domain-set of 'states' in a Kripkean frame, we take a set of points. For the accessibility relations, we assign a collection of (based on a 'parameter set') ternary relations, which talk about the magnitude of angles in triangles formed by each trio of states. Generally speaking, the modalities of our language which will correspond to these accessibility relations have the semantics in the form of 'φ holds at somewhere and ψ holds at somewhere else, with (less or greater than) x degrees of angle in between them about here.'

Triangles are especially important mathematical objects since one can construct any kind of complex polygonal object by using mere triangles. Therefore, a formalism which has the capability of talking about triangles, will naturally likely to have the ability to talk about any polygonal object as well.

However, the presented formalism lacks in its formation the necessary tools for handling the metric information. Hence, it can not be considered as a "trigonometric logic". From one perspective, this is considered a positive feature since in this way the proposed formalism gains a qualitative character: We can represent knowledge about the *shapes* of triangles (or any polygon), e.g., whether a triangle is obtuse, right-angled or equilateral, but the knowledge about the size of triangles is out of reach. In other words, there is no way of distinguishing between 'similar triangles' in the proposed formalism.

The useful qualitative notions of 'collinearity' and 'betweenness' are easily covered by the expressive power of the proposed formalism. Defining a modality with the semantics 'here is between somewhere at which φ holds and somewhere at which ψ holds' is a trivial task as we will show in the coming sections. Similarly, defining a modality with the semantics that 'here is collinear with somewhere at which φ holds and somewhere else at which ψ holds' is also easily possible.

There are enormous number of applications of trigonometry. For example, 'triangulation' is a widely used technique in astronomy, geography and satellite navigation systems. An interesting application of a trigonometric deduction system is the 'Canadarm2' robotic arm on the International Space Station, which is operated by controlling the angles of its joints. Calculating the final position of the tip of the arm requires repeated use of the trigonometric functions of those angles [70]. See Section 3.4 for more on the matter.

This chapter is organised as follows: In Section 3.2, we introduce the relational structure in detail which constitutes the basic semantical part of this study. In Section 3.3, we define a modal language and introduce the 'angular modal logic' using a semantical approach. Section 3.4 gives an impression of the expressive capabilities of the formalism. Section 3.5 is the part where we give the first halve of the main research results of this chapter and establish the finite model property using multiple model repair methods and conclude with the decidability of angular modal logic. In Section 3.6, we discuss the possible future research topics and propose an idea for an extension logic, where distance information can be handled in combination with angle information. We summarise our results in Section 3.7.

3.2 Trigonometric Relational Structures

Let $\mathbb{A} \subseteq [0, 180] \cap \mathbb{Q}^+$ be a set such that,

- $\{0, 180\} \subseteq \mathbb{A}$;
- if $x, y \in \mathbb{A}$ and $x + y \leq 180$, then $(x + y) \in \mathbb{A}$;
- if $x \in \mathbb{A}$, then $(180 - x) \in \mathbb{A}$.

We call $\mathbb{A}$ the set of (angle) parameters. In addition, let $\mathbb{O} = \{<, \leq, >, \geq\}$ be the set of operators. Operators are defined with Definition 3.2.1 below in more detail. An angular frame $\mathfrak{F}$ with parameters from $\mathbb{A}$ is a tuple,

$$\mathfrak{F}[\mathbb{A}] := \left\langle \mathsf{W}, \left\{\mathsf{ANG}_{\blacktriangleleft x}\right\}\right\rangle_{\blacktriangleleft \in \mathbb{O}, x \in \mathbb{A}}$$

where W is a set (of points or states, depending on the context), $\mathsf{ANG}_{\blacktriangleleft x}$ is a ternary relation over $\mathsf{W} \times \mathsf{W} \times \mathsf{W}$ ('the angle relation') where $\blacktriangleleft \in \mathbb{O}$ and $x \in \mathbb{A}$ and $\mathfrak{F}$ satisfies the conditions **CNT**1-**CNT**10 laid out below. In what follows, we will try to be efficient with our notation and simply write $\mathfrak{F}$ instead of $\mathfrak{F}[\mathbb{A}]$ for the sake of the simplicity and as long as the parameter set is clear from the context.

Let $w, u, v \in \mathsf{W}$ be three points. Accessibility relations in the form of $\mathsf{ANG}_{\blacktriangleleft x}$, where $\blacktriangleleft \in \mathbb{O}$ and $x \in \mathbb{A}$, are used to represent the angle information available at the any one of every three points in the space of W. For example, $\mathsf{ANG}_{<x}(w, u, v)$ states that the angle at point w of the triangle of points w, u and v is less than x. In other words, these relations talk about the interior angles of triangles formed by every three points.

Before we continue any further, let us first agree on some very basic notation:

Definition 3.2.1. We call the elements of the set $\mathbb{O} = \{<, \leq, >, \geq\}$ as 'operators.' In order to be able to speak in general terms, we often use expressions such as "$\mathsf{ANG}_{\blacktriangleleft x}$ for every $\blacktriangleleft \in S$," where S is some subset of $\mathbb{O}$. To enhance our ability to make even more general expressions for the sake of simplifying the technical parts of the text, we often use 'operator converters'.

Operator converters are mappings from $\mathbb{O}$ to $\mathbb{O}$ and they are denoted by the following superscripts: $\cdot^s$ 'strict order,' $\cdot^d$ 'dual order' and $\cdot^e$ 'weak dual order.' Table 3.1 gives the definition of each converter mapping over the set of operators.

	$<$	$\leq$	$>$	$\geq$
$\cdot^s$	$<$	$<$	$>$	$>$
$\cdot^d$	$\geq$	$>$	$\leq$	$<$
$\cdot^e$	$\leq$	$\leq$	$\geq$	$\geq$

Table 3.1: The definition of converter mappings over operator symbols.

For every $x, y, z \in \mathbb{A}$ and every $\blacktriangleleft \in \mathbb{O}$, an angular frame satisfies the following conditions **CNT**1-**CNT**10:

(**CNT**1) $\forall w \forall u [\mathsf{ANG}_{\leq 0}(w, w, u)]$ and $\forall w \forall u [\mathsf{ANG}_{\leq 0}(w, u, u)]$;

(**CNT**2) $\forall w \forall u \forall v [\mathsf{ANG}_{\blacktriangleleft x}(w, u, v) \Rightarrow \mathsf{ANG}_{\blacktriangleleft x}(w, v, u)]$;

(**CNT**3) $\mathsf{ANG}_{<x} \cup \mathsf{ANG}_{\geq x} = \mathsf{ANG}_{>x} \cup \mathsf{ANG}_{\leq x} = \mathsf{W} \times \mathsf{W} \times \mathsf{W}$;

(**CNT**4) $\mathsf{ANG}_{<x} \subseteq \mathsf{ANG}_{\leq x}$, $\mathsf{ANG}_{>x} \subseteq \mathsf{ANG}_{\geq x}$;

(**CNT**5) $\mathsf{ANG}_{\leq x} \cap \mathsf{ANG}_{>x} = \mathsf{ANG}_{<x} \cap \mathsf{ANG}_{\geq x} = \emptyset$;

(**CNT**6) $x \leq y \Rightarrow [\mathsf{ANG}_{\blacktriangleleft x} \subseteq \mathsf{ANG}_{\blacktriangleleft y}]$, where $\blacktriangleleft \in \{<, \leq\}$;

(**CNT**7) $x \leq y \Rightarrow [\mathsf{ANG}_{\blacktriangleleft y} \subseteq \mathsf{ANG}_{\blacktriangleleft x}]$, where $\blacktriangleleft \in \{>, \geq\}$;

(**CNT**8) $\forall w \forall u \forall v [\mathsf{ANG}_{\leq 180}(w, u, v) \wedge \mathsf{ANG}_{\geq 0}(w, u, v)]$;

(**CNT**9) $\forall w \forall u \forall v [[\neg(w = u \vee w = v \vee u = v) \wedge \mathsf{ANG}_{\blacktriangleleft x}(w, u, v) \wedge \mathsf{ANG}_{\blacktriangleleft y}(u, w, v)] \Rightarrow \mathsf{ANG}_{\blacktriangleleft^e 180-(x+y)}(v, w, u)]$ where $\blacktriangleleft \in \{\leq, \geq\}$ and whenever $(x + y) \leq 180$;

(**CNT**10) $\forall w \forall u \forall v [\mathsf{ANG}_{\leq 0}(w, u, v) \Rightarrow [\mathsf{ANG}_{\leq 0}(u, w, v) \vee \mathsf{ANG}_{\geq 180}(u, w, v)]]$;

Constraints **CNT**1 and **CNT**2 should be self-explanatory. The constraints from **CNT**3 to **CNT**7 establish the same natural ordering over $\mathbb{A}$ in the relational context. Constraint **CNT**8 states the obvious limits of the angles within a triangle. The basic fundamental rule of trigonometry concerning angles within a triangle is that the sum of all three inner angles of a triangle is equal to 180 degrees. Constraint **CNT**9 imposes this property on the triangles of the relational structure.

By constraint **CNT**10, we deal with the "extreme" configuration of angles in a triangle. More precisely, we allow the parameters 0 and 180 to be assigned for angles in which case we are no longer talking about a triangle in the classical sense, but only three collinear points. Even though the classical understanding of a triangle does not consider collinear points as a triangle, our understanding will allow this. Our "triangles" may very well be just three collinear points.

3.3 A Modal Logic of Trigonometry

3.3.1 Language and Semantics

In this section, we define the modal language $\mathcal{L}[\mathbb{A}]$ for talking about angular frames. Let the set of parameters $\mathbb{A}$ be as defined in Section 3.2. In what follows, we will omit $\mathbb{A}$ from our notation and simply write $\mathcal{L}$ instead of $\mathcal{L}[\mathbb{A}]$, as long as the relevant parameter set is clear from the context.

$\mathcal{L}$ contains a denumerably infinite set of proposition letters $\mathscr{P}$ and nominal letters $\mathscr{N}$, which we denote the elements by using lower case Latin letters $p, q, r, \ldots$ and $\mathsf{i}, \mathsf{j}, \mathsf{k}, \ldots$, respectively; the standard boolean connectives $\wedge, \neg$ together with the propositional constant $\top$; a nominal satisfaction operator @ and finally the following modal operators:

- For every $x \in \mathbb{A}$ and $\blacktriangleleft \in \mathbb{O}$, a binary (polyadic) modal operator in the form $\langle \mathsf{ANG}_{\blacktriangleleft x} \rangle$ for talking about the inner angles of triangles of points. For example, $\langle \mathsf{ANG}_{<x} \rangle(\varphi, \psi)$ says that 'φ holds at somewhere and ψ holds at somewhere else, with less than x degrees of angle in between about here.' This operator is illustrated in Figure 3.1. Note that the number of such operators is determined by the size of the set of angle parameters $\mathbb{A}$.

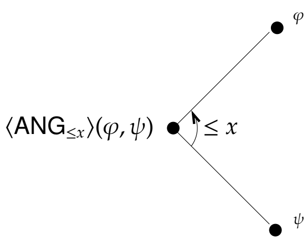

Figure 3.1: An illustration of the polyadic angle operator $\langle \mathsf{ANG}_{\leq x} \rangle$.

An angular model is the usual pair of a frame and a valuation: Let $\mathfrak{F}[\mathbb{A}]$ be an angular frame as introduced in Section 3.2 and $V\colon \mathscr{P} \cup \mathscr{N} \to 2^{\mathsf{W}}$ be a valuation function mapping the propositional letters to the subsets of W and nominal letters only to the singleton subsets of W. Then, an angular model with angles from $\mathbb{A}$ is a pair such as,

$$\mathfrak{M}[\mathbb{A}] := \left\langle \mathfrak{F}[\mathbb{A}], V \right\rangle.$$

Most of the time we will simplify the notation as we have done with $\mathfrak{F}$ and $\mathcal{L}$ and write $\mathfrak{M}$ instead of $\mathfrak{M}[\mathbb{A}]$. We denote the class of all angular models by T and the class of all such finite models by T^f.

Formulas of $\mathcal{L}$ are interpreted over angular models through the relation of truth $\models$, which is defined recursively as follows: Let $\mathfrak{M}$ be an angular model, $w \in \mathsf{W}$ be a state, α be an atomic (propositional or nominal) letter, $x \in \mathbb{A}$ be a parameter, $\blacktriangleleft \in \mathbb{O}$ be an operator symbol and φ, ψ be formulas of $\mathcal{L}$. Then we have the following:

Non-modal elements of $\mathcal{L}$ are interpreted in the usual way as follows:

- $\mathfrak{M}, w \models \top$,
- $\mathfrak{M}, w \models \alpha$ iff $w \in V(\alpha)$,
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$,
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$.

The more interesting parametric polyadic modal operators of angles are interpreted as follows:

- $\mathfrak{M}, w \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\varphi, \psi)$ iff $\exists u \exists v[\mathsf{ANG}_{\blacktriangleleft x}(w, u, v) \wedge \mathfrak{M}, u \models \varphi \wedge \mathfrak{M}, v \models \psi]$.

Finally, we interpret the nominal satisfaction operator as follows:

- $\mathfrak{M}, w \models @_{\mathsf{i}}\varphi$ iff $\exists u[\mathfrak{M}, u \models \mathsf{i} \wedge \varphi]$.

The accustomed reader will not have any difficulty in generating the interpretation of the dual operators. We denote the duals of $\langle \mathsf{ANG}_{\blacktriangleleft x} \rangle$ with $[\mathsf{ANG}_{\blacktriangleleft x}]$ for every $x \in \mathbb{A}$ and $\blacktriangleleft \in \mathbb{O}$.

We denote the logic of all $\mathcal{L}$ formulas which are true on every angular model with **TL**. We will investigate **TL**'s computational and logical properties in the following sections.

3.3.2 About Collinearity and Betweenness

The notions of collinearity and betweenness are very important, especially from the perspective of qualitative spatial reasoning. With our approach to represent angles and allow *any* three points to be regarded as a triangle, we can define collinearity as follows:

$$\mathsf{COL}(w, u, v) \equiv_{def} \mathsf{ANG}_{\leq 0}(w, u, v) \vee \mathsf{ANG}_{\geq 180}(w, u, v).$$

Henceforth, we can easily define a polyadic modal 'collinearity operator' in terms of modal angle operators as follows:

$$\langle \mathrm{COL} \rangle(\varphi, \psi) := \langle \mathsf{ANG}_{\leq 0} \rangle(\varphi, \psi) \vee \langle \mathsf{ANG}_{\geq 180} \rangle(\varphi, \psi).$$

It is obvious that we have,

$$\mathfrak{M}, w \models \langle \mathrm{COL} \rangle(\varphi, \psi) \text{ iff } \exists u \exists v [\mathsf{COL}(w, u, v) \wedge \mathfrak{M}, u \models \varphi \text{ and } \mathfrak{M}, v \models \psi].$$

So, the intended meaning of $\langle \mathrm{COL} \rangle(\varphi, \psi)$ is 'at some points collinear with here, φ and ψ holds.'

In the very same way, we can define a modal operator of 'betweenness,' if we observe that the interpretation of the relation $\mathsf{ANG}_{\geq 180}(w, u, v)$ actually states that w lies between the points u and v. Therefore, we can immediately adapt the following modality for betweenness:

$$\langle \mathrm{BT} \rangle(\varphi, \psi) := \langle \mathsf{ANG}_{\geq 180} \rangle(\varphi, \psi),$$

which obviously has the semantics that 'here is between a point where φ holds and another point where ψ holds.'

3.3.3 Defining Global Modality

Another consequence of the frame constraints is that we are able to define the 'global modality' almost trivially. It is easy to see that, from constraint **CNT**3 it follows that for any $x \in \mathbb{A}$ and any $w, u, v \in \mathsf{W}$ we have that,

$$\mathsf{ANG}_{\leq x}(w, u, v) \vee \mathsf{ANG}_{> x}(w, u, v).$$

In plain words, we can "see" any state from any other state in our frames. From here, we can easily provide a definition for the global modal operator $\exists\varphi$, with the intended meaning that 'somewhere in the entire model φ holds,' as follows:

$$\exists\varphi := \langle \mathsf{ANG}_{\leq x}\rangle(\varphi, \top) \vee \langle \mathsf{ANG}_{>x}\rangle(\varphi, \top).$$

3.4 Expressive Capabilities and Fields of Application

Logic **TL** can be used primarily for deal with the polygonal-shape based spatial reasoning tasks. The basic elements of the angular frames are triangles and since every arbitrary polygon can be constructed by using only triangles, the modal logic **TL** can deal with any polygon. There are many fields of application for shape based spatial reasoning and we will mention some of them here.

3.4.1 Shape Analysis

Shape analysis is mainly the automatic analysis of geometric shapes, for example using a computer to detect similarly shaped objects in a database or parts that fit together. For a computer to automatically analyse and process geometric shapes, the objects have to be represented in a digital form. Most commonly a boundary representation is used to describe the object with its boundary (usually the outer shell). However, other volume based representations (e.g., constructive solid geometry) or point based representations (e.g., point clouds) can be used to represent shape [36].

Once the objects are given, either by modelling (computer-aided design), by scanning (3D scanner) or by extracting shape from 2D or 3D images, they have to be simplified before a comparison can be achieved. The simplified representation is often called a shape descriptor. These simplified representations try to carry most of the important information, while being easier to handle, to store and to compare than the shapes directly. A complete shape descriptor is a representation that can be used to completely reconstruct the original object. Different shape descriptors target different aspects of shape and can be used for a specific application. Therefore, depending on the application, it is necessary to analyse how well a descriptor captures the features of interest.

Shape analysis is used in many application fields:

- In Archaeology, to find similar objects or missing parts;
- In Architecture, to identify objects that spatially fit into a specific space;
- Medical imaging to understand shape changes related to illness or aid surgical planning;
- Virtual environments or on the 3D model market to identify objects for copyright purposes;
- Security applications such as face recognition;
- Entertainment industry (movies, games) to construct and process geometric models or animations;
- Computer-aided design and computer-aided manufacturing to process and to compare designs of mechanical parts or design objects.

3.4.2 Molecular Geometry

Understanding the shapes of molecules is an important first step in being able to discuss and predict chemical properties. This topic has important applications in understanding the behaviour of much larger molecules. Much of biochemistry is now being discussed based on how macromolecules are shaped and how different molecules "fit" together [51, 25].

Molecular geometry or molecular structure is the three-dimensional arrangement of the atoms that constitute a molecule. It determines several properties of a substance including its reactivity, polarity, phase of matter, colour, magnetism and biological activity. There are six *basic* shape types for molecules:

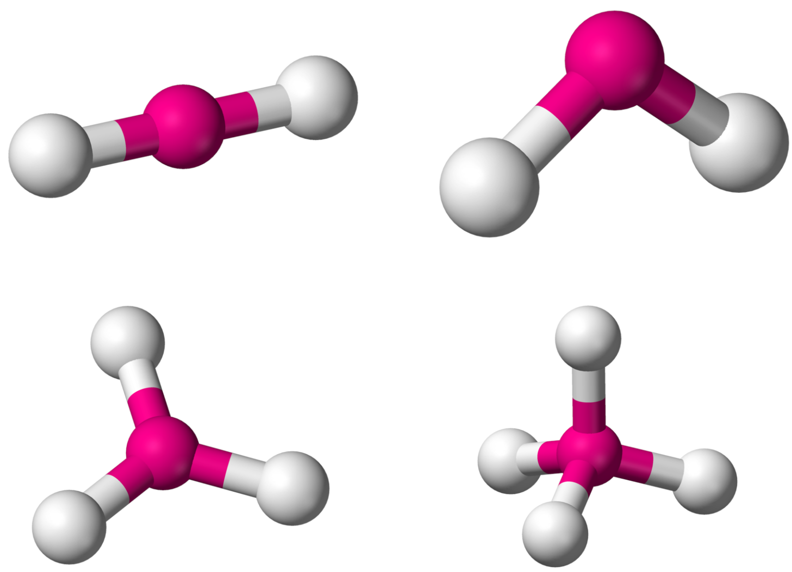

Figure 3.2: Illustrations of some basic types of molecule shapes. Top left is the linear type. An example of this is the carbon dioxide molecule. Bottom left is the trigonal type, an example of this type molecule is the boron trifluoride. Top right is an illustration of the bent type, water molecule is a good example here. Bottom right is the tetrahedral type for which we can give methane as an example.

- Linear: In a linear model, atoms are connected in a straight line: The bond angles are set at 180°. A bond angle is the angle between two adjacent bonds. For example, carbon dioxide (CO_2) has a linear molecular shape.

- Trigonal: Just from its name, it can easily be said that molecules with the trigonal shape are somewhat triangular. Consequently, the bond angles are set at 120°. An example of this is boron trifluoride (BF_3).

- Tetrahedral ("four surfaces"): This is when there are four bonds all on one central atom and the bond angles between the electron bonds are arccos(1/3) = 109.47°. An example of a tetrahedral molecule is methane (CH_4).

- Octahedral ("eight surfaces"): The bond angle is 90 degrees. An example of an octahedral molecule is sulfur hexafluoride (SF_6).

- Pyramidal: Pyramidal-shaped molecules have pyramid-like shapes. Unlike the linear and trigonal shapes but similar to the tetrahedral orientation, pyramidal shapes requires three dimensions in order to fully separate the electrons. An example is ammonia (NH_3).

- Bent: The final basic shape of a molecule is the non-linear shape, also known as bent or angular. One of the most unquestionably important molecules any chemist studies is water (H_2O) and it is an example of bent shape molecules. Bond angle is 106°.

Figure 3.2 illustrates some of these types.

3.4.3 Application Areas of Trigonometric Reasoning

Given the complete lack of metric information in our formalism, **TL** can not be characterized as a complete trigonometric reasoning platform. For more on this discussion, the reader is advised to look at Section 3.6 for planned future research. But for now, let us remind the reader that the main purpose of this study is not to develop a formalism of angles alone, but to create a step-stone to a formalism which will combine reasoning with angles and reasoning with metrics and become a formalism for trigonometric reasoning. So, let us now talk about the areas of application which could utilise such a formalism.

There are enormous number of applications of trigonometry. For instance, the technique of triangulation is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks and in satellite navigation systems. Other fields which make use of trigonometry include astronomy (especially, for locating the apparent positions of celestial objects, in which spherical trigonometry is essential), navigation systems (on the oceans, in aircraft, and in space) and robotics.

A concrete application that has been actively used since April 2001 is the robotic arm called 'Canadarm2' on the International Space Station, which is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the tip of the arm requires repeated use of the trigonometric functions of those angles [70]. Figure 3.3 depicts the working robotic arm mounted on the ISS.

Figure 3.3: Canadarm2 in action at the International Space Station.

3.5 Finite Model Property

In this section, we establish that **TL** has the finite model property. In other words, we show that for every formula which is satisfiable over an angular model, there is also a finite angular model over which it is satisfiable.

3.5.1 Overview

Unfortunately, the proof that **TL** possesses the finite model property is far from being trivial. It encompasses the standard filtration technique and two similar "split & repair" procedures. This technique has been applied before in various finite model property proofs [46].

Firstly, we construct a filtration (see Definition 5.0.5) of angular models through finite and subformula closed sets of formulas. This turns out to be a standard filtration construction. However, this construction gives a frame where two of the angular frame conditions, **CNT**5 and **CNT**9, can not be guaranteed to hold. We call the abnormal situations in frames where configurations (of relations, states and parameters) violating frame conditions exist as 'defects.' With the proof, we will apply two separate repair procedures in order to remove such defects. Each procedure will address the defect caused by the violations of one of the aforementioned frame conditions. In general, we call any three states as 'triangle.' Therefore, a 'triangle with a defect' is a set of three states over which a frame condition is violated.

The underlying idea in both repair procedures is the same "splitting" notion: Whenever some relations can not coexist over some group of states, affected states are split into separate states to form a new frame. In this way, conflicting configurations are isolated from each other while no information is lost, satisfiability is preserved and a frame without defects is constructed.

As already mentioned above, in the proof, there are three main stages to

$$\mathfrak{F} \xrightarrow[\textit{Filtration}]{} \mathfrak{F}^f \xrightarrow[\textit{First repair}]{} \mathfrak{F}^* \xrightarrow[\textit{Second repair}]{} \mathfrak{F}^+$$

Figure 3.4: The way to finite angular frames: Overview of the filtration and repair procedures. Given an arbitrary angular frame $\mathfrak{F}$, first we construct its filtration frame $\mathfrak{F}^f$. Then, we construct frame $\mathfrak{F}^*$ by repairing the **CNT**5 defects in $\mathfrak{F}^f$. Finally, we repair the **CNT**9 defects in frame $\mathfrak{F}^*$ to obtain finite angular frame $\mathfrak{F}^+$.

obtain a finite angular frame from an arbitrary angular frame. Each stage can be characterized by the frames constructed at those stages: Filtration frame is denoted by $\mathfrak{F}^f$. The second stage frame with repaired **CNT**5 defects will be named as $\mathfrak{F}^*$. The defect-free frame produced at the final stage, where **CNT**9 defects are removed, is denoted by $\mathfrak{F}^+$. Figure 3.4 illustrates these main stages in the proof.

