DEFORMATION QUANTIZATION OF OPEN SYSTEMS

AND

ALGEBRAIC PROPERTIES OF COMPLETELY POSITIVE MAPS

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Introduction

In this thesis, we explore the deformation quantization of open systems with special regard to the conservation of the complete positivity of the time evolution maps at the transition from classical to quantum systems. Additionally, we investigate some interesting algebraic and categorial properties of these completely positive maps in the general framework of $\ast$-algebras, culminating in the right-unital $\ast$-bicategory $\text{CP}$ of completely positive maps between $\ast$-algebras and an embedding of the $\ast$-bicategory $\text{Bimod}^{\text{idem}}$, which encodes the strong Morita equivalence of the underlying $\ast$-algebras, into $\text{CP}$.

Motivation

Usually, we get an intuitive grasp of how to construct mathematical models of quantum physical systems from analogous classical systems. Therefore, we may view quantization as a mathematical process, generally used in order to obtain some intuition with respect to the construction of quantum models, while the classical limit seems to be a genuinely physical phenomenon.

Even today, it is still important to get a good understanding of the transition between the classical and the quantum physical description of systems, especially as, since the beginning of quantum mechanics, many attempts have been made at a mathematically precise and physically consistent quantization of non-conservative systems. For examples, see Brittin [21], Dekker [31], and Razavy [69] among many others. For a current overview on non-conservative quantum systems, see Breuer and Petruccione [19, 20] and Razavy [70]. By a conservative system, we mean a system with conserved total energy usually formulated in the framework of Hamiltonian systems, both classically and quantum mechanically. In contrast, the total energy of a non-conservative system is not conserved.

Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer [6] formulated formal deformation quantization, building on the ideas of Weyl [82], Moyal [61], and Berezin [12]. Deformation quantization is a quantization scheme which is mostly applied to systems in classical mechanics. The central object of deformation quantization is the algebra of observables, which in the classical case is given by the smooth complex-valued functions $C^\infty(M)$ on a phase space $M$. The phase space is usually given by a symplectic or Poisson manifold, whence the Poisson brackets originate. States are regarded as a derived concept in the sense of normalized positive linear functionals on the algebra of observables in the classical as well as in the quantum case. In order to obtain a quantum mechanical associative and noncommutative algebra of observables, the classical associative and commutative algebra is deformed with respect to the point-wise product into a star product algebra, whose star product should correspond to the operator product of the corresponding quantum observables, see [2, 34, 47, 74, 80] for an overview.

The ansatz for this star product is an expansion in powers of $\hbar$, where the zeroth order is just the commutative product of smooth functions on the classical phase space. Thus, the classical limit of the associative algebra containing the quantum mechanical observables is just the original classical algebra of observables. Physically, this means that, in the classical limit, $\hbar$ can be neglected compared to factors with the physical dimension of an action inherent to the system. Allowing for the correspondence principle, the commutator of star products has to correspond in first order of $\hbar$ to $i$ times the classical Poisson bracket. The true algebraic imposition on the higher orders in $\hbar$ is the call for associativity of the star product. The mathematical theory
underlying the algebraic aspects of deformation quantization is the theory of deformations of associative
algebras by Gerstenhaber [42–46].

Trying to find a deformed product on the algebra of smooth functions fulfilling the above conditions, it
quickly becomes clear that the formal series in \( \hbar \) describing such a product will, in general, not converge
as a power series. Convergence can only be achieved for certain subalgebras. The mathematical reason is
that in each order of \( \hbar \), the star product is a bidifferential operator with an order of differentiation increasing
with the order of \( \hbar \). Physically, this is to be expected for dimensional reasons, as the physical dimensions of
the orders of \( \hbar \) and of the differentiations with respect to coordinates on the phase space have to compensate
in order to give a dimensionless product. For arbitrary smooth functions, such a star product can only
be a formal series leading to the algebra \((\mathcal{C}^\infty(M)[[\hbar]], \star)\) of formal series with coefficients in the smooth
functions on the phase space \( M \) with the star product \( \star \) as the appropriate algebra of observables.

Nevertheless, star products can be viewed as asymptotic expansions in \( \hbar \) of the convergent structures
used for “quantum mechanics on phase space” with Wigner functions, cf. Rieffel [72], Neumaier [64],
Plebanski et al. [68], Bordemann et al. [15], Becher [7], and Waldmann [78]. In fact, the expansions are not
only asymptotic point-wise, but usually they are also asymptotic with respect to the relevant topologies of
the algebras of functions used for the convergent formulation.

The advantage over more conventional quantizations using “quantization maps”, which assign an op-
erator on some Hilbert space to some classical observable, is that one has an intrinsic and consistent construc-
tion of the observable algebra and a well-defined and physically meaningful classical limit. Such a
star product algebra can be represented by operators on a pre-Hilbert space, thus directly connecting to the
usual formulation of quantum mechanics on a Hilbert space. This approach of conceptually separating the
algebra of observables and its representations has the benefit of separating purely algebraic properties like
symmetries and positivity from functional-analytic questions like super-selection rules and spectral values.

Using the mathematically succinct and powerful framework of deformation quantization, the study of
the quantization of non-conservative systems has renewed, cf. Dito and Turrubiates [35] and subsequently
Belchev and Walton [10] for the deformation quantization of the damped harmonic oscillator.

The possibly most fundamental approach to non-conservative systems is to consider the system of in-
terest to be interacting with a second system, usually called environment, reservoir, or bath, where both
systems together form a conservative system. Then, the usual quantization procedures are applied to the
total system, and appropriate time evolution operators can be found by “integrating” over the degrees of
freedom of the bath. For the purpose of this thesis, we denote such systems by the name of open systems.

Results and Open Questions

Open Systems

In the present work, we investigate the deformation quantization of open systems as defined at the end
of the previous section. To this end, in Definition 3.1.11, we give a notion of classical open systems on
Poisson manifolds as phase spaces appropriate for deformation quantization. Such open systems result
from two classical Hamiltonian systems on Poisson manifolds by forming the completed tensor product
of the algebras of observables, which is again an algebra of observables on the product manifold. The
coupling of the systems then occurs in the total Hamiltonian of the total system, which usually consists of
the Hamiltonians of the constituting systems and an interaction term.

The classical open time evolution \( (\Phi^{\omega}_t)^\circ \) is achieved by mapping observables of the system S into the
total system, then using the Hamiltonian time evolution of the total system, and finally integrating over the
bath degrees of freedom by using a positive linear functional on the algebra of observables of the bath B.

The “integrating out” of the degrees of freedom of the bath is done by using the completed tensor product
\( \text{id}_S \otimes \omega_B \) of the identity map \( \text{id}_S \) on the system algebra of observables and a positive linear functional \( \omega_B \) on the
algebra of the bath. In a generalization of the famous Riesz Representation Theorem, see e.g. [25, App. B],
every positive linear functional on an algebra of smooth complex-valued functions on a smooth manifold is a positive Borel measure with compact support. Therefore, by applying the completed tensor product of the identity map and a positive linear functional on the bath, we literally integrate over the degrees of freedom of the bath with respect to that Borel measure. The notation \( (\Phi^\omega)^t \) is of course only symbolic as there is no underlying map of manifolds, due to the “integration” on the bath. The time evolution of a classical Hamiltonian system is a \(^*\)-automorphism of the classical algebra of observables. We show in Proposition 3.1.8 that the only algebraic feature of the time evolution of a classical open system in the sense of Definition 3.1.11, which remains in comparison to classical Hamiltonian systems, is complete positivity. For a formal definition of complete positivity see Section 1.3.

Thus, complete positivity forms a minimal requirement for the deformation quantized open time evolution. Also, from a modelling point of view, complete positivity of the time evolution of the open systems is desirable, as it makes the subsequent coupling of systems in the form of algebras of observables possible. Let us clarify this: First, positivity is required in order to conserve the positivity of elements of the algebra of observables, when applying the time evolution maps. Let us belatedly couple another algebra via the tensor product to our original algebra. Then, complete positivity assures that the tensor product of the dynamical map of the original algebra with the identity map on the second coupled algebra is a positive map for all possible additionally coupled algebras. This feature is important as it allows, at least in principle, a formulation of independent subdynamics within the ex post coupled total system.

In [29], Bursztyn and Waldmann show that every classical state \( \omega_0 \), that is every normalized positive linear functional on the smooth functions on a Poisson manifold, can be deformed into a state \( \omega \) on a Hermitian star product algebra by a certain type of deformation. We generalize this result in Section 2.3 to more general deformations \( S \) of classical states called \textit{maps preserving squares}. In Theorem 2.3.5, we immediately get that for every Hermitian star product \( \star \) on a Poisson manifold there exists an equivalent star product \( \star' \) with the property that every classically positive linear functional \( \omega_0 \) is also positive with respect to \( \star' \). A by-product of independent interest is that for every Hermitian star product on a Poisson manifold there is a completely positive map into the undeformed algebra of formal series of smooth functions deforming the identity map.

Subsequently, we state a deformation quantized version of open systems in Definition 3.2.2 defined analogously to classical open system by using the deformation quantization of Hamiltonian systems presented in Section 2.2. The star products used to deform the classical algebra of observables in this definition are Hermitian star products. The obtained quantum open time evolution \( A^\omega_t : C^\infty(M)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]] \) actually is a deformation of a classical open time evolution \( A^t = (\Phi^\omega)^t \circ 0(\hbar) \). Here, \( 0(\hbar) \) denotes the terms of order \( \hbar \) or higher. In general, the quantized open time evolution \( A^\omega \) is no \(^*\)-automorphism of \( (C^\infty(S))[[\hbar]], \star \). Moreover, \( A^\omega_t \circ A^\omega_{t'} \neq A^\omega_{t+t'} \), as expected for a microscopic system.

Then, considering the existence of maps preserving squares for Hermitian star products, it becomes clear that any classical open system can be deformed to a quantum open system. We argue in Theorem 3.2.4 that any classical open time evolution with respect to a classical state can be deformation quantized into a quantized open time evolution \( A^\omega_t \) with respect to a deformed state. Conversely, by construction, the classical limit of any quantized open time evolution is a classical open time evolution for the classical limit of the quantum state, as \( A^\omega_t = (\Phi^\omega)^t + 0(\hbar) \).

It is tempting to believe that the quantized open time evolution \( A^\omega_t \) is \textit{completely positive}, as the algebraic tensor product of the completely positive maps \( \id \) and \( \omega \) is again completely positive, and so is the composition with the completely positive \(^*\)-homomorphisms \( A_t \) and \( pr^\omega \). However, the Fréchet topology of the smooth functions and the \( \hbar \)-adic topology originating from the ring ordering of \( C^\infty(M)[[\hbar]] \) are not very well compatible. It is actually not clear whether the completed tensor product is, in general, completely positive or not. In Theorem 3.2.5 it is shown that every classical state \( \omega_0 \) can be deformed into a state \( \omega = \omega_0 \circ S \) on a Hermitian star product algebra by using a map \( S \) preserving squares, such that any quantized open time evolution with respect to \( \omega \) is completely positive. Thereby, it follows that all classical open systems can be quantized in a way such that complete positivity is preserved. By this, we mean that for any classical state
on the bath a consistent simultaneous deformation quantization of observables, states, and time evolution of a classical open system is possible such that the quantum open time evolution is a completely positive map.

Not all quantum states can be reached that way, but the assertion of Theorem 3.2.5 is actually true for more quantum states than the ones of type (3.27) implied in the theorem. Proposition 3.4.4, shows, that square preserving maps are not the only possible deformations of classical states leading to normalized positive linear functionals on a Hermitian star product algebra that additionally leave the quantum open time evolution with respect to the deformed state completely positive. From this point of view, Theorem 3.2.5 is a proof of existence.

Furthermore, the example of a harmonic oscillator in a thermal state in Subsection 3.4.2 illustrates that for certain convergent models the same results as in the usual formulation of quantum mechanics are reached. In Subsection 3.4.2, we recover the quantum mechanical results of a harmonic oscillator in a thermal state. In many cases deformation quantization does not lead to convergent models by the lack of appropriate functional analytic properties. In this case, deformation quantization does not only give a convergent model. From a mathematical point of view, the use of the star product formalism yields a great simplification of the computations. In the end, no “higher math” than Gaussian integrals are necessary for the concrete computations concerning the harmonic oscillator in a thermal state.

Deformation quantization usually has no good notion of a spectrum as it is purely algebraic. The only partially successful way of extracting spectra is based on the Weyl-Moyal star product on a flat phase space, see The [76] for the counter-example of the free motion on the unit circle $S^1$ as configuration space and see Waldmann [78, Subsec. 6.3.4] for a discussion. Nonetheless, the example of Subsection 3.4.2 shows that in the deformation quantization of coupled harmonic oscillators, no spectrum is needed. This is a contrast to the usual quantum mechanical treatment, where the relation $\text{tr}(e^{-\beta H}) = Z$ is nontrivial. There, one first has to prove that $e^{-\beta H}$ is trace-class at all by an explicit calculation of the spectrum of $H$, which, in the end, is difficult both for technical and conceptual reasons.

Some of the results concerning open systems in Section 2.3 and Chapter 3 have already been made available in the preprint [8] and are submitted to Letters in Mathematical Physics for publication.

**Completely Positive Maps**

Motivated by the importance of complete positivity of the time evolution in open systems, we further consider the algebraic properties of completely positive maps especially with respect to representations on pre-Hilbert modules and to category theory. For this part, the setting of $\ast$-algebras is chosen as it includes the smooth, complex-valued functions on a Poisson manifold, star product algebras, and $C^\ast$-algebras at the same time as, on the one hand, $\ast$-algebras generalize $C^\ast$-algebras and star product algebras and thus the algebras of observables both in the setting of Hilbert spaces over $\mathbb{C}$ and in the setting of deformation quantization. On the other hand, $\ast$-algebras generalize the smooth complex-valued functions on a Poisson manifold and thus the classical algebras of observables. Thus, we can view $\ast$-algebras as generalized algebras of observables. All $\ast$-algebras are assumed to be unital.

As we are especially concerned with the relations of completely positive maps of $\ast$-algebras to $\ast$-representations on pre-Hilbert modules and to category theory, we give a more or less straightforward adaption of some generalizations of the GNS construction given by Stinespring [75] and Kasparov [52] regarding representations of $C^\ast$-algebras in the adjointable maps on Hilbert modules induced by completely positive maps from $C^\ast$-algebras into adjointable maps on different Hilbert modules.

Furthermore, we introduce $\ast$-categories motivated by the constructions of Stinespring and Kasparov. First, by $\ast\text{-SKRep}_B(A)$, we denote the subcategory of $\ast\text{-Rep}_B(A)$ consisting of the minimal Stinespring-Kasparov representations from $A$ on pre-Hilbert modules over $B$ with, not necessarily unitary, intertwiners as morphisms. By $\ast\text{-SKRep}_B'(A)$, we denote the subcategory of the category $\ast\text{-Rep}_B(A)$ consisting of all minimal Stinespring-Kasparov representations of a completely positive $\ast$-map $\rho$ with unitary intertwin-
ers. Then, we take a close look at the properties of Stinespring-Kasparov representations in the sense of *-representation theory, especially at the internal tensor product of Stinespring-Kasparov representation bimodules, and show in Theorem 4.2.5 that the minimalized internal tensor product of certain canonical Stinespring-Kasparov representation bimodules is isomorphic to the canonical Stinespring-Kasparov representation bimodule of the composition of the completely positive maps.