Given a frame at a certain stage ($\mathfrak{F}, \mathfrak{F}^f, \mathfrak{F}^*$ or $\mathfrak{F}^+$), we refer to the frame from a previous stage as the 'ancestor frame.' Similarly, a 'successor frame' is the frame which has the given frame as the ancestor frame. For example, $\mathfrak{F}$ is the ancestor frame of $\mathfrak{F}^f$ and in the opposite direction, $\mathfrak{F}^f$ is the successor frame of $\mathfrak{F}$, et cetera.

We use the same notation with the states: The 'successor state(s)' of a given state are the states in the successor frame which are the results of a splitting or other similar operation we applied to that state. For example, the state (or equivalence class) $[w] \in \mathfrak{F}^f$ is the successor state of $w \in \mathfrak{F}$ and in the opposite direction, w is the ancestor state of $[w]$.

The first repair procedure takes the filtration frame $\mathfrak{F}^f$ and applies the necessary repairs to it in order to ensure the construction of a frame ($\mathfrak{F}^*$) satisfying frame constraint **CNT**5. For example, for some parameter x and a triangle of w, u and v, it repairs defects which are in the form of $\mathsf{ANG}_{\leq x}(w,u,v) \wedge \mathsf{ANG}_{>x}(w,u,v)$ or in the form of $\mathsf{ANG}_{\geq x}(w,u,v) \wedge \mathsf{ANG}_{<x}(w,u,v)$.

The repair procedure works by splitting the affected states and isolating incompatible relations from each other. In the example from the previous paragraph, it would add two states with different indices ($\langle w, 0\rangle, \langle w, 1\rangle$ and $\langle u, 0\rangle, \langle u, 1\rangle$ and $\langle v, 0\rangle, \langle v, 1\rangle$) to the new model for every state affected by the defect (w, u and v) in the old model. Then, to isolate the two problematic relations from each other ($\mathsf{ANG}_{\leq x}$ and $\mathsf{ANG}_{>x}$), it simply assigns one of them to the triangles of the successor states having equal indices and the other one to the triangles of the successor states with unequal indices. In this way, while the successor states collectively carry the exact same information from the old frame, we also get the incompatible relations isolated. Figure 3.5 illustrates how the "split & repair" procedure works.

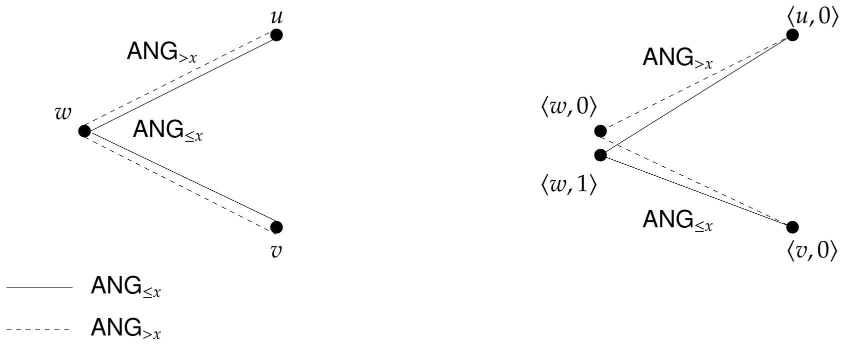

Figure 3.5: Split and repair method: The state w is involved in a defect (on the left). To repair the defect, w is split in two new states and the conflicting relations are isolated (on the right).

Second repair procedure deals with the defects which require a more complicated repair procedure. It repairs defects in frame $\mathfrak{F}^*$ resulting from the existence of configurations which violate the frame constraint **CNT**9 and outputs an angular frame $\mathfrak{F}^+$. For some parameters x, y and z such that $x + y + z = 180$ and a triangle of states w, u and v, these defects are in the form $\mathsf{ANG}_{\leq x}(w, u, v) \wedge \mathsf{ANG}_{\leq y}(u, w, v) \wedge \mathsf{ANG}_{<z}(v, w, u)$ or in the form $\mathsf{ANG}_{\geq x}(w, u, v) \wedge \mathsf{ANG}_{\geq y}(u, w, v) \wedge$

$\mathsf{ANG}_{>z}(v, w, u)$.

First, let us define a new notion: We call the modification of a frame configuration by replacing a relation over a certain triangle with its dual (see Definition 3.2.1) as 'flipping.' For example, if the relation $\mathsf{ANG}_{\leq x}(w, u, v)$ is flipped, then we get $\mathsf{ANG}_{>x}(w, u, v)$.

Roughly speaking, once more we will be using the "split & repair" technique from the point of basic strategy. However, this time within a more complicated methodology. Given a triangle with an appropriate defect, this procedure splits each state into three separate states in the new model. In other words, each state which is related to a defect in the old model, is represented by three states in the new model. This produces nine states or twenty seven triangles in the new model for only three states or one triangle in the old model.

For the sake of an example, fix a triangle with states w, u, v and some parameters x, y, z such that $x + y + z = 180$. Assume that we have the appropriate defect as follows: $\mathsf{ANG}_{\leq x}(w, u, v) \wedge \mathsf{ANG}_{\leq y}(u, w, v) \wedge \mathsf{ANG}_{<z}(v, w, u)$. In order to repair this defect, our procedure seeks to find a way to flip at least one and at most two of the relations involved in the defect. Note that, if we flip all of the three relations, although we will manage to remove the defect in question, we would cause a defect of the dual kind. For example, $\mathsf{ANG}_{>\mathbf{x}}(w, u, v) \wedge \mathsf{ANG}_{\leq y}(u, w, v) \wedge \mathsf{ANG}_{<z}(v, w, u)$ is one of the six ways (only at the maximum) of successfully repairing the defect in this example by flipping only one relation. However, we need to make sure that such a flipping does not "sacrifice satisfiability" by adding or removing information. To achieve this extra care, the procedure relies on two main rules:

Firstly, a flipping operation over a triangle can only be made if the needed dual relation for the flipping is already present over an ancestor state and triangle in one of the previous frames in the construction chain (for this, the procedure simply checks the existence at the filtration frame $\mathfrak{F}^f$, see Construction 3.5.3 for details).

Secondly, flipping relations must not remove any information from the old frame. Therefore, when a relation is flipped, the original relation must still be present in the new model over the successors of the respective states.

We now need to establish that there always will be a flipping option available to the procedure for the first rule above. However, as we will establish by Lemma 3.5.8, any defect over a triangle also guarantees the existence of two dual relations (of the relations involved in the defect) at the filtration frame $\mathfrak{F}^f$. Therefore, if the procedure attempts to flip any two relations of a defected triangle, it can be guaranteed that it will succeed with at least one of the attempts. Hence, the defect will certainly be repaired.

To summarise, we now explain how the "splitting and flipping" works altogether. Consider a defect triangle with states w, u and v. First of all, the procedure splits each of the states into three distinct states. Then, it assigns relations to the nine new states in such a way that three types of triangles are produced. These triangles are all possible constructions that can be achieved by flipping only two of the relations of the triangle of w, u and v at a time and keeping the original (non-flipped) relation at the other state. Given a triangle, this process can obviously produce at most three types of triangles. More precisely, one type inherits original relations on (w, u, v) but flipped relations on (u, w, v) and (v, w, u). Other type inherits original relations on (u, w, v) but flipped relations on (w, u, v) and (v, w, u). Finally, the last type inherits original relations on (v, w, u) but flipped relations on (u, w, v) and (w, u, v). In this way, the procedure succeeds in isolating conflicting configurations while satisfying the second rule above and hence, producing triangles without any defects and preserving the satisfiability. See Figure 3.6 for an illustration of the underlying

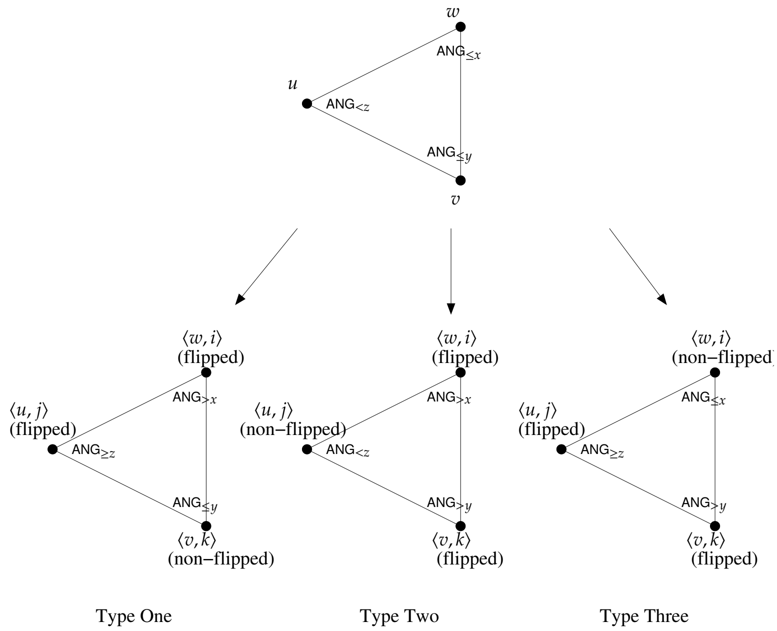

Figure 3.6: Split, flip and repair procedure: When a defect occurs over a triangle (at the top), each state involved in the defect is split in three. Then, by flipping the relations of only two of the states at a time, three types of triangles are produced (at the bottom). In this way, while the conflicting relations are isolated from each other, all the information is preserved after the repair.

idea.

3.5.2 The Proof

Let $\mathfrak{M} = \langle \mathfrak{F}[\mathbb{A}], V\rangle$ be an angular model such that,

$$\mathfrak{F}[\mathbb{A}] = \left\langle \mathsf{W}, \left\{\mathsf{ANG}_{\blacktriangleleft x}\right\}\right\rangle_{\blacktriangleleft \in \mathbb{O}, x \in \mathbb{A}}$$

and let Σ be a finite and subformula-closed set of formulas (see Definition 5.0.1). The first part of the proof is about constructing the filtration of $\mathfrak{M}$ through Σ. Let $\mathbb{A}^f$ denote the set of all parameters that occur in Σ.

Construction 3.5.1 (Filtration)**.** Define a relation $\equiv$ such that for any $w, u \in \mathsf{W}$, we have,

$$w \equiv u \Leftrightarrow \forall\varphi \in \Sigma[\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{M}, u \models \varphi].$$

It is a trivial exercise to verify that $\equiv$ is an equivalence relation. We denote the equivalence class of a $w \in \mathsf{W}$ induced by $\equiv$ with $[w]$. Now we construct the filtration of $\mathfrak{M}$ through Σ as follows. For every $x \in \mathbb{A}^f$, $\blacktriangleleft \in \mathbb{O}$ and $\alpha \in \mathscr{P} \cup \mathscr{N}$ set:

- $\mathsf{W}^{\mathrm{Fin}} = \{[w] \mid w \in \mathsf{W}\}$,
- $\mathsf{ANG}^f_{\blacktriangleleft x}([w], [u], [v])$ iff $\exists w' \in [w] \exists u' \in [u] \exists v' \in [v][\mathsf{ANG}_{\blacktriangleleft x}(w', u', v')]$,
- $V^f(\alpha) = \{[w] \mid \mathfrak{M}, w \models \alpha\}$.

Now set,

$$\mathfrak{F}^f[\mathbb{A}^f] := \left\langle \mathsf{W}^{\mathrm{Fin}}, \left\{\mathsf{ANG}^f_{\blacktriangleleft x}\right\}\right\rangle_{\blacktriangleleft \in \mathbb{O}, x \in \mathbb{A}^f}.$$

Hence, the filtration of $\mathfrak{M}$ through Σ can be given by the following pair:

$$\mathfrak{M}^f := \left\langle \mathfrak{F}^f[\mathbb{A}^f], V^f \right\rangle.$$

It should be a trivial task to verify that the above construction satisfies the conditions laid out by Definition 5.0.5. Hence, $\mathfrak{M}^f$ is clearly a filtration of $\mathfrak{M}$ though Σ. Moreover, $\mathsf{ANG}^f_{\blacktriangleleft x}$ is the smallest filtration (see Definition 5.0.6) of $\mathsf{ANG}_{\blacktriangleleft x}$. We will address the issues related to the size of the models later in this section. For now, we will address frame conditions and satisfiability issues.

It can be seen from the following lemma that the filtration frame $\mathfrak{F}^f$ does not satisfy two frame conditions **CNT**5 and **CNT**9, which is also the source of complexity behind the long finite model property proof. We will apply two separate repair procedures in order to obtain a frame which satisfies all of the frame conditions **CNT**1-**CNT**10.

Lemma 3.5.1. *$\mathfrak{F}^f$ satisfies all conditions CNT1-CNT10 except* **CNT5** *and* **CNT9**.

Proof. Pick $\langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi,\psi) \in \Sigma$ and $[w],[u],[v],[y] \in \mathsf{W}^{\mathrm{Fin}}$ and fix some $\blacktriangleleft \in \mathbb{O}$.

In order to see **CNT**1, first of all note that we have $\mathsf{ANG}_{\leq 0}(w,w,u)$ and $\mathsf{ANG}_{\leq 0}(w,u,u)$ since $\mathfrak{F}$ is an angular frame. But then by the filtration (refer Definition 5.0.5) it follows that we have $\mathsf{ANG}^f_{\leq 0}([w],[w],[u])$ and $\mathsf{ANG}^f_{\leq 0}([w],[u],[u])$. So, $\mathfrak{F}^f$ satisfies **CNT**1.

We continue by showing that $\mathfrak{F}^f$ satisfies **CNT**2. Assume that we have $\mathsf{ANG}^f_{\blacktriangleleft x}([w],[u],[v])$. From here it follows that we have $\exists w' \in [w] \exists u' \in [u] \exists v' \in [v][\mathsf{ANG}_{\blacktriangleleft x}(w',u',v')]$. Since $\mathfrak{F}$ satisfies **CNT**2, it follows that $\mathsf{ANG}_{\blacktriangleleft x}(w',v',u')$. Hence, we get $\mathsf{ANG}^f_{\blacktriangleleft x}([w],[v],[u])$ from Construction 3.5.1 as desired.

To see that **CNT**3 holds, it is sufficient to observe that, since $\mathfrak{F}$ satisfies **CNT**3, we have $\mathsf{ANG}_{\blacktriangleleft x}(w,u,v) \vee \mathsf{ANG}_{\blacktriangleleft^d x}(w,u,v)$. From here and the standard filtration properties as mentioned in Definition 5.0.5, it follows that $\mathsf{ANG}^f_{\blacktriangleleft x}([w],[u],[v]) \vee \mathsf{ANG}^f_{\blacktriangleleft^d x}([w],[u],[v])$.

Next, we will establish that $\mathfrak{F}^f$ satisfies **CNT**4. Assume that we have $\mathsf{ANG}^f_{<x}([w],[u],[v])$. From here we get that $\exists w' \in [w] \exists u' \in [u] \exists v' \in [v]$ such that $\mathsf{ANG}_{<x}(w',u',v')$. Since $\mathfrak{F}$ satisfies **CNT**4, it follows that $\mathsf{ANG}_{\leq x}(w',u',v')$. Hence, $\mathsf{ANG}^f_{\leq x}([w],[u],[v])$ as desired. The other half of the proof that $\mathsf{ANG}^f_{>x}([w],[u],[v])$ implies $\mathsf{ANG}^f_{\geq x}([w],[u],[v])$ follows in a very similar way.

We move on to show that **CNT**6 is satisfied. Let $x,y \in \mathbb{A}^f$ such that $x \leq y$ and assume that $\mathsf{ANG}^f_{<x}([w],[u],[v])$. Therefore we get that $\exists w' \in [w] \exists u' \in [u] \exists v' \in [v][\mathsf{ANG}_{<x}(w',u',v')]$. Since $\mathfrak{F}$ satisfies **CNT**6, it follows that $\mathsf{ANG}_{<y}(w',u',v')$. Hence, $\mathsf{ANG}^f_{<y}([w],[u],[v])$ as desired. It can be shown in a similar way that $\mathsf{ANG}^f_{\leq x}([w],[u],[v])$ implies $\mathsf{ANG}^f_{\leq y}([w],[u],[v])$. **CNT**7 follows along similar lines.

CNT8 is straightforward. Finally, to see that $\mathfrak{F}^f$ satisfies **CNT**10, assume that $0, 180 \in \mathbb{A}^f$ and $\mathsf{ANG}^f_{\leq 0}([w],[u],[v])$. Then, $\exists w' \in [w] \exists u' \in [u] \exists v' \in$

$[v][\mathsf{ANG}_{\leq 0}(w', u', v')]$. Now, using the fact that $\mathfrak{F}$ satisfies **CNT**10, it follows that $\mathsf{ANG}_{\leq 0}(u', w', v') \vee \mathsf{ANG}_{\geq 180}(u', w', v')$. But this implies that $\mathsf{ANG}^f_{\leq 0}([u], [w], [v]) \vee \mathsf{ANG}^f_{\geq 180}([u], [w], [v])$. □

The following lemma shows that satisfiability is preserved through the filtration process which constructs $\mathfrak{F}^f$ from $\mathfrak{F}$.

Lemma 3.5.2. *For every $\varphi \in \Sigma$ and every $w \in \mathsf{W}$, we have that $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}^f, [w] \models \varphi$.*

Proof. Let $\varphi \in \Sigma$ and a $w \in \mathsf{W}$. The proof is by induction on the complexity of φ. Let $\alpha \in \mathscr{P} \cup \mathscr{N}$. The base case, when $\varphi = \alpha$, is straightforward: From Construction 3.5.1, we have that $\mathfrak{M}, w \models \alpha \Leftrightarrow w \in V(\alpha) \Leftrightarrow [w] \in V^f(\alpha) \Leftrightarrow \mathfrak{M}^f, [w] \models \alpha$. Cases for the boolean connectives are standard.

Now let $x \in \mathbb{A}^f$ and fix $\blacktriangleleft \in \mathbb{O}$. Assume that $\varphi = \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$ and $\mathfrak{M}, w \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$. This means that, $\exists u \exists v[\mathsf{ANG}_{\blacktriangleleft x}(w, u, v) \wedge \mathfrak{M}, u \models \beta \wedge \mathfrak{M}, v \models \psi]$. From the induction hypothesis and Construction 3.5.1, it follows that $\mathsf{ANG}^f_{\blacktriangleleft x}([w], [u], [v])$ and $\mathfrak{M}^f, [u] \models \beta \wedge \mathfrak{M}^f, [v] \models \psi$. In other words, $\mathfrak{M}^f, [w] \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$ or $\mathfrak{M}^f, [w] \models \varphi$ as desired.

In the opposite direction, assume that $\mathfrak{M}^f, [w] \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$. Then, $\exists [u] \exists [v][\mathsf{ANG}^f_{\blacktriangleleft x}([w], [u], [v]) \wedge \mathfrak{M}^f, [u] \models \beta \wedge \mathfrak{M}^f, [v] \models \psi]$. Since $\varphi \in \Sigma$ and Σ is subformula-closed, it follows that $\beta, \psi \in \Sigma$. Together with the induction hypothesis, it follows that $\exists w' \in [w] \exists u' \in [u] \exists v' \in [v][\mathsf{ANG}_{\blacktriangleleft x}(w', u', v') \wedge \mathfrak{M}, u' \models \beta \wedge \mathfrak{M}, v' \models \psi]$. Hence, $\mathfrak{M}, w' \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$. Since w and w' are modally equivalent with respect to Σ, it follows that $\mathfrak{M}, w \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\beta, \psi)$ as desired.

Now suppose that $\varphi = @_{\mathsf{i}}\psi$ for some nominal letter i and $\mathfrak{M}, w \models @_{\mathsf{i}}\psi$. Then, $\exists u[\mathfrak{M}, u \models \mathsf{i} \wedge \psi]$. Since $\mathsf{i} \in \Sigma$ and it is the name for the state u, by construction it follows that $[u] \in \mathsf{W}^{\mathrm{Fin}}$ and $\mathfrak{M}^f, [u] \models \mathsf{i}$ [3]. By the induction hypothesis, we also

[3]Actually, for any state which is named by a nominal from Σ, its equivalence class consists of only by itself.

get that $\mathfrak{M}^f, [u] \models \psi$. Putting it altogether, we have that $\mathfrak{M}^f, [w] \models @_{\mathsf{i}}\psi$.

Conversely, suppose that $\mathfrak{M}^f, [w] \models @_{\mathsf{i}}\psi$. Then, we have $\exists[u][\mathfrak{M}^f, [u] \models \mathsf{i} \wedge \psi]$. It is easy too see by Construction 3.5.1 that we have $\mathfrak{M}, u \models \mathsf{i}$. On the other hand, by the induction hypothesis we have that $\mathfrak{M}, u \models \psi$. Hence, $\mathfrak{M}, w \models @_{\mathsf{i}}\psi$. □

We will now begin to address the issues arising from the fact that constraints **CNT**5 and **CNT**9 are not satisfied over the filtrated frame.

First, let us enrich our vocabulary with some new and useful notions. We call the abnormal situations in frames where configurations (of relations, states or parameters) violate frame conditions as 'defects.' We will perform two separate repair procedures in order to remove such defects from frames. Each procedure will address one kind of defect, caused by the violations of one of the two aforementioned frame conditions. First, we begin dealing with defects related to the constraints **CNT**5.

The underlying idea of the first repair procedure is to create a new frame $\mathfrak{F}^*$ from the filtrated frame $\mathfrak{F}^f$ by means of splitting the states which are related to a defect in the filtrated frame, into multiple states in the new frame. In this way, the procedure is given "enough room" to isolate the incompatible relations from each other in the new frame and thereby repairing the defects.

Unfortunately, a procedure which merely splits the states in two or more states and isolates incompatible relations in this way will not produce frames where **CNT**10 can be guaranteed to hold, even though there are no problems in $\mathfrak{F}^f$ related to **CNT**10 (see Lemma 3.5.1). In other words, a simple splitting technique will create other problems even tough it solves the original one.

To address this issue simultaneously, we enhance the splitting procedure to handle the "relevant cases": It will be sufficient to increase the amount of splitting of the states which are in the range of constraint **CNT**10 in $\mathfrak{F}^f$ and devote a special part of the procedure to handle these states. In this way, such situations can be addressed in isolation, without interacting with the main purpose of the procedure, which is repairing **CNT**5 defects.

We will use the following shorthands in order to make the formalisation of the procedure easier:

- $\mathsf{EQ\_0} \equiv_{\text{def}} \blacktriangleleft = \leq \vee (x \neq 0 \wedge \blacktriangleleft = <) \vee (x = 0 \wedge \blacktriangleleft = \geq)$
- $\mathsf{EQ\_180} \equiv_{\text{def}} \blacktriangleleft = \geq \vee (x \neq 180 \wedge \blacktriangleleft = >) \vee (x = 180 \wedge \blacktriangleleft = \leq)$
- $\mathsf{DISPUTE}(w, u, v) \equiv_{\text{def}} \mathsf{ANG}^{f}_{\leq 0}(w, y, z) \wedge \mathsf{ANG}^{f}_{\geq 180}(w, y, z)$
- $\mathsf{NO\_DISPUTE\_0}(w, u, v) \equiv_{\text{def}} \mathsf{ANG}^{f}_{\leq 0}(w, y, z) \wedge \neg\mathsf{ANG}^{f}_{\geq 180}(w, y, z)$
- $\mathsf{NO\_DISPUTE\_180}(w, u, v) \equiv_{\text{def}} \neg\mathsf{ANG}^{f}_{\leq 0}(w, y, z) \wedge \mathsf{ANG}^{f}_{\geq 180}(w, y, z)$

We call the conjunction of the following two implications as CHECK:

- $\mathsf{NO\_DISPUTE\_0}(w, u, v) \wedge \mathsf{ANG}^{f}_{>0}(w, u, v) \Rightarrow odd(k + l + i)$
- $\mathsf{NO\_DISPUTE\_180}(w, u, v) \wedge \mathsf{ANG}^{f}_{<180}(w, u, v) \Rightarrow odd(k + l + i)$

One final note on the notation: Given a model, we say that a state is 'named' iff there is a nominal letter which holds at that state.