Dynamical systems in physics are usually defined on particular *-algebras like the smooth functions $C^\infty(M)$ or the continuous operators $\mathfrak{B}(H)$ on a Hilbert space over $\mathbb{C}$. Therefore, it is a simple act to generalize the usual notion of a dynamical system to general *-algebras. Furthermore, in Proposition 4.3.3, we show an interesting connection between the composition of dynamical maps and tensorial properties of their canonical Stinespring-Kasparov representations, as, by Theorem 4.2.5, the minimalized internal tensor product of the canonical Stinespring-Kasparov representation bimodules of two time evolution maps is isomorphic to the canonical Stinespring-Kasparov representation bimodule of the composition of the time evolution maps.

After having seen a certain compatibility between the internal tensor product and Stinespring-Kasparov representations, in Proposition 4.4.1 we show a similar compatibility with the external tensor product. Furthermore, in Corollary 4.4.10 we give some conditions regarding the non-degeneracy of the ring-theoretic tensor product of Stinespring-Kasparov representations.

We find a generalization *-CP$_B$(A) of the *-category *-Rep$_B$(A) of *-representations of A on pre-Hilbert right B-modules by using unital completely positive *-maps instead of *-homomorphisms in Section 4.5. In order to assure the existence of such maps, we combine Corollary 1.2.3 and Lemma 1.3.2 to show in Lemma 1.3.3 that a unital completely positive map between *-algebras is a *-map if the target algebra has sufficiently many positive linear functionals. For *-algebras with sufficiently many positive linear functionals over quadratic extensions of ordered rings where 2 is invertible, we show that the *-algebra of adjointable maps on any pre-Hilbert module over such an algebra has sufficiently many positive linear functionals. Furthermore, in an abstraction of the adaptations of the constructions by Stinespring [75] and Kasparov [52], we give three examples of functors. In Theorem 4.5.3, we present the composition functor

$$\otimes_B : \text{*-CP}_B(A) \times \text{*-CP}_C(B) \longrightarrow \text{*-CP}_C(A).$$

In Theorem 4.5.7, the canonical Stinespring-Kasparov functor

$$c\text{SK} : \text{*-CP}_B(A) \longrightarrow \text{*-SKRep}_B(A),$$

mapping completely positive maps onto their canonical Stinespring-Kasparov representations is given. The functor

$$i\text{SK} : \text{*-Rep}_B(A) \longrightarrow \text{*-SKRep}_B(A),$$

mapping *-representations onto their identical Stinespring-Kasparov representations is used in Proposition 4.5.10 in order to prove the naturality of an isomorphism between iSK and the restriction of cSK from *-CP$_B$(A) to *-Rep$_B$(A).

Using the *-category *-CP$_B$(A) and the functor $\otimes$, in preparation of Theorem 4.6.1 we construct the right-unital *-bicategory *CP and embed the *-bicategory *Bimod$^{\text{str}}$ into *CP. The bicategory *Bimod$^{\text{str}}$ is closely related to *-representations. As we are only considering unital *-algebras, in our case, the objects of *Bimod$^{\text{str}}$ are the unital *-algebras. The 1-morphisms and the 2-morphisms are given by *-representations and intertwiners.

The bicategory *Bimod$^{\text{str}}$ encodes the strong Morita equivalence of the underlying unital *-algebras. It is of interest for deformation quantization as Bursztyn, Dolgushev, and Waldmann [22] have described the Morita equivalence of two star product algebras on a Poisson manifold by their Kontsevich classes. Furthermore, they have shown that for symplectic manifolds $M$, their result recovers the established result of [26],

...
where two star product algebras $\star$ and $\star'$ are Morita equivalent if and only if there exists a symplectomorphism $\Psi : M \rightarrow M$ such that

$$\frac{1}{2\pi i} \langle (\star', \Psi^*(\star)) \rangle \in H^2_{\text{dR}}(M, \mathbb{Z})$$

for the relative class $t$. The question of equivalence classes of star products $\star$ and $\star'$ on a manifold $M$ is of great interest. Two star products are equivalent if there exists an algebra homomorphism $S$ between them beginning in zeroth order of $\hbar$ with the identity map $\text{id}_{\mathcal{C}^\infty(M)}$. This means that the transition from a star product to an equivalent star product via an equivalence transformation leaves alone the classical limits.

Some interesting questions remain: How well do the constructions above work for concrete examples? Can we adapt the deformation theory of equivalence bimodules, cf. Bursztyn and Waldmann [27], to the right-unital bicategory $\mathcal{C}_\mathrm{P}$, and what are the classical limits? Are Hopf algebra symmetries, cf. Jansen and Waldmann [50], implementable to the category $\ast\mathcal{C}_\mathrm{P}(\mathcal{B})$? In particular, are there notions of $\mathbb{H}$-covariance for $\ast\mathcal{C}_\mathrm{P}(\mathcal{B})$?

Outline

This thesis is organized into four chapters and one appendix.

Chapter 1 gives some definitions, properties, and concepts of the theory of $\ast$-algebras over ordered rings. In particular, in Section 1.1, $\ast$-algebras over ordered rings are introduced. In Section 1.2, we define appropriate notions of states on $\ast$-algebras and of the positivity of elements of $\ast$-algebras and give some first interesting relations between positive algebra elements. These relations are used to induce further relations between certain maps in Section 1.3, after giving the necessary definitions and properties of the maps between $\ast$-algebras involved. The final Section 1.4 of this chapter presents basic notions and definitions of $\ast$-representation theory and of category theory.

Chapter 2 introduces deformation quantization. In Section 2.1, basic notions of star products and star product algebras on Poisson manifolds necessary for Chapter 3 are sketched. In Section 2.2, we briefly recall the notion of a quantum time evolution of a Hamiltonian system with regard to a Hermitian star product. Finally, in Section 2.3, we specialize the notion of states on a $\ast$-algebra to star product algebras. In addition, original results on the definition and existence of square preserving maps are given. In particular, we prove in Theorem 2.3.3 that for every Hermitian star product one has a completely positive map deforming the identity into a formal series of smooth functions with respect to the undeformed product.

Chapter 3 is arranged in the following way: In Section 3.1, a notion of classical dynamical systems in general and the notion of classical open systems used for deformation quantization in particular are defined. In Section 3.2, we use Theorem 2.3.3 as the main tool to show in Theorem 3.2.5 that every classical open system can be deformation quantized into a quantum open system with completely positive open time evolution. In Section 3.3, as an illustration, we give the standard example of the total time evolution of two one-dimensional linearly coupled harmonic oscillators in the setting of deformation quantization. Section 3.4 contains the open time evolutions of a coupled harmonic oscillator with respect to states on the bath oscillator corresponding to deformed initial values and to KMS states.

Chapter 4 contains $\ast$-representational and categorial results connected to completely positive maps on $\ast$-algebras. In the preface, it is argued that the open time evolutions of Chapter 3 fulfill all requirements necessary for completely positive maps in this chapter. In Section 4.1, an adaption of a construction given by Stinespring [75] and Kasparov [52] regarding representations induced by completely positive maps from $\mathcal{C}$-algebras into adjointable maps on different Hilbert modules is given. Additionally, some further properties of this construction for $\ast$-algebras are proved. Section 4.2 shows a certain compatibility of this adapted construction with the internal tensor product and introduces two $\ast$-categories $\ast\mathcal{SKRep}(\mathcal{B})$ and $\ast\mathcal{SKRep}'(\mathcal{B})$ connected to the Stinespring-Kasparov construction. The results of Section 4.2 are used in Section 4.3 in
order to show a relation between the internal tensor product of canonical Stinespring-Kasparov representations and the dynamical systems on $\ast$-algebras introduced in this section. In Section 4.4, the compatibility of the external tensor product with the Stinespring-Kasparov representations of completely positive maps is shown, together with conditions of non-degeneracy of the tensor product. Section 4.5 gives definitions and properties of the $\ast$-category $\ast\CP_B(A)$, the functor $\otimes$, the functor $\iSK$, and the functor $\cSK$, as well as a natural isomorphism between $\iSK$ and $\cSK$. These results are used in Section 4.6 in order to define the right-unital $\ast$-bicategory $\CP$ related to completely positive maps and an embedding of the $\ast$-bicategory $\bimod^{str}$, which encodes the strong Morita equivalence of the underlying $\ast$-algebras, as a non unital $\ast$-bicategory into $\CP$.

Finally, Appendix A contains some introductory definitions, relations, and results of the theory of rings, algebras, and modules in order to facilitate the understanding of Chapter 1.
Chapter 1

Associative ∗-Algebras, States, Maps, and Representations

In this first chapter, we introduce some definitions, properties and concepts of the theory of ∗-algebras over ordered rings, see e.g. [28, 77] for an overview and [78, Chap. 7] for an introduction and further references. The content of this chapter is not only a necessary prerequisite for Chapter 4, it also specializes to Hermitian star product algebras and thus lays the algebraic foundation for Chapters 2 and 3. The theory of ∗-algebras is of interest for modelling physical systems, as it generalizes the algebraic properties both of classical and of quantum mechanical algebras of observables. Some basic notions of rings, algebras, and modules are given in Appendix A.

1.1 Associative ∗-Algebras

Consider an ordered ring \(R\), see also Definition A.1.11, which will usually be given either by the real numbers \(\mathbb{R}\) together with their usual ordering structure or by the ring of real formal power series \(\mathbb{R}[[\hbar]]\). A non-zero real formal power series \(a = \sum_{r=0}^{\infty} \hbar^r a_r \in \mathbb{R}[[\hbar]]\) is called positive, if its lowest non-zero component is positive, \(a_0 > 0\). In this case we write \(a > 0\). The positivity in the ordered ring \(R\) then induces notions of positivity for linear functionals and algebra elements later on. As the replacement for \(\mathbb{C}[[\hbar]]\) or \(\mathbb{C}[[\hbar]]\) we take the quadratic ring extension \(\mathbb{C} = \mathbb{R}(i)\) by a square root of \(-1\).

The algebras we contemplate are unital ∗-algebras \(A\) over \(\mathbb{C}\), i.e. associative algebras over \(\mathbb{C}\) with an antilinear involutive antiautomorphism, i.e. a ∗-involution \(a \mapsto -\overline{a}\) for \(a \in A\) and with a neutral element 1 with respect to multiplication.

**Definition 1.1.1 (∗-Algebras)**
A ∗-algebra \(A\) over \(\mathbb{C}\) is an associative algebra with a ∗-involution \(a \mapsto a^*\), that is with an involutive \(\mathbb{C}\)-antilinear antiautomorphism. A morphism of ∗-algebras, or ∗-homomorphism, is an algebra homomorphism \(\phi : A \rightarrow B\) with \(\phi(a^*) = \phi(a)^*\). A ∗-ideal \(J \subseteq A\) is an ideal closed under the operation of ∗-involution. A ∗-algebra with unit element 1 with regard to multiplication is called unital.

**Remark 1.1.2**
In this thesis, we will consider all abstract ∗-algebras to be unital unless stated otherwise.

**Example 1.1.3 (∗-Algebras)**
The following are examples of ∗-algebras over ordered rings:

\(i.)\) the bounded operators on a complex Hilbert space with the composition as product and the adjoint as ∗-involution,
ii.) the complex-valued smooth functions $(C^\infty(M), \cdot)$ on a smooth manifold $M$ with the point-wise product and the complex conjugation as $^*$-involution,

iii.) a star product algebra $(C^\infty(M)[[\lambda]], \star, -)$ over $\mathbb{C}[[\lambda]]$ with a Hermitian star product $\star$ and the complex conjugation as $^*$-involution, see Chapter 2,

iv.) the algebra of square matrices $M_n(\mathbb{C})$ over a quadratic extension $\mathbb{C}$ of an ordered ring for all $n \in \mathbb{N}$ with the $^*$-involution $(c^*)_{ij} = \overline{c_{ji}}$,

v.) the algebra of square matrices $M_n(A)$ over a $^*$-algebra $A$ for all $n \in \mathbb{N}$ with the $^*$-involution $(a^*)_{ij} = (a_{ji})^*$,

vi.) for $^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$, the algebra given by the ring-theoretic tensor product $\mathcal{A} \otimes \mathcal{B}$ by setting

\[(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2 \quad \text{and} \quad (a \otimes b)^* = a^* \otimes b^* \quad (1.1)\]

and its (anti-)linear extension to arbitrary tensors. In this sense, we have $M_n(\mathcal{A}) \cong M_n(\mathbb{C}) \otimes \mathcal{A}$.

It is possible and even useful to translate the notions of isometric, unitary, Hermitian, and normal elements from the theory of bounded operators on a complex Hilbert space to the more general situation of $^*$-algebras. An element $a \in \mathcal{A}$ of a $^*$-algebra is called

i.) isometric, if $a^*a = 1$,

ii.) unitary, if $a^*a = 1 = aa^*$,

iii.) Hermitian, if $a^* = a$,

iv.) and normal, if $a^*a = aa^*$.

v.) An element $p \in \mathcal{A}$ is called a projector, if $p^2 = p = p^*$.

### 1.2 States and Positivity

As we have already mentioned, $^*$-algebras generalize important properties of common algebras of observables. Having introduced $^*$-algebras in the previous section, we give an analogous algebraic generalization of states.

**Definition 1.2.1 (State)**

Let $\mathcal{A}$ be a $^*$-algebra over $\mathbb{C}$. A linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is called positive, if

\[\omega(a^*a) \geq 0 \quad (1.2)\]

for all $a \in \mathcal{A}$ with respect to the ordering of $\mathbb{R} \subseteq \mathbb{C}$. Furthermore, $\omega$ is called a state if $\mathcal{A}$ is unital and $\omega(1) = 1$.

Elements of a $^*$-algebra $\mathcal{A}$ of the form $\sum a_i a_i^*$, where $a_i \in \mathcal{A}$ with $a_i > 0$ and $a_i \in \mathcal{A}$, are called algebraically positive algebra elements. An algebra element $a \in \mathcal{A}$ is called positive if $\omega(a) \geq 0$ for all positive linear functionals $\omega$. The set of all positive algebra elements is denoted by $\mathcal{A}^+$, the subset of algebraically positive elements is denoted by $\mathcal{A}^{++}$ and we have $\mathcal{A}^{++} \subseteq \mathcal{A}^+$. Furthermore, the positive and the algebraically positive elements form convex cones in $\mathcal{A}$.

An important question in describing physics is whether two different observables can actually be discerned when measured in some state. For $C^*$-algebras this question can be answered positively. For general $^*$-algebras this is not true. Therefore, both for physical and mathematical reasons, we will need the following definition.
A *-algebra $\mathcal{A}$ has sufficiently many positive linear functionals if for every non-zero Hermitian element \(0 \neq a = a^* \in \mathcal{A}\) there exists a positive linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$ such that $\omega(a) \neq 0$.

In order to show that a *-algebra $\mathcal{A}$ has sufficiently many positive linear functionals if all *-algebras $M_n(\mathcal{A})$ have sufficiently many positive linear functionals and vice versa, we can recombine several well-known results.

**Corollary 1.2.3**

Let $\mathcal{A}$ be a *-algebra over $\mathbb{C}$. Then, $\mathcal{A}$ has sufficiently many positive linear functionals if and only if, for all $n \in \mathbb{N}$, $M_n(\mathcal{A})$ has sufficiently many positive linear functionals.

**Proof:** First, note that $\mathcal{A}$ and $M_n(\mathcal{A})$ are strongly Morita equivalent for all $n \in \mathbb{N}$. Then, use [24, Cor. 5.5] in order to see that the minimal ideal of $\mathcal{A}$ is zero if and only if the minimal ideal of $M_n(\mathcal{A})$ is zero. Next, remember that the minimal ideal of a *-algebra is zero if and only if the algebra has a faithful *-representation on some pre-Hilbert module over $\mathbb{C}$. Finally, by [25, Prop. 2.8] we know that a *-algebra has sufficiently many positive linear functionals if and only if there exists a faithful *-representation on some pre-Hilbert module over $\mathbb{C}$.

Positive linear functionals on a unital *-algebra have further useful properties. For a positive linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$, we have $\omega(a^*) = \omega(a)$ for all $a \in \mathcal{A}$. Furthermore, $\omega(1) = 0$ implies $\omega = 0$.

**Remark 1.2.4**

If $\mathcal{A}$ has sufficiently many positive linear functionals, then $0 \neq a \in \mathcal{A}^*$ implies $a^* = a$, because with $\omega(a) \in \mathbb{R}$ we have $\omega(a^*) = \omega(a) = \omega(a)$. Therefore, as $0 = \omega(a^*) - \omega(a) = \omega(a^*-a)$ for all positive linear functionals $\omega$, $a^*-a = 0$ because $\mathcal{A}$ has sufficiently many positive linear functionals.

On a *-algebra $\mathcal{A}$ with sufficiently many positive linear functionals it is possible to discern two elements $a, b$ with respect to positive linear functionals and thus with respect to the structure defining positivity. Hence, we can define a partial order on $\mathcal{A}^+$ by setting

\[
a \geq b \iff a - b \in \mathcal{A}^+
\]

for all $a, b \in \mathcal{A}^+$. This partial order connects to the usual algebraic structures, because $a \geq b$ and $b \geq a$ imply $a = b$ in the algebraic sense. We can see this by evaluating $\omega(a-b) \geq 0$ and $-\omega(a-b) = \omega(b-a) \geq 0$, whence $\omega(a-b) = 0$ for all $\omega$ and hence $a - b = 0$ as $\mathcal{A}$ has sufficiently many positive linear functionals.

### 1.3 Maps

A $\mathbb{C}$-linear map $\Phi$ between *-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$ is called unital if $\Phi(1) = 1$. It is called a *-map if additionally $\Phi(a^*) = \Phi(a)^*$ for all $a \in \mathcal{A}$. A $\mathbb{C}$-linear map $\Phi$ between *-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$ is called positive if $\Phi(\mathcal{A}^+) \subseteq \mathcal{B}^+$. It is sufficient to test the positivity of a linear map on algebraically positive elements. Also, a $\mathbb{C}$-linear map $\Phi$ between *-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$ is called n-positive if its continuation to matrices $\Phi^{(n)} : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ is a positive map, that is $\Phi^{(n)}(M_n(\mathcal{A})^+) \subseteq M_n(\mathcal{B})^+$. An n-positive map is automatically $(n-1)$-positive.

Furthermore, a $\mathbb{C}$-linear map $\Phi$ between *-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{C}$ is called completely positive if it is n-positive for all $n \in \mathbb{N}$. We also write this as $\Phi \in \text{CP}(\mathcal{A}, \mathcal{B})$. Note that positive linear functionals are completely positive and that an equivalent definition of complete positivity is given by defining a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ as positive if $\mu \circ \Phi : \mathcal{A} \rightarrow \mathbb{C}$ is a positive linear functional for all positive linear functionals $\mu : \mathcal{B} \rightarrow \mathbb{C}$. From this follows that it is sufficient to check the positivity of a $\mathbb{C}$-linear map $\Phi$ on algebraically positive elements. Furthermore, any positive map from a *-algebra into the underlying ring is a positive linear functional and hence completely positive.
Also, \(\ast\)-homomorphisms are completely positive. Moreover, the composition of completely positive maps as well as convex combinations of completely positive maps are again completely positive. Finally, less evident but nevertheless true is the fact that the algebraic tensor product of completely positive maps is again completely positive, see e.g. Waldmann [79, Prop. 1.5.5]. In general, this last statement is wrong for positive maps.

**Lemma 1.3.1**

Let \(\mathcal{A}\) be a unital \(\ast\)-algebra over \(\mathbb{C}\). Then, the map

\[
\Phi : M_n(\mathcal{A}) \rightarrow \mathcal{A}
\]

\[
\Phi : A \mapsto \sum_{i,j=1}^{n} a_i^* A_{ij} a_j,
\]

where \(A \in M_n(\mathcal{A})\) and \(a_i \in \mathcal{A}\) for all \(i = 1, \ldots, n\), is positive for all \(n \in \mathbb{N}\).

**Proof:** We check this on algebraically positive elements. In fact, because of the \(\mathbb{C}\)-linearity of \(\Phi\), it is sufficient to check on squares \(A^* A\) with \(A \in M_n(\mathcal{A})\).

\[
\Phi(A^* A) = \sum_{i,j,k=1}^{n} a_i^* A_{ik}^* A_{kj} a_j = \sum_{i,j,k=1}^{n} A_{ik} a_i^* A_{kj} a_j = \sum_{k=1}^{n} \left( \sum_{i,j=1}^{n} A_{ik} a_i^* \right) a_j \left( \sum_{j=1}^{n} A_{kj} a_j \right) \in \mathcal{A}^+
\]

for all \(A \in M_n(\mathcal{A})\) and all \(a_i \in \mathcal{A}\) for all \(i = 1, \ldots, n\). \(\Box\)

Next, we use the map of Lemma 1.3.1 in order to obtain interesting relations between components of matrices over \(\ast\)-algebras.

**Lemma 1.3.2**

Let \(\mathcal{A}\) be a unital \(\ast\)-algebra over \(\mathbb{C}\) with sufficiently many positive linear functionals. Let

\[
Q = \begin{pmatrix} 1 & t' \\ t & q \end{pmatrix}
\]

with \(t, t', q \in \mathcal{A}\) be a positive matrix \(Q \in M_2(\mathcal{A})^+\). Then

\[
q \in \mathcal{A}^+, \quad t' = t', \quad \text{and} \quad q \geq t't.
\]

**Proof:** First of all, from \(Q^* = Q\), we get \(q = q^*\) and \(t' = t'\). Furthermore, note that, by Lemma 1.3.1, for \(Q \in M_2(\mathcal{A})^+\) we get

\[
\sum_{i,j=1}^{2} a_i^* Q_{ij} a_j \in \mathcal{A}^+
\]

for all \(a_1, a_2 \in \mathcal{A}\). Therefore,

\[
0 \leq a_1^* a_1 + a_2^* a_2 + a_1^* t a_2 + a_2^* t' a_1,
\]

wherefore, by setting \(a_1 = 0\) and \(a_2 = 1\), we see that \(q \geq 0\) and thence \(q \in \mathcal{A}^+\). The case \(a_1 = -t\) and \(a_2 = 1\) shows that necessarily \(q \geq t't\). \(\Box\)

Analogously to Relation (1.3), we introduce a partial order on completely positive maps from \(\mathcal{A}\) to \(\mathcal{B}\) by setting

\[
\Psi \leq \Phi \iff \Phi - \Psi \in \text{CP}(\mathcal{A}, \mathcal{B})
\]

for all completely positive maps \(\Phi, \Psi \in \text{CP}(\mathcal{A}, \mathcal{B})\).
Lemma 1.3.3
Let \( A, B \) be unital \(^\ast\)-algebras over \( \mathbb{C} \) where \( B \) has sufficiently many positive linear functionals. If \( \Phi : A \to B \) is a unital 2-positive map, then for all \( a \in A \) we get
\[
\Phi(a^*a) \geq \Phi(a^*)\Phi(a), \\
\Phi(a^*) = \Phi(a)^*.
\]

Proof: Notice that for \( a \in A \)
\[
\begin{pmatrix}
1 & a \\
{a^*} & {a^*a}
\end{pmatrix}
= \begin{pmatrix}
1 & a^* \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & a \\
0 & 0
\end{pmatrix}
\in M_2(A)^{++}.
\]
Thus,
\[
\begin{pmatrix}
\Phi(1) & \Phi(a) \\
\Phi(a^*) & \Phi(a^*a)
\end{pmatrix}
\in M_2(B)^{+}.
\]
Therefore, by Lemma 1.3.2, we get the expressions (1.8) and (1.9).

Remark 1.3.4
As \( M_n(M_m(A)) \cong M_{mn}(A) \), we additionally see that for all unital \((2k)\)-positive linear maps the relation
\[
\Phi^{(n)}(A^*A) \geq \Phi^{(n)}(A^*) \Phi^{(n)}(A)
\]
holds for all \( n = 1, \ldots, k \).

1.4 Representations

Next, we need representation spaces with extra structure reflecting the \(^\ast\)-involution of the \(^\ast\)-algebras. As there is a subtle difference in the definitions of pre-Hilbert modules over rings and of pre-Hilbert modules over algebras, we will quickly give the definition of pre-Hilbert modules over rings before turning to the more general case. We follow mostly the notation of [25] and related articles.

Definition 1.4.1 (Inner product module)
Let \( \mathcal{H} \) be a right \( \mathbb{C} \)-module. A \( \mathbb{C} \)-valued inner product on \( \mathcal{H} \) is a map
\[
\langle \cdot, \cdot \rangle^{c} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}
\]
with the following properties:

i.) \( \langle \phi, c\psi \rangle^{c} = c \langle \phi, \psi \rangle^{c} \) for all \( \phi, \psi \in \mathcal{H} \) and \( c \in \mathbb{C} \).

ii.) \( \langle \phi, \psi \rangle^{c} = \overline{\langle \psi, \phi \rangle^{c}} \) for all \( \phi, \psi \in \mathcal{H} \).

\( (\mathcal{H}, \langle \cdot, \cdot \rangle^{c}) \) is called an inner product module over \( \mathbb{C} \) if \( \langle \cdot, \cdot \rangle^{c} \) is non-degenerate, that is if \( \langle \phi, \psi \rangle^{c} = 0 \) for all \( \phi \in \mathcal{H} \) implies that \( \psi = 0 \). The inner product is called positive semi-definite if \( \langle \phi, \phi \rangle^{c} \geq 0 \) for all \( \phi \in \mathcal{H} \), and positive definite if \( \langle \phi, \phi \rangle^{c} > 0 \) for \( \phi \neq 0 \). The notion of positive definiteness is well-defined due to the notion of positivity for ordered rings, as \( \langle \phi, \phi \rangle^{c} \in \mathbb{R} \) for all \( \phi \).

Remark 1.4.2 (Degeneracy submodule)
The degeneracy submodule of a module with inner product \( \langle \cdot, \cdot \rangle^{c} \) is denoted by
\[
\mathcal{H}^\bot = \{ \phi \in \mathcal{H} \mid \langle \phi, \psi \rangle^{c} = 0 \text{ for all } \psi \in \mathcal{H} \}.
\]
It is a submodule of \( \mathcal{H} \) because of the sesquilinearity of the inner product. A positive definite inner product obviously is non-degenerate, which can easily be seen by contradiction. If \( \langle \cdot, \cdot \rangle^{c} \) is positive semi-definite, the notions of non-degeneracy and positive definiteness concur.
Definition 1.4.3 (Pre-Hilbert module)
Let \((\mathcal{H}, \langle \cdot, \cdot \rangle^\mathcal{H})\) be an inner product module. Then, \((\mathcal{H}, \langle \cdot, \cdot \rangle^\mathcal{H})\) is called a pre-Hilbert module if \(\langle \cdot, \cdot \rangle^\mathcal{H}\) is positive definite.

We call an element \(h \in \mathcal{H}\) of a \(\mathbb{C}\)-module a torsion element if there exists a non-zero element \(c \in \mathbb{C}\) such that \(hc = 0\). The set of torsion elements is denoted by \(\text{tor}(\mathcal{H})\). \(\text{tor}(\mathcal{H})\) is a submodule of \(\mathcal{H}\), the torsion submodule, see also Appendix A.

Lemma 1.4.4
A pre-Hilbert module \((\mathcal{H}, \langle \cdot, \cdot \rangle^\mathcal{H})\) over \(\mathbb{C}\) is torsion-free, i.e. \(\text{tor}(\mathcal{H}) = \{0\}\).

Proof: Let \(0 \neq z \in \mathbb{C}\) and \(0 \neq \psi \in \mathcal{H}\). Then,

\[
\langle \psi z, \psi z \rangle^\mathcal{H} = \langle \psi, \psi \rangle^\mathcal{H} \overbrace{\mathbb{C}}^{>0} \overbrace{\mathbb{C}}^{>0} > 0.
\] (1.12)

Definition 1.4.5 (Inner product module with algebra-valued inner product)
An inner product module \(\mathcal{E}_A\) over \(A\) is a right \(A\)-module with inner product

\[
\langle \cdot, \cdot \rangle^\mathcal{E}_A : \mathcal{E}_A \times \mathcal{E}_A \rightarrow A,
\]
such that for all \(x, y \in \mathcal{E}_A\) and \(a \in A\)

i.) \(\langle \cdot, \cdot \rangle_A\) is \(\mathbb{C}\)-linear in the second argument,

ii.) \(\langle x, y \rangle_A = \langle y, x \rangle_A^\ast\),

iii.) \(\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a,

iv.) \(\langle \cdot, \cdot \rangle_A\) is non-degenerate, that is \(\langle x, y \rangle_A = 0\) for all \(y \in \mathcal{E}_A\) implies \(x = 0\).

As above, the superscript denoting the relevant \(A\)-module will be dropped if no ambiguities arise. The submodule \(\mathcal{E}_A^\mathcal{E}_A = \{x \in \mathcal{E}_A | \langle x, y \rangle_A = 0\text{ for all } y \in \mathcal{E}_A\}\) of an \(A\)-module \(\mathcal{E}_A\) with inner product consisting of all degenerate elements is called the degeneracy submodule.

Having an inner product module, we can consider the following \(\mathbb{C}\)-linear operators on it. We call a map \(A : \mathcal{E}_A \rightarrow \mathcal{F}_A\) from one inner product right \(A\)-module to another adjointable if there exists a map \(A^\ast : \mathcal{F}_A \rightarrow \mathcal{E}_A\) with

\[
\langle x', Ax \rangle^\mathcal{F}_A = \langle A^\ast x', x \rangle^\mathcal{E}_A
\] (1.13)
for all \(x \in \mathcal{E}_A\) and \(x' \in \mathcal{F}_A\). Then the maps \(A\) and \(A^\ast\) are necessarily right \(A\)-linear, \(A^\ast\) is uniquely determined by \(A\), and \(A^\ast\) is adjointable as well with \((A^\ast)^\ast = A\). Denoting the set of all adjointable operators by \(\mathcal{B}(\mathcal{E}_A, \mathcal{F}_A)\), we have the usual properties of linear combinations and compositions of adjointable operators again being adjointable with adjoints given in the usual way. In particular, \(\mathcal{B}(\mathcal{E}_A, \mathcal{E}_A) = \mathcal{B}(\mathcal{E}_A, \mathcal{E}_A)\) is a unital \(\ast\)-algebra itself.