Construction 3.5.2 (First Repair)**.** Now we are ready to give the details of the procedure. The easy part is to determine the states to be split. This is done as follows:

$$\begin{aligned} D^{f} = \Big\{ w \in \mathsf{W}^{\text{Fin}} \mid & \neg\exists \mathsf{i} \in \Sigma[\mathfrak{M}^{\text{Fin}}, w \models \mathsf{i}] \wedge \\ & \exists u \exists v \exists x \exists \blacktriangleleft [(\mathsf{ANG}^{f}_{\blacktriangleleft x}(w, u, v) \wedge \mathsf{ANG}^{f}_{\blacktriangleleft^{d} x}(w, u, v)) \vee \\ & \mathsf{ANG}^{f}_{\leq 0}(w, u, v) \vee \mathsf{ANG}^{f}_{\geq 180}(w, u, v)] \Big\}. \end{aligned}$$

Note that we do not split named-states. Other than that, we split any state which is involved in a collinear trio or any trio with a dispute. Domain set of the new frame, after the splitting operations which are explained above, is formed as follows:

$$\mathsf{W}^* := \left\{\langle w, 0\rangle \mid w \in \mathsf{W}^{\mathrm{Fin}} - D^f\right\} \cup \left\{\langle w, k\rangle \mid w \in D^f, k \in \{0,1,2,3,4\}\right\}$$

Now we continue with the more challenging part of the procedure. Our construction continues with a procedure which will determine how to carry over the relations from the old frame $\mathfrak{F}^f$ to the new frame $\mathfrak{F}^*$. Let three arbitrary states from W^*, such as $\langle w,k\rangle, \langle u,l\rangle, \langle v,i\rangle \in \mathsf{W}^*$ where $w,u,v \in \mathsf{W}^{\mathrm{Fin}}$ and $k,l,i \in \{0,1,2,3,4\}$. The following procedure determines the relations on this triangle for every $\blacktriangleleft \in \mathbb{O}$ and every $x \in \mathbb{A}^f$:

Step 1 deals with the relations on triangles containing identical states. Steps 2 and 3 are dedicated to ensure that the resulting frame will not violate constraint **CNT**10. Step 4 is the main part of the procedure where incompatible relations are isolated from each other, so that **CNT**5 is satisfied over $\mathfrak{F}^*$.

1. If $\langle w,k\rangle = \langle u,l\rangle \vee \langle u,l\rangle = \langle v,i\rangle \vee \langle w,k\rangle = \langle v,i\rangle$, then:

$$\mathsf{ANG}^*_{\blacktriangleleft x}(\langle w,k\rangle, \langle u,l\rangle, \langle v,i\rangle) \text{ iff } \mathsf{EQ\_0}$$

2. If $\mathsf{ANG}^f_{\leq 0}(w,u,v) \vee \mathsf{ANG}^f_{\geq 180}(w,u,v)$ and *exactly two* of the states w, u and v are named and we also have CHECK, then:

   Firstly, we read/set a global variable[4]:

   If $\mathsf{VAR}(w, \{w,u,v\}) = \mathit{odd}/\mathit{null}$, then set $\mathsf{VAR}(w, \{w,u,v\}) = \mathit{even}$ otherwise, set $\mathsf{VAR}(w, \{w,u,v\}) = \mathit{odd}$.

[4]In certain situations when different triangles share the same states which can not be split because they are named, we must know which states have been assigned which relations in different triangles (i.e., previous iterations), so that we can ensure all relations from $\mathfrak{F}^f$ are carried over to $\mathfrak{F}^*$ while keeping the constraint **CNT**10 satisfied. In situations when states can be split we employ index values to achieve this. So, a global variable is used only in this step. For variable reading and setting we use a function VAR, which maps the input to *even*, *odd* or *null*. This should be self-explanatory.

For the sake of simplicity, let † to stand for "$y \in \{u, v\}$ and w, y are named."

$\mathsf{ANG}^{*}_{\sphericalangle x}(\langle w, k\rangle, \langle u, l\rangle, \langle v, i\rangle)$ iff either of the following holds,

- $\mathsf{NO\_DISPUTE\_0}(w, u, v) \wedge \mathsf{EQ\_0}$
- $\mathsf{NO\_DISPUTE\_180}(w, u, v) \wedge \mathsf{EQ\_180}$
- $\mathsf{DISPUTE}(w, u, v) \wedge even(k + l + i) \wedge \dagger \wedge \mathsf{VAR}(y, \{w, u, v\}) = odd/null \wedge \mathsf{EQ\_180}$
- $\mathsf{DISPUTE}(w, u, v) \wedge odd(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y, \{w, u, v\}) = even \wedge \mathsf{EQ\_180}$
- $\mathsf{DISPUTE}(w, u, v) \wedge odd(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y, \{w, u, v\}) = odd/null \wedge \mathsf{EQ\_0}$
- $\mathsf{DISPUTE}(w, u, v) \wedge even(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y, \{w, u, v\}) = even \wedge \mathsf{EQ\_0}$

3. If $\mathsf{ANG}^{f}_{\leq 0}(w, u, v) \vee \mathsf{ANG}^{f}_{\geq 180}(w, u, v)$ and *at most one* of the states w, u and v are named and we also have CHECK, then:

   For the sake of simplicity, let † to stand for "either u or v is named."

   $\mathsf{ANG}^{*}_{\sphericalangle x}(\langle w, k\rangle, \langle u, l\rangle, \langle v, i\rangle)$ iff either of the following holds,

   - $\mathsf{NO\_DISPUTE\_0}(w, u, v) \wedge \mathsf{EQ\_0}$
   - $\mathsf{NO\_DISPUTE\_180}(w, u, v) \wedge \mathsf{EQ\_180}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge k = 0 \wedge \dagger \wedge even(k + l + i) \wedge \mathsf{EQ\_180}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge k \neq l = i = 0 \wedge \dagger \wedge even(k + l + i) \wedge \mathsf{EQ\_0}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge (k > l > i \vee k > i > l) \wedge \mathsf{EQ\_180}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge (k < l \vee k < i) \wedge k \neq l \neq i \neq k \wedge \mathsf{EQ\_0}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge k \neq l = i \wedge \mathsf{EQ\_180}$
   - $\mathsf{DISPUTE}(w, u, v) \wedge (k = l \neq i \vee k = i \neq l) \wedge \mathsf{EQ\_0}$

4. If $\neg[\mathsf{ANG}^{f}_{\leq 0}(w, u, v) \vee \mathsf{ANG}^{f}_{\geq 180}(w, u, v)]$ or $\neg\mathsf{CHECK}$, then:

   $\mathsf{ANG}^{*}_{\sphericalangle x}(\langle w, k\rangle, \langle u, l\rangle, \langle v, i\rangle)$ iff either of the following holds,

- $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \neg \mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$

- $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge x \neq 0 \wedge \blacktriangleleft \in \{<, \leq\} \wedge (k = l = i \vee k \neq l \neq i \neq k)$

- $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge x \neq 180 \wedge \blacktriangleleft \in \{>, \geq\} \wedge (k = l \neq i \vee k = i \neq l \vee l = i \neq k)$

- $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge x = 0 \wedge \blacktriangleleft \in \{>, \geq\}$

- $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge x = 180 \wedge \blacktriangleleft \in \{<, \leq\}$

Now it remains to set up the new valuation function. For every $\alpha \in \mathscr{P} \cup \mathscr{N}$, the valuation function can be defined by setting:

$$V^*(\alpha) = \{\langle w, k\rangle \in \mathsf{W}^* \mid w \in V^f(\alpha), k \in \{0,1,2,3,4\}\}.$$

Recall that the named states are not split and hence, the nominal letters continue to hold at unique states. Our frame and model can now be put together as follows:

$$\mathfrak{F}^*[\mathbb{A}^f] := \left\langle \mathsf{W}^*, \left\{\mathsf{ANG}^*_{\blacktriangleleft x}\right\}\right\rangle_{\blacktriangleleft \in \mathbb{O}, x \in \mathbb{A}^f}$$

and

$$\mathfrak{M}^* := \left\langle \mathfrak{F}^*[\mathbb{A}^f], V^*\right\rangle.$$

Lemma 3.5.3. *Let $w, u, v \in \mathfrak{F}^f$ such that u and v are named-states. Then the following statements hold:*

- *w is dispute-free, i.e., there is no $x \in \mathbb{A}^f$ and $\blacktriangleleft \in \mathbb{O}$ such that $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$.*

- *If we have $\mathsf{DISPUTE}(u,w,v)$, then we also have $\mathsf{DISPUTE}(v,w,u)$.*

Proof. Let $w, u, v \in \mathfrak{F}^f$ such that u and v are named-states. For the sake of a contradiction, suppose that we have $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$ for some $x \in \mathbb{A}^f$ and $\blacktriangleleft \in \mathbb{O}$. By the filtration, it follows that there are $w', w'' \in w$, $u', u'' \in u$

and $v', v'' \in v$ such that $\mathsf{ANG}_{\blacktriangleleft x}(w', u', v') \wedge \mathsf{ANG}_{\blacktriangleleft^d x}(w'', u'', v'')$ in $\mathfrak{F}$. However, since u and v are named-states, we must have $u' = u''$ and $v' = v''$.

It follows that w' and w'' are distinct states but modally-equivalent with respect to Σ. For otherwise, this would contradict with the fact that $\mathfrak{F}$ is an angular frame and satisfies **CNT**5. From here, we also have $\neg\mathsf{ANG}_{\blacktriangleleft x}(w'', u', v') \wedge \neg\mathsf{ANG}_{\blacktriangleleft^d x}(w', u'', v'')$. Let j and k be the names of u' and v', respectively. It follows that we have $\mathfrak{M}, w' \models \langle\mathsf{ANG}_{\blacktriangleleft x}\rangle(\mathsf{j}, \mathsf{k}) \wedge \neg\langle\mathsf{ANG}_{\blacktriangleleft^d x}\rangle(\mathsf{j}, \mathsf{k})$ and $\mathfrak{M}, w'' \models \langle\mathsf{ANG}_{\blacktriangleleft^d x}\rangle(\mathsf{j}, \mathsf{k}) \wedge \neg\langle\mathsf{ANG}_{\blacktriangleleft x}\rangle(\mathsf{j}, \mathsf{k})$, contradicting with the fact that w' and w'' are modally-equivalent.

In order to see the second claim, suppose we have $\mathsf{DISPUTE}(u, w, v)$. From here, we must have either $\mathsf{DISPUTE}(w, u, v)$ or $\mathsf{DISPUTE}(v, w, u)$. Since we can not have $\mathsf{DISPUTE}(w, u, v)$ as shown in the other half of this lemma, it follows that we must have $\mathsf{DISPUTE}(v, w, u)$. $\square$

Lemma 3.5.4 shows that the repair procedure works correctly. In other words, defects related to constraint **CNT**5 are removed as a result of Construction 3.5.2. Moreover, as the Lemma 3.5.6 establishes, the satisfiability is preserved through this repair procedure.

Lemma 3.5.4. *$\mathfrak{F}^*$ satisfies all conditions **CNT**1-**CNT**10 except **CNT**9.*

Proof. Let $W, U, V, Y \in \mathsf{W}^*$ such that $W = \langle w, k\rangle$, $U = \langle u, l\rangle$, $V = \langle v, i\rangle$ and $Y = \langle y, j\rangle$ for some $w, u, v, y \in \mathsf{W}^{\mathrm{Fin}}$. Also fix some $x \in \mathbb{A}^{\mathrm{Fin}}$ and $\blacktriangleleft \in \mathbb{O}$. First of all, note that the case of **CNT**1 follows immediately from step 1 of Construction 3.5.2.

We continue by showing that **CNT**2 holds over $\mathfrak{F}^*$. Assume that we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$. If all of the states w, u and v are named, then we have from step 4 that $\mathsf{ANG}^f_{\blacktriangleleft x}(w, u, v)$. On the other hand, from Lemma 3.5.3 we get that $\neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, u, v)$. Since $\mathfrak{F}^f$ satisfies **CNT**2, we have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, v, u) \wedge \neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, v, u)$. Now, it follows from step 4 that we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ as desired. Now suppose that at most two of the states in question are named. In Construction 3.5.2, we have the following cases to be considered:[5]

- (**C**A) $\mathsf{ANG}^f_{\leq 0}(w,u,v) \vee \mathsf{ANG}^f_{\geq 180}(w,u,v)$ and the following implications:
  - If $\mathsf{NO\_DISPUTE\_0}(w,u,v) \wedge \mathsf{ANG}^f_{>0}(w,u,v)$, then $odd(k+l+i)$
  - If $\mathsf{NO\_DISPUTE\_180}(w,u,v) \wedge \mathsf{ANG}^f_{<180}(w,u,v)$, then $odd(k+l+i)$
- (**C**B) $\neg[\mathsf{ANG}^f_{\leq 0}(w,u,v) \vee \mathsf{ANG}^f_{\geq 180}(w,u,v)]$ or one of the following:
  - $\mathsf{NO\_DISPUTE\_0}(w,u,v) \wedge \mathsf{ANG}^f_{>0}(w,u,v) \wedge even(k+l+i)$
  - $\mathsf{NO\_DISPUTE\_180}(w,u,v) \wedge \mathsf{ANG}^f_{<180}(w,u,v) \wedge even(k+l+i)$

Let us first consider (**C**A). If exactly two of the states are named, then it means that we have the following sub-cases: For the sake of simplicity, let † to stand for "$y \in \{u,v\}$ and w,y are named."

- (**C**A.1.1) $\mathsf{NO\_DISPUTE\_0}(w,u,v) \wedge \mathsf{EQ\_0}$
- (**C**A.1.2) $\mathsf{NO\_DISPUTE\_180}(w,u,v) \wedge \mathsf{EQ\_180}$
- (**C**A.1.3) $\mathsf{DISPUTE}(w,u,v) \wedge even(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y,\{w,u,v\}) = odd/null \wedge \mathsf{EQ\_180}$
- (**C**A.1.4) $\mathsf{DISPUTE}(w,u,v) \wedge odd(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y,\{w,u,v\}) = even \wedge \mathsf{EQ\_180}$
- (**C**A.1.5) $\mathsf{DISPUTE}(w,u,v) \wedge odd(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y,\{w,u,v\}) = odd/null \wedge \mathsf{EQ\_0}$
- (**C**A.1.6) $\mathsf{DISPUTE}(w,u,v) \wedge even(k+l+i) \wedge \dagger \wedge \mathsf{VAR}(y,\{w,u,v\}) = even \wedge \mathsf{EQ\_0}$

[5] Since we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W,U,V)$ by the assumption, it must be assigned by one of the steps of Construction 3.5.2. Initially, we consider two possibilities. One possibility is that, this relation is set by one of step 2 or step 3 and the other possibility is that it is assigned by step 4. Here, (**C**A) and (**C**B) represent these two possibilities. Notice that, in the construction procedure (**C**A) is shortly referred as CHECK for the sake of simplicity. (**C**B) is referred as ¬CHECK.

First consider (**C**A.1.1). We get $\mathsf{NO\_DISPUTE\_0}(w, u, v)$. So, we must have $\blacktriangleleft \in \{<, \leq\}$ or $\blacktriangleleft = \geq \wedge x = 0$. Since $\mathfrak{F}^f$ satisfies **CNT**2, it follows that we have $\mathsf{NO\_DISPUTE\_0}(w, v, u)$. This implies that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ through step 3. (**C**A.1.2) is similar. If we have (**C**A.1.3), then we also have $\mathsf{DISPUTE}(w, v, u)$ since $\mathfrak{F}^f$ satisfies **CNT**2. Moreover, we have that $\mathsf{VAR}(y, \{w, u, v\}) = odd/null$ for some $y \in \{u, v\}$. Since $k + i + l$ is even, we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ as desired. The rest of the cases (**C**A.1.4) - (**C**A.1.6) follow in a very similar way.

Now suppose that at most one of the states is named. This means that we have the following sub-cases to consider: For the sake of simplicity, let † to stand for "either u or v is named."

- (**C**A.2.1) $\mathsf{NO\_DISPUTE\_0}(w, u, v) \wedge \mathsf{EQ\_0}$
- (**C**A.2.2) $\mathsf{NO\_DISPUTE\_180}(w, u, v) \wedge \mathsf{EQ\_180}$
- (**C**A.2.3) $\mathsf{DISPUTE}(w, u, v) \wedge k = 0 \wedge \dagger \wedge even(k + l + i) \wedge \mathsf{EQ\_180}$
- (**C**A.2.4) $\mathsf{DISPUTE}(w, u, v) \wedge k \neq l = i = 0 \wedge \dagger \wedge even(k + l + i) \wedge \mathsf{EQ\_0}$
- (**C**A.2.5) $\mathsf{DISPUTE}(w, u, v) \wedge (k > l > i \vee k > i > l) \wedge \mathsf{EQ\_180}$
- (**C**A.2.6) $\mathsf{DISPUTE}(w, u, v) \wedge (k < l \vee k < i) \wedge k \neq l \neq i \neq k \wedge \mathsf{EQ\_0}$
- (**C**A.2.7) $\mathsf{DISPUTE}(w, u, v) \wedge k \neq l = i \wedge \mathsf{EQ\_180}$
- (**C**A.2.8) $\mathsf{DISPUTE}(w, u, v) \wedge (k = l \neq i \vee k = i \neq l) \wedge \mathsf{EQ\_0}$

(**C**A.2.1) and (**C**A.2.2) are very similar to the proof of (**C**A.1.1) above. So now consider (**C**A.2.3). First of all, we easily get that $\mathsf{DISPUTE}(w, v, u)$ since $\mathfrak{F}^f$ satisfies **CNT**2. Moreover, we get that either u or v is named and $k = 0$. But then putting it altogether, we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ as desired. (**C**A.2.4) follows in a similar way. Consider (**C**A.2.5). First, note that we get $\mathsf{DISPUTE}(w, v, u)$ and that k is strictly greater than l and i. But this implies

$\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$. (**C**A.2.6) follows similarly. Now consider case (**C**A.2.7). We again get $\mathsf{DISPUTE}(w, v, u)$ since $\mathfrak{F}^f$ satisfies **CNT**2. On the other hand, we have $k \neq l = i$. Hence, $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ as desired. Finally, consider (**C**A.2.8). We have $\mathsf{DISPUTE}(w, v, u)$ and either $k = l \neq i$ or $k = i \neq l$. In both cases we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ from step 4 once again.

Now let us consider case (**C**B). First of all, it immediately follows that we have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, u, v)$ from step 4. Moreover, since $\mathfrak{F}^f$ satisfies **CNT**2 by Lemma 3.5.1, we also have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, v, u)$. In addition, one of the following must be the case:

- (**C**B.1) $\neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, u, v)$
- (**C**B.2) $x \neq 0 \wedge \blacktriangleleft \in \{<, \leq\} \wedge (k = l = i \vee k \neq l \neq i \neq k)$
- (**C**B.3) $x \neq 180 \wedge \blacktriangleleft \in \{>, \geq\} \wedge (k = l \neq i \vee k = i \neq l \vee l = i \neq k)$
- (**C**B.4) $x = 0 \wedge \blacktriangleleft \in \{>, \geq\}$
- (**C**B.5) $x = 180 \wedge \blacktriangleleft \in \{<, \leq\}$

If (**C**B.1), then we clearly have that $\neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, v, u)$ since $\mathfrak{F}^f$ satisfies **CNT**2 by Lemma 3.5.1. But since we also have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, v, u)$, this implies that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$ by step 4. In the case of (**C**B.2), since we have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, v, u)$ and $x \neq 0 \wedge \blacktriangleleft \in \{<, \leq\}$ and either $k = l = i$ or $k \neq l \neq i \neq k$, we immediately get the desired result from step 4. (**C**B.3) follows similarly. In case of (**C**B.4), it follows that we have $x = 0 \wedge \blacktriangleleft \in \{>, \geq\}$. But then again from here we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, V, U)$, which is what we want. (**C**B.5) follows similarly.

Now we show that $\mathfrak{F}^*$ satisfies **CNT**3, i.e., $\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V) \vee \mathsf{ANG}^*_{\blacktriangleleft^d x}(W, U, V)$. Note that, since $\mathfrak{F}^f$ satisfies **CNT**3 by Lemma 3.5.1, we have $\mathsf{ANG}^f_{\blacktriangleleft x}(w, u, v) \vee \mathsf{ANG}^f_{\blacktriangleleft^d x}(w, u, v)$. Now, in Construction 3.5.2, there are two cases to be considered regarding the triple W, U, V. These are again the cases (**C**A) and (**C**B) listed in the above, in the proof of constraint **CNT**2.

First of all, note that there is nothing to show in the case that all of the states w, u and v are named, since the relations on the triangle of W, U and V are identical to its predecessor triangle in $\mathfrak{F}^f$ and $\mathfrak{F}^f$ satisfies **CNT**3.

Let us first consider case (**C**A). First, the easy part. If $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \neg\mathsf{ANG}^f_{\geq 180}(w, u, v)$ or in other words, $\mathsf{NO\_DISPUTE\_0}(w, u, v)$, and we also have $x \neq 0$, we get $\mathsf{ANG}^*_{<x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$ regardless of the number of named states. This gives us what we want. On the other hand, if $x = 0$ then $\mathsf{ANG}^*_{\geq x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$, which also gives us the desired result. The case when we have $\neg\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \mathsf{ANG}^f_{\geq 180}(w, u, v)$ follows similarly.

Now suppose that we have $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \mathsf{ANG}^f_{\geq 180}(w, u, v)$. Suppose only w and u are named-states and $k + l + i$ is odd. If $\mathsf{VAR}(u, \{w, u, v\}) = odd/null$, then we get $\mathsf{ANG}^*_{<x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$. On the other hand, if $\mathsf{VAR}(u, \{w, u, v\}) = even$, then we have $\mathsf{ANG}^*_{>x}(W, U, V)$ and also $\mathsf{ANG}^*_{\geq x}(W, U, V)$. Now suppose $k + l + i$ is even. Then we get $\mathsf{ANG}^*_{<x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$ when $\mathsf{VAR}(u, \{w, u, v\}) = even$ and on the other hand, $\mathsf{ANG}^*_{>x}(W, U, V)$ and $\mathsf{ANG}^*_{\geq x}(W, U, V)$ when $\mathsf{VAR}(u, \{w, u, v\}) = odd/null$. The case when w and v are named-states follows similarly. Now suppose u and v are named. But from Lemma 3.5.3 it follows that this is impossible since we have $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \mathsf{ANG}^f_{\geq 180}(w, u, v)$.

Alternatively, suppose that at most one of w, u and v is named. We can safely assume that $\neg(k = l = i)$. In this case, either the indexes are mutually distinct or exactly two of them are equal to each other. In the former case, we have $\mathsf{ANG}^*_{>x}(W, U, V)$ and $\mathsf{ANG}^*_{\geq x}(W, U, V)$ when k is the maximum of all three and $\mathsf{ANG}^*_{<x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$ otherwise. In the case of the latter, we have $\mathsf{ANG}^*_{<x}(W, U, V)$ and $\mathsf{ANG}^*_{\leq x}(W, U, V)$ if k is equal to one of the other two and $\mathsf{ANG}^*_{>x}(W, U, V)$ and $\mathsf{ANG}^*_{\geq x}(W, U, V)$ if k is not equal to the other two, which must be equal to each other.

Let us now consider (**C**B). We have the following sub-cases to be considered:

- (**C**B.1) $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$,
- (**C**B.2) $\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$,
- (**C**B.3) $\neg\mathsf{ANG}^f_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^f_{\blacktriangleleft^d x}(w,u,v)$.

Consider (**C**B.2). It is obvious from step 4 of Construction 3.5.2 that we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W,U,V)$. Similarly, (**C**B.3) implies $\mathsf{ANG}^*_{\blacktriangleleft^d x}(W,U,V)$.

So, consider case (**C**B.1). Assume that $\blacktriangleleft \in \{<,\leq\}$. The case when $\blacktriangleleft \in \{>,\geq\}$ can be established using similar arguments. Firstly, suppose we have either $k=l=i$ or $k \neq l \neq i \neq k$. Assume $x \neq 0$. Now, if $\blacktriangleleft = <$, then we have $\mathsf{ANG}^f_{<x}(w,u,v)$ and hence, $\mathsf{ANG}^f_{\leq x}(w,u,v)$ since $\mathfrak{F}^f$ satisfies **CNT**4 by Lemma 3.5.1. From step 4, we get $\mathsf{ANG}^*_{<x}(W,U,V)$ and $\mathsf{ANG}^*_{\leq x}(W,U,V)$, which gives us what we want. Alternatively, suppose that $\blacktriangleleft = \leq$. Clearly, we have $\mathsf{ANG}^*_{\leq x}(W,U,V)$ by the construction. Moreover, it is easy to see that we have $\mathsf{ANG}^f_{>x}(w,u,v)$ and hence, $\mathsf{ANG}^f_{\geq x}(w,u,v)$ since $\mathfrak{F}^f$ satisfies **CNT**4. So, if $\neg\mathsf{ANG}^f_{<x}(w,u,v)$, then we have $\mathsf{ANG}^*_{\geq x}(W,U,V)$. Otherwise, obviously $\mathsf{ANG}^*_{<x}(W,U,V)$. Now suppose we have $x=0$. So, we must have $\blacktriangleleft = \leq$. Therefore, we also have $\mathsf{ANG}^f_{>x}(w,u,v)$ and henceforth, $\mathsf{ANG}^f_{\geq x}(w,u,v)$. From here we get $\mathsf{ANG}^*_{\geq x}(W,U,V) \wedge \mathsf{ANG}^*_{>x}(W,U,V)$ by step 4 of Construction 3.5.2.