An \(A\)-valued inner product is called positive if \(\langle x, x \rangle_A \in \mathbb{A}^+\) for all \(x \in \mathcal{E}_A\). Furthermore, an \(A\)-valued inner product is called completely positive, if \((\langle x_i, x_j \rangle_A) \in M_n(A)^\ast\) for all \(x_1, \ldots, x_n \in \mathcal{E}_A\) and for all \(n \in \mathbb{N}\).

Analogously to the situation of Hilbert modules over \(\mathbb{C}\)-algebras, an inner product right \(A\)-module \(\mathcal{E}_A\) is called a pre-Hilbert module over \(A\) if the inner product is completely positive. If \(A = \mathbb{C}\) we use the stronger condition of positive definiteness of the inner product in order to obtain a pre-Hilbert module over \(\mathbb{C}\). In contrast to the situation of Hilbert modules over \(\mathbb{C}\)-algebras, it is not sufficient to only demand the positivity of the inner product, as for general \(\ast\)-algebras the complete positivity does not follow automatically from positivity.

A rather generic example of a pre-Hilbert module over a unital \(\ast\)-algebra \(A\) over an ordered ring \(\mathbb{C}\) is the algebra \(A\) itself, equipped with the inner product \(\langle a, b \rangle_A = a^\ast b\) for all \(a, b \in A\).
**Example 1.4.6**

Let $A$ be a unital $^*$-algebra. Let $\langle a, b \rangle_A := L_a' b = a' b$ for all $a, b \in A$. Then, $(A, (\cdot, \cdot)_A)$ is a pre-Hilbert module over $A$, which in the remainder of this thesis we will simply denote by $A$ for the sake of convenience. By the unitality of $A$, we finally get the identification $A = \mathcal{B}(A)$, as for all right $A$-linear $t \in \mathcal{B}(A)$ we have $t = L_t(1)$, and as $(L_a a, b)_A = (a, L_t' b)_A$ we can see that $L_a = L_a^*$ for all $a \in A$.

By Example 1.4.6, any unital $^*$-algebra $A$ of sufficiently many positive linear functionals passes to the unital $^*$-algebra of adjointable maps on any pre-Hilbert module over $A$.

**Lemma 1.4.7**

Let $A$ be a unital $^*$-algebra over $\mathbb{C}$ and let $(\mathcal{H}_A, (\cdot, \cdot)_A)$ be a pre-Hilbert module over $A$. Then, the map

$$
\Phi : M_n(\mathcal{B}(\mathcal{H}_A)) \rightarrow A
$$

$$
\Phi : T \mapsto \sum_{i,j=1}^n \langle h_i, T_{ij} h_j \rangle_A
$$

where $T \in M_n(\mathcal{B}(\mathcal{H}_A))$ and $h_i \in \mathcal{H}_A$ for $i = 1, \ldots, n$, is a positive map for all $n \in \mathbb{N}$.

**Proof:** Let $T \in M_n(\mathcal{B}(\mathcal{H}_A))$ and $h_i \in \mathcal{H}_A$ for $i = 1, \ldots, nT \in \mathcal{B}(\mathcal{H}_A)$. Then,

$$
\Phi(T^* T) = \sum_{i,j,k=1}^n \langle h_i, T_{ik}^* T_{kj} h_j \rangle_A = \sum_{k=1}^n \left( \sum_{i=1}^n T_{ki} h_i, \sum_{j=1}^n T_{kj} h_j \right)_A \in A^+.
$$

For the next proposition, it is required that $2a = 0$ implies $a = 0$ for all elements $a$ of a $^*$-algebra. This is given if $2 \in \mathbb{R}$ is invertible, which, for example, holds if $\mathbb{Q} \subseteq \mathbb{R}$.

**Proposition 1.4.8**

Let $\mathbb{C}$ be such that $2 \in \mathbb{R}$ is invertible. Let $A$ be a unital $^*$-algebra over $\mathbb{C}$ where $A$ has sufficiently many positive linear functionals. Let $(\mathcal{H}_A, (\cdot, \cdot)_A)$ be a pre-Hilbert module over $A$. Then, $\mathcal{B}(\mathcal{H}_A)$ has sufficiently many positive linear functionals.

**Proof:** Let $T = T^* \in \mathcal{B}(\mathcal{H}_A)$ be Hermitian. We first prove that if $\langle h, Th \rangle_A = 0$ for all $h \in \mathcal{H}_A$, then $T = 0$. Let $\langle h, Th \rangle_A$ be the $^*$-representation of $h$ in $\mathcal{B}(\mathcal{H}_A)$.

$$
\langle h, Th \rangle_A = 0 \quad \forall h \in \mathcal{H}_A, \quad T \in \mathcal{B}(\mathcal{H}_A), \quad a \in A.
$$

by evaluating (1.14) for $\lambda = 1$ and $a = 1$, we see that $2\Re \langle \lambda, Th \rangle_A = 0$ and $2\Im \langle \lambda, Th \rangle_A = 0$ for all $h, g \in \mathcal{H}_A$, wherefore $2T = 0$ and $T = 0$, as 2 is invertible. The real part $\Re$ and the imaginary part $\Im$ of an algebra element are defined as usual by $\Re(a) = \frac{a + a^*}{2}$ and $\Im(a) = \frac{a - a^*}{2i}$.

Furthermore, $\langle h, Th \rangle_A = \langle h, Th \rangle_A^*$. Let $\omega$ be a positive linear functional on $A$. Then, $\omega(\langle h, h \rangle_A^* A)$ is a positive linear functional on $\mathcal{B}(\mathcal{H}_A)$ for all $h \in \mathcal{H}_A$ as $\langle h, h \rangle_A^* A$ is, by Lemma 1.4.7, a positive map from $\mathcal{B}(\mathcal{H}_A)$ to $A$ for all $h \in \mathcal{H}_A$. Remembering that $A$ has sufficiently many positive linear functionals, we see that if $\omega(\langle h, h \rangle_A^* A) = 0$ for all positive linear functionals $\omega$ and for all $h$, then, necessarily, $\langle h, Th \rangle_A = 0$ for all $h$, which means that $T = 0$. Thus, $\mathcal{B}(\mathcal{H}_A)$ has sufficiently many positive linear functionals.
adjointable $\mathcal{A}$-module map $T : \mathcal{E}_\mathcal{A} \to \mathcal{F}_\mathcal{A}$ is called an intertwiner if $T \pi(b) = \pi'(b)T$ for all $b \in \mathcal{B}$. If $\mathcal{E}_\mathcal{A}$ is a pre-Hilbert module, the representation is actually strongly non-degenerate, that is $\pi(\mathcal{B})\mathcal{E}_\mathcal{A} = \mathcal{E}_\mathcal{A}$ as $\mathcal{B}$ is unital and $\pi(1) = \text{id}_\mathcal{E}$.

Furthermore, we need the notion of a category. A category $C$ consists of the following three mathematical entities:

i.) a class $\text{Obj}(C)$, whose elements are called objects, 

ii.) for each pair $a, b$ of objects a set $\text{Morph}(a, b)$, whose elements are called morphisms from $a$ to $b$. Each morphism $f$ has a unique source object $a$ and target object $b$ and we write $f : a \to b$.

iii.) A binary operation $\circ$, called composition of morphisms, such that for any three objects $a, b, c$, we have $\circ : \text{Morph}(a, b) \times \text{Morph}(b, c) \to \text{Morph}(a, c)$. The composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ and satisfies the following conditions:

(a) associativity condition: if $f : a \to b, g : b \to c$, and $h : c \to d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
(b) identity condition: for every object $a$, there exists a morphism $1_a : a \to a$, called the identity morphism for $a$, such that for every morphism $f : a \to b$, we have $1_a \circ f = f = f \circ 1_a$.

Actually, one can prove that there is exactly one identity morphism for every object.

A functor $F$ is a map between categories $C$ and $D$, which to every object $a \in C$ associates an object $F(a) \in D$, and to every morphism $f : a \to b$ in $C$ a morphism $F(f) : F(a) \to F(b)$ in $D$, such that the following two conditions hold:

i.) functors preserve identity morphisms: $F(\text{id}_a) = \text{id}_{F(a)}$ for every object $a \in C$.

ii.) functors preserve compositions of morphisms: $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : a \to b$ and $g : b \to c$.

For a good reference for categories, functors, and bicategories, see the book of MacLane [59].

The category of $^\ast$-representations of $\mathcal{B}$ on inner product right $\mathcal{A}$-modules with intertwiners as morphisms is called $^\ast\text{-Mod}_\mathcal{A}(\mathcal{B})$. The sub-category of $^\ast$-representations of $\mathcal{B}$ on pre-Hilbert modules over $\mathcal{A}$ is denoted by $^\ast\text{-Rep}_\mathcal{A}(\mathcal{B})$. An inner product $\mathcal{A}$-module $\mathcal{E}_\mathcal{A}$ together with a $^\ast$-representation of $\mathcal{B}$ is called a $(\mathcal{B}, \mathcal{A})$-inner-product bimodule. If $\mathcal{E}_\mathcal{A}$ is a pre-Hilbert module, it is a $(\mathcal{B}, \mathcal{A})$-pre-Hilbert bimodule.

Furthermore, we will need two different concepts of tensor products, the external tensor product and the internal tensor product. By the external tensor product $\otimes_{\text{ext}}$ we mean the ring-theoretic tensor product $\otimes_{\mathbb{C}}$ of algebras, modules, and linear maps on them, in the case of inner product modules divided by the ensuing degeneracy module. The inner product of the external tensor product of pre-Hilbert modules is defined as the $\mathbb{C}$-sesquilinear extension of the tensor product of the constituent inner products. Actually, $\otimes_{\text{ext}}$ is a functor

$$
\otimes_{\text{ext}} : ^\ast\text{-Mod}_{\mathcal{A}_1}(\mathcal{B}_1) \times ^\ast\text{-Mod}_{\mathcal{A}_2}(\mathcal{B}_2) \to ^\ast\text{-Mod}_{\mathcal{A}_1 \otimes \mathcal{A}_2}(\mathcal{B}_1 \otimes \mathcal{B}_2),
$$

which is not difficult to see. Additionally,

$$
\otimes_{\text{ext}} : ^\ast\text{-Rep}_{\mathcal{A}_1}(\mathcal{B}_1) \times ^\ast\text{-Rep}_{\mathcal{A}_2}(\mathcal{B}_2) \to ^\ast\text{-Rep}_{\mathcal{A}_1 \otimes \mathcal{A}_2}(\mathcal{B}_1 \otimes \mathcal{B}_2),
$$

which is more non-trivial as especially the complete positivity of the resulting inner products is not simple to show.

The internal tensor product is the algebraic tensor product $\otimes_{\mathbb{B}}$ of elements $\mathcal{E}_\mathcal{B} \in ^\ast\text{-Mod}_{\mathbb{B}}(\mathcal{A})$ and $\mathcal{F}_\mathcal{C} \in ^\ast\text{-Mod}_{\mathbb{C}}(\mathcal{B})$ divided by the degeneracy submodule with respect to the inner product defined below. This, we denote by $\otimes_{\mathbb{B}}$. The internal tensor product of inner product modules is defined cascadingly as the $\mathbb{C}$-sesquilinear extension of

$$
\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{\mathcal{E} \otimes \mathcal{F}} = \{ f_1, \langle e_1, e_2 \rangle_{\mathcal{B}} f_2 \}_{\mathcal{E} \otimes \mathcal{F}}.
$$

(1.15)
Again, we get functors
\[ \hat{\otimes}_B : \ast \text{-Mod}_B(A) \times \ast \text{-Mod}_C(B) \longrightarrow \ast \text{-Mod}_C(A) \]
and
\[ \hat{\otimes}_B : \ast \text{-Rep}_B(A) \times \ast \text{-Rep}_C(B) \longrightarrow \ast \text{-Rep}_C(A) \]
where, again, the second relation, involving complete positivity, is much more difficult to show. A continuous treatment can e.g. be found in [25].

A quite abstract concept of algebra is the bicategory. As we will use bicategories (weak 2-categories) in Section 4.6, we introduce the most basic terms and definitions and refer to Benabou [11], Leinster [58], or MacLane [59] for further (and more complete) information on the subject.

A bicategory $B$ consists of the following data:

1. A class $B_0$, the objects of $B$.
2. For each two objects $a, b \in B_0$, we have a category $B(a, b)$. The objects $B_1(a, b) = \text{Obj}(B(a, b))$ of this category are called 1-morphisms from $a$ to $b$. The morphisms $T : f \longrightarrow f'$ for two 1-morphisms $f, f' \in B_1(a, b)$ are called 2-morphisms from $f$ to $f'$. The set of these 2-morphisms is denoted by $B_2(f, f')$.
3. For each three objects $a, b, c \in B_0$, we have a functor $\otimes : B(a, b) \times B(b, c) \longrightarrow B(a, c)$, called the composition of 1-morphisms or the tensor product. We simply write $\otimes$ if the context is clear.
4. For each object $a \in B_0$, we have a 1-morphism $\text{Id}_a \in B_1(a, a)$, called the identity at $a$.
5. For each four objects $a, b, c, d \in B_0$, we have a natural isomorphism $a : \otimes_c \circ (\otimes_b \times \text{id}) \longrightarrow \otimes_b \circ (\text{id} \times \otimes_c)$, called the associativity.
6. For each two objects $a, b \in B_0$, we have natural isomorphisms
   \[
   \text{left} : \otimes_a \circ (\text{Id}_a \times \text{id}) \longrightarrow \text{id},
   \]
   \[
   \text{right} : \otimes_b \circ (\text{id} \times \text{Id}_b) \longrightarrow \text{id},
   \]
called the left identity and the right identity. Here, $\otimes_a \circ (\text{Id}_a \times \text{id})$ is viewed as a functor from the category $1 \times B(a, b)$ to the category $B(a, b)$ and $\text{Id}_a$ is the unique functor sending the object 1 of 1 to $\text{Id}_a$, and analogously for right.

Additionally, this data has to fulfill two coherence conditions, the associativity coherence and the identity coherence which essentially guarantee that combinations of the natural transformations $\text{left}$, $\text{right}$, and $a$ and of $\otimes$ do not lead to new natural isomorphisms.

Actually, we need even more general notions. A right-unital bicategory (left-unital bicategory) is almost a bicategory, as the conditions i.) to vi.) of the definition of a bicategory are fulfilled, except that there is no left identity isomorphism (right identity isomorphism). Therefore, the 1-morphism $\text{Id}_a \in B_1(a, a)$, is only a right (left) identity at $a$ and no identity coherence condition is demanded.
It is clear that a left-unital, right-unital bicategory fulfilling an identity coherence condition is a bicategory.

For a non-unital bicategory, the conditions i.) to iii.) and v.) of the definition of a bicategory are fulfilled, together with the associativity coherence condition.