Now suppose we have either of $k=l\neq i$ or $k=i\neq l$ or $l=i\neq k$. If $x \neq 180$ and $\blacktriangleleft = <$, this means that we have $\mathsf{ANG}^f_{\geq x}(w,u,v)$ and so, $\mathsf{ANG}^*_{\geq x}(W,U,V)$. On the other hand, suppose we have $\mathsf{ANG}^f_{>x}(w,u,v)$. It easily follows that we have $\mathsf{ANG}^*_{>x}(W,U,V)$ and we get what we want. Alternatively, suppose that $\neg\mathsf{ANG}^f_{>x}(w,u,v)$. Since we have $\mathsf{ANG}^f_{<x}(w,u,v)$, we must also have that $\mathsf{ANG}^f_{\leq x}(w,u,v)$ since $\mathfrak{F}^f$ satisfies **CNT**4. But then from step 4, we conclude that $\mathsf{ANG}^*_{\leq x}(W,U,V)$ and once again we are through. On the other hand, suppose we have $\blacktriangleleft = \leq$. So it follows that we have $\mathsf{ANG}^f_{>x}(w,u,v)$ and hence, $\mathsf{ANG}^f_{\geq x}(w,u,v)$

since $\mathfrak{F}^f$ satisfies **CNT**4. This gives $\mathsf{ANG}^*_{>x}(W, U, V) \wedge \mathsf{ANG}^*_{\geq x}(W, U, V)$ which gives us what we want. Finally, suppose $x = 180$. If $\blacktriangleleft=<$, then we have that $\mathsf{ANG}^f_{<x}(w, u, v)$ and hence, $\mathsf{ANG}^f_{\leq x}(w, u, v)$ since $\mathfrak{F}^f$ satisfies **CNT**4. Now, it follows from step 4 of Construction 3.5.2 that we have $\mathsf{ANG}^*_{<x}(W, U, V) \wedge \mathsf{ANG}^*_{\leq x}(W, U, V)$. On the other hand, $\blacktriangleleft=\leq$ is not a possibility since it would imply that $\mathsf{ANG}^f_{>180}(w, u, v)$, which is absurd.

To see that **CNT**4 holds over $\mathfrak{F}^*$, assume that $\mathsf{ANG}^*_{<x}(W, U, V)$. We have to show that $\mathsf{ANG}^*_{\leq x}(W, U, V)$. We leave it to the reader, in the case that all of the states w, u and v are named. It is straightforward once we observe that $\mathfrak{F}^f$ satisfies **CNT**4. In Construction 3.5.2, we have the cases (**C**A) and (**C**B) listed in the proof of constraint **CNT**2 to be considered.

Let us first consider (**C**A). If exactly two of the states in question are named, this means that we have the sub-cases (**C**A.1.1) - (**C**A.1.6) listed above. By the assumption, we have $\blacktriangleleft=<$. This means that the cases (**C**A.1.2), (**C**A.1.3) and (**C**A.1.4) are impossible. On the other hand, for the cases (**C**A.1.1), (**C**A.1.5) and (**C**A.1.6), there is not much to show and it immediately follows that we have $\mathsf{ANG}^*_{\leq x}(W, U, V)$ as desired from step 2.

Now suppose that at most one of the states w, u and v is named. In this case we have one of the sub-cases (**C**A.2.1) - (**C**A.2.8) listed above. Now, due to the fact that $\blacktriangleleft=<$, the cases (**C**A.2.2), (**C**A.2.3), (**C**A.2.5) and (**C**A.2.7) are impossible. For the rest of the cases, the desired result follows immediately in the same way.

Now, let us now consider (**C**B). Firstly, from Construction 3.5.2 one can easily see that we have $\mathsf{ANG}^f_{<x}(w, u, v)$. Moreover, since $\mathfrak{F}^f$ satisfies **CNT**4 by Lemma 3.5.1, it follows that $\mathsf{ANG}^f_{\leq x}(w, u, v)$. In addition, we have one of the sub-cases listed as (**C**B.1) - (**C**B.5) in the above.

If (**C**B.1), then we have $\neg\mathsf{ANG}^f_{>x}(w, u, v)$ since $\mathfrak{F}^f$ satisfies **CNT**4 by Lemma

3.5.1. Since we also have that $\mathsf{ANG}^f_{\leq x}(w, u, v)$, it follows from step 4 of Construction 3.5.2 that we have $\mathsf{ANG}^*_{\leq x}(W, U, V)$ as desired. Now consider (**C**B.2). But then there is nothing to show and we get $\mathsf{ANG}^*_{\leq x}(W, U, V)$ from step 4. Note that cases (**C**B.3) and (**C**B.4) are impossible since we have $\blacktriangleleft = <$. Finally, if (**C**B.5) is the case, then we immediately get what we want. It can be established along very similar lines that we have $\mathsf{ANG}^*_{>x}(W, U, V) \Rightarrow \mathsf{ANG}^*_{\geq x}(W, U, V)$ as well.

Let us now show that **CNT**5 holds over $\mathfrak{F}^*$. Suppose $\mathsf{ANG}^*_{\leq x}(W, U, V)$. If all of the states w, u and v are named, then this triangle of states must be defect free in $\mathfrak{F}^f$ by Lemma 3.5.3. This means that **CNT**5 is satisfied on this particular triangle. Hence, it is satisfied on the triangle W, U and V, which has the identical relations to its predecessor. Now, we have the cases (**C**A) and (**C**B) listed in the proof of constraint **CNT**2 to be considered.

Let us first consider (**C**A). If exactly two of the states in question are named, this means that we have the sub-cases (**C**A.1.1) - (**C**A.1.6) listed above. It is easy to see that in either of the cases (**C**A.1.1) or (**C**A.1.5) or (**C**A.1.6), we have $\neg\mathsf{ANG}^*_{>x}(W, U, V)$ from step 2. On the other hand, if either of (**C**A.1.2) or (**C**A.1.3) or (**C**A.1.4) is the case, then we must have $x = 180$. Therefore, in any case we will have $\neg\mathsf{ANG}^*_{>x}(W, U, V)$ from step 2 as desired.

Now suppose that at most one of the states w, u and v is named. In this case we have one of the sub-cases (**C**A.2.1) - (**C**A.2.8) listed above. However, this is very similar to the reasoning in the previous paragraph: In cases (**C**A.2.1), (**C**A.2.4), (**C**A.2.6) and (**C**A.2.8), we get $\neg\mathsf{ANG}^*_{>x}(W, U, V)$ from step 3 as desired. For the rest of the cases, it follows that we must have $x = 180$, leading to the desired result.

Now let us consider case (**C**B). First of all, we get that $\mathsf{ANG}^f_{\leq x}(w, u, v)$. In addition, we have one of the sub-cases listed as (**C**B.1) - (**C**B.5) in the above. In case of (**C**B.1), we easily get that $\neg\mathsf{ANG}^*_{>x}(W, U, V)$. On the other hand, suppose that we have (**C**B.2). For the sake of a contradiction, suppose $\mathsf{ANG}^*_{>x}(W, U, V)$. From here, the only way for this to happen from step 4 is when we have $\neg\mathsf{ANG}^f_{\leq x}(w, u, v)$, which is a contradiction. The cases of (**C**B.3) and (**C**B.4) are not possible since $\blacktriangleleft = \leq$. Finally, consider (**C**B.5). We get $x = 180$. But then this means that $\mathsf{ANG}^*_{>x}(W, U, V)$, which is impossible.

Next, we show that **CNT**6 holds over $\mathfrak{F}^*$. Let $y \in \mathbb{A}^f$ such that $x \leq y$ and suppose that $\mathsf{ANG}^*_{<x}(W, U, V)$. We will show that $\mathsf{ANG}^*_{<y}(W, U, V)$. As usual, we leave the case when all of w, u and v are named to the reader. The case that $\mathsf{ANG}^*_{\leq x}(W, U, V) \Rightarrow \mathsf{ANG}^*_{\leq y}(W, U, V)$ follows in a similar way. In Construction 3.5.2, there are two cases to be considered which were listed above as (**C**A) and (**C**B).

Note that we must have $y > 0$, since we have $\mathsf{ANG}^*_{<x}(W, U, V)$ and $x \leq y$. First, we consider case (**C**A). If exactly two of the states in question are named, this means that we have the sub-cases (**C**A.1.1) - (**C**A.1.6) listed above. Since $\blacktriangleleft = <$, it follows that none of the cases (**C**A.1.2), (**C**A.1.3) and (**C**A.1.4) is possible. On the other hand, in all of the rest of the cases we get $\mathsf{ANG}^*_{<y}(W, U, V)$ immediately. Moreover, if at most one of the states are named, we are through with the very same arguments.

Now let us consider case (**C**B). First of all, we have $\mathsf{ANG}^f_{<x}(w, u, v)$. Moreover, since $\mathfrak{F}^f$ satisfies **CNT**6 from Lemma 3.5.1, we get $\mathsf{ANG}^f_{<y}(w, u, v)$. In addition, we have one of the cases (**C**B.1) - (**C**B.5). First consider (**C**B.1). From here and since $\mathfrak{F}^f$ satisfies **CNT**6, it is easy to see that we have $\neg\mathsf{ANG}^f_{\geq y}(w, u, v)$. Hence, from step 4 of Construction 3.5.2 we get $\mathsf{ANG}^*_{<y}(W, U, V)$. Now let us have a look at (**C**B.2). But then we immediately get $\mathsf{ANG}^*_{<y}(W, U, V)$ from step 4. On the other hand, note that (**C**B.3) and (**C**B.4) are not possible since we have $\blacktriangleleft = <$. Finally, in case of (**C**B.5), it follows that $x = y = 180$ and the desired result is immediate. **CNT**7 can be established in a very similar way.

To establish that **CNT**8 holds over $\mathfrak{F}^*$, we will show that $\mathsf{ANG}^*_{\leq 180}(W, U, V)$ and leave the part that $\mathsf{ANG}^*_{\geq 0}(W, U, V)$ to the reader. For this, it is sufficient to establish that $\neg\mathsf{ANG}^f_{>180}(w, u, v)$. We will show that this implies $\neg\mathsf{ANG}^*_{>180}(W, U, V)$ and since we have already established that $\mathfrak{F}^*$ satisfies **CNT**3, we would have our result. Since $\mathfrak{F}^f$ satisfies **CNT**8, we have $\mathsf{ANG}^f_{\leq 180}(w, u, v)$. For sake of a contradiction, assume that $\mathsf{ANG}^f_{>180}(w, u, v)$. But now, from Construction 3.5.1 it follows that there are $w', u', v' \in \mathsf{W}$ such that $\mathsf{ANG}_{>180}(w, u, v)$, which is a contradiction since $\mathfrak{F}$ satisfies **CNT**8 and **CNT**5. In other words, $\neg\mathsf{ANG}^f_{>180}(w, u, v)$ as desired.

So, it remains to show that $\neg\mathsf{ANG}^f_{>180}(w, u, v)$ implies $\neg\mathsf{ANG}^*_{>180}(W, U, V)$. For sake of a contradiction, suppose that we have $\mathsf{ANG}^*_{>180}(W, U, V)$. In Construction 3.5.2, we have the cases (**C**A) and (**C**B) to be considered.

First, we consider case (**C**A). Assume that exactly two of the states w, u and v are named. So, we have the sub-cases (**C**A.1.1) - (**C**A.1.6). Since $\blacktriangleleft$=>, none of the cases (**C**A.1.1), (**C**A.1.5) or (**C**A.1.6) is possible. However, neither the rest of the cases is possible, since we have the combination $\blacktriangleleft$=> and $x = 180$. Hence, we get the contradiction we want. The case when there is at most one named state among w, u and v follows similarly.

Now consider case (**C**B). We have one of the sub-cases (**C**B.1) - (**C**B.5). But all of them imply that we have $\mathsf{ANG}^f_{>180}(W, U, V)$, which was shown in the above not to be the case. Hence, a contradiction again.

Finally, to see that **CNT**10 holds over $\mathfrak{F}^*$, assume that $\mathsf{ANG}^*_{\leq 0}(W, U, V)$. The case when all of w, u and v are named is as usual. In Construction 3.5.2, we must consider the cases (**C**A) and (**C**B) listed in the above.

First, we consider case (**C**A). Suppose exactly two of the states in question are named. Then the sub-cases we have to consider are (**C**A.1.1) - (**C**A.1.6). If (**C**A.1.1) is the case, then we have that $\mathsf{ANG}^f_{\leq 0}(w, u, v)$. Since $\mathfrak{F}^f$ satisfies **CNT**10 by Lemma 3.5.1, we get $\mathsf{ANG}^f_{\leq 0}(u, w, v) \vee \mathsf{ANG}^f_{\geq 180}(u, w, v)$. Now there are three sub-cases to be considered:

- (**C**A.1.1.1) $\mathsf{ANG}^f_{\leq 0}(u, w, v) \wedge \neg\mathsf{ANG}^f_{\geq 180}(u, w, v)$
- (**C**A.1.1.2) $\mathsf{ANG}^f_{\geq 180}(u, w, v) \wedge \neg\mathsf{ANG}^f_{\leq 0}(u, w, v)$
- (**C**A.1.1.3) $\mathsf{ANG}^f_{\leq 0}(u, w, v) \wedge \mathsf{ANG}^f_{\geq 180}(u, w, v)$

If (**C**A.1.1.1) is the case, then we have $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ from step 2. Similarly, If (**C**A.1.1.2) is the case, we have $\mathsf{ANG}^*_{\geq 180}(U, W, V)$. Now, suppose that we have (**C**A.1.1.3). So, we have $\mathsf{DISPUTE}(u, w, v)$. It follows from Lemma 3.5.3 and the fact that exactly two of the states are named, that u must be named. Otherwise, w and v are named with a dispute on u. This contradicts with Lemma 3.5.3. Now, it follows from step 2 that we must have either $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ or $\mathsf{ANG}^*_{\geq 180}(U, W, V)$. Note that (**C**A.1.2), (**C**A.1.3) and (**C**A.1.4) are impossible. For the rest of the cases, the proof is almost identical to the case of (**C**A.1.1) above.

Now suppose that at most one of the states in question are named. Then the sub-cases we have to consider now are (**C**A.2.1) - (**C**A.2.8). Of these cases, only (**C**A.2.1), (**C**A.2.4), (**C**A.2.6) and (**C**A.2.8) are possible. Suppose we have (**C**A.2.1). We get $\mathsf{ANG}^f_{\leq 0}(w, u, v)$. Since $\mathfrak{F}^f$ satisfies **CNT**10 by Lemma 3.5.1, we get $\mathsf{ANG}^f_{\leq 0}(u, w, v) \vee \mathsf{ANG}^f_{\geq 180}(u, w, v)$. The rest of the proof is similar to the above where we proceed by considering three cases (**C**A.1.1.1) - (**C**A.1.1.3).

Now suppose we have (**C**A.2.4). So, by the assumption we have the following: $\mathsf{DISPUTE}(w, u, v)$, $k \neq l = i = 0$, that either of u or v is named and, that $k+l+i$ is even. First of all, we have that $\mathsf{ANG}^f_{\leq 0}(u, w, v)$. If we have $\neg\mathsf{ANG}^f_{\geq 180}(u, w, v)$, then we are easily through and we get $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ from step 3. So, suppose we have $\mathsf{ANG}^f_{\geq 180}(u, w, v)$. In other words, $\mathsf{DISPUTE}(u, w, v)$. If u is not named, then v is and we get $\mathsf{ANG}^*_{\geq 180}(U, W, V)$ since we also have $l = 0$. On the other hand, if u is named, it follows that neither w nor v is named, since we have at most one named state. It follows from step 3 that we have $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ since we have $l = i \neq k$.

Now suppose we have (**C**A.2.6). Then, k, l and i are mutually distinct and k is not the maximum of all three. Moreover, we have $\mathsf{DISPUTE}(w, u, v)$. We obviously have that $\mathsf{ANG}^f_{\leq 0}(u, w, v)$. If we have $\neg\mathsf{ANG}^f_{\geq 180}(u, w, v)$, then we are easily through and we get $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ from step 3. So, suppose we have $\mathsf{ANG}^f_{\geq 180}(u, w, v)$. In other words, $\mathsf{DISPUTE}(u, w, v)$. Now, if l is the maximum of all three, then we get $\mathsf{ANG}^*_{\geq 180}(U, W, V)$ otherwise, $\mathsf{ANG}^*_{\leq 0}(U, W, V)$ from step 3. (**C**A.2.8) follows in a similar way.

Finally, we consider case (**C**B). We will show that this is not possible. It follows that we have one of the sub-cases (**C**B.1) - (**C**B.5). But having $\mathsf{ANG}^*_{\leq 0}(W, U, V)$ by the assumption, the only possible case is actually (**C**B.1). From here we get that $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \neg\mathsf{ANG}^f_{>0}(w, u, v)$. But for case (**C**B) to get applied for this triangle, this implies that we have either $\mathsf{NO\_DISPUTE\_0}(w, u, v) \wedge \mathsf{ANG}^f_{>0}(w, u, v) \wedge even(k + l + i)$ or $\mathsf{NO\_DISPUTE\_180}(w, u, v) \wedge \mathsf{ANG}^f_{<180}(w, u, v) \wedge even(k + l + i)$ by the construction. A contradiction. Hence, (**C**A) is the only possible case. □

The following is a helper lemma for the following Lemma 3.5.6 and later for Lemma 3.5.7.

Lemma 3.5.5. $\mathsf{ANG}^*_{\leq 0}(W, U, V) \wedge \mathsf{ANG}^*_{\leq 0}(U, W, V) \Rightarrow \mathsf{ANG}^*_{\geq 180}(V, W, U)$.

Proof. The proof of the lemma is very easy and left to the reader. □

Lemma 3.5.6. *For every Φ and $\langle w, k\rangle \in \mathsf{W}^*$, we have that $\mathfrak{M}^f, w \models \Phi$ iff $\mathfrak{M}^*, \langle w, k\rangle \models \Phi$.*

Proof. Let Φ be a formula and $\langle w, k\rangle \in \mathsf{W}^*$. Call $W = \langle w, k\rangle$. The proof is by induction on the complexity of Φ. Base case is trivial from the construction of

$\mathfrak{F}^*$ and the boolean cases are standard. Now let $x \in \mathbb{A}^f$ and fix $\blacktriangleleft \in \mathbb{O}$. Assume that $\Phi = \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\varphi, \psi)$.

In order to see the claim from left to right, assume $\mathfrak{M}^f, w \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\varphi, \psi)$. Then, $\exists u \exists v[\mathsf{ANG}^f_{\blacktriangleleft x}(w, u, v) \wedge \mathfrak{M}^f, u \models \varphi \wedge \mathfrak{M}^f, v \models \psi]$. From the induction hypothesis, it follows that $\mathfrak{M}^*, U \models \varphi \wedge \mathfrak{M}^*, V \models \psi$ for every $U, V \in \mathsf{W}^*$ such that $U = \langle u, l \rangle$, $V = \langle v, i \rangle$ and $l, i \in \{0, 1, 2, 3, 4\}$. Therefore, we would be able to complete the proof of this case if we could only show that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$ for some $U, V \in \mathsf{W}^*$ such that $U = \langle u, l \rangle$, $V = \langle v, i \rangle$ and $l, i \in \{0, 1, 2, 3, 4\}$. The proof proceeds by considering the following cases:

- (**C**A) $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \mathsf{ANG}^f_{\geq 180}(w, u, v)$
- (**C**B) $\mathsf{ANG}^f_{\leq 0}(w, u, v) \wedge \neg\mathsf{ANG}^f_{\geq 180}(w, u, v)$
- (**C**C) $\mathsf{ANG}^f_{\geq 180}(w, u, v) \wedge \neg\mathsf{ANG}^f_{\leq 0}(w, u, v)$
- (**C**D) $\neg[\mathsf{ANG}^f_{\leq 0}(w, u, v) \vee \mathsf{ANG}^f_{\geq 180}(w, u, v)]$

Let us first consider (**C**A). It follows from Lemma 3.5.3 that at most two of the states w, u and v could be named-states. First, assume that w and u are named-states. The case with w and v follows in a very similar way. It follows that $k = 0$. On the other hand, from step 2 of Construction 3.5.2 we get that either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0 \rangle, \langle v, 2 \rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0 \rangle, \langle v, 3 \rangle)$. This gives us what we want. Note that under the assumption (**C**A), it is not possible that u and v are both named-states by Lemma 3.5.3.

Now assume that at most one of w, u and v is named. Suppose $\blacktriangleleft \in \{<, \leq\}$. If $k \neq 0$, then since at most one of u and v could be named, we have either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0 \rangle, \langle v, k \rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, k \rangle, \langle v, 0 \rangle)$ from step 3. On the other hand, if $k = 0$, then we have either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0 \rangle, \langle v, 3 \rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 3 \rangle, \langle v, 0 \rangle)$.

Now suppose that we have $\blacktriangleleft \in \{>, \geq\}$. If $k \neq 0$ and k is odd, then we have that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0 \rangle, \langle v, 0 \rangle)$ from step 3. If k is even, then either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 1 \rangle, \langle v, 0 \rangle)$

or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 1\rangle)$. Now suppose that we have $k = 0$. If neither u nor v is named, then we obviously have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 1\rangle, \langle v, 1\rangle)$. On the other hand, if one of u and v is named, then we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 2\rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 2\rangle, \langle v, 0\rangle)$ from step 3.

Next, we will consider (**C**B) and leave (**C**C) to the reader since they have very similar proofs. If all of the states w, u and v are named-states, then it follows from Lemma 3.5.3 that this triangle is defect-free. However, in this case the desired result follows immediately from step 3 of Construction 3.5.2.

Now assume that only w and u are named. The case with w and v follows in a very similar way. If $\neg\mathsf{ANG}^f_{>0}(w, u, v)$, then it trivially follows that we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, l\rangle, \langle v, i\rangle)$ for any $l, i \in \{0, 1, 2, 3, 4\}$ from step 2 of Construction 3.5.2. Alternatively, suppose that $\mathsf{ANG}^f_{>0}(w, u, v)$. If $\blacktriangleleft \in \{<, \leq\}$, then $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 1\rangle)$ (notice the odd sum of indexes). On the other hand, if $\blacktriangleleft \in \{>, \geq\}$, then $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 2\rangle)$ (this time notice the even sum of indexes) from step 4 of Construction 3.5.2. Now suppose that u and v are named-states. This implies that we have $\neg\mathsf{ANG}^f_{>0}(w, u, v)$ and the desired result follows immediately from step 2.

Now assume that at most one of w, u and v is named. First, suppose we have $\neg\mathsf{ANG}^f_{>0}(w, u, v)$. It immediately follows that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, l\rangle, \langle v, i\rangle)$ for any $l, i \in \{0, 1, 2, 3, 4\}$ from step 3 and we are through. Now suppose that we have $\mathsf{ANG}^f_{>0}(w, u, v)$. If $\blacktriangleleft \in \{<, \leq\}$, then we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, l\rangle, \langle v, i\rangle)$ for any $l, i \in \{0, 1, 2, 3, 4\}$ such that $k + l + i$ is odd. On the other hand, if $\blacktriangleleft \in \{>, \geq\}$ and $k = 0$, then we have either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 2\rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 2\rangle, \langle v, 0\rangle)$ from step 4. Alternatively, if $k \neq 0$, then we have either $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, k\rangle, \langle v, 0\rangle)$ or $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, k\rangle, \langle v, 0\rangle)$ from step 4.

Finally, let us consider (**C**D). First of all, if we have $\neg\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, u, v)$, then we clearly have that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, l\rangle, \langle v, i\rangle)$ for any $l, i \in \{0, 1, 2, 3, 4\}$ such that

$\langle u, l\rangle, \langle v, i\rangle \in \mathsf{W}^*$.

Now assume that $\mathsf{ANG}^f_{\blacktriangleleft^d x}(w, u, v)$. Notice that it is impossible that all of w, u and v are named. First, suppose that only w and u are named. If $\blacktriangleleft \in \{<, \leq\}$, then we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 0\rangle)$ from step 4 of Construction 3.5.2. On the other hand, if $\blacktriangleleft \in \{>, \geq\}$, then we get from step 4 that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, i\rangle)$ for any $i \in \{0, 1, 2, 3, 4\}$ such that $i \neq 0$. The case for w and v is as usual very similar. Note that, from Lemma 3.5.3 and the assumptions, it follows that u and v can not be both named-states.

Now assume that w is the only named-state. From here we get that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 0\rangle)$ whenever $\blacktriangleleft \in \{<, \leq\}$ and $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 1\rangle, \langle v, 1\rangle)$ whenever $\blacktriangleleft \in \{>, \geq\}$. Finally, assume that u is the only named-state and $\blacktriangleleft \in \{<, \leq\}$. In this case, if $k = 0$, then we have that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 0\rangle)$. On the other hand, if $k \neq 0$, then we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, i\rangle)$ for any $i \in \{0, 1, 2, 3, 4\} - \{0, k\}$. Now suppose $\blacktriangleleft \in \{>, \geq\}$. If $k = 0$, then we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, 1\rangle)$ and if $k \neq 0$, then we get $\mathsf{ANG}^*_{\blacktriangleleft x}(W, \langle u, 0\rangle, \langle v, k\rangle)$.

We have established that we can always find appropriate $U, V \in \mathsf{W}^*$ such that $\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$ and since we also have that $\mathfrak{M}^*, U \models \varphi$ and $\mathfrak{M}^*, V \models \psi$ for every such $U, V \in \mathsf{W}^*$, we conclude that $\mathfrak{M}^*, W \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$ as desired.