Furthermore, a bicategory $B$ is called a $\ast$-bicategory over $C$, if for any two objects $a, b \in B_0$ and two 1-morphisms $f, f' \in B_1(a, b)$, the 2-morphisms $B_2(f, f')$ are a $C$-module and if there is a map

\[ \ast : B_2(f, f') \rightarrow B_2(f', f), \]

called the $\ast$-involution, such that the following properties are fulfilled:

i.) The composition of 2-morphisms is $C$-bilinear.

ii.) The $\ast$-involution is $C$-antilinear, involutive and

\[ (T \circ S)^\ast = S^\ast \circ T^\ast \]

for $S \in B_2(f, f')$ and $T \in B_2(f', f'')$.

iii.) The tensor product of 2-morphisms is $C$-bilinear.

iv.) For $S \in B_2(g, g')$ and $T \in B_2(f, f')$ with $g, g' \in B_1(b, c)$ and $f, f' \in B_1(a, b)$, we have

\[ (T \otimes S)^\ast = T^\ast \otimes S^\ast. \]

v.) The natural isomorphisms a, left, and right are unitary.

This notion of a $\ast$-bicategory over $C$ follows Waldmann [79]. A right-unital bicategory (left-unital, non-unital) is called a right-unital (left-unital, non-unital) $\ast$-bicategory over $C$ if the conditions i.) to v.) for a $\ast$-bicategory are fulfilled except the condition of unitarity of left (right, left and right) for lack of existence.
Chapter 2

Deformation Quantization

The concept of deformation quantization, first introduced in [6], is used to construct the quantum observable algebras of physical systems in terms of the classical data. The central object of deformation quantization is the algebra of observables. States are regarded as a derived concept in the sense of normalized positive linear functionals on the algebra of observables in the classical as well as in the quantum case.

The classical algebra of observables is given by the smooth, complex-valued functions \((C^\infty(M), \cdot)\) with the pointwise product on some Poisson manifold \((M, \pi)\) where the Poisson tensor \(\pi\) induces a Poisson bracket \(\{\cdot, \cdot\}\) on the smooth functions. Deformation quantization is about algebraically deforming this commutative associative algebra \((C^\infty(M), \cdot)\) into a non-commutative \(\hbar\)-dependent associative algebra, the star product algebra. The star product algebra carries most of the features of the usual quantum mechanics on Hilbert spaces.

As both the algebras of smooth functions \((C^\infty(M), \cdot)\) and the algebras of Hermitian deformations of \((C^\infty(M), \cdot)\) by star products are \(*\)-algebras, this chapter makes considerable use of the results of Chapter 1.

2.1 Deformation Quantization of Observables

We can only give a basic introduction containing the results most necessary for the Chapters 2 and 3. The original results regarding existence and classification can be found in DeWilde and Lecomte [33], Fedosov [38–41], and Omori, Maeda, and Yoshioka [66] for the existence on symplectic manifolds, in Nest and Tsygan [62, 63], Deligne [32], Bertelson, Cahen, and Gutt [13], and Weinstein and Xu [81] as well as Deligne [32] and Neumaier [65] for the classification on symplectic manifolds, and in Kontsevich [53, 54] for the existence and classification on Poisson manifolds. See also Waldmann [78] for a well-arranged introduction.

**Definition 2.1.1 (Formal star product)**

Let \((M, \pi)\) be a Poisson manifold. A formal star product \(\star\) for \((M, \pi)\) is a \(\mathbb{C}[[\hbar]]\)-bilinear multiplication

\[ \star : C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] \]  

of the form

\[ f \star g = \sum_{r=0}^{\infty} \hbar^r C_r(f, g) \]  

with \(\mathbb{C}\)-bilinear maps \(C_r : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)\), which can be continued \(\mathbb{C}[[\hbar]]\)-bilinearly the usual way, such that \(\star\) possesses the following properties:

1. \(\star\) is associative.
Chapter 2. Deformation Quantization

ii.) $C_0(f, g) = fg$.

iii.) $C_1(f, g) - C_1(g, f) = i \{f, g\}$.

iv.) $1 \star f = f \star 1 = f$.

v.) The $C_r$ are bidifferential operators for all $r \in \mathbb{N}$.

A star product $\star$ is called Hermitian, if

$$f \star g = \overline{g} \star f$$

(2.3)

for all $f, g \in C^\infty(M)[[\hbar]]$ where $\overline{\hbar} = \hbar$ is treated as a real quantity. This $\star$-involution will be necessary to have the honest interpretation of the algebra $(C^\infty(M)[[\hbar]], \star)$ as observable algebra of the quantum system corresponding to the classical system.

In order to compare different star products, we need a notion of equivalence.

Definition 2.1.2

Two star products $\star$ and $\star'$ for $(M, \pi)$ are called equivalent, if there exists a formal power series $S = id + \sum_{r=1}^{\infty} \hbar^r S_r$ with linear maps $S_r : C^\infty(M) \rightarrow C^\infty(M)$ such that

$$f \star' g = S^{-1}(S f \star S g) \quad \text{and} \quad S 1 = 1$$

for all $f, g \in C^\infty(M)[[\hbar]]$. Such a map $S$ is also called an equivalence transformation.

The existence of star products on arbitrary Poisson manifolds was proven by Kontsevich [54] and is formulated in the following theorem.

Theorem 2.1.3

On every Poisson manifold (Hermitian) star products exist.

2.2 Deformation Quantization of Hamiltonian Dynamics

In this section, we introduce a notion of deformation quantization of Hamiltonian dynamics, closely following [78]. For details on this quantized version of the classical time evolution, we refer to [78, Sect. 6.3.4] and references therein.

In the sequel, a Hermitian star product $\star$ is used. Given a Hamiltonian $H \in C^\infty(M)[[\hbar]]$, where we might even allow for some $\hbar$-dependent correction terms, consider the Heisenberg equation

$$\frac{d}{dt} f(t) = i\hbar [H, f(t)]_\star$$

(2.4)

for $f(t) \in C^\infty(M)[[\hbar]]$. The right-hand side of the equation (2.4) is a well-defined formal power series as the commutator vanishes in zeroth order. For simplicity, assume that the Hamiltonian vector field corresponding to the zeroth order of $H$ has a complete flow $\Phi_t$. In this case, one can show that (2.4) has a solution for all times with the following properties: There exists a formal series of time-dependent differential operators $T_t = id + \sum_{r=1}^{\infty} \hbar^r T_t^r$ on $M$ such that

$$A_t = \Phi_t^* \circ T_t : C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]$$

(2.5)

is a one-parameter group of automorphisms of $\star$ with $f(t) = A_t f$ being the unique solution of (2.4) with the initial condition $f(0) = f$. Moreover, $A_t$ commutes with the commutator $[H, \cdot]_\star$ and we have conservation of energy $A_t H = H$ as usual. Finally, for a Hermitian star product $\star$ and a real Hamiltonian $H = \overline{H}$, $A_t$ is a $\star$-automorphism for each $t$.

The concrete form $A_t = \Phi_t^* \circ T_t$ shows that the quantum time evolution really is a deformation of the classical time evolution, and that therefore the classical limit of the quantum time evolution is given by the classical time evolution.
2.3 Deformation Quantization of States

Given a classical algebra of observables \((C^\omega(M), \cdot, \cdot)\), the \(\delta\)-functional naturally corresponds to points in the phase space \(M\). Thus, classical point states can be viewed as positive linear functionals. The algebraic definition of states in Section 1.2 implies to look for more general states among the positive linear functionals on \(C^\omega(M)\).

Recall from Section 1.2 that a linear functional \(\omega_0 : C^\omega(M) \rightarrow \mathbb{C}\) is positive if \(\omega_0(f^2) \geq 0\) for all functions \(f \in C^\omega(M)\). Similarly, one defines a positive functional on matrix-valued functions \(M_\delta(C^\omega(M))\). Having the notion of positive linear functionals, positive algebra elements are defined by setting that \(f \in C^\omega(M)\) is positive if \(\omega_0(f) \geq 0\) for all positive functionals \(\omega_0\). This purely algebraic notion of the positivity of functions nicely translates to the usual notion of positivity of a function, as it is a true but non-trivial fact that \(f\) is positive if and only if \(f(p) \geq 0\) for all points \(p \in M\). The same holds for matrix-valued functions: A function \(F \in M_\delta(C^\omega(M))\) is positive if and only if \(F(p)\) is a positive semi-definite matrix for all \(p \in M\) (cf. [25, App. B] for a discussion).

For star product algebras, we can proceed analogously to the classical case by using positive linear functionals. From Section 1.1, we remember what a positive formal series is: A non-zero real formal power series \(a = \sum_{r=0}^\infty \hbar^r a_r \in \mathbb{R}[[\hbar]]\) is called positive, if its lowest non-zero component is positive, \(a_0 > 0\), and in this case we write \(a > 0\). This is a good definition for many reasons: If we view formal series as arising from asymptotic expansions, then this is what remains from a positive function. More algebraically, \(\mathbb{R}[[\hbar]]\) becomes an ordered ring by this definition, making applicable the relations in Chapter 1.

Define a \(\mathbb{C}[[\hbar]]\)-linear functional \(\omega : C^\omega(M)[[\hbar]] \rightarrow \mathbb{C}[[\hbar]]\) to be positive if

\[\omega(f^2) \geq 0\]  \hspace{1cm} (2.6)

for all \(f \in C^\omega(M)[[\hbar]]\). It can be shown that it suffices to check (2.6) for \(f \in C^\omega(M)\) without higher orders of \(\hbar\). Analogously, we define positive linear functionals for matrix-valued functions \(F \in M_\delta(C^\omega(M)[[\hbar]]\)) where the star product is extended to matrices in the usual way. Having positive functionals, we define \(f \in C^\omega(M)[[\hbar]]\) or \(F \in M_\delta(C^\omega(M)[[\hbar]]\)) to be a positive algebra element if

\[\omega(f) \geq 0 \quad \text{and} \quad \Omega(F) \geq 0\]  \hspace{1cm} (2.7)

for all positive functionals \(\omega\) and \(\Omega\), respectively, see Chapter 1. Analogously to the general *-algebraic case, a \(\mathbb{C}[[\hbar]]\)-linear map \(\phi : C^\omega(M)[[\hbar]] \rightarrow C^\omega(N)[[\hbar]]\) between two star product algebras on possibly different underlying manifolds is called positive, if \(\phi\) maps positive elements to positive elements. Equivalently, \(\phi\) is called positive if \(\omega \circ \phi\) is a positive functional on \(C^\omega(M)[[\hbar]]\) for all positive functionals \(\omega\) on \(C^\omega(N)[[\hbar]]\). The map \(\phi\) is called completely positive, if this is also true for arbitrary matrix-valued functions, i.e. if \(\phi : M_n(C^\omega(M)[[\hbar]]) \rightarrow M_n(C^\omega(N)[[\hbar]]\)) is positive for all \(n \in \mathbb{N}\). Note that even though these definitions are in complete analogy to the classical situation, it is nevertheless crucial to have a good notion of positive formal power series in \(\mathbb{R}[[\hbar]]\).

In order to describe the positive \(\mathbb{C}[[\hbar]]\)-linear functionals of \((C^\omega(M)[[\hbar]], \star)\), one first notes that \(\omega\) is necessarily of the form

\[\omega = \sum_{r=0}^\infty \hbar^r \omega_r\]  \hspace{1cm} (2.8)

with linear maps \(\omega_r : C^\omega(M) \rightarrow \mathbb{C}\). Then, the positivity \(\omega(f^2) \geq 0\) in the sense of formal power series immediately implies that \(\omega_0(f^2) \geq 0\) classically, i.e. \(\omega_0\) is a positive \(C\)-linear functional. This raises the question whether every classical state \(\omega_0\) can be “quantized” into a state \(\omega\) with respect to the star product. In other words, we ask whether every classical state is the classical limit of some quantum state. Physically, this is absolutely necessary as quantum theory is believed to be the more fundamental description of nature. Fortunately, we can rely on
Theorem [29, Thm. 2.1], which states that any Hermitian star product on a Poisson manifold is a completely positive deformation. This means that we can not only deform every classical positive linear functional into a positive linear functional of the Hermitian star product algebra, but that the same holds true for the respective matrix algebras. The proof of Theorem [29, Thm. 2.1] turns out to be quite constructive, and in the following, we generalize the maps used in that proof, in order to get a more general concrete construction of deformed positive linear functionals, see also [8].

**Definition 2.3.1 (Square preserving map)**
A $\mathbb{C}[[\hbar]]$-linear map $S = \text{id} + \sum_{r=1}^{\infty} \hbar^r S_r : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ with differential operators $S$, $S(1) = 1$, and $S(f) = \overline{S(f)}$ is called *preserving squares* with respect to $\star$, if there are formal series of differential operators $D_{r,l} : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$ for $r \in \mathbb{N}_0$ and $l$ running over a finite range (possibly depending on $r$) such that

$$S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_{l} D_{r,l}(f)D_{r,l}(g) \quad (2.9)$$

for all $f, g \in C^\infty(M)[[\hbar]]$.

**Remark 2.3.2**
It is fairly simple to see that a map preserving squares according to Definition 2.3.1 is in fact a completely positive map from the quantized algebra $(C^\infty(M)[[\hbar]], \star)$ to the algebra $(C^\infty(M)[[\hbar]], \cdot)$ with the *undeformed* product.

**Theorem 2.3.3**
*Given a Hermitian star product, there exists a globally defined map $S$ preserving squares with respect to this star product.*

**Proof:** By [29] we know that for a Hermitian star product $\star$ on an open subset $U \subseteq \mathbb{R}^n$ there exists a map preserving squares with respect to $\star$, denoted by

$$S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_{l} D_{r,l}(f)D_{r,l}(g). \quad (*)$$

For the Poisson manifold $M$ with star product $\star$ we choose a *finite* atlas. Note that we can always find an atlas consisting of $\dim(M) + 1$ not necessarily connected charts. Denote the domains of the charts by $U_\alpha \subseteq M$. Next, we choose a corresponding subordinate *finite* quadratic partition of unity $\chi_\alpha \in C^\infty(M)$, i.e. $\text{supp} \chi_\alpha \subseteq U_\alpha$ and $\sum_\alpha \chi_\alpha = 1$. Now let $S_\alpha$ be the locally available maps preserving squares with respect to $\star|_{U_\alpha}$ with corresponding locally defined differential operators $D_{r,l,a}$. Then, we set

$$S(f) = \sum_\alpha \chi_\alpha S_\alpha \left( f|_{U_\alpha} \right).$$

Clearly, this gives a globally well-defined formal series of differential operators with $\overline{S(f)} = S(f)$ and $S(1) = 1$. Moreover, since the star product is bidifferential, we have $(f \star g)_{|U_\alpha} = f|_{U_\alpha} \star g|_{U_\alpha}$, and hence we can apply $(*)$ to obtain

$$S(f \star g) = \sum_{r=0}^{\infty} \hbar^r \sum_{l,\alpha} \chi_\alpha D_{r,l,a}(f)\chi_\alpha D_{r,l,a}(g).$$

**Remark 2.3.4**
Recently, a $C^*$-algebraic version of this theorem was obtained for particular strict deformation quantizations in [51].
The proof of Theorem 2.3.3 immediately leads to the following consequence.

**Theorem 2.3.5**

*For every Hermitian star product \( \star \) on a Poisson manifold there exists an equivalent star product \( \star' \) with the property that every classically positive linear functional \( \omega_0 \) is also positive with respect to \( \star' \).*

**Proof:** Take a map \( S \) preserving squares with respect to \( \star \). Then the star product \( f \star' g = S(S^{-1}(f) \star S^{-1}(g)) \) is easily shown to do the job.

**Remark 2.3.6**

Rephrasing the result from [29] in terms of Theorem 2.3.3 says that every classical positive linear functional \( \omega_0 \) can be deformed into a positive linear functional with respect to a Hermitian star product. Indeed, \( \omega_0 \circ S \) is such a deformation, even universal for all \( \omega_0 \) once \( S \) is specified. In general, correction terms in higher orders of \( \hbar \) are necessary to obtain positivity. Moreover, they are by far not unique and neither is the map \( S \). This is of course to be expected, both from a physical and mathematical point of view. Finally, note that each term \( \omega_0 \circ S_r \) is continuous in the smooth topology, since the classical functional \( \omega_0 \) is continuous and the differential operators \( S_r \) are continuous as well.
Chapter 3

Deformation Quantization of Open Systems

In this chapter, a consistent quantization of open systems is given as in [8]. In the manner of speaking of [20], we get an open system (classical and quantum mechanical) by constructing a microscopic model and non-selectively integrating the degrees of freedom of the environment.

As a first step, we give a consistent and general definition of what a classical and quantum open system in the sense of deformation quantization should be, relying on the notion of completely positive time evolutions in both cases. The main result we then prove is that, for any classical state on the environment, or bath as we shall call it henceforth, a consistent deformation quantization of observables, states, and time evolution of a classical open system is possible, such that the resulting quantum open time evolution is completely positive. Therefore, every such classical open system can be deformation quantized preserving complete positivity of the evolution map. Finally, the general formalism is exemplified for two coupled harmonic oscillators for two different states of the bath.

3.1 Classical Open Systems

There are many ways to specify the notion of open dynamical systems. A fairly general approach is obtained as follows: We start with a subsystem whose pure states are described by a smooth manifold $S$ and a bath which is described analogously by a smooth manifold $B$. The combined total system has the Cartesian product $S \times B$ as space of pure states.

An open dynamical system is now a time evolution of (pure) states in $S \times B$ where we only look at the $S$-part “ignoring” the $B$-part. More precisely, this is obtained as follows:

On the total system we specify an ordinary dynamical system, i.e. a vector field $X \in \Gamma^\infty(T(S \times B))$ with flow $\Psi_t : S \times B \to S \times B$. For simplicity, we may assume that the flow $\Psi_t$ is complete, otherwise we have to restrict to certain neighbourhoods in $S \times B$ and finite times in the usual way. With this assumption, $\Psi_t$ is a smooth one-parameter group of diffeomorphisms of $S \times B$ with

$$\frac{d}{dt} \Psi_t = X \circ \Psi_t \quad \text{for all } t \in \mathbb{R}. \quad (3.1)$$

Next, we consider the canonical projection maps

$$S \xleftarrow{pr_S} S \times B \xrightarrow{pr_B} B, \quad (3.2)$$

which allow to decompose the tangent bundle $T(S \times B)$ into

$$T(S \times B) \cong pr_S^\# TS \oplus pr_B^\# TB, \quad (3.3)$$

where $pr_S^\# TS$ and $pr_B^\# TB$ denote the pull-backs of the tangent bundles of $S$ and $B$, respectively.
Clearly, the map $\text{pr}_S$ forgets the degrees of freedom of the bath and thus corresponds precisely to the idea that we want to ignore the B-part. However, for the time evolution of $S$ we still have to specify an initial condition for the bath as well. For the moment, we restrict ourselves to pure states and allow for mixed states later on. Thus, let $x_0 \in B$ be a point, wherefore we have the embedding

$$\iota_{xB} : S \ni x_s \mapsto (x_s, x_0) \in S \times B,$$

which is clearly a diffeomorphism onto its image such that $\text{pr}_S \circ \iota_{xB} = \text{id}_S$ and $\text{pr}_B \circ \iota_{xB} = x_0$ is the constant map.

**Definition 3.1.1 (Open time evolution, pure case)**

For any $x_0 \in B$ the open time evolution $\Phi_t^{xB} : S \to S$ of $S$ with respect to the total time evolution $\Psi_t$ of $S \times B$ and the pure state $x_0$ of the bath is given by

$$\Phi_t^{xB} = \text{pr}_S \circ \Psi_t \circ \iota_{xB}. \quad (3.5)$$

Of course, we have to justify this definition and examine some consequences as well as properties of $\Phi_t^{xB}$. First of all, the map

$$\Phi_t^{xB} : \mathbb{R} \times S \ni (t, x_0) \mapsto \Phi_t^{xB}(x_0) \in S \quad (3.6)$$

is clearly smooth. However, it does not have the usual properties of an ordinary time evolution: For a fixed time $t$ the map $\Phi_t^{xB}$ needs not to be a diffeomorphism, not even for small times. We only have the following “evolution property” which easily follows from the one-parameter group property of $\Psi_t$:

**Proposition 3.1.2**

*For the open time evolution we have*

$$\Phi_0^{xB} = \text{id}_S \quad \text{and} \quad \Phi_s^{pr_B(\Psi_s(x_0, x_0))} \circ \Phi_t^{xB}(x_0) = \Phi_{s+t}^{xB}(x_0) \quad (3.7)$$

*for all $x_0 \in S$, $x_0 \in B$, and $s, t \in \mathbb{R}$.*

**Example 3.1.3**

Let $S = \mathbb{R} = B$ and consider the time evolution

$$\Psi_t(x_s, x_0) = (x_s \cos(\nu t) - x_0 \sin(\nu t), x_s \sin(\nu t) + x_0 \cos(\nu t)) \quad (3.8)$$

on $S \times B$.

i.) The simplest case is obtained for $\nu \in \mathbb{R}$ being a non-zero constant. Then the open time evolution for $x_0 \in B$ is given by

$$\Phi_t^{xB}(x_0) = x_s \cos(\nu t) - x_0 \sin(\nu t) \quad (3.9)$$

which is also a diffeomorphism for small $t$ but the constant map for $\nu t \in \frac{\pi}{2} + \pi \mathbb{Z}$.

ii.) We can also consider the case where $\nu$ is a function on $S \times B$ depending only on the radius, e.g. $\nu(x_s, x_0) = x_s^2 + x_0^2$. Then $\Psi_t$ is still a one-parameter group of diffeomorphisms, and the flow lines are still concentric circles around $(0, 0)$. However, the points in $S \times B$ spin faster the further away from $(0, 0)$ they are. Now the open time evolution is

$$\Phi_t^{xB}(x_0) = x_s \cos(\sqrt{x_s^2 + x_0^2} t) - x_0 \sin(\sqrt{x_s^2 + x_0^2} t). \quad (3.10)$$

For $x_0 = 0$, for example, this gives $\Phi_t^0(x_0) = x_s \cos(\frac{\pi}{2} t)$ which yields

$$\Phi_0^0 \left( \sqrt{\frac{\pi}{2t}} \right) = 0 \quad (3.11)$$

for all $t > 0$. Since also $\Phi_t^0(0) = 0$ for all $t$, we see that $\Phi_t^0$ cannot be a diffeomorphism, even for arbitrarily small times $t > 0$. 
From this example we conclude that the open time evolution $\Phi^B_t$ in general is not a solution to a probably time-dependent differential equation on $S$ alone, i.e. in general there is no time-dependent vector field $X_t \in \Gamma^\infty(TS)$ with
\[
\frac{d}{dt} \Phi^B_t = X_t \circ \Phi^B_t.
\] Nevertheless, this situation of a time-dependent vector field is a particular case of an open time evolution as the next example shows.

**Example 3.1.4**

Let $X_t \in \Gamma^\infty(TS)$ be a smooth time-dependent vector field on $S$, and let $\overline{X} \in \Gamma^\infty(T(S \times \mathbb{R}))$ be the corresponding time-independent vector field
\[
\overline{X}(x_s, t) = \left( X_t(x_s), \frac{d}{dt} \right),
\]
where we use the splitting (3.3) of $T(S \times \mathbb{R})$ and the canonical constant vector field on the “bath” $B = \mathbb{R}$. For simplicity, we assume that $\overline{X}$ has a complete flow $\Psi_t$. Then the open time evolution for the initial condition $x_B = 0$ of the bath is
\[
\Phi^0_t(x_s) = \text{pr}_S(\Psi_t(x_s, 0)).
\]
But this is precisely the time evolution of the time-dependent vector field $X_t$, i.e. we have
\[
\frac{d}{dt} \Phi^0_t = X_t \circ \Phi^0_t,
\]
as an easy and well-known computation shows. Thus, the ordinary time evolution of a time-dependent vector field can be viewed as a particular case of an open time evolution in the sense of Definition 3.1.1 for a very particular bath.

In view of the yet to be found quantization of open dynamical systems, we now consider the effect of an open time evolution on the functions $C^\infty(S)$, as these will play the role of the observables later. Remembering that $\iota^*_B = \text{id} \hat{\otimes} \delta_{x_B}$, the following statement is obvious.

**Proposition 3.1.5**

Let $x_B \in B$. Then $(\Phi^B_t)^* : C^\infty(S) \to C^\infty(S)$ is a *-homomorphism for every $t \in \mathbb{R}$, and we have
\[
(\Phi^B_t)^* = (\text{id} \hat{\otimes} \delta_{x_B}) \circ \Psi^*_t \circ \text{pr}^*_S.
\] Here $\delta_{x_B} : C^\infty(S) \to \mathbb{C}$ denotes the $\delta$-functional at $x_B$, i.e. the evaluation of a function at the point $x_B$. Moreover, $\text{id} \hat{\otimes} \delta_{x_B}$ is the induced map
\[
\text{id} \hat{\otimes} \delta_{x_B} : C^\infty(S) \hat{\otimes} C^\infty(B) = C^\infty(S \times B) \to C^\infty(S),
\]
where $\hat{\otimes}$ denotes the completed projective tensor product. Note that the involved Fréchet spaces are nuclear anyway.

Though Proposition 3.1.5 is a trivial reformulation of the definition of $\Phi^B_t$, it gives a new point of view. While $\Phi^B_t$ and $\text{pr}^*_S$ are canonically given *-homomorphisms of the *-algebras of smooth functions and hence completely positive maps, the map $\text{id} \hat{\otimes} \delta_{x_B}$ can also be interpreted as a positive (and in fact completely positive) map which coincides with a *-homomorphism $\iota^*_B$ “by accident”. In particular, we can replace the positive functional $\delta_{x_B}$ by any, not necessarily pure, state $\omega_0$ of $C^\infty(B)$, that is, by a positive linear normalized functional $\omega_0 : C^\infty(B) \to \mathbb{C}$. This yields the following, more general definition of an open time evolution.
**Def. 3.1.6 (Open time evolution, general case)**
For any state $\omega_0 : C^\infty(B) \rightarrow C$ of the bath, the open time evolution of $S$ with respect to the total time evolution $\Psi_t$ and the state $\omega_0$ is given by

$$(\Phi^{\omega_0}_t)^* = (\text{id} \otimes \omega_0) \circ \Psi_t^* \circ \text{pr}_s^*.$$  (3.18)

**Rem. 3.1.7**
Any positive functional $\omega_0 : C^\infty(B) \rightarrow C$ is actually a positive Borel measure with compact support, see e.g. [25, App. B]. For continuous functions this is the famous Riesz Representation Theorem, see e.g. [73, Thm. 2.14], which can be shown to extend to the smooth setting. Therefore, any state $\omega_0 : C^\infty(B) \rightarrow C$ is automatically continuous with respect to the smooth topology. Thus, the map $\text{id} \otimes \omega_0$ extends to the completed tensor product making the above expression in (3.18) well-defined.

The notation $(\Phi^{\omega_0}_t)^*$ is of course only symbolic as there clearly is no longer an underlying map of manifolds. With this definition we have shifted the focus to the algebra of observables rather than the underlying geometry.

**Prop. 3.1.8**
For any state $\omega_0$ of the bath, the open time evolution $(\Phi^{\omega_0}_t)^* : C^\infty(S) \rightarrow C^\infty(S)$ is a completely positive map.

**Proof:** Since $\Psi_t^*$ and $\text{pr}_s^*$ are $^*$-homomorphisms, we only have to show that $\text{id} \otimes \omega_0$ is a completely positive map from $C^\infty(S \times B)$ to $C^\infty(S)$. Thus, let $F \in M_n(C^\infty(S \times B))$ be given, and let $x_0 \in S$. Then, we have the embedding $\iota_{x_0} : B \rightarrow S \times B$, wherefore

$$\delta_{x_0} \circ (\text{id} \otimes \omega_0) = \omega_0 \circ (\delta_{x_0} \otimes \text{id}) = \omega_0 \circ \iota_{x_0}^*.$$  (3.19)

Since $\iota_{x_0}^*$ is a $^*$-homomorphism, the composition $\omega_0 \circ \iota_{x_0}^*$ is still a positive functional and hence a completely positive map. Thus, applied to $F^* F$, we get a positive semi-definite matrix $\omega_0 \circ \iota_{x_0}^*(F^* F) = \omega_0 \circ (\text{id} \otimes \omega_0)(F^* F)$. Since this is true for every point $x_0 \in S$, we have a positive element $(\text{id} \otimes \omega_0)(F^* F) \in M_n(C^\infty(S))$ proving the claim. ■

**Rem. 3.1.9**
Since any positive functional $\omega_0 : C^\infty(B) \rightarrow C$ is actually a positive Borel measure with compact support, the map $\text{id} \otimes \omega_0$ indeed means to integrate over the bath degrees of freedom with respect to a measure specified by $\omega_0$.

**Rem. 3.1.10**
Note also that in the case of a $\delta$-functional instead of an arbitrary state $\omega_0$ the open time evolution actually is a $^*$-homomorphism, in contrast to the case of arbitrary states. However, in general, $(\Phi^{\omega_0}_t)^*$ is just a completely positive map without any further nice algebraic features.

While, up to now, we have considered arbitrary dynamical systems, we shall now concentrate on more specific ones: We assume to have a Hamiltonian dynamics on the total space of the system and the bath. In more detail, we choose the rather general setting of Poisson geometry to formulate Hamiltonian dynamics. This framework contains in particular any symplectic phase space such as coadjoint orbits, cotangent bundles, or Kähler manifolds. However, also the dual of a Lie algebra is a (linear) Poisson manifold which is important when dealing with symmetries, see e.g. [78, Chap. 3 & Chap. 4] for an introduction.

Thus, let the state space of the system $(S, \pi_s)$ and the one of the bath $(B, \pi_B)$ in addition be Poisson manifolds with Poisson structures $\pi_s$ and $\pi_B$. On the total system $S \times B$, we choose the product Poisson structure

$$\pi = \text{pr}_S^*\pi_s + \text{pr}_B^*\pi_B.$$  (3.20)
This means that for functions $f_s, g_S \in C^\infty(S)$ and $f_B, g_B \in C^\infty(B)$ the factorizing functions $f = f_s \otimes f_B$ and $g = g_S \otimes g_B$ have the Poisson bracket

$$\{f, g\} = \{f_s, g_S\}_B \otimes \{f_B, g_B\}_B.$$  

(3.21)

The dynamics of the total system is given by the Hamiltonian vector field $X_H \in \Gamma^\infty(T(S \times B))$ with respect to the total Hamiltonian $H \in C^\infty(S \times B)$. Recall that the Hamiltonian vector field is defined by $X_H = \{\cdot, H\}$. In typical situations, the total Hamiltonian contains three parts: We have the Hamiltonian $H \in C^\infty(S)$ of the system alone, the Hamiltonian $H_B \in C^\infty(B)$ of the bath alone, and an interaction Hamiltonian $H_I \in C^\infty(S \times B)$ such that the total Hamiltonian is

$$H = \text{pr}^*_S H_S + \text{pr}^*_B H_B + H_I.$$  

(3.22)

Then, the total Hamiltonian time evolution is the flow $\Psi_t : S \times B \longrightarrow S \times B$ of $X_H$ which we assume to be complete for simplicity, and, analogously to Definition 3.1.6, the classical open time evolution with respect to a given state of the bath is defined as follows.