In the opposite direction, assume that $\mathfrak{M}^*, W \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$. Then, $\exists U \exists V[\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V) \wedge \mathfrak{M}^*, U \models \varphi \wedge \mathfrak{M}^*, V \models \psi]$ such that U and V are in the form $\langle u, l\rangle$ and $\langle v, i\rangle$, respectively, for some $u, v \in \mathsf{W}^{\mathrm{Fin}}$ and $l, i \in \{0, 1, 2, 3, 4\}$. From here and the induction hypothesis, we have $\mathfrak{M}^f, u \models \varphi$ and $\mathfrak{M}^f, v \models \psi$. In order to conclude the argument, it is sufficient to observe that $\mathsf{ANG}^*_{\blacktriangleleft x}(\langle w, k\rangle, \langle u, l\rangle, \langle v, i\rangle)$ implies that $\mathsf{ANG}^f_{\blacktriangleleft x}(w, u, v)$ by Construction 3.5.2. This implies that $\mathfrak{M}^f, w \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$ as desired.

It remains to see the hybrid satisfaction operator. However, this is straightforward: $\mathfrak{M}^f, w \models @_{\mathsf{i}}\varphi$ iff $\exists u[\mathfrak{M}^f, u \models \mathsf{i} \wedge \varphi]$ iff (since u is named and from the induction hypothesis) $\mathfrak{M}^*, \langle u, 0\rangle \models \mathsf{i} \wedge \varphi$ iff $\mathfrak{M}^*, W \models @_{\mathsf{i}}\varphi$. This completes the proof. □

It remains to repair the defects related to constraint **CNT**9. This repair procedure is more complicated when compared to the previous one, but it fundamentally uses the same idea of "splitting" from the previous repair procedure. Defects we will be dealing with in this part of the proof consist of configurations in $\mathfrak{F}^*$ with the following form:

$$\mathsf{ANG}^*_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft y}(u,w,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft^s z}(v,w,u)$$

for any $\blacktriangleleft \in \{\leq, \geq\}$ and $x, y, z \in \mathbb{A}^f$ such that $x + y + z = 180$. For the sake of compactness and simplicity of the rest of the proof, we define the following shorthand which formally checks for the defects on the triangles in $\mathfrak{F}^*$:[6]

$$\begin{aligned}\mathbf{DFCT}(w,u,v) \equiv \exists x \exists y \exists z \exists \blacktriangleleft\ [&(x+y+z) = 180 \wedge \blacktriangleleft \in \{\leq, \geq\} \wedge \\ &\mathsf{ANG}^*_{\blacktriangleleft x}(w,u,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft y}(u,w,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft z}(v,w,u) \wedge \\ &[\mathsf{ANG}^*_{\blacktriangleleft^s x}(w,u,v) \vee \mathsf{ANG}^*_{\blacktriangleleft^s y}(u,w,v) \vee \mathsf{ANG}^*_{\blacktriangleleft^s z}(v,w,u)]]\end{aligned}$$

The states that are involved in a defect and need to be repaired can be identified with the following set:

$$D^* = \left\{ w \in \mathsf{W}^* \mid \neg\exists \mathsf{i}[\mathfrak{M}^*, w \models \mathsf{i}] \wedge \exists u \exists v [\mathbf{DFCT}(w,u,v)] \right\}$$

It is important to note that the named-states are not a part of this set. This means that, a named-state will never be split into multiple states by the repair procedure below.

Before we start with the details of the repair procedure, we discuss some

[6] The definition of the notation $\blacktriangleleft^s$ ('strict operator') can be found in Definition 3.2.1.

notions and basic arguments that are used by this procedure.

One of the most important notions we will be using is called ‘flipping.’ We call the change of a relation on a triangle to its dual (relation) as flipping. For example, if we have $\mathsf{ANG}_{\blacktriangleleft x}(w, u, v)$, then flipping this configuration will give us $\mathsf{ANG}_{\blacktriangleleft^d x}(w, u, v)$. This is a simple but very important notion that we will use frequently in this repair procedure.

The procedure can be divided into two main components: The first component is the one where the states are split (when possible) into new states for the new frame and it prepares the appropriate “infrastructure” for the more complicated second component. The second component assigns relations of the new frame $\mathfrak{F}^+$ by using the relations from frames $\mathfrak{F}^f$ and $\mathfrak{F}^*$ as guidance and in such a way that, all of the defects are removed. Naturally, we pay more attention to the second component of the repair procedure. The following overview is devoted to the second component.

Roughly speaking, to repair a defect over a triangle it suffices to flip some of the relations involved in the defect. This is the main argument of this entire repair procedure. Let us now explain in detail how this is done.

In order to prevent any loss of information while constructing $\mathfrak{F}^+$ from $\mathfrak{F}^*$, the procedure generates three types of triangles for each triangle in $\mathfrak{F}^*$[7] where in each of these triangle types, the relations on a different pair of states are flipped. Figure 3.6 illustrates this situation. Let us analyse this deeper with the following four observations:

Firstly, flipping relations of a triangle can only be made if the resulting dual-relation by the flipping is already present over the ancestor of this triangle in the frame $\mathfrak{F}^f$. In order to verify this, the procedure simply checks the existence of the needed dual-relations at the filtration frame $\mathfrak{F}^f$. Hence, we will generally

[7]There will be always at least three triangles generated, but in general a triangle from $\mathfrak{F}^*$ can generate up to 27 triangles in $\mathfrak{F}^+$. What we care about here is the *type* of triangles. No matter how many triangles are generated, there will be always precisely three types of triangles.

talk about the procedure 'attempting to flip' a relation, since there might not be always an available relation to achieve this.

Secondly, as established by Lemma 3.5.8, the existence of a defect over a triangle guarantees the existence of dual-relations on at least two of its states in the frame $\mathfrak{F}^f$. Therefore, if the procedure attempts to flip relations on any two states of a defect triangle, we can guarantee that it will succeed with at least one of these attempts. Hence, the defect will certainly be repaired.

Thirdly, the procedure does not flip relations on all of the three states of a defect triangle at once. Although this will manage to remove the defect, it will cause a defect of the dual kind.

Fourthly, since the relations on one different state in each triangle type does not get flipped, every relation over every state in $\mathfrak{F}^*$ is guaranteed to exist in $\mathfrak{F}^+$ over the respective successor state(s). This will guarantee the preservation of the satisfiability from $\mathfrak{F}^*$ to $\mathfrak{F}^+$.

Unfortunately, there is one side effect of the method described so far: The repair procedure might flip the relations of a non-defect configuration and cause an initially non-existent defect. For example, consider a triangle of states w, u and v with the following defect:

$$\mathsf{ANG}^*_{\leq x'}(w,u,v) \wedge \mathsf{ANG}^*_{\leq y'}(u,w,v) \wedge \mathsf{ANG}^*_{<z'}(v,w,u)$$

for some x', y' and z' such that $x' + y' + z' = 180$. For some other parameters x, y and z such that $x + y + z = 180$, assume that we have:

$$\mathsf{ANG}^*_{\leq x}(w,u,v) \wedge \mathsf{ANG}^*_{\leq y}(u,w,v) \wedge \mathsf{ANG}^*_{>z}(v,w,u).$$

Since the triangle in question has a defect, eventually there will be a successor triangle in $\mathfrak{F}^+$ where the relations on (w,u,v) and (u,w,v) will be possibly flipped. This will result with the following configuration:

$$\mathsf{ANG}^{+}_{>x}(w,u,v) \wedge \mathsf{ANG}^{+}_{>y}(u,w,v) \wedge \mathsf{ANG}^{+}_{>z}(v,w,u).$$

But, this is a defect!

Flipping relations in this kind of configurations can be avoided by determining a 'critical parameter value' for one of the two states that are going to be flipped: We claim that not flipping the relations with parameters above or below (depending on the type of defect) this critical parameter value, prevents all of the situations described in the above paragraph without preventing the repair process of any defects. By adapting this critical parameter value into the repair procedure, we can finally guarantee the repair of all the defects. Figure 3.7 illustrates the repair procedure.

Roughly speaking, the critical parameter value of a state in a particular triangle is calculated as the minimum or maximum value (depending on the type of the defect) of certain parameters involved in a defect over that state and triangle. First of all let us define the set of those parameters: For every $W, U, V \in \mathsf{W}^{*}$ such that $W = \langle w, k\rangle$, $U = \langle u, l\rangle$ and $V = \langle v, i\rangle$ for some k, l, i, we set:

$$\begin{aligned}\mathbf{M}^{\blacktriangleleft}(W,U,V) = \Big\{x \in \mathbb{A}^{f} \mid \exists y \exists z[(x+y+z) = 180 \wedge \\ [\neg \mathsf{ANG}^{f}_{\blacktriangleleft^{d} y}(u,w,v) \vee \neg \mathsf{ANG}^{f}_{\blacktriangleleft^{d} z}(v,w,u)] \wedge \\ \mathsf{ANG}^{*}_{\blacktriangleleft x}(W,U,V) \wedge \mathsf{ANG}^{*}_{\blacktriangleleft y}(U,W,V) \wedge \mathsf{ANG}^{*}_{\blacktriangleleft z}(V,W,U) \wedge \\ [\mathsf{ANG}^{*}_{\blacktriangleleft^{s} x}(W,U,V) \vee \mathsf{ANG}^{*}_{\blacktriangleleft^{s} y}(U,W,V) \vee \mathsf{ANG}^{*}_{\blacktriangleleft^{s} z}(V,W,U)]]\Big\}\end{aligned}$$

Using this set of parameters we will produce two critical parameter values, one for each type of defect. These are $\max \mathbf{M}^{\leq}(W,U,V)$ and $\min \mathbf{M}^{\geq}(W,U,V)$. Additionally, note that we have $\mathbf{M}^{\blacktriangleleft}(W,U,V) = \mathbf{M}^{\blacktriangleleft}(W,V,U)$ for any $\blacktriangleleft$.

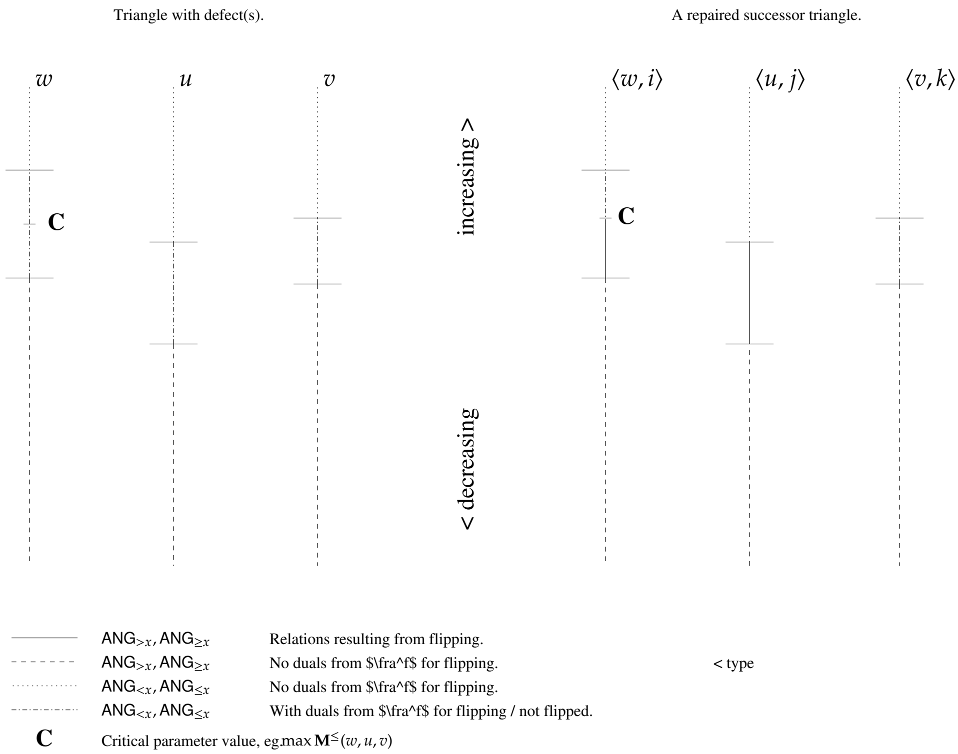

Figure 3.7: Two triangles are represented by the ordering of the parameters on each of their states -in increasing order from bottom to top. Different slices represent different relation types mentioned in the key and hold for parameters in those intervals. Only one of the three types of triangles produced by the repair procedure is depicted (on the right). In this case, the first two states w and u of the defect triangle (on the left) are being attempted for the flipping of their relations. Among the states w and u, the flipping on w is restricted with the critical parameter value denoted by **C**. All of the relations that could be flipped are flipped on state u and no flipping at all is applied to the relations on state v.

Lemma 3.5.7. *Let $W, U, V \in \mathsf{W}^*$. Then we have that,*

- $0 < \max \mathbf{M}^{\leq}(W, U, V) < 180$ *and,*
- $0 < \min \mathbf{M}^{\geq}(W, U, V) < 180$.

Proof. Let $W, U, V \in \mathsf{W}^*$. Let us prove the first claim. It is sufficient to establish that $0, 180 \notin \mathbf{M}^{\leq}(W, U, V)$. For sake of a contradiction, assume that

$0 \in \mathbf{M}^{\leq}(W, U, V)$. Then we have that, $\exists y \exists z[(y + z = 180) \wedge \mathsf{ANG}^{*}_{\leq 0}(W, U, V) \wedge \mathsf{ANG}^{*}_{\leq y}(U, W, V) \wedge \mathsf{ANG}^{*}_{<z}(V, W, U)]$.

Since $\mathfrak{F}^{*}$ satisfies **CNT**10 from Lemma 3.5.4 and also from Lemma 3.5.5, it follows that we have either $\mathsf{ANG}^{*}_{\geq 180}(U, W, V)$ or $\mathsf{ANG}^{*}_{\geq 180}(V, W, U)$. Now, if $\mathsf{ANG}^{*}_{\geq 180}(U, W, V)$, then this means that we have $y = 180$ and $z = 0$. Hence, we get $\mathsf{ANG}^{*}_{<0}(V, W, U)$. A contradiction. On the other hand, assume that $\mathsf{ANG}^{*}_{\geq 180}(V, W, U)$. But then this contradicts with the fact that $\mathsf{ANG}^{*}_{<z}(V, W, U)$ since we obviously also have that $z \leq 180$ and $\mathfrak{F}^{*}$ satisfies **CNT**5 from Lemma 3.5.4.

Now assume that $180 \in \mathbf{M}^{\leq}(W, U, V)$. Then we have $\mathsf{ANG}^{*}_{\leq 180}(W, U, V) \wedge \mathsf{ANG}^{*}_{\leq 0}(U, W, V) \wedge \mathsf{ANG}^{*}_{<0}(V, W, U)$ or we have $\mathsf{ANG}^{*}_{\leq 0}(W, U, V) \wedge \mathsf{ANG}^{*}_{\leq 0}(U, W, V) \wedge \mathsf{ANG}^{*}_{<180}(V, W, U)$. The former case is obviously a contradiction since it says $\mathsf{ANG}^{*}_{<0}(V, W, U)$. For the latter case, it follows from Lemma 3.5.5 that we must have $\mathsf{ANG}^{*}_{\geq 180}(V, W, U)$ when we have $\mathsf{ANG}^{*}_{\leq 0}(W, U, V) \wedge \mathsf{ANG}^{*}_{\leq 0}(U, W, V)$. So, a contradiction. Establishing that $0 < \min \mathbf{M}^{\geq}(W, U, V) < 180$ is along very similar lines. □

The following Lemma 3.5.8 and Lemma 3.5.9, although straightforward, they establish the basis of the underlying idea of the repair procedure. More precisely, they establish the fact that whenever we have a defect over a triangle of states, then we must also have the dual-relations of at least two of the relations involved in the defect holding over the ancestor states in $\mathfrak{F}^{f}$. As already explained above, this enables us to flip relations without adding any new information into the newly constructed frame.

Lemma 3.5.8. *Let $W, U, V \in \mathsf{W}^{Fin}$ and $x, y, z \in \mathbb{A}^{f}$ such that $x + y + z = 180$. For every $\blacktriangleleft \in \{\leq, \geq\}$, if we have that,*

$$\mathsf{ANG}^{f}_{\blacktriangleleft x}(W, U, V) \wedge \mathsf{ANG}^{f}_{\blacktriangleleft y}(U, W, V) \wedge \mathsf{ANG}^{f}_{\blacktriangleleft^{s} z}(V, W, U),$$

then at least one of the following holds:

- $\mathsf{ANG}^f_{\blacktriangleleft^d x}(W, U, V) \wedge \mathsf{ANG}^f_{\blacktriangleleft^d y}(U, W, V)$,
- $\mathsf{ANG}^f_{\blacktriangleleft^d y}(U, W, V) \wedge \mathsf{ANG}^f_{\blacktriangleleft^{sd} z}(V, W, U)$,
- $\mathsf{ANG}^f_{\blacktriangleleft^d x}(W, U, V) \wedge \mathsf{ANG}^f_{\blacktriangleleft^{sd} z}(V, W, U)$.

Proof. We will provide a proof only for $\blacktriangleleft = \leq$. Let $W, U, V \in \mathsf{W}^{\mathrm{Fin}}$ and $x, y, z \in \mathbb{A}^f$ be as in the hypothesis of the lemma. Assume that $\mathsf{ANG}^f_{\leq x}(W, U, V) \wedge \mathsf{ANG}^f_{\leq y}(U, W, V) \wedge \mathsf{ANG}^f_{<z}(V, W, U)$. We have to show that dual-relations hold for at least two of these relations. So, for the sake of a contradiction, suppose we have $\neg[\mathsf{ANG}^f_{>x}(W, U, V) \vee \mathsf{ANG}^f_{>y}(U, W, V)]$. From Construction 3.5.1, it follows that we have $\neg[\mathsf{ANG}_{>x}(w, u, v) \vee \mathsf{ANG}_{>y}(u, w, v)]$ for every $w \in W, u \in U, v \in V$. Since $\mathfrak{F}$ satisfies **CNT**3, this means that $\mathsf{ANG}_{\leq x}(w, u, v) \wedge \mathsf{ANG}_{\leq y}(u, w, v)$ for every $w \in W, u \in U, v \in V$. On the other hand, since we have $\mathsf{ANG}^f_{<z}(V, W, U)$, it follows that $\exists w' \in W \exists u' \in U \exists v' \in V$ such that $\mathsf{ANG}_{<z}(v', w', u')$. But since we also have that $\mathsf{ANG}_{\leq x}(w', u', v') \wedge \mathsf{ANG}_{\leq y}(u', w', v')$, this contradicts with the fact that $\mathfrak{F}$ satisfies **CNT**9. □

Lemma 3.5.9. *Let $W, U, V \in \mathsf{W}^*$ such that $W = \langle w, k \rangle$, $U = \langle u, l \rangle$ and $V = \langle v, i \rangle$ for some k, l, i and also let $x, y, z \in \mathbb{A}^f$ such that $x + y + z = 180$. Then,*

- *If $k = l = i \vee k \neq l \neq i \neq k$, then there are no defects such that $\mathsf{ANG}^*_{\geq x}(W, U, V) \wedge \mathsf{ANG}^*_{\geq y}(U, W, V) \wedge \mathsf{ANG}^*_{>z}(V, W, U)$.*
- *If $k = l \neq i \vee k = i \neq l \vee l = i \neq k$, then there are no defects such that $\mathsf{ANG}^*_{\leq x}(W, U, V) \wedge \mathsf{ANG}^*_{\leq y}(U, W, V) \wedge \mathsf{ANG}^*_{<z}(V, W, U)$.*

Proof. We will only establish the first claim. Let states W, U, V and parameters x, y, z be as in the hypothesis of the lemma. For sake of a contradiction, assume that either $k = l = i$ or $k \neq l \neq i \neq k$. On top of this, assume that we have $\mathsf{ANG}^*_{\geq x}(W, U, V) \wedge \mathsf{ANG}^*_{\geq y}(U, W, V) \wedge \mathsf{ANG}^*_{>z}(V, W, U)$. From

Construction 3.5.2, it follows that we have $\neg\mathsf{ANG}^f_{<x}(w,u,v)$, $\neg\mathsf{ANG}^f_{<y}(u,w,v)$, $\neg\mathsf{ANG}^f_{\leq z}(v,w,u)$ and $\mathsf{ANG}^f_{\geq x}(w,u,v)$, $\mathsf{ANG}^f_{\geq y}(u,w,v)$, $\mathsf{ANG}^f_{>z}(v,w,u)$. On the other hand, from Lemma 3.5.8 we have either $\mathsf{ANG}^f_{<x}(w,u,v) \wedge \mathsf{ANG}^f_{<y}(u,w,v)$ or $\mathsf{ANG}^f_{<y}(u,w,v) \wedge \mathsf{ANG}^f_{\leq z}(v,w,u)$ or $\mathsf{ANG}^f_{<x}(w,u,v) \wedge \mathsf{ANG}^f_{\leq z}(v,w,u)$. Hence, we get the contradiction we were looking for. $\square$

Now we are ready to begin constructing the final frame of the proof. The domain of the new frame is defined as follows:

$$\mathsf{W}^+ := \Big\{\langle W,0\rangle \mid W \in \mathsf{W}^* - D^*\Big\} \cup \Big\{\langle W,k\rangle \mid W \in D^*, k \in \{0,1,2\}\Big\}.$$

With the following definition, we formally define the three types of triangles which were discussed before and the main proof argument relies on. Triangle types are defined in different ways depending on whether states in the given triangle are named or not. However, in general the type of a triangle is determined by looking at the indexes of its states. Types are denoted by $\mathsf{TRI\_1}$, $\mathsf{TRI\_2}$ and $\mathsf{TRI\_3}$.

Definition 3.5.1. Given a triangle $\langle W,j\rangle, \langle U,g\rangle$ and $\langle V,h\rangle$ from W^+, the type of this triangle is determined as follows:

- If none of W, U or V is named:
  - $\mathsf{TRI\_1}$ when $j+g+h=3$,
  - $\mathsf{TRI\_2}$ when $j+g+h<3$,
  - $\mathsf{TRI\_3}$ when $j+g+h>3$.
- If precisely one of W, U or V is named: (suppose W is named)
  - $\mathsf{TRI\_1}$ when $g=h$,
  - $\mathsf{TRI\_2}$ when $g \neq h \wedge g \neq 0 \neq h$,

	TRI_1	TRI_2	TRI_3
$W, \{W, U, V\}^\dagger$	All	None	Some
$U, \{W, U, V\}$	Some	All	None
$V, \{W, U, V\}$	None	Some	All

†: when only one state is named, it must be W.

Table 3.2: This table shows which flipping action a state will get in different triangle types. Each column represents a one of three triangle types and each row represents one of the three states in a triangle. When the triangle in question has precisely one named state, then how to flip the relations of this state in different triangle types is fixed. In such a situation, the first row will always represent the named state. Other than that, the states are chosen arbitrarily. We use this table in accordance with Construction 3.5.3 with the help of the function VAR. It should be clear how the function VAR is defined using this table.

  - TRI_3 when $g \neq h \wedge (g = 0 \vee h = 0)$.

- If precisely two of W, U and V are named: (suppose W is unnamed)
  - TRI_1 when $j = 0$,
  - TRI_2 when $j = 1$,
  - TRI_3 when $j = 2$.

In the following repair procedure below, step 1 deals with the triangles which are defect free. These could be triangles that are in collinear configuration, relations of which were already dealt with in the step 3 of Construction 3.5.2. The main action takes place in step 2 where the relations are flipped by considering certain factors. Now we can finally provide our repair procedure formally:

Construction 3.5.3 (Second Repair)**.** Fix some $\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle \in \mathsf{W}^+$ such that $W, U, V \in \mathsf{W}^*$ and $j, g, h \in \{0, 1, 2\}$ and such that $W = \langle w, k\rangle$, $U = \langle u, l\rangle$ and $V = \langle v, i\rangle$ for some $k, l, i \in \{0, 1, 2, 3, 4\}$ and $w, u, v \in \mathsf{W}^{\mathrm{Fin}}$.

Before we begin with the procedure, for the sake of simplicity and compactness of the formalisation of the procedure, let us define the following shorthand:

Definition 3.5.2. Let VAR be the function defined by Table 3.2 for every trio $W, U, V \in \mathsf{W}^*$. For any $\mathsf{type} \in \{\mathsf{All}, \mathsf{None}, \mathsf{Some}\}$, we say that 'flipping type of triangle $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ (order is important) is type' whenever one of the following holds:

- $\mathsf{TRI\_1} \wedge \mathsf{VAR}(W, \{W, U, V\}, \mathsf{TRI\_1}) = \mathsf{type}$ or,
- $\mathsf{TRI\_2} \wedge \mathsf{VAR}(W, \{W, U, V\}, \mathsf{TRI\_2}) = \mathsf{type}$ or,
- $\mathsf{TRI\_3} \wedge \mathsf{VAR}(W, \{W, U, V\}, \mathsf{TRI\_3}) = \mathsf{type}$.

In most of the use cases, we simply write '$\mathsf{FLIP} = \mathsf{type}$' when the triangle in question is clear from the context.

For every $\blacktriangleleft \in \mathbb{O}$ and every $x \in \mathbb{A}^f$, the repair procedure is as follows:

1. If $\mathsf{ANG}^*_{\leq 0}(W, U, V) \vee \mathsf{ANG}^*_{\geq 180}(W, U, V) \vee \neg\mathbf{DFCT}(W, U, V)$, then:

$$\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle) \text{ iff } \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$$

2. For all of the rest of the cases, we have:

   $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ iff either of the following holds,

   - $\mathsf{FLIP} = \mathsf{None} \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$
   - $\mathsf{FLIP} = \mathsf{Some} \wedge (k = l = i \vee k \neq l \neq i \neq k)$ and either of:
     - $\blacktriangleleft \in \{>, \geq\} \wedge x \leq \max \mathbf{M}^{\leq}(W, U, V)$
     - $x > \max \mathbf{M}^{\leq}(W, U, V) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$
   - $\mathsf{FLIP} = \mathsf{Some} \wedge (k = l \neq i \vee k = i \neq l \vee l = i \neq k)$ and either of:
     - $\blacktriangleleft \in \{<, \leq\} \wedge x \geq \min \mathbf{M}^{\geq}(W, U, V)$
     - $x < \min \mathbf{M}^{\geq}(W, U, V) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$
   - $\mathsf{FLIP} = \mathsf{All} \wedge (k = l = i \vee k \neq l \neq i \neq k)$ and either of:

- $\mathsf{ANG}^f_{>x}(w,u,v) \wedge \blacktriangleleft \in \{>, \geq\}$
- $\neg\mathsf{ANG}^f_{>x}(w,u,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W,U,V)$

- $\mathsf{FLIP} = \mathsf{All} \wedge (k = l \neq i \vee k = i \neq l \vee l = i \neq k)$ and either of:
  - $\mathsf{ANG}^f_{<x}(w,u,v) \wedge \blacktriangleleft \in \{<, \leq\}$
  - $\neg\mathsf{ANG}^f_{<x}(w,u,v) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W,U,V)$

For every $\alpha \in \mathscr{P} \cup \mathscr{N}$, define the valuation function by setting:

$$V^+(\alpha) = \{\langle w, k\rangle \in \mathsf{W}^+ \mid w \in V^*(\alpha), k \in \{0,1,2\}\}.$$

Our final frame and model can now be given as:

$$\mathfrak{F}^+[\mathbb{A}^f] := \left\langle \mathsf{W}^+, \left\{\mathsf{ANG}^+_{\blacktriangleleft x}\right\}\right\rangle_{\blacktriangleleft \in \mathbb{O}, x \in \mathbb{A}^f}$$

and

$$\mathfrak{M}^+ := \left\langle \mathfrak{F}^+[\mathbb{A}^f], V^+\right\rangle.$$

Finally we are ready to establish that we have an angular model:

Lemma 3.5.10. *$\mathfrak{F}^+$ satisfies all conditions **CNT**1-**CNT**10, i.e., $\mathfrak{F}^+$ is an angular frame.*

Proof. Let $W^+, U^+, V^+ \in \mathsf{W}^+$ such that $W^+ = \langle W, j\rangle$, $U^+ = \langle U, g\rangle$, $V^+ = \langle V, h\rangle$ for some $W, U, V \in \mathsf{W}^*$ and $j, g, h \in \{0,1,2\}$ such that $W = \langle w, k\rangle$, $U = \langle u, l\rangle$ and $V = \langle v, i\rangle$ for some $w, u, v \in \mathsf{W}^{\mathrm{Fin}}$ and $k, l, i \in \{0,1,2,3,4\}$. Then we have the following possibilities to consider regarding the triangle $\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle$:

- (**C**A) $\mathsf{ANG}^*_{\leq 0}(W,U,V) \vee \mathsf{ANG}^*_{\geq 180}(W,U,V) \vee \neg\mathbf{DFCT}(W,U,V)$.
- (**C**B) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is None.
- (**C**C) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is Some.
- (**C**D) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is All.

We will assume that $k = l = i \vee k \neq l \neq i \neq k$ and leave out the sub-cases for $k = l \neq i \vee k = i \neq l \vee l = i \neq k$. It should be obvious that the cases which are left out consist of repetitive and very similar arguments that are used below.

We begin with **CNT**2. Assume $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, U^{+}, V^{+})$. Consider (**C**A). Then, $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V)$ from Construction 3.5.3. Since $\mathfrak{F}^{*}$ satisfies **CNT**2, we get that $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, V, U)$. From Construction 3.5.3, it follows that we have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, V^{+}, U^{+})$ as desired. Case (**C**B) is very similar. Next, consider (**C**C). Then either of the following holds:

- (**C**C.1) $\blacktriangleleft \in \{>, \geq\} \wedge x \leq \max \mathbf{M}^{\leq}(W, U, V)$
- (**C**C.2) $x > \max \mathbf{M}^{\leq}(W, U, V) \wedge \mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V)$

It follows from Construction 3.5.3 that (**C**C.1) implies $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, V^{+}, U^{+})$, so we have what we want. On the other hand, consider (**C**C.2). Then we have $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V)$ and since $\mathfrak{F}^{*}$ satisfies **CNT**2, we get $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, V, U)$. Moreover, we have $x > \max \mathbf{M}^{\leq}(W, V, U)$ and hence, from Construction 3.5.3 we get $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, V^{+}, U^{+})$ as desired.

Now, let us consider case (**C**D). Then we have either of the following:

- (**C**D.1) $\mathsf{ANG}^{f}_{>x}(w, u, v) \wedge \blacktriangleleft \in \{>, \geq\}$
- (**C**D.2) $\neg\mathsf{ANG}^{f}_{>x}(w, u, v) \wedge \mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V)$

Consider (**C**D.1). First of all, note that we get $\mathsf{ANG}^{f}_{>x}(w, v, u)$ since $\mathfrak{F}^{f}$ satisfies **CNT**2. From here and Construction 3.5.3, it immediately follows that $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W, V, U)$ just as we want. Alternatively, consider (**C**D.2). Since $\mathfrak{F}^{f}$ satisfies **CNT**2, we get $\neg\mathsf{ANG}^{f}_{>x}(w, v, u)$. In addition, since $\mathfrak{F}^{*}$ satisfies **CNT**2 as well, we have $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, V, U)$. Hence, we get $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, V^{+}, U^{+})$ from Construction 3.5.3 as desired.

Now lets show that $\mathfrak{F}^{+}$ satisfies **CNT**3. Fix some $x \in \mathbb{A}^{f}$. First of all, since $\mathfrak{F}^{*}$ satisfies **CNT**3, we have $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V) \vee \mathsf{ANG}^{*}_{\blacktriangleleft^{d} x}(W, U, V)$. If either (**C**A) or (**C**B), then we clearly have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, U^{+}, V^{+}) \vee \mathsf{ANG}^{+}_{\blacktriangleleft^{d} x}(W^{+}, U^{+}, V^{+})$ from Construction 3.5.3.

Now consider case (**C**C). If $x \leq \max \mathbf{M}^{\leq}(W, U, V)$, then we clearly have that $\mathsf{ANG}^{+}_{>x}(W^{+}, U^{+}, V^{+}) \wedge \mathsf{ANG}^{+}_{\geq x}(W^{+}, U^{+}, V^{+})$. It is easy to see that this implies the desired result. Alternatively, suppose that $x > \max \mathbf{M}^{\leq}(W, U, V)$. Since we also have that $\mathsf{ANG}^{*}_{\blacktriangleleft x}(W, U, V) \vee \mathsf{ANG}^{*}_{\blacktriangleleft^{d} x}(W, U, V)$, it follows from Construction 3.5.3 that $\mathsf{ANG}^{+}_{\blacktriangleleft x}(W^{+}, U^{+}, V^{+}) \vee \mathsf{ANG}^{+}_{\blacktriangleleft^{d} x}(W^{+}, U^{+}, V^{+})$ as desired.

Now let us consider (**C**D). If $\neg\mathsf{ANG}^{f}_{>x}(w, u, v)$, then we are through from Construction 3.5.3 and from the fact that $\mathfrak{F}^{*}$ satisfies **CNT**3. Now assume that $\mathsf{ANG}^{f}_{>x}(w, u, v)$. But then, we get $\mathsf{ANG}^{+}_{>x}(W^{+}, U^{+}, V^{+}) \wedge \mathsf{ANG}^{+}_{\geq x}(W^{+}, U^{+}, V^{+})$ which implies that $\mathfrak{F}^{+}$ satisfies **CNT**3.

To see that **CNT**4 holds over $\mathfrak{F}^{+}$, assume that $\mathsf{ANG}^{+}_{<x}(W^{+}, U^{+}, V^{+})$. We will show that $\mathsf{ANG}^{+}_{\leq x}(W^{+}, U^{+}, V^{+})$. If either (**C**A) or (**C**B), then through the fact that $\mathfrak{F}^{*}$ satisfies **CNT**4 and Construction 3.5.3, we get $\mathsf{ANG}^{+}_{\leq x}(W^{+}, U^{+}, V^{+})$. Now, let us proceed with the more interesting cases. Consider case (**C**C). If $x \leq \max \mathbf{M}^{\leq}(W, U, V)$, then we must have $<\in \{>, \geq\}$, which is absurd. So we must have that $x > \max \mathbf{M}^{\leq}(W, U, V) \wedge \mathsf{ANG}^{*}_{<x}(W, U, V)$. Since $\mathfrak{F}^{*}$ satisfies constraint **CNT**4, it follows that $\mathsf{ANG}^{*}_{\leq x}(W, U, V)$. Hence, we get $\mathsf{ANG}^{+}_{\leq x}(W^{+}, U^{+}, V^{+})$ from Construction 3.5.3 as desired.

Finally consider (**C**D). If $\mathsf{ANG}^{f}_{>x}(w, u, v)$, then $<\in \{>, \geq\}$ from Construction 3.5.3, which is absurd. So assume that $\neg\mathsf{ANG}^{f}_{>x}(w, u, v) \wedge \mathsf{ANG}^{*}_{<x}(W, U, V)$. But then we are through from Construction 3.5.3 and from the fact that $\mathfrak{F}^{*}$ satisfies **CNT**4.

Next, we will demonstrate that $\mathfrak{F}^{+}$ satisfies **CNT**5. Suppose we have $\mathsf{ANG}^{+}_{<x}(W^{+}, U^{+}, V^{+})$. We need to establish that $\neg\mathsf{ANG}^{+}_{\geq x}(W^{+}, U^{+}, V^{+})$. For sake of a contradiction, assume that $\mathsf{ANG}^{+}_{\geq x}(W^{+}, U^{+}, V^{+})$. If either (**C**A) or (**C**B), then through Construction 3.5.3, we get $\mathsf{ANG}^{*}_{<x}(W, U, V) \wedge \mathsf{ANG}^{*}_{\geq x}(W, U, V)$ which is

a contradiction since $\mathfrak{F}^*$ satisfies **CNT**5.

Now assume that we have (**C**C). If $x \leq \max \mathbf{M}^{\leq}(W, U, V)$, then this contradicts with the fact that $\mathsf{ANG}^+_{<x}(W^+, U^+, V^+)$. So we must have that $x > \max \mathbf{M}^{\leq}(W, U, V)$. Then, we get that $\mathsf{ANG}^*_{<x}(W, U, V) \wedge \mathsf{ANG}^*_{\geq x}(W, U, V)$, which is also a contradiction since $\mathfrak{F}^*$ satisfies **CNT**5.

Finally consider (**C**D). If we assume $\mathsf{ANG}^f_{>x}(w, u, v)$, then we immediately get a contradiction from Construction 3.5.3. So we must have $\neg\mathsf{ANG}^f_{>x}(w, u, v)$. From here and Construction 3.5.3, it follows that we have $\mathsf{ANG}^*_{<x}(W, U, V) \wedge \mathsf{ANG}^*_{\geq x}(W, U, V)$, which is a contradiction since $\mathfrak{F}^*$ satisfies **CNT**5. It can be established in a similar way that $\mathsf{ANG}^+_{>x}(W^+, U^+, V^+) \wedge \mathsf{ANG}^+_{\leq x}(W^+, U^+, V^+)$ is also impossible.

Now we will show that **CNT**6 holds over $\mathfrak{F}^+$. The other half of **CNT**6 can be seen using similar arguments. Let $x, y \in \mathbb{A}^f$ such that $x \leq y$. Suppose that $\mathsf{ANG}^+_{<x}(W^+, U^+, V^+)$. We will show that $\mathsf{ANG}^+_{<y}(W^+, U^+, V^+)$. Cases (**C**A) and (**C**B) are immediate as usual.

Consider (**C**C). If $x \leq \max \mathbf{M}^{\leq}(W, U, V)$, then this contradicts with the fact that $\mathsf{ANG}^+_{<x}(W^+, U^+, V^+)$. So we must have that $x > \max \mathbf{M}^{\leq}(W, U, V)$. Then we get through Construction 3.5.3 that $\mathsf{ANG}^*_{<x}(W, U, V)$. From here and the fact that $\mathfrak{F}^*$ satisfies **CNT**6, we get $\mathsf{ANG}^*_{<y}(W, U, V)$. Since $y \geq x > \max \mathbf{M}^{\leq}(W, U, V)$, it follows from the construction that $\mathsf{ANG}^+_{<y}(W^+, U^+, V^+)$ as desired.

Now lets have a look at (**C**D). It follows that we have $\neg\mathsf{ANG}^f_{>x}(w, u, v) \wedge \mathsf{ANG}^*_{<x}(W, U, V)$. Moreover, since $x \leq y$ and $\mathfrak{F}^f$ satisfies **CNT**6, we also have that $\neg\mathsf{ANG}^f_{>y}(w, u, v)$. On the other hand, since $\mathfrak{F}^*$ satisfies the constraint as well, we have $\mathsf{ANG}^*_{<y}(W, U, V)$. Putting it all together, it follows from Construction 3.5.3 that $\mathsf{ANG}^+_{<y}(W^+, U^+, V^+)$ as desired. **CNT**7 can be established in a very similar way.

Lets show that **CNT**8 holds over $\mathfrak{F}^+$. Both of the cases (**C**A) and (**C**B) are trivial. So consider (**C**C). Since we have $0 < \max \mathbf{M}^{\leq}(W, U, V)$ from Lemma 3.5.7, it follows from Construction 3.5.3 that we have $\mathsf{ANG}^{+}_{\geq 0}(W^{+}, U^{+}, V^{+})$ as desired. On the other hand, again from Lemma 3.5.7 we have that $180 > \max \mathbf{M}^{\leq}(W, U, V)$. Moreover, since $\mathfrak{F}^{*}$ satisfies **CNT**8, we have $\mathsf{ANG}^{*}_{\leq 180}(W, U, V)$. From Construction 3.5.3 we get $\mathsf{ANG}^{+}_{\leq 180}(W^{+}, U^{+}, V^{+})$ as desired.

Now consider (**C**D). If we have $\mathsf{ANG}^{f}_{>0}(w, u, v)$, then we clearly also have that $\mathsf{ANG}^{+}_{\geq 0}(W^{+}, U^{+}, V^{+})$ from the construction. On the other hand, if we have $\neg\mathsf{ANG}^{f}_{>0}(w, u, v)$, this time using the fact that $\mathsf{ANG}^{*}_{\geq 0}(W, U, V)$, we get $\mathsf{ANG}^{+}_{\geq 0}(W^{+}, U^{+}, V^{+})$ as desired. Moreover, since $\neg\mathsf{ANG}^{f}_{>180}(w, u, v)$ and also since $\mathsf{ANG}^{*}_{\leq 180}(W, U, V)$, it follows from Construction 3.5.3 that $\mathsf{ANG}^{+}_{\leq 180}(W^{+}, U^{+}, V^{+})$.

In order to establish **CNT**10 over $\mathfrak{F}^{+}$, suppose $\mathsf{ANG}^{+}_{\leq 0}(W^{+}, U^{+}, V^{+})$. If (**C**A), then we have $\mathsf{ANG}^{*}_{\leq 0}(W, U, V)$ and since $\mathfrak{F}^{*}$ satisfies **CNT**10, we get $\mathsf{ANG}^{*}_{\leq 0}(U, W, V) \vee \mathsf{ANG}^{*}_{\geq 180}(U, W, V)$. From here and Construction 3.5.3, we get $\mathsf{ANG}^{+}_{\leq 0}(U^{+}, W^{+}, V^{+}) \vee \mathsf{ANG}^{+}_{\geq 180}(U^{+}, W^{+}, V^{+})$ as desired. Case (**C**B) is very similar.

(**C**C) is completely absurd under our assumption, so now consider case (**C**D). Then it must be the case that $\neg\mathsf{ANG}^{f}_{>0}(w, u, v)$ and $\mathsf{ANG}^{*}_{\leq 0}(W, U, V)$. From here we get that $\mathsf{ANG}^{*}_{\leq 0}(U, W, V) \vee \mathsf{ANG}^{*}_{\geq 180}(U, W, V)$ since $\mathfrak{F}^{*}$ satisfies **CNT**10. Now from step 1 of Construction 3.5.3, it follows that $\mathsf{ANG}^{+}_{\leq 0}(U^{+}, W^{+}, V^{+}) \vee \mathsf{ANG}^{+}_{\geq 180}(U^{+}, W^{+}, V^{+})$ as desired.

Finally, we move on to prove the most interesting case of the proof. We establish that $\mathfrak{F}^{+}$ satisfies **CNT**9. Assume that we have $\mathsf{ANG}^{+}_{\leq x}(U^{+}, W^{+}, V^{+}) \wedge \mathsf{ANG}^{+}_{\leq y}(V^{+}, W^{+}, U^{+})$ for some $x, y \in \mathbb{A}^{f}$ such that $x + y < 180$. We will show that this implies $\mathsf{ANG}^{+}_{\geq 180-(x+y)}(W^{+}, U^{+}, V^{+})$. The other half of the constraint **CNT**9, i.e., that $\mathsf{ANG}^{+}_{\geq x}(U^{+}, W^{+}, V^{+}) \wedge \mathsf{ANG}^{+}_{\geq y}(V^{+}, W^{+}, U^{+}) \Rightarrow \mathsf{ANG}^{+}_{\leq 180-(x+y)}(W^{+}, U^{+}, V^{+})$, can be easily established in a similar way. Since we have already established that $\mathfrak{F}^{+}$ satisfies **CNT**5 and **CNT**3 in the above, we can safely assume for sake of a contradiction that $\mathsf{ANG}^{+}_{<180-(x+y)}(W^{+}, U^{+}, V^{+})$.

First of all, let us consider case (**C**A). If $\neg\mathbf{DFCT}(W,U,V)$, then from Construction 3.5.3 and the hypothesis, it follows that we have $\mathsf{ANG}^*_{\leq x}(U,W,V) \wedge \mathsf{ANG}^*_{\leq y}(V,W,U)$ and $\mathsf{ANG}^*_{<180-(x+y)}(W,U,V)$. This means that $\mathbf{DFCT}(W,U,V)$, which contradicts with our assumption.

Now, assume that $\mathsf{ANG}^*_{\leq 0}(W,U,V)$. Since $\mathfrak{F}^*$ satisfies **CNT**10, we get that $\mathsf{ANG}^*_{\leq 0}(U,W,V) \vee \mathsf{ANG}^*_{\geq 180}(U,W,V)$. If $\mathsf{ANG}^*_{\leq 0}(U,W,V)$, then from Lemma 3.5.5 we get that $\mathsf{ANG}^*_{\geq 180}(V,W,U)$. From here and Construction 3.5.3, it follows that $\mathsf{ANG}^+_{\geq 180}(V^+,W^+,U^+)$. Since $\mathsf{ANG}^+_{\leq y}(V^+,W^+,U^+)$, we must have that $y = 180$. But then it follows that we have $\mathsf{ANG}^+_{<0}(W^+,U^+,V^+)$, which is a contradiction. On the other hand, if $\mathsf{ANG}^*_{\geq 180}(U,W,V)$, then since $\mathsf{ANG}^+_{\leq x}(U^+,W^+,V^+)$, we get that $x = 180$. Hence we get $\mathsf{ANG}^+_{<0}(W^+,U^+,V^+)$ again, which is a contradiction.

Now assume that $\mathsf{ANG}^*_{\geq 180}(W,U,V)$. From Construction 3.5.3, this implies that $\mathsf{ANG}^+_{\geq 180}(W^+,U^+,V^+)$, which clearly contradicts with our hypothesis that $\mathsf{ANG}^+_{<180-(x+y)}(W^+,U^+,V^+)$.

Let us now consider (**C**B). Then we have that $\mathsf{ANG}^*_{<180-(x+y)}(W,U,V)$. Moreover, the flipping type of (U^+,W^+,V^+) is either All or Some (which implies that the flipping type of (V^+,W^+,U^+) is either Some or All, respectively).

Suppose the flipping type of (U^+,W^+,V^+) is All. Since we have that $\mathsf{ANG}^+_{\leq x}(U^+,W^+,V^+)$, we get $\neg\mathsf{ANG}^f_{>x}(u,w,v) \wedge \mathsf{ANG}^*_{\leq x}(U,W,V)$ from Construction 3.5.3. On the other hand, since $\mathsf{ANG}^+_{\leq y}(V^+,W^+,U^+)$, we get $y > \max \mathbf{M}^{\leq}(V,W,U) \wedge \mathsf{ANG}^*_{\leq y}(V,W,U)$ from Construction 3.5.3. It follows that $y \in \mathbf{M}^{\leq}(V,W,U)$. But this contradicts with the fact that $y > \max \mathbf{M}^{\leq}(V,W,U)$.

Alternatively, suppose the flipping type of (U^+,W^+,V^+) is Some. Since $\mathsf{ANG}^+_{\leq x}(U^+,W^+,V^+)$, we get $x > \max \mathbf{M}^{\leq}(U,W,V) \wedge \mathsf{ANG}^*_{\leq x}(U,W,V)$. On the other hand, since we have $\mathsf{ANG}^+_{\leq y}(V^+,W^+,U^+)$, we get $\neg\mathsf{ANG}^f_{>y}(v,w,u) \wedge \mathsf{ANG}^*_{\leq y}(V,W,U)$. So, by definition we must have $x \in \mathbf{M}^{\leq}(U,W,V)$. However, this is impossible since we also have $x > \max \mathbf{M}^{\leq}(U,W,V)$.

Next, consider (**C**C). Since we have $\mathsf{ANG}^+_{<180-(x+y)}(W^+, U^+, V^+)$ by the assumption, we get $180 - (x+y) > \max \mathbf{M}^{\leq}(W, U, V) \wedge \mathsf{ANG}^*_{<180-(x+y)}(W, U, V)$. Moreover, the flipping type of (U^+, W^+, V^+) is either All or None. First, suppose the flipping type is All.

Since $\mathsf{ANG}^+_{\leq x}(U^+, W^+, V^+)$, we get $\neg\mathsf{ANG}^f_{>x}(u, w, v) \wedge \mathsf{ANG}^*_{\leq x}(U, W, V)$. On the other hand, since $\mathsf{ANG}^+_{\leq y}(V^+, W^+, U^+)$ and the flipping type of (V^+, W^+, U^+) must be None, we derive that $\mathsf{ANG}^*_{\leq y}(V, W, U)$. Therefore, it follows that $(180 - (x+y)) \in \mathbf{M}^{\leq}(W, U, V)$. This is a contradiction since we already have $(180 - (x+y)) > \max \mathbf{M}^{\leq}(W, U, V)$ in the above. The case when the flipping type of (U^+, W^+, V^+) is either None follows similarly.

Finally, consider (**C**D). First of all, since $\mathsf{ANG}^+_{<180-(x+y)}(W^+, U^+, V^+)$, it follows that $\neg\mathsf{ANG}^f_{>180-(x+y)}(w, u, v) \wedge \mathsf{ANG}^*_{\leq 180-(x+y)}(W, U, V)$ from Construction 3.5.3. Moreover, the flipping type of (U^+, W^+, V^+) is either Some or None. First, suppose the flipping type is Some.

Since we have $\mathsf{ANG}^+_{\leq x}(U^+, W^+, V^+)$ by the assumption, we derive that $x > \max \mathbf{M}^{\leq}(U, W, V) \wedge \mathsf{ANG}^*_{\leq x}(U, W, V)$. On the other hand, since we have that $\mathsf{ANG}^+_{\leq y}(V^+, W^+, U^+)$ and the flipping type of (V^+, W^+, U^+) must be None, we get $\mathsf{ANG}^*_{\leq y}(V, W, U)$. This implies that $x \in \mathbf{M}^{\leq}(U, W, V)$, which is a contradiction since we also have $x > \max \mathbf{M}^{\leq}(U, W, V)$ in the above. This completes the proof of the lemma. $\square$

Lemma 3.5.11. *Let $W, U, V \in \mathsf{W}^*$. If U and V are named-states, then we have that*

$$\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle) \text{ iff } \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$$

for every $j, g, h \in \{0, 1, 2\}$ such that $\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle \in \mathsf{W}^+$. In other words, Construction 3.5.3 performs no flipping on such triangles.