**Definition 3.1.11 (Classical open time evolution, Hamiltonian case)**

The classical open time evolution of the system $S$ with respect to a total Hamiltonian time evolution $\Psi_t$ of $S \times B$ and a given state $\omega_0$ of the bath is given as the open time evolution

$$(\Phi^{\omega_0}_t)^* : C^\infty(S) \longrightarrow C^\infty(S)$$  

(3.23)

according to Definition 3.1.6.

**Remark 3.1.12**

Again, unless we have special circumstances, the open time evolution is just a completely positive map without any further algebraic features. In particular, there is no reason that $(\Phi^{\omega_0}_t)^*$ should preserve Poisson brackets, even for $\omega_0 = \delta_{\eta_B}$ being a pure state.

### 3.2 Deformation Quantization of Open Systems

In this section, we establish the deformation quantized version of the open time evolution with respect to a total Hamiltonian time evolution of a total system. To this end, we take a Hermitian formal star product $\star$ on a Poisson manifold $(M, \pi)$. The reason for choosing formal star products, where a priori no convergence in $\hbar$ is controlled, is that for this situation we have the powerful existence and classification theorems of deformation quantization at hand, see Section 2.1. Physically, of course, one would like to have convergence or at least some asymptotic statements. In many examples, as in Section 3.4, this is possible, but we shall not illuminate this rather technical issue here any further.

Now, let us come back to our original situation of a coupled total system $S \times B$. As there already is a nice separation of the total Poisson structure into the Poisson structure of the system and the one of the bath, we shall require the same feature also for the quantization. Thus, we assume to have Hermitian star products $\star_S$ on $S$ and $\star_B$ on $B$, respectively. This induces a Hermitian star product $\star = \star_S \otimes \star_B$ on $S \times B$ in such a way that

$$\begin{align*}
(C^\infty(S)[[\hbar]], \star_S) &\xrightarrow{\text{pr}^*_S} (C^\infty(S \times B)[[\hbar]], \star) \\
(C^\infty(B)[[\hbar]], \star_B) &\xleftarrow{\text{pr}^*_B} (C^\infty(S \times B)[[\hbar]], \star_B)
\end{align*}$$  

(3.24)

are both $\star$-homomorphisms of the involved star products. On factorizing functions, we have

$$f \star g = (f_s \star_S g_S) \otimes (f_B \star_B g_B),$$  

(3.25)

where $f = f_s \otimes f_B$ and $g = g_S \otimes g_B$ for $f_s, g_S \in C^\infty(S)[[\hbar]]$ and $f_B, g_B \in C^\infty(B)[[\hbar]]$. Clearly, (3.21) becomes the first order limit of (3.25) in the commutators.
Remark 3.2.1
It is crucial for the approach that the algebraic structure of the observables is a priori given and will stay untouched. The physical interpretation is that whatever the time evolution will be, the way how certain quantities, the observables, are measured is independent of any sort of dynamics but a purely kinematical property of the physical system. Thus, the star products $\ast$, $\ast_\delta$, and $\ast_\hbar$ will be given once and for all and will not be changed by the open time evolution. Note that this is not the only possibility to deal with open systems. In [35], the star product itself was modified in order to describe a damped harmonic oscillator.

A good definition of a quantized open time evolution in deformation quantization can now be given analogously to the classical case using the deformation quantized Hamiltonian time evolution of Section 2.2.

Definition 3.2.2 (Quantized open time evolution)
Let $H \in C^\omega(S \times B)[[\hbar]]$ be a Hamiltonian with complete time evolution $A_h$, and let $\omega : C^\omega(B)[[\hbar]] \to \mathbb{C}[[\hbar]]$ be a positive $\mathbb{C}[[\hbar]]$-linear functional. Then, the quantized open time evolution of $S$ with respect to $\omega$ is

$$A_h^\omega = (\text{id} \otimes \omega) \circ A_h \circ \text{pr}_S : C^\omega(S)[[\hbar]] \to C^\omega(S)[[\hbar]].$$

(3.26)

Remark 3.2.3
The above completed tensor product is understood order by order in $\hbar$. Thus, we have to require that $\omega = \sum_{r=0}^\infty \hbar^r \omega_r$ is continuous in each order of $\hbar$, i.e. each $\omega_r$ is a continuous linear functional with respect to the smooth topology. In view of Theorem 2.3.3 and Remark 2.3.6, this seems to be a very reasonable assumption.

Theorem 3.2.4
Any classical open time evolution can be deformation quantized into a quantized open time evolution. Conversely, the classical limit of any quantized open time evolution is a classical open time evolution for the classical limit of the quantum state.

Proof: Putting Theorem 2.3.3, Remark 2.3.6, the existence of Hermitian star products in [54], and the existence of the quantum time evolution of Equation (2.5) together, it is easy to see that any classical open time evolution can be deformation quantized into a quantized open time evolution. Conversely, by construction, the classical limit of any quantized open time evolution is a classical open time evolution for the classical limit of the quantum state, as $A_h^\omega = (\Phi_t^\omega)^* + O(\hbar)$. Here, $O(\hbar)$ denotes terms of order $\hbar$ or higher. ■

In view of Definition 3.2.2, it is tempting to believe that the quantized open time evolution $A_h^\omega$ is completely positive. Indeed, if we had used the algebraic tensor product in (3.26) instead of the completed one $\hat{\otimes}$ in every order of $\hbar$, then this would be a trivial statement: The algebraic tensor product of the completely positive maps $\text{id}$ and $\omega$ is again completely positive, and so is the composition with the completely positive homomorphisms $A_h$ and $\text{pr}_S$. However, the crucial point is that the Fréchet topology of the smooth functions and the $\hbar$-adic topology originating from the ring ordering are not very well compatible. In fact, it is not clear whether the completed tensor product is completely positive or not. Note that this is rather different from the $C^*$-algebraic case where the completed projective tensor product of completely positive maps is always completely positive. Considering these technical difficulties, the following principal result on the quantized open time evolution is non-trivial.

Theorem 3.2.5
Let $\omega$ be a positive $\mathbb{C}[[\hbar]]$-linear functional on $(C^\omega(B)[[\hbar]], \ast_\delta)$ of the form

$$\omega = \omega_0 \circ S$$

(3.27)

with $S$ preserving squares with respect to $\ast_\delta$. Then, any quantized open time evolution with respect to $\omega$ is completely positive.
Proof: As \( pr_i^* \) and \( A_i \) are \(^*\)-homomorphisms, the only thing left to show is that \( id \otimes \omega \) is completely positive. We extend \( S \) to matrices as usual. For \( F \in M_n(C^\infty(S \times B)[[\hbar]]) \), we have

\[
\tau^*_n \left( (id \otimes \delta S)(F^* \ast F) \right) = \sum_{r=0}^{\infty} \hbar^r \sum_t \left( \tau^*_n \left( D_{r,t}(F) \right) \right)^* \ast_s \left( \tau^*_n \left( D_{r,t}(F) \right) \right),
\]

since the restriction to \( x_0 \in B \) commutes with the pointwise products in Equation (2.9). Furthermore, let \( \mu : M_n(C^\infty(S)[[\hbar]]) \rightarrow \mathbb{C}[[\hbar]] \) be a positive \( \mathbb{C}[[\hbar]] \)-linear functional with respect to \( \ast_s \). Then for every \( x_0 \)

\[
\tau^*_n \left( (\mu \otimes \delta S)(F^* \ast F) \right) = \sum_{r=0}^{\infty} \hbar^r \sum_t \mu \left( \tau^*_n \left( D_{r,t}(F) \right) \right)^* \ast_s \left( \tau^*_n \left( D_{r,t}(F) \right) \right) \in \mathbb{C}[[\hbar]]
\]
is positive. So if \( \omega_0 \) is classically positive, we conclude that \( \omega_0 \circ (\mu \otimes \delta S)(F^* \ast F) \geq 0 \). But \( \omega_0 \circ (\mu \otimes \delta S) = \mu \circ (id \otimes (\omega_0 \otimes \delta S)) = \mu \circ (id \otimes \omega) \). Thus, \( \mu \circ (id \otimes \omega)(F^* \ast F) \) is positive for all positive functionals \( \mu \). This implies that \( (id \otimes \omega)(F^* \ast F) \) is a positive algebra element for all matrices \( F \), and hence \( id \otimes \omega \) is a completely positive map as claimed.

\[ \square \]

**Remark 3.2.6**

The assertion of Theorem 3.2.5 is actually true for more quantum states than the ones of type (3.27): We will see examples later on in Proposition 3.4.4. We also note that a possible failure of the complete positivity of \( A_i^\omega \) should be seen as an artifact of the rather fine (and not too physical) \( \hbar \)-adic topology of formal power series in \( \hbar \). One would expect reasonable behaviour as soon as one enters a convergent regime such as strict deformation quantization.

**Remark 3.2.7**

In general, the quantized open time evolution \( A_i^\omega \) is no \(^*\)-automorphism of \( (C^\infty(S)[[\hbar]], \ast_s) \). Furthermore, a close look at Equation (3.26) shows that usually \( A_i^\omega \circ A_i^\omega \neq A_{s+i}^\omega \), as expected from a microscopic system.

**Remark 3.2.8**

The question of the implementability of spin into the quantization scheme for open systems developed in this section can be answered positively. Using the notions of super manifolds and star products on super symplectic manifolds according to [14, 36], one can easily extend our formalism to this framework.

### 3.3 Linearly Coupled Harmonic Oscillators I: Generalities

In this section, we shall describe the deformation quantization of linearly coupled harmonic oscillators and the corresponding construction of the open time evolution for arbitrary states on the bath.

Consider the well-known linear coupling of two one-dimensional harmonic oscillators. We shall describe a one-dimensional harmonic oscillator as a Hamiltonian system \((M, \{ \cdot, \cdot \}_M, H)\), given by a phase space \( M = T^\ast \mathbb{R}^2_q \cong \mathbb{R}^2_{q,p} \) with Hamiltonian \( H(q, p) = \frac{1}{2m} p^2 + \frac{m \omega^2}{2} q^2 \), where \( m, \nu \in \mathbb{R}^+ \). The Poisson bracket is then determined by

\[
\{ q, p \} = 1, \quad \{ q, q \} = 0 = \{ p, p \}.
\]

Now, take as system and bath two one-dimensional harmonic oscillators, that is set \( S = M = B \). The Hamiltonian system \((S \times B, \{ \cdot, \cdot \}_M, H)\) describing the linearly coupled identical harmonic oscillators is then given by the smooth manifold \( S \times B = \mathbb{R}^2_{q_s, p_s, q_b, p_b} \times \mathbb{R}^2_{q_s, p_s, q_b, p_b} \) with the corresponding Poisson bracket as given by Equation (3.21). In the following, we shall use the same symbols \( q_s, p_s, q_b, p_b \) for the coordinate functions on \( S, B, \) and \( S \times B \), respectively, in order to simplify our notation. In the same spirit, we simply write \( H = H_s + H_b + H_i \) for the total Hamiltonian without the explicit use of \( p_s^i \) and \( p_b^i \). For the linearly coupled harmonic oscillators the interaction term is given by \( H_i = \frac{\kappa}{2} (q_s - q_b)^2 \), with \( \kappa \in \mathbb{R}^+ \) being the coupling constant.
Using the new and still global coordinate functions

\[ \tilde{q}_1 = \frac{1}{\sqrt{2}}(q_s + q_b), \quad \tilde{p}_1 = \frac{1}{\sqrt{2}}(p_s + p_b), \quad \tilde{q}_2 = \frac{1}{\sqrt{2}}(q_s - q_b), \quad \tilde{p}_2 = \frac{1}{\sqrt{2}}(p_s - p_b), \]  

(3.28)

we can bring the total Hamiltonian into normal form and get the well-known expression

\[ H = \frac{1}{2m} \left( \tilde{p}_1^2 + \tilde{p}_2^2 \right) + \frac{m v^2}{2} \tilde{q}_1^2 + \frac{m v^2}{2} \tilde{q}_2^2 \quad \text{with} \quad v^2 = \nu^2 + \frac{2\kappa}{m} \]  

(3.29)

The classical time evolution \( \Phi_t \) is known to be a linear map for all \( t \), which can be expressed in matrix form as

\[ \Phi_t = \frac{1}{2} \begin{pmatrix} \cos(vt) + \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} + \frac{\sin(v\nu t)}{m
\nu} & \cos(vt) - \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} - \frac{\sin(v\nu t)}{m
\nu} \\ -m(v \sin(vt) + v \sin(v\nu t)) & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) - v \sin(v\nu t)) & \cos(vt) - \cos(v\nu t) \\ \cos(vt) - \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} - \frac{\sin(v\nu t)}{m
\nu} & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) + v \sin(v\nu t)) \\ -m(v \sin(vt) - v \sin(v\nu t)) & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) + v \sin(v\nu t)) & \cos(vt) - \cos(v\nu t) \end{pmatrix} \]

(3.30)

with respect to the global linear coordinates \( q_s, p_s, q_b, p_b \). Thus, the open system time evolution \( \Phi_t^{\omega_0} \) of the open subsystem with regard to the state \( \omega_0 \) of the bath takes the form

\[ \left( \Phi_t^{\omega_0} \right)^*(q_s, p_s) = \frac{1}{2} \begin{pmatrix} \cos(vt) + \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} + \frac{\sin(v\nu t)}{m
\nu} & \cos(vt) - \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} - \frac{\sin(v\nu t)}{m
\nu} \\ -m(v \sin(vt) + v \sin(v\nu t)) & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) - v \sin(v\nu t)) & \cos(vt) - \cos(v\nu t) \\ \cos(vt) - \cos(v\nu t) & \frac{\sin(vt)}{m
\nu} - \frac{\sin(v\nu t)}{m
\nu} & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) + v \sin(v\nu t)) \\ -m(v \sin(vt) - v \sin(v\nu t)) & \cos(vt) + \cos(v\nu t) & -m(v \sin(vt) + v \sin(v\nu t)) & \cos(vt) - \cos(v\nu t) \end{pmatrix} \left( \begin{pmatrix} q_s \\ p_s \end{pmatrix} \right) \]

(3.31)

Analogously to the classical case, we shall use the normal coordinates (3.28) in order to simplify the computation of the quantum time evolution of the total system. Moreover, it will be advantageous to combine the real \( \tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \) and \( \tilde{p}_2 \) into complex coordinates, which will play the role of annihilation and creation “operators” later on. We set

\[ z_k = \sqrt{m v_k} \tilde{q}_k + i \sqrt{m v_k} \tilde{p}_k, \quad \text{and hence} \quad \tilde{q}_k = \frac{1}{2 \sqrt{m v_k}} (z_k + \bar{z}_k), \quad \tilde{p}_k = \frac{1}{2 \sqrt{m v_k}} (z_k - \bar{z}_k), \]