Proof. The proof is straightforward and left to the reader. $\square$

Lemma 3.5.12. *For every Φ and $\langle W, j\rangle \in \mathsf{W}^+$, we have that $\mathfrak{M}^*, W \models \Phi$ iff $\mathfrak{M}^+, \langle W, j\rangle \models \Phi$.*

Proof. Let Φ be a formula and $\langle W, j\rangle \in \mathsf{W}^+$ for some $j \in \{0, 1, 2\}$ and $W \in \mathsf{W}^*$ such that $W = \langle w, k\rangle$ for some $w \in \mathsf{W}^{\mathrm{Fin}}$ and $k \in \{0, 1, 2, 3, 4\}$. The proof is by induction on the complexity of Φ. The base case is trivial by the construction of $\mathfrak{F}^+$. Boolean cases are standard as usual.

Now, let $x \in \mathbb{A}^f$ and fix $\blacktriangleleft \in \mathbb{O}$. Assume that $\mathfrak{M}^*, W \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$. Then, $\exists U \exists V[\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V) \wedge \mathfrak{M}^*, U \models \varphi \wedge \mathfrak{M}^*, V \models \psi]$ such that $U = \langle u, l\rangle$ and $V = \langle v, i\rangle$ for some $u, v \in \mathsf{W}^{\mathrm{Fin}}$ and $l, i \in \{0, 1, 2, 3, 4\}$. From the induction hypothesis we get that $\mathfrak{M}^+, \langle U, g\rangle \models \varphi$ and $\mathfrak{M}^+, \langle V, h\rangle \models \psi$ for any $g, h \in \{0, 1, 2\}$. Therefore, we would be able to complete this half of the proof if we could show that $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ for some $g, h \in \{0, 1, 2\}$.

If $\mathsf{ANG}^*_{\leq 0}(W, U, V)$ or $\mathsf{ANG}^*_{\geq 180}(W, U, V)$ or $\neg\mathbf{DFCT}(W, U, V)$ is the case, then we are through since either of these cases implies that $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ for any $g, h \in \{0, 1, 2\}$ by step 1 of Construction 3.5.3. So, assume $\mathbf{DFCT}(W, U, V)$ and $\neg[\mathsf{ANG}^*_{\leq 0}(W, U, V) \vee \mathsf{ANG}^*_{\geq 180}(W, U, V)]$.

If U and V are named-states, then the desired result follows from Lemma 3.5.11. For the rest of the cases, we rely on the following simple fact: Table 3.2 implies that, every state is guaranteed to appear in one of the three triangle types in such a way that its relations are not flipped at all.

Now suppose that W and U are named-states (case when W and V are named-states follows similarly). Then, we have $j = 0$ and $g = 0$. By construction, all the different triangles that are based on the triangle W, U, V are in the form of $(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, h\rangle)$ for every $h \in \{0, 1, 2\}$. So, by the observation in the above paragraph, $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$ or $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 1\rangle)$ or $\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 2\rangle)$.

Now suppose that exactly one of the states is named. Note that in Table 3.2, there is a mark showing how the named states are flipped in different triangle types (they are represented by the first row). Firstly, suppose W is named and U, V are not. So, we have $j = 0$. In this case we have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 1\rangle, \langle V, 2\rangle)$ since this is a triangle of type $\mathsf{TRI\_2}$.

Secondly, suppose exactly one of U and V is named. Since W is not named, the flipping type on $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$, for any j, g, h, is None when it is either triangle type $\mathsf{TRI\_1}$ or triangle type $\mathsf{TRI\_3}$ as Table 3.2 suggests. So, we are through if we find triangles of both type.

First, triangles of type $\mathsf{TRI\_1}$: If $j = 0$, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$. If $j = 1$, then there are two sub-cases: First of all, if U is a named-state, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 0\rangle, \langle V, 1\rangle)$. On the other hand, if V is a named-state, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 1\rangle, \langle V, 0\rangle)$. If $j = 2$, then there are two sub-cases again: If U is named, then we have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 0\rangle, \langle V, 2\rangle)$ and if V is named, then we have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 2\rangle, \langle V, 0\rangle)$.

Now, we find triangles of type $\mathsf{TRI\_3}$: If $j = 0$, then there are two cases: If U is named, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 1\rangle)$. On the other hand, if V is named, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 1\rangle, \langle V, 0\rangle)$. If $j = 1$, then we have $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$. If $j = 2$, then $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$.

Finally, suppose that none of the states are named. If $j = 0$, then we have either $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 1\rangle, \langle V, 2\rangle)$ (type $\mathsf{TRI\_1}$) or $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$ (type $\mathsf{TRI\_2}$) or $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 0\rangle, \langle U, 2\rangle, \langle V, 2\rangle)$ (type $\mathsf{TRI\_3}$). If $j = 1$, then we have either $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 0\rangle, \langle V, 2\rangle)$ or $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$ or we have that $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 1\rangle, \langle U, 2\rangle, \langle V, 2\rangle)$. If $j = 2$, then either $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 1\rangle, \langle V, 0\rangle)$ or $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 0\rangle, \langle V, 0\rangle)$ or $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, 2\rangle, \langle U, 2\rangle, \langle V, 2\rangle)$.

So, we have established that, one can always find some $g, h \in \{0, 1, 2\}$ such that $\mathsf{ANG}^{+}_{\blacktriangleleft x}(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$. From a previous paragraph, we have $\mathfrak{M}^{+}, \langle U, g\rangle \models \varphi$ and $\mathfrak{M}^{+}, \langle V, h\rangle \models \psi$. Thus, we have that $\mathfrak{M}^{+}, \langle W, j\rangle \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$

as desired. The case of nominal satisfaction operator is trivial.

In the opposite direction, assume that $\mathfrak{M}^+, \langle W, j\rangle \models \langle \mathsf{ANG}_{\blacktriangleleft x}\rangle(\varphi, \psi)$. Then, $\exists U^+ \exists V^+[\mathsf{ANG}^+_{\blacktriangleleft x}(\langle W, j\rangle, U^+, V^+) \wedge \mathfrak{M}^+, U^+ \models \varphi \wedge \mathfrak{M}^+, V^+ \models \psi]$ such that $U^+ = \langle U, g\rangle$ and $V^+ = \langle V, h\rangle$ for some $U, V \in \mathsf{W}^*$ and $g, h \in \{0,1,2\}$, where $U = \langle u, l\rangle$ and $V = \langle v, i\rangle$ for some $u, v \in \mathsf{W}^{\mathrm{Fin}}$ and $l, i \in \{0,1,2,3,4\}$. From the induction hypothesis, it follows that we have $\mathfrak{M}^*, U \models \varphi$ and $\mathfrak{M}^*, V \models \psi$. Moreover, we have the following possibilities to consider regarding the triangle $\langle W, j\rangle$, $\langle U, g\rangle$, $\langle V, h\rangle$:

- (**C**A) $\mathsf{ANG}^*_{\leq 0}(W, U, V) \vee \mathsf{ANG}^*_{\geq 180}(W, U, V) \vee \neg\mathbf{DFCT}(W, U, V)$.
- (**C**B) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is None.
- (**C**C) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is Some.
- (**C**D) Flipping type of $(\langle W, j\rangle, \langle U, g\rangle, \langle V, h\rangle)$ is All.

We will assume that $k = l = i \vee k \neq l \neq i \neq k$ and leave out the sub-cases for $k = l \neq i \vee k = i \neq l \vee l = i \neq k$. It should be obvious that the cases which are left out consist of repetitive and very similar arguments that are used below.

If we have either case (**C**A) or (**C**B), then clearly we have $\mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$ from Construction 3.5.3 and we are easily through. Now let us consider (**C**C). There are two sub-cases to be considered:

- (**C**C.1) $\blacktriangleleft \in \{>, \geq\} \wedge x \leq \max \mathbf{M}^{\leq}(W, U, V)$
- (**C**C.2) $x > \max \mathbf{M}^{\leq}(W, U, V) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$

If (**C**C.2), then we are clearly through. So now suppose that (**C**C.1). For the sake of simplicity, call $z = \max \mathbf{M}^{\leq}(W, U, V)$. Obviously, we have $z \in \mathbf{M}^{\leq}(W, U, V)$. From here and from Lemma 3.5.8, it is not so hard to see that we have $\mathsf{ANG}^f_{>z}(w, u, v)$. Moreover, since $x \leq z$ and $\mathfrak{F}^f$ satisfies **CNT**7, we get $\mathsf{ANG}^f_{>x}(w, u, v)$. Since it also satisfies **CNT**4, we get $\mathsf{ANG}^f_{\geq x}(w, u, v)$.

On the other hand, since $\mathfrak{M}^*, U \models \varphi \wedge \mathfrak{M}^*, V \models \psi$, it follows from Lemma 3.5.6 that, $\mathfrak{M}^f, u \models \varphi \wedge \mathfrak{M}^f, v \models \psi$. Therefore, we get $\mathfrak{M}^f, w \models \langle \mathsf{ANG}_{>x} \rangle(\varphi, \psi) \wedge \langle \mathsf{ANG}_{\geq x} \rangle(\varphi, \psi)$. From here and Lemma 3.5.6, it follows that we have $\mathfrak{M}^*, W \models \langle \mathsf{ANG}_{>x} \rangle(\varphi, \psi) \wedge \langle \mathsf{ANG}_{\geq x} \rangle(\varphi, \psi)$, which is what we want.

We finally address case (**C**D). If (**C**D), then we have the following sub-cases:

- (**C**D.1) $\mathsf{ANG}^f_{>x}(w, u, v) \wedge \blacktriangleleft \in \{>, \geq\}$
- (**C**D.2) $\neg\mathsf{ANG}^f_{>x}(w, u, v) \wedge \mathsf{ANG}^*_{\blacktriangleleft x}(W, U, V)$

Obviously, there is nothing to show for case (**C**D.2). So assume that we have (**C**1). But from here it can be derived in a very similar way to the (**C**C) above by using Lemma 3.5.6 that we have $\mathfrak{M}^*, W \models \langle \mathsf{ANG}_{\blacktriangleleft x} \rangle(\varphi, \psi)$. The case of nominal satisfaction operator is trivial. This completes the proof of the lemma. □

At this point, it only remains to establish that W^+ has finite cardinality. We deal with this in the following lemma:

Lemma 3.5.13. *We have that* $|\mathsf{W}^+| \leq 15 \cdot 2^{|\Sigma|}$.

Proof. First, define a function $f \colon \mathsf{W}^{\mathrm{Fin}} \to 2^{\Sigma}$ such that,

$$f([w]) = \{\varphi \in \Sigma \mid \mathfrak{M}, w \models \varphi\}.$$

It is a straightforward task to show that f is a well-defined and injective function: Let $[w], [u] \in \mathsf{W}^{\mathrm{Fin}}$ and assume that $[w] = [u]$. This means that w and u are modally equivalent with respect to Σ. Hence, $f([w]) = f([u])$, i.e., f is well-defined. Now suppose that $f([w]) = f([u])$. But this implies that w and u are modally equivalent with respect to Σ. Hence, $[w] = [u]$. Henceforth, we have that $|\mathsf{W}^{\mathrm{Fin}}| \leq 2^{|\Sigma|}$.

Moreover, at the worst case scenario the number of states grow by a factor of five and three in Constructions 3.5.2 and 3.5.3, respectively. Therefore, we have that $|\mathsf{W}^+| \leq 15 \cdot 2^{|\Sigma|}$ where $|\Sigma|$ and $|\mathsf{W}^+|$ denote the size of the sets Σ and W^+, respectively. $\square$

Now we present our main results:

Theorem 3.5.14 (Finite Model Property)**.** *Let Φ be a formula. If Φ is satisfiable over an angular model, then it is satisfiable over a finite angular model of size at most $15 \cdot 2^{|\Phi|}$, where $|\Phi|$ denotes the size of the set of subformulas of Φ. In other words,* **TL** *has the finite model property with respect to* T.

Actually, we have not only shown that **TL** has the finite model property, but also that it has the strong finite model property, since we established a computable upper bound, albeit an exponential one, on the size of models.

Now, if a formula Φ is satisfiable on at least one of such models, then obviously Φ is **TL**-satisfiable, i.e., there is a model from T over which Φ is satisfied. On the other hand, if Φ is not satisfied on any of such models, then it is not **TL**-satisfiable since **TL** has the finite model property with respect to T. Or shortly,

Theorem 3.5.15 (Decidability)**.** **TL** *has a decidable satisfiability problem.*

Let us also note the following important result which follows directly from the above proofs:

Theorem 3.5.16. *The satisfiability problem of the logic* **TL** *is in NEXPTIME.*

3.6 Discussion & Future Research

Our biggest goal is to turn **TL** into a truly trigonometric logic, which can take advantage of combining the reasoning with angles and metrics. This means that, we should be looking into ways of incorporating metric information into angular frames and the modal language $\mathcal{L}$ used to talk about angular frames.

Thanks to the work of Kutz et al. [46], we actually have a very strong literature background on how to bring metric information into the realm of modal logics. This means that, the main discussion concerning a trigonometric logic will evolve around finding ways to have these two sorts of information to interact.

The 'law of sines' can be used to compute the remaining sides of a triangle when two angles and a side or when two sides and an angle are known (also known as the technique of triangulation). In other words, using the law of sines, we can establish the desired interaction between metric information and angle information. Let us first remember the law of sines, which can be found in any textbook of basic geometry. Let d be a metric function and w, u and v be three points. The notation $\angle wuv$ denotes the angle at the corner w of the triangle formed by the points w, u and v. The law of sines is as follows:

$$\frac{d(w,u)}{\sin(\angle vwu)} = \frac{d(u,v)}{\sin(\angle wuv)} = \frac{d(w,v)}{\sin(\angle uwv)}$$

We can adapt the law of sines as frame constraints and add it to the collection of angular frame constraints as follows: For every w, u and v,

- $\mathsf{ANG}_{=x}(w,u,v) \wedge \mathsf{DIS}_{=a}(u,v) \wedge \mathsf{ANG}_{=y}(u,w,v) \Rightarrow \mathsf{DIS}_{=\frac{a \cdot \sin(y)}{\sin(x)}}(w,v)$,
- $\mathsf{ANG}_{=x}(w,u,v) \wedge \mathsf{DIS}_{=a}(u,v) \wedge \mathsf{DIS}_{=b}(w,v) \Rightarrow \mathsf{ANG}_{=\arcsin(\frac{b \cdot \sin(x)}{a})}(u,w,v)$.

Naturally, we will also need an appropriately equipped language in order to be able to talk about such a semantic structure. Using a set of parametric unary modalities such as $\langle \mathsf{DIS}_{\leq a} \rangle \varphi$ (or such as $\langle \mathsf{DIS}_{<a} \rangle \varphi$, $\langle \mathsf{DIS}_{\geq a} \rangle \varphi$ and $\langle \mathsf{DIS}_{>a} \rangle \varphi$) which has the semantics 'somewhere less than a units from here, φ holds' to correspond the relation of distance DIS, we can talk about metric information within the new language. The hybrid character of $\mathcal{L}$ is actually very helpful for the interaction of two kinds of information.

Consider the example where an observer is standing on the edge of a river and can measure the angle to a "point of interest" on the edge of the other side of the river. If the observer knows the width of the river, then he can figure out the distance to the point of interest using trigonometry. This is illustrated in Figure 3.8. The contents of the figure and the example can be formalised by the following formula:

$$@_{\mathsf{i}}(\langle \mathsf{ANG}_{=60}\rangle(\mathsf{j},\mathsf{k}) \wedge \langle \mathsf{DIS}_{=40}\rangle \mathsf{j}) \wedge @_{\mathsf{j}}\langle \mathsf{ANG}_{=90}\rangle(\mathsf{i},\mathsf{k})$$

which would imply that we have:

$$@_{\mathsf{i}}\langle \mathsf{DIS}_{=\frac{40}{\sin(30)}}\rangle \mathsf{k}$$

which gives the distance from the observer to the point of interest.

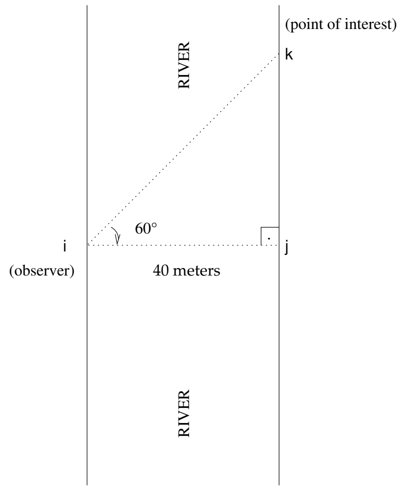

Figure 3.8: Each one of the corners of the triangle are named with nominal letters i, j and k. The width of the river is 40 meters and the angle between the opposite side of the river to the point of interest is 60°.

There is also the discussion that, whether we should be only focusing on Euclidean trigonometry. There are alternative kinds of trigonometric theories based on non-Euclidean spaces. For example, 'spherical geometry' is one of the most important such non-Euclidean theories. It is the geometry of the two-dimensional surface of a sphere (e.g., the face of the earth). Some of the practical applications of the spherical geometry include (earth-surface, orbital and space) navigation systems and astronomy [69].

In the classical Euclidean geometry, the basic concepts are points and lines. On the sphere, points are defined in the usual sense. However, the equivalents of lines are not defined in the usual sense of "straight line," but in the sense of "the shortest paths between points" which is called a geodesic. On the sphere the geodesics are the great circles, so the other geometric concepts are defined like in plane geometry but with lines replaced by great circles. Thus, in spherical geometry angles are defined between great circles, resulting in a spherical trigonometry that differs from ordinary trigonometry in many respects (e.g., the sum of the interior angles of a triangle exceeds 180 degrees!).

A sphere is not a Euclidean space, but locally the laws of Euclidean geometry are good approximations. In appropriate mathematical terms, it is locally Euclidean: Every point has a neighbourhood which "resembles" (i.e., is homeomorphic to) Euclidean space. In a small triangle on the face of the earth, the sum of the angles is very nearly 180 degrees and a sphere can be represented by a collection of two dimensional maps.

3.7 Conclusion

By using a classical modal logic approach, we have presented a modal logic formalism which can talk about the interior angles of triangles or angles in general. Describing the polygonal shapes or the configurations of points by using angles have applications in the fields like molecular geometry. We adopted a language which is based on a parameter set of numeric values. By altering the parameter set, it gives us the possibility to variate between qualitative and quantitative logics for talking about angles. For example, if we set the parameter set to the set of values $\{0, 180\}$, then we simply get a modal logic which can talk about collinearity and betweenness.

By using various and rather complicated model repairing methods, we established that this logic has the finite model property and also concluded that it has a decidable satisfiability problem. While the proof of the finite model property establishes the NEXPTIME upper bound on the complexity of the problem, the question of the lower bound of the complexity remains an open problem.

This work aims to be a step-stone in developing a true modal logic of trigonometry. In this work, we have only dealt with the interior angles of triangles induced by any three points in space. In Section 3.6, we have discussed in detail on the possible ways of incorporating metric information into our relational structures, which also gave some clues as to how an appropriate trigonometric modal logic language could be designed. We proposed the use of 'the law of sines' as the basic principle on how distances and angles could interact inside the relational frames.

Chapter 4

Conclusion

In each of the main chapters of this thesis, the reader will find a section where a detailed discussion on the results and future research topics of that chapter are contained. Our conclusions for each chapter are presented in a dedicated section of those chapters as well. Nonetheless, we will succinctly present an overall conclusion of the thesis by summarising for each of the main chapters.

In Chapter 2, we presented a first-order and a modal logic formalism which have the ability to talk about distance information in a comparative manner. With these two logics, we aimed to contribute common-sense knowledge representation and reasoning with an alternative framework for distances, compared to the available techniques which use a set of basic symbols to talk about distances, e.g., $\mathbb{R}$ (for a quantitative approach) or simply $\{\text{close}, \text{far}, \text{very far}\}$ (for a qualitative approach). With the logics utilising this framework for distances, it becomes possible to make assertions of the form 'if my arm can reach the tea cup but not the desk lamp, then the lamp is farther away from me than the cup,' which makes this approach more cognitively plausible compared to other approaches where distances between objects are measured in absolute terms.

We established that, while the first-order comparative distance logic is finitely axiomatisable, the modal comparative distance logic is also finitely axiomatisable and moreover, it has the finite model property and it is decidable, with an NP-complete satisfiability problem.

In Chapter 3, we studied the properties of a multi-modal logic which can talk about the interior angles of triangles induced by every trio of points in space. Our general purpose from a higher point of view is to contribute the development of modal logics which can perform trigonometric reasoning. Various logics of distances have been studied in the literature. However, in order to perform trigonometric reasoning, which has an enormous variety of applications in many different domains, reasoning about distances must be combined with reasoning about angles. In Chapter 3, we mainly studied the properties of modal reasoning about angles and shortly discussed building a combined formalism of distances and angles. We presented a modal logic formalism with modal operators having intended meanings such as 'φ holds at somewhere and ψ holds at somewhere else, with less than a degrees of angle in between about here,' and established that this formalism has the finite model property and it is decidable. The complexity of the satisfiability problem is known to be in NEXPTIME via the proof of the finite model property, but unfortunately a lower bound could not be determined.

Bibliography

[1] M. Aiello and J. van Benthem. A modal walk through space. *Journal of Applied Non-Classical Logics*, 12(3-4):319–364, 2002.

[2] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: The modal way. *J. Log. Comput.*, 13(6):889–920, 2003.

[3] N. Asher and L. Vieu. Toward a geometry for common sense: A semantics and a complete axiomatization for mereotopology. In C. Mellish, editor, *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence*, pages 846–852, San Francisco, 1995. Morgan Kaufmann.

[4] P. Balbiani and V. Goranko. Modal logics of parallelism, ortogonality and affine geometries. *Journal of Applied Non-Classical Logics*, 12:365–397, 2002.

[5] P. Balbiani and T. Tinchev. First-order theories and modal logics of line-based affine planes. In J. J. Alferes and J. Leite, editors, *Logics in Artificial Intelligence, Ninth European Conference, JELIA'04, Lecture Notes in Computer Science*. Springer-Verlag, 2004.

[6] B. Bennett. Spatial reasoning with propositional logics. In J. Doyle, E. Sandewall, and P. Torasso, editors, *Principles of Knowledge Representation and Reasoning (KR94)*, pages 51–62, San Francisco, CA., 1994. Morgan Kaufmann.

[7] B. Bennett. Carving up space: steps towards construction of an absolutely complete theory of spatial regions. In L. P. J.J. Alfres and E. Orlowska, editors, *Proceedings of JELIA'96*, pages 337–353, 1996.

[8] B. Bennett. Modal logics for qualitative spatial reasoning. *Bulletin of the Interest Group in Pure and Applied Logic (IGPL)*, 4(1):23–45, 1996.

[9] B. Bennett. *Logical Representations for Automated Reasoning about Spatial Relationships*. PhD thesis, School of Computer Studies, The University of Leeds, 1997.

[10] B. Bennett. A categorical axiomatisation of region-based geometry. *Fundamenta Informaticae*, 46:145–158, 2001.

[11] B. Bennett, A. G. Cohn, P. Torrini, and S. M. Hazarika. A foundation for region-based qualitative geometry. In W. Horn, editor, *Proceedings of ECAI-2000*, pages 204–208, 2000.

[12] B. Bennett, A. G. Cohn, F. Wolter, and M. Zakharyaschev. Multi-dimensional modal logic as a framework for spatio-temporal reasoning. *Applied Intelligence*, 17(3):239–251, 2002.

[13] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2002.

[14] B. L. Clarke. Individuals and points. *Notre Dame Journal of Formal Logic*, 26:61–75, 1985.

[15] E. Clementini and P. D. Felice. Approximate topological relations. *International Journal of Approximate Reasoning*, 16:173–204, 1997.

[16] E. Clementini and P. D. Felice. A global framework for qualitative shape description. *GeoInformatica*, 1(1):1–17, 1997.

[17] E. Clementini, P. D. Felice, and D. Hernández. Qualitative representation of positional information. *Artificial Intelligence*, 95(2):317–356, 1997.

[18] P. Clote and R. Backofen. *Computational Molecular Biology*. John Wiley & Sons, Chichester, 2000.

[19] A. G. Cohn. A hierarchical representation of qualitative shape based on connection and convexity. In A. M. Frank, editor, *Proc COSIT95*, LNCS, pages 311–326. Springer Verlag, 1995.

[20] A. G. Cohn. Qualitative spatial representation and reasoning techniques. In G. Brewka, C. Habel, and B. Nebel, editors, *Proceedings of KI-97*, volume 1303 of *LNAI*, pages 1–30. Springer Verlag, 1997.

[21] A. G. Cohn, B. Bennett, J. M. Gooday, and N. Gotts. RCC: a calculus for region based qualitative spatial reasoning. *GeoInformatica*, 1:275–316, 1997.

[22] A. G. Cohn, B. Bennett, J. M. Gooday, and N. M. Gotts. Representing and reasoning with qualitative spatial relations about regions. In O. Stock, editor, *Temporal and spatial reasoning*. Kluwer, 1997.

[23] A. G. Cohn and S. M. Hazarika. Qualitative spatial representation and reasoning: An overview. *Fundamenta Informaticae*, 46(1-2):1–29, 2001.

[24] A. G. Cohn, D. A. Randell, Z. Cui, and B. Bennett. Qualitative spatial reasoning and representation. In N. P. Carreté and M. G. Singh, editors, *Qualitative Reasoning and Decision Technologies*, pages 513–522, Barcelona, 1993. CIMNE.

[25] F. Cotton. *Advanced Inorganic Chemistry*. Wiley, New York, 1999.

[26] T. de Laguna. The nature of space -part one. *The Journal of Philosophy*, 19:393–407, 1922.

[27] T. de Laguna. The nature of space -part two. *The Journal of Philosophy*, 19:421–440, 1922.

[28] T. de Laguna. Point, line and surface as sets of points. *The Journal of Philosophy*, 19:449–461, 1922.

[29] D. Dubois, H. Prade, F. Esteva, P. Garcia, and L. Godo. A logical approach to interpolation based on similarity relations. *International Journal of Approximate Reasoning*, 17:1–36, 1997.

[30] M. Egenhofer. A model for detailed binary topological relationships. *Geomatica*, 47(3–4):261–273, 1993.

[31] M. J. Egenhofer. A formal definition of binary topological relationships. In *FODO 1989*, number 347 in LNCS, pages 457–472. Springer, 1989.

[32] M. J. Egenhofer and R. D. Franzosa. Point set topological relations. *International Journal of Geographical Information Systems*, 5:161–174, 1991.

[33] M. J. Egenhofer and D. M. Mark. Naive geography. In *Spatial Information Theory*, pages 1–15, 1995.

[34] M. T. Escrig and F. Toledo. A framework based on CLP extended with CHRs for reasoning with qualitative orientation and positional information. *Journal of Visual Languages and Computing*, 9:81–101, 1998.

[35] F. Esteva, P. Garcia, L. Godo, and R. Rodriguez. A modal account of similarity-based reasoning. *International Journal of Approximate Reasoning*, 16:235–260, 1997.

[36] L. Floriani. *Shape Analysis and Structuring*. Springer, Berlin, 2008.

[37] A. U. Frank. Qualitative spatial reasoning: Cardinal directions as an example. *International Journal of Geographical Information Science*, 10(3):269–290, 1996.

[38] A. Gerevini and B. Nebel. Qualitative spatio-temporal reasoning with rcc-8 and allen's interval calculus: Computational complexity. In *Proceedings of the 15th European Conference on Artificial Intelligence (ECAI'02)*. Springer-Verlag, 2002.

[39] N. M. Gotts, J. M. Gooday, and A. G. Cohn. A connection based approach to common-sense topological description and reasoning. *The Monist*, 79(1):51–75, 1996.

[40] A. Grzegorczyk. Undecidability of some topological theories. *Fundamenta Mathematicae*, 38:137–152, 1951.

[41] A. Grzegorczyk. Axiomatizability of geometry without points. *Synthese*, 12:228–235, 1960.

[42] D. Hernández, E. Clementini, and P. D. Felice. Qualitative distances. In W. Kuhn and A. Frank, editors, *Spatial Information Theory: a theoretical basis for GIS*, volume 988 of *LNCS*, pages 45–58, Berlin. Springer Verlag.

[43] A. Isli and A. G. Cohn. An algebra for cyclic ordering of 2d orientations. In *Proceedings of the 15th American Conference on Artificial Intelligence (AAAI-98)*, pages 643–649, Madison, WI, 1998. AAAI/MIT Press.

[44] R. Jansana. Some logics related to von wright's logic of place. *Notre Dame Journal of Formal Logic*, 35(1):88–98, 1994.

[45] E. Jungert. Symbolic spatial reasoning on object shapes for qualitative matching. In A. U. Frank and I. Campari, editors, *Spatial Information Theory, Proceedings of the COSIT93*, pages 444–462. Springer Verlag, 1993.

[46] O. Kutz, H. Sturm, N. Suzuki, F. Wolter, and M. Zakharyaschev. Logics of metric spaces. *ACM Transactions on Computational Logic*, 4(2):260–294, 2003.

[47] L. J. Latecki and R. Röhrig. Orientation and qualitative angle for spatial reasoning. In *IJCAI*, pages 1544–1549, 1993.

[48] O. J. Lemon. Semantical foundations of spatial logics. In L. C. Aiello, J. Doyle, and S. Shapiro, editors, *Principles of Knowledge Representation and Reasoning (KR96)*, pages 212–219, San Francisco, CA., 1996. Morgan Kaufmann.

[49] J. Liu. A method of spatial reasoning based on qualitative trigonometry. *Artificial Intelligence*, 98(1-2):137–168, 1998.

[50] M. Marx and M. Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9(6):897–914, 1999.

[51] J. Mcmurry. *Organic Chemistry*. Brooks/Cole Pub, Pacific Grove, 1992.

[52] E. Mendelson. *Introduction to Mathematical Logic*. D. Van Nostrand, Princeton, New Jersey, 1968.

[53] I. Pratt and O. Lemon. Ontologies for plane, polygonal mereotopology. *Notre Dame Journal of Formal Logic*, 38(2):225–245, 1997.

[54] M. O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969.

[55] O. Raiman. Order of magnitude reasoning. In *AAAI*, pages 100–104, 1986.

[56] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In B. Nebel, C. Rich, and W. Swartout, editors, *Principles of*

Knowledge Representation and Reasoning (KR92), pages 165–176, San Mateo, 1992. Morgan Kaufmann.

[57] N. Rescher and J. Garson. Topological logic. *Journal of Symbolic Logic*, 33:537–548, 1968.

[58] R. Röhrig. A theory for qualitative spatial reasoning based on order relations. In *AAAI*, pages 1418–1423, 1994.

[59] R. Röhrig. Representation and processing of qualitative orientation knowledge. In G. Brewka, C. Habel, and B. Nebel, editors, *KI-97: Advances in Artificial Intelligence, 21st Annual German Conference on Artificial Intelligence*, pages 219–230, 1997.

[60] K. Segerberg. A note on the logic of elsewhere. *Theoria*, 46:183–187, 1980.

[61] P. Simons. *Parts: A Study in Ontology*. Clarendon Press, Oxford, 1987.

[62] A. Tarski. Foundations of the geometry of solids. In J. H. Woodger, editor, *Logic, Semantics, Metamathematics*, pages 24–29. Oxford Clarendon Press, 1956.

[63] A. Tarski. Sentential calculus and topology. In J. H. Woodger, editor, *Logic, Semantics, Metamathematics*. Oxford Clarendon Press, 1956.

[64] A. Tarski. What is elementary geometry? In H. L., S. P., and T. A., editors, *The Axiomatic Method (with special reference to geometry and physics)*, pages 16–29, Amsterdam, 1959. North-Holland.

[65] A. Tarski and J. McKinsey. Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, 13:1–15, 1948.

[66] Y. Venema. Expressiveness and completeness of an interval tense logic. *Notre Dame Journal of Formal Logic*, 31:529–547, 1990.

[67] G. H. von Wright. A modal logic of place. In E. Sosa, editor, *The Philosophy of Nicholas Rescher*, pages 65–73. Reidel, Dordrecht, 1974.

[68] A. Whitehead. *Process and Reality*. Macmillan, Newyork, 1929.

[69] Wikipedia. Spherical geometry — wikipedia, the free encyclopedia, 2009. [Online; accessed 19-January-2009].

[70] Wikipedia. Trigonometry — wikipedia, the free encyclopedia, 2009. [Online; accessed 16-January-2009].

[71] F. Wolter and M. Zakharyaschev. Spatial reasoning in rcc-8 with boolean region terms. In W. Horn, editor, *ECAI 2000, Proceedings of the 14th European Conference on Artificial Intelligence, Berlin, Germany, August 20-25 2000*, pages 244–250. IOS Press, 2000.

[72] F. Wolter and M. Zakharyaschev. A logic of metric and topology. *Journal of Symbolic Logic*, 2005.

[73] M. Worboys. Metrics and topologies for geographic space. In M. J. Kraak and M. Molenaar, editors, *Advances in Geographic Information Systems Research II: Proceedings of the International Symposium on Spatial Data Handling*, pages 7A.1–7A.11, Delft, 1996.

[74] K. Zimmermann. Measuring without measures: The delta-calculus. In *COSIT*, pages 59–67, 1995.

Chapter 5

Appendix

Definition 5.0.1 (Subformula Closure)**.** A finite set of formulas Σ is subformula-closed iff for all formulas φ and ψ we have that:

- $\varphi \wedge \psi \in \Sigma \Rightarrow \varphi, \psi \in \Sigma$;
- $\neg\varphi \in \Sigma \Rightarrow \varphi \in \Sigma$;
- For any n-ary modal operator $\Diamond$: $\Diamond(\varphi_1, \ldots, \varphi_n) \in \Sigma \Rightarrow \varphi_1, \ldots, \varphi_n \in \Sigma$;
- For any hybrid satisfaction operator @: $@_{\mathsf{i}}\varphi \in \Sigma \Rightarrow \mathsf{i}, \varphi \in \Sigma$.

The subformula closure of a set of formulas Σ is the smallest subformula-closed set of formulas which contains Σ. It is generally denoted by $SCL(\Sigma)$.

Definition 5.0.2 (Degree of a Formula)**.** The degree of a modal formula is defined recursively as follows:

- $deg(\bot) = 0$,
- $deg(p) = 0$ if p is a propositional letter,
- $deg(\neg\varphi) = deg(\varphi)$,

- $deg(\varphi \wedge \psi) = \max\{deg(\varphi), deg(\psi)\}$,
- For *any* modality $\diamond$, $deg(\diamond(\varphi_1, \ldots, \varphi_n)) = 1 + \max\{deg(\varphi_1), \ldots, deg(\varphi_n)\}$.

If Σ is a set of formulas, then the degree of Σ is simply

$$deg(\Sigma) = \max\{deg(\varphi) \mid \varphi \in \Sigma\}.$$

Definition 5.0.3 (Height of a State)**.** Let $\mathfrak{M}$ be a rooted model with a root r. We define the height of a state w in $\mathfrak{M}$ (denoted by $hgt(w)$) inductively as follows:

- The height of r is 0;
- A state w is at height $n + 1$ iff it is a successor of a state u of height n and have not been assigned a height smaller than $n + 1$.

The height a model $\mathfrak{M}$ is the maximum n such that there is a state of height n in $\mathfrak{M}$.

Definition 5.0.4 (Restriction of a Model)**.** Let $\mathfrak{M}$ be a rooted model. The restriction of model $\mathfrak{M}$ to n for some $n \in \mathbb{N}$ (denoted by $\mathfrak{M} \upharpoonright n$) is the following model:

$$\mathfrak{M} \upharpoonright n = \langle \mathsf{W} \upharpoonright n, R_k \upharpoonright n, \mathsf{V} \upharpoonright n \rangle.$$

whereas for every $k \in I$,

- $\mathsf{W} \upharpoonright n = \{w \in \mathsf{W} \mid hgt(w) \leq n\}$;
- $R_k \upharpoonright n = R_k \cap (\mathsf{W} \upharpoonright n)$;
- $\mathsf{V} \upharpoonright n = \mathsf{V} \cap (\mathsf{W} \upharpoonright n)$.

Definition 5.0.5 (Filtrations)**.** Let I be an index set and $\mathfrak{M} = \langle \mathsf{W}, \{R_k\}_{k \in I}, \mathsf{V} \rangle$ be a model and Σ be a subformula-closed set of formulas. Define an equivalence relation $\equiv$ as follows:

$$w \equiv u \text{ iff } \forall \varphi \in \Sigma[\mathfrak{M}, w \models \varphi \Leftrightarrow \mathfrak{M}, u \models \varphi].$$

We denote the equivalence class of a $w \in \mathsf{W}$ induced by $\equiv$ with $[w]$. Let $\mathfrak{M}^f = \langle \mathsf{W}^f, \{R_k^f\}_{k \in I}, \mathsf{V}^f \rangle$ such that,

- $\mathsf{W}^f = \{[w] \mid w \in \mathsf{W}\}$,
- For every $k \in I$, $R_k(w_0, \ldots, w_n) \Rightarrow R_k^f([w_0], \ldots, [w_n])$,
- $R_k^f([w_0], \ldots, [w_n]) \Rightarrow \forall \Diamond_k(\varphi_1, \ldots, \varphi_n) \in \Sigma[\bigwedge_{1 \le k \le n}(\mathfrak{M}, w_k \models \varphi_k) \Rightarrow \mathfrak{M}, w_0 \models \Diamond_k(\varphi_1, \ldots, \varphi_n)]$,
- For every propositional letter p, $\mathsf{V}^f(p) = \{[w] \mid w \in \mathsf{V}(p)\}$.

Then, $\mathfrak{M}^f$ is a filtration of $\mathfrak{M}$ through Σ.

Definition 5.0.6 (Smallest and Largest Filtrations)**.** Let $\mathfrak{M} = \langle \mathsf{W}, R, \mathsf{V} \rangle$ be a model and fix a subformula-closed set of formulas Σ. Let $\equiv$ be defined as in Definition 5.0.5. Define relations R^s and R^l as follows:

- $R^s([w_0], \ldots, [w_n])$ iff $\exists u_0 \in [w_0] \ldots \exists u_n \in [w_n][R(u_0, \ldots, u_n)]$.
- $R^l([w_0], \ldots, [w_n])$ iff $\forall \Diamond(\varphi_1, \ldots, \varphi_n) \in \Sigma[\bigwedge_{1 \le k \le n}(\mathfrak{M}, w_k \models \varphi_k) \Rightarrow \mathfrak{M}, w_0 \models \Diamond_k(\varphi_1, \ldots, \varphi_n)]$.

The relations R^s and R^l are called the smallest and largest filtrations respectively. In other words, for any filtration R^f of R, we have that $R^s \subseteq R^f \subseteq R^l$.

Definition 5.0.7 (Tree and Tree Model)**.** A tree is a pair $\langle N, E \rangle$ such that N is a set of 'nodes' and E is a binary relation of 'edges' over $N \times N$ such that,

1. $\exists r \in N$ such that $\forall n \in N[E^*(r, n)]$ where E^* is the reflexive and transitive closure of E (r is called the 'root' of the tree);

2. $\forall n \in N[n \neq r \Rightarrow \exists n' \in N[E(n', n)]]$, i.e. every non-root node has a predecessor;

3. $\forall n \in N[\neg E^+(n, n)]$ where E^+ is the transitive closure of E, i.e. a tree is an acyclic structure.

A model $\mathfrak{M} = \langle \mathsf{W}, R, \mathsf{V} \rangle$ is said to be "tree-like" or that it is a "tree model" whenever $\langle \mathsf{W}, R \rangle$ is a tree.

Definition 5.0.8 (Bounded Morphism)**.** Let I be an index set and let $\mathfrak{M}$ and $\mathfrak{M}'$ be two models. A function $f \colon \mathfrak{M} \to \mathfrak{M}'$ is called a bounded morphism iff for every $w, u \in \mathsf{W}$ and $u' \in \mathsf{W}'$ we have that,

- w and $f(w)$ satisfy the same atoms of the language;
- For every $k \in I$, $R_k(w, u) \Rightarrow R'_k(f(w), f(u))$, i.e., f is a homomorphism;
- For every $k \in I$, $R'_k(f(w), u') \Rightarrow \exists v[R_k(w, v) \wedge f(v) = u']$ (also known as the "back condition").

If f is also surjective, then $\mathfrak{M}'$ is called as the bounded morphic image of $\mathfrak{M}$.

Definition 5.0.9 (n-Bisimulation)**.** Let $\mathfrak{M} = \langle \mathsf{W}, R, \mathsf{V} \rangle$ and $\mathfrak{M}' = \langle \mathsf{W}', R', \mathsf{V}' \rangle$ be two models and let $w \in \mathsf{W}$ and $w' \in \mathsf{W}'$. Fix some $n \in \mathbb{N}$. Then w in $\mathfrak{M}$ bisimulates with to w' in $\mathfrak{M}'$ up to depth n (denoted by $\mathfrak{M}, w \leftrightarrows_n \mathfrak{M}', w'$) iff there is a sequence of relations $Z_n \subseteq \cdots \subseteq Z_0$ such that:

- $Z_n(w, w')$;
- For every atom α, $v \in \mathsf{W}$ and $v' \in \mathsf{W}'$,

$$Z_0(v, v') \Rightarrow [\mathfrak{M}, v \models \alpha \Leftrightarrow \mathfrak{M}', v' \models \alpha].$$

- For every $k \in \mathbb{N}$ such that $k + 1 \leq n$, $v \in \mathsf{W}$ and $v' \in \mathsf{W}'$,

$$Z_{k+1}(v, v') \wedge R(v, u) \Rightarrow \exists u' \in \mathsf{W}'[Z_k(u, u') \wedge R'(v', u')].$$

- For every $k \in \mathbb{N}$ such that $k + 1 \leq n$, $v \in \mathsf{W}$ and $v' \in \mathsf{W}'$,

$$Z_{k+1}(v, v') \wedge R'(v', u') \Rightarrow \exists u \in \mathsf{W}\ [Z_k(u, u') \wedge R(v, u)].$$

Definition 5.0.10 (Maximal Consistent Sets)**.** Let **L** be a normal modal logic. A set of formulas Σ is called a maximal (**L**-)consistent set whenever Σ is **L**-consistent and for any other **L**-consistent set of formulas Γ such that $\Sigma \subseteq \Gamma$, we have $\Sigma = \Gamma$.

Lemma 5.0.1 (Properties of Maximal Consistent Sets)**.** *Let* **L** *be a normal modal logic and* Σ *be a maximal (***L***-)consistent set. Then we have the following:*

- Σ *is closed under modus ponens, i.e. if* $\varphi \to \psi \in \Sigma$ *and* $\varphi \in \Sigma$*, then* $\psi \in \Sigma$*;*
- $\mathbf{L} \subseteq \Sigma$*;*
- *For any formula* φ*, either* $\varphi \in \Sigma$ *or* $\neg\varphi \in \Sigma$*;*
- *For all formulas* φ *and* ψ*,* $\varphi \vee \psi \in \Sigma$ *iff* $\varphi \in \Sigma$ *or* $\psi \in \Sigma$*.*

Lemma 5.0.2 (Lindenbaum's Lemma)**.** *Every consistent set of (modal or first-order) formulas can be extented into a maximal consistent set of formulas.*

Theorem 5.0.3 (Canonical Model Theorem)**.** *Any normal modal logic is strongly complete with resprect to its canonical model.*

Proof. For a proof, see Blackburn et al., Theorem 4.22, pg. 199 [13]. □

Definition 5.0.11 (Metric Space)**.** Let W be a set and $d \colon \mathsf{W} \times \mathsf{W} \to \{0\} \cup \mathbb{R}^{+}$ be a function. The function d is called a metric and the pair $\mathcal{M} = \langle \mathsf{W}, d \rangle$ a metric space iff for every $x, y, z \in \mathsf{W}$ we have that:

1. $d(x, y) = 0$ iff $x = y$ (identity of indiscernibles),
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

Definition 5.0.12 (Topological Space)**.** Let X be a non-empty set. A collection of subsets $\mathfrak{T}$ of X is a topology over X iff,

- $\emptyset, X \in \mathfrak{T}$;
- If $S, R \in \mathfrak{T}$ then $S \cap R \in \mathfrak{T}$, i.e. $\mathfrak{T}$ is closed under finite intersections;
- For any collection of sets S_n, if $S_n \in \mathfrak{T}$, then $\bigcup S_n \in \mathfrak{T}$, i.e. $\mathfrak{T}$ is closed under arbitrary unions.

The pair $\langle X, \mathfrak{T} \rangle$ is called a topological space.

Definition 5.0.13 (Topological Space via Kuratowski Axioms)**.** A topological space is a pair $\langle X, \mathbf{C} \rangle$ where X is a set and $\mathbf{C} \colon 2^X \to 2^X$ is a function ('closure operator') such that for every $S, R \in 2^X$ we have that,

1. $S \subseteq \mathbf{C}(S)$;
2. $\mathbf{C}(\mathbf{C}(S)) = \mathbf{C}(S)$;
3. $\mathbf{C}(S \cup R) = \mathbf{C}(S) \cup \mathbf{C}(R)$;
4. $\mathbf{C}(\emptyset) = \emptyset$.

Definition 5.0.14. Let $\langle X, \mathfrak{S}, \mathsf{M} \rangle$ be a measure space and $\langle X, \mathfrak{T} \rangle$ be a topological space. The σ-algebra generated by the topology $\mathfrak{T}$ is called the Borel algebra generated by $\mathfrak{T}$. More precisely, $\mathfrak{B}$ is a Borel algebra generated by $\mathfrak{T}$ iff,

$$\mathfrak{B} = \bigcap\{\mathfrak{S} \mid \mathfrak{T} \subseteq \mathfrak{S} \wedge \mathfrak{S} \text{ is a } \sigma\text{-algebra}\}.$$

Obviously, every Borel algebra is itself a σ-algebra.

Definition 5.0.15 (Metrizable Space)**.** A metrizable space is a topological space that is homeomorphic to a metric space. That is, a topological space $\langle X, \mathfrak{T}\rangle$ is said to be metrizable if there is a metric $d\colon X \to \{0\} \cup \mathbb{R}^+$ such that the topology induced by d is $\mathfrak{T}$.

Definition 5.0.16 (Recursive Set)**.** Given a set S, S is a recursive set iff there is a Turing machine such that for any given input it terminates after finite number of computation steps and correctly returns whether the input is an element of S or not. In short, a set is called recursive iff it has a decidable membership problem.

Definition 5.0.17 (Polysize model property)**.** Let $\mathbf{L}$ be a normal modal logic, M a set of finitely based models such that the set of all formulas that are true on every model in M generates $\mathbf{L}$ and f a function mapping natural numbers to natural numbers. $\mathbf{L}$ has the $f(n)$-size model property with respect to M if every $\mathbf{L}$-consistent formula φ is satisfiable in a model in M containing at most $f(|\varphi|)$ states. $\mathbf{L}$ has the polysize model property with respect to M if there is a polynomial f such that $\mathbf{L}$ has the $f(n)$-size model property with respect to M.

Lemma 5.0.4. *Let τ be a finite similarity type. Let Λ be a consistent normal modal logic over τ with the polysize model property with respect to some class of models M. If the problem of deciding whether $\mathfrak{M} \in \mathsf{M}$ is computable in time polynomial in $|\mathfrak{M}|$, then Λ has an NP-complete satisfiability problem.*

Lemma 5.0.5. *If F is a class of frames definable by a first-order sentence, then the* *problem of deciding whether $\mathfrak{F}$ belongs to* F *is decidable in time polynomial in the size of* $\mathfrak{F}$.

Definition 5.0.18 (SnS, Monadic Second-Order Theory of Trees of Infinite Depth)**.** Let $\mathcal{A}$ be a set of symbols ('the alphabet'). $\mathcal{A}^*$ denotes the set of all finite sequences ('words') that can be produced from the alphabet $\mathcal{A}$. We define the following relations over $\mathcal{A}^*$: For every $w, u \in \mathcal{A}^*$,

- $w \ll u$ iff u extends w, i.e., $\exists v \in \mathcal{A}^*$ such that $u = wv$, where wv denotes the concatination of the word v at the end of the word w. This is called the 'initial segment of' relation.
- $w \leq u$ iff either $w \ll u$ or $\exists p \exists v \exists z \in \mathcal{A}^*$ and $\exists a \exists b \in \mathcal{A}$ such that $w = pav$, $u = pbz$ and $a < b$ where pav and pbz denote the concatinatons of the respective elements and $<$ denotes a total ordering relation over $\mathcal{A}$. We say that $\leq$ is the total lexiographic order over $\mathcal{A}^*$ induced by $<$.
- For every $a \in \mathcal{A}$, the function $f_a \colon \mathcal{A}^* \to \mathcal{A}^*$ such that $f_a(w) = wa$ is called the a-th successor function over $\mathcal{A}^*$.

Fix some $n \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{A}_n = \{k \in \mathbb{N} \mid k < n\}$ be an alphabet. Let $\leq$ be the lexiographic order over $\mathcal{A}_n^*$ induced by the usual ordering relation over $\mathbb{N}$, f_k be the kth successor function over $\mathcal{A}_n^*$ for every $k \in \mathcal{A}_n$ and $\ll$ be the 'initial segment of' relation over $\mathcal{A}_n^*$ as defined above. Then,

$$\mathcal{S}_n = \langle \mathcal{A}_n, \ll, \leq, \{f_k\}_{k<n} \rangle$$

is called the structure of n successor functions.

Let $\mathcal{L}^2$ be a monadic second-order language for $\mathcal{S}_n$ consisting of a denumerably infinite set of variable symbols, a denumerably infinite set of predicate symbols, a function symbol f_k for every $k < n$ corresponding to the kth successor function of $\mathcal{S}_n$, two relation symbols $\ll$ and $\leq$ corresponding to the respective relations of $\mathcal{S}_n$, the standard boolean connectives and the usual quantifiers with the well-known second-order interpretations. Note that this entails quantification not only over the variables but also over the predicates.

The monadic second-order theory of trees of infinite depth (S*n*S for short) is the theory obtained by the second-order interpretation of $\mathcal{L}^2$ over $\mathcal{S}_n$.

For a proof of the following theorem, see [54].

Theorem 5.0.6 (Rabin's Theorem)**.** *For every $n \in \mathbb{N} \cup \{\infty\}$, the satisfiability problem of SnS is decidable.*