(3.32)

for \( k = 1, 2 \) and \( \nu_1 = \nu, \nu_2 = \nu_k \). Respect to these global coordinate functions on \( M \), the total Hamiltonian can be written as

\[ H = \frac{\nu^2}{16 v_k} (z_2 + \bar{z}_2)^2 - \frac{v_k}{16} (z_2 - \bar{z}_2)^2 + \frac{v^2}{8 \sqrt{v_k}} (z_1 + \bar{z}_1)(z_2 + \bar{z}_2) - \frac{\sqrt{v v_k}}{8} (z_1 - \bar{z}_1)(z_2 - \bar{z}_2). \]

(3.33)

On the other hand, in order to be able to apply a state to “trace out” bath degrees of freedom we will also need “factorizing” complex coordinates with respect to the original Darboux coordinates on the system \( S \) and the bath \( B \). Hence, we set

\[ z_s = \sqrt{m v_s} q_s + i \sqrt{m v_s} p_s, \quad z_b = \sqrt{m v_b} q_b + i \sqrt{m v_b} p_b. \]

(3.34)

In these coordinates, the Hamiltonians of the system and the bath are given by \( H_s = \frac{\nu}{2} z_s \bar{z}_s \) and \( H_b = \frac{\nu}{2} z_b \bar{z}_b \). The interaction term now reads as \( H = \frac{\nu v_k}{8 m v_s} (z_s + \bar{z}_s - z_b + \bar{z}_b)^2 \). Again, for the Poisson brackets one finds \( \{z_k, z_l\} = 0 = \{\bar{z}_k, \bar{z}_l\} \) and \( \{z_k, \bar{z}_l\} = \frac{1}{2} \delta_{k, l} \) for all \( k, l = s, b \).
After these preparations, we can specify the star product on the total algebra of observables. We take the Weyl-Moyal star product on the total system $S \times B$ defined by

$$f \star_{\text{Weyl}} g = \sum_{r=0}^{\infty} \sum_{l=0}^{r} \left( -\frac{i}{2} \right)^{r-l} \frac{1}{l! (r-l)!} \sum_{i_1, \ldots, i_l=1}^{2} \frac{\partial^r f}{\partial q_{i_1} \cdots \partial q_{i_l}} \frac{\partial^r g}{\partial p_{i_1} \cdots \partial p_{i_l}} \right)$$

for $f, g \in C^\infty(S \times B)[[\hbar]]$, see e.g. [6] and [78, Chap. 5].

**Remark 3.3.1**

The Weyl-Moyal star product on a flat symplectic phase space $\mathbb{R}^{2n}$ is uniquely determined by the requirement of invariance under the affine symplectic group. Under the usual quantization map into differential operators it corresponds to the total symmetrization, see e.g. [78, Chap. 5] for a detailed discussion. We also note that $\star_{\text{Weyl}} = \star_{\text{Weyl}} \otimes \star_{\text{Weyl}}$, as required by our general framework.

We then choose the Weyl-Moyal star product to be our Hermitian star product and set $\star = \star_{\text{Weyl}}$. While the Weyl-Moyal star product is the most natural one with respect to phase space symmetries, it has certain disadvantages: When dealing with harmonic oscillators, for technical reasons, it will be more convenient to employ a Wick star product. Such a Wick star product is no longer unique, but depends on the choice of a compatible linear complex structure on the phase space, which is nothing but the choice of a harmonic oscillator. Therefore, we will have different Wick star products adapted to the various harmonic oscillators at hand: either with or without the coupling. In detail, one passes from the Weyl-Moyal star product to a Wick star product by means of the equivalence transformation explicitly given by

$$S = \exp(\hbar \Delta) \quad \text{with} \quad \Delta = \sum_{k=1}^{2} \frac{\partial^2}{\partial z_k \partial \overline{z}_k}.$$  \hspace{1cm} (3.35)

Then this Wick star product $\star_{\text{Wick}}$ is defined by $f \star_{\text{Wick}} g = S^{-1}(f) \star_{\text{Weyl}} S^{-1}(g)$. Explicitly, $\star_{\text{Wick}}$ is given by

$$f \star_{\text{Wick}} g = \sum_{r=0}^{\infty} (2\hbar)^r \sum_{i_1, \ldots, i_l=1}^{2} \frac{\partial^r f}{\partial z_{i_1} \cdots \partial z_{i_l}} \frac{\partial^r g}{\partial \overline{z}_{i_1} \cdots \partial \overline{z}_{i_l}}$$

for $f, g \in C^\infty(\mathbb{R}^{4})[[\hbar]]$. Alternatively, we ignore the coupling term and use the complex coordinates $z_s, z_B$ for the system and $\overline{z}_s, \overline{z}_B$ for the bath. This gives the two equivalence transformations

$$S_s = \exp\left(\hbar \frac{\partial^2}{\partial z_s \partial \overline{z}_s}\right) \quad \text{and} \quad S_B = \exp\left(\hbar \frac{\partial^2}{\partial z_B \partial \overline{z}_B}\right),$$

acting on functions on S and B, respectively. Analogously to (3.36), we get Wick star products $\star_{\text{Wick}}^s$ and $\star_{\text{Wick}}^B$ for the system and the bath, respectively. Since we ignored the coupling terms in the definition of the latter two Wick star products, we have

$$S \neq S_s \otimes S_B \quad \text{and} \quad \star_{\text{Wick}} \neq \star_{\text{Wick}}^s \otimes \star_{\text{Wick}}^B.$$  \hspace{1cm} (3.37)

The total time evolution with respect to $\star$ and $H$ can actually be calculated in a much easier way than by solving the corresponding evolution equation (2.4). We first compute the time evolution with respect to the Wick star product $\star_{\text{Wick}}$, which turns out to be simple, and then transform the time evolved observables back using $S$.

The total time evolution $A_{\text{Wick}}$ with respect to the Wick star product is determined by

$$\frac{d}{dt} A_{\text{Wick}}\ f = \frac{i}{\hbar} [H, A_{\text{Wick}}\ f]_{\text{Wick}} = [A_{\text{Wick}}\ f, H]$$ \hspace{1cm} (3.39)
for $f \in C^\infty(S \times B)[h]$ due to the fact that $H = \frac{\nu}{2}z_1 + \frac{\nu\kappa}{2}z_2$. It immediately follows that the time evolution is just the \textit{classical} one, i.e. $A_t^{\text{Wick}} = \Phi_t^*$, and no higher order correction terms arise. But then it is clear that the time evolution with respect to $\star$ is given by conjugation with $S$, since $SH = H + c$ with a constant $c = \frac{\hbar^2 \nu}{2}$. Hence, we have

$$A_t = S^{-1} \circ \Phi_t^* \circ S. \quad (3.40)$$

As a consequence, we immediately obtain the following result for the open time evolution with respect to the Weyl-Moyal star product.

\textbf{Proposition 3.3.2}

The quantized open time evolution of the open subsystem with respect to the functional $\omega$ is given by

$$A_t^\omega = (\text{id} \otimes \omega) \circ S^{-1} \circ \Phi_t^* \circ S \circ \text{pr}_s^*. \quad (3.41)$$

\textbf{Remark 3.3.3}

The $\star$-automorphism $A_t$ obviously restricts to the polynomials $\text{Pol}(S \times B)[h]$. Thus, being only interested in polynomial observables leads to a convergent formulation of the deformed time evolution of the open harmonic oscillator if the quantized state $\omega$ used to reduce the total dynamics gives a finite order (or convergence) in $\hbar$ for every polynomial on the bath. This will be the case for the deformed $\delta$-functionals as well as for the KMS functionals in Section 3.4. Thence, here we recover the usual quantum mechanical formulation including the convergence in $\hbar$.

To further illustrate the above situation, we compute the open time evolution of some specific observables of the system still allowing for a general state $\omega$.

As a first step, we calculate the total quantum time evolutions of the total system for $q_s$ and $p_s$. To do so, we have to evaluate the chain of maps (3.41) applied to these observables. First, we note that

$$S^*q_s = q_s = S^{-1}q_s \quad \text{and} \quad S^*p_s = p_s = S^{-1}p_s. \quad (3.42)$$

Then, the classical time evolution is \textit{linear}, wherefore applying the transformation $S^{-1}$ again does not give additional terms. We conclude that

$$A_tq_s = \Phi_t^*q_s \quad \text{and} \quad A_tp_s = \Phi_t^*p_s. \quad (3.43)$$

For the Hamiltonian $H_s$ of the system, the calculation is slightly more complicated. First we note that applying $S$ yields an additional constant, namely

$$SH_s = H_s + \frac{\hbar}{4} \left( \nu + \nu_k - \frac{\kappa}{mv_x} \right). \quad (3.44)$$

The classical open time evolution of $H_s$ is quite complicated and can be computed most easily from $\Phi_t^*z_1 = \exp(-i\nu t)z_1$, $\Phi_t^*z_2 = \exp(-i\nu_k t)z_2$, and (3.32). The remarkable fact is that

$$\Delta\Phi_t^*H_s = \Delta H_s = \frac{1}{4} \left( \nu + \nu_k - \frac{\kappa}{mv_x} \right) \quad (3.45)$$

for all $t$. Thus, applying $S^{-1}$ to $\Phi_t^*H_s$ gives $\Phi_t^*H_s$ minus the same constant as we obtained in (3.44). We conclude that also for the Weyl star product

$$A_tH_s = \Phi_t^*H_s. \quad (3.46)$$

Replacing the complex coordinates and their (simple) time evolution by the original real coordinates, we get the explicit total classical and hence also quantum time evolutions for $q_s$, $p_s$, and $H_s$. 

Proposition 3.3.4
The explicit total quantum time evolutions for $q_s$, $p_s$, and $H_s$ are given by

$$A_s q_s = \Phi_t^* q_s = \frac{1}{2} (\cos(\nu t) + \cos(v_k t))q_s + \left(\frac{\sin(\nu t)}{2mv} + \frac{\sin(v_k t)}{2mv_k}\right)p_s$$

$$+ \frac{1}{2} (\cos(\nu t) - \cos(v_k t))q_b + \left(\frac{\sin(\nu t)}{2mv} - \frac{\sin(v_k t)}{2mv_k}\right)p_b, \quad (3.47)$$

$$A_s p_s = \Phi_t^* p_s = -\frac{m}{2} (\nu \sin(\nu t) + v_k \sin(v_k t))q_s + \frac{1}{2} (\cos(\nu t) + \cos(v_k t))p_s$$

$$- \frac{m}{2} (\nu \cos(\nu t) - v_k \cos(v_k t))q_b + \frac{1}{2} (\cos(\nu t) - \cos(v_k t))p_b, \quad (3.48)$$

and

$$A_s H_s = \Phi_t^* H_s = \left(\frac{m}{8} (\nu \sin(\nu t) + v_k \sin(v_k t))^2 + \frac{mv^2}{8} (\cos(\nu t) + \cos(v_k t))^2\right)q_s^2$$

$$+ \left(\frac{1}{8m} (\cos(\nu t) + \cos(v_k t))^2 + \frac{mv^2}{8} \left(\frac{\sin(\nu t)}{mv} + \frac{\sin(v_k t)}{mv_k}\right)^2\right)p_s^2$$

$$+ \left(\frac{m}{8} (\nu \sin(\nu t) - v_k \sin(v_k t))^2 + \frac{mv^2}{8} (\cos(\nu t) - \cos(v_k t))^2\right)q_b^2$$

$$+ \left(\frac{1}{8m} (\cos(\nu t) - \cos(v_k t))^2 + \frac{mv^2}{8} \left(\frac{\sin(\nu t)}{mv} - \frac{\sin(v_k t)}{mv_k}\right)^2\right)p_b^2$$

$$+ \left(-\frac{1}{4} (\nu \sin(\nu t) + v_k \sin(v_k t)) (\nu \sin(\nu t) + v_k \sin(v_k t)) (\cos(\nu t) + \cos(v_k t))\right)q_s p_s$$

$$+ \left(\frac{m}{4} (\nu \sin(\nu t) + v_k \sin(v_k t))(\nu \sin(\nu t) - v_k \sin(v_k t))\right)q_s p_b$$

$$+ \left(\frac{mv^2}{4} (\cos(\nu t) + \cos(v_k t))(\cos(\nu t) - \cos(v_k t))\right)q_s q_b$$

$$+ \left(-\frac{1}{4} (\cos(\nu t) - \cos(v_k t)) (\sin(\nu t) + \sin(v_k t))^2\right)q_s p_b$$

$$+ \left(\frac{mv^2}{4} (\cos(\nu t) + \cos(v_k t))(\sin(\nu t) + \sin(v_k t))\right)q_s q_b$$

$$+ \left(-\frac{1}{4} (\cos(\nu t) + \cos(v_k t))(\nu \sin(\nu t) - v_k \sin(v_k t))\right)q_s q_b$$

$$+ \left(\frac{mv^2}{4} (\cos(\nu t) - \cos(v_k t)) (\sin(\nu t) + \sin(v_k t))^2\right)p_s q_b$$

$$+ \left(\frac{1}{4m} (\cos(\nu t) + \cos(v_k t))(\cos(\nu t) - \cos(v_k t))\right)q_s p_b$$

$$+ \left(\frac{mv^2}{4} (\sin(\nu t) + \sin(v_k t)) (\sin(\nu t) - \sin(v_k t))\right)p_s p_b$$

$$+ \left(-\frac{1}{4} (\nu \sin(\nu t) - v_k \sin(v_k t))(\cos(\nu t) - \cos(v_k t))\right)q_s p_b$$

$$+ \left(\frac{mv^2}{4} (\cos(\nu t) - \cos(v_k t)) (\sin(\nu t) - \sin(v_k t))\right)q_s p_b, \quad (3.49)$$
Proof: The result is an application of Equation (3.49). For \( q_s \) and \( p_s \) this is an easy calculation. For \( H_s \), the calculation is equally simple but tedious. First, we use the transformation to normal coordinates (3.29), then to complex coordinates (3.32). Second, apply the equivalence transformation (3.44), the time evolution in complex coordinates, and the inverse equivalence transformation, the first order of which is given in (3.45). Finally, transform back to normal coordinates and from there to the original Darboux coordinates.

The reason for transforming the time evolved observables back to the Darboux coordinate functions \( q_s, p_s, q_b, \) and \( p_b \) is not just an addiction to extensive exercise. It is in these variables where we can apply the final map \( \omega \) needed for the open time evolution, where \( \omega \) is a state of the bath with respect to \( \star_{\text{Weyl}} \). The procedure is very simple: We will have to replace all bath variables by their expectation values with respect to \( \omega \), i.e. \( q_b \) is to be replaced by \( \omega(q_b) \), \( q_b p_b \) is replaced by \( \omega(q_b)p_b \), etc. We will not write down the explicit formulas as these are now obtained from (3.47), (3.48), and (3.49) just by copying.

Remark 3.3.5
Note that for these observables, the open time evolutions in the classical and quantum regime only differ by the (possibly) different expectation values with respect to \( \omega \) and its classical limit \( \omega_0 \). In general, we have to expect additional quantum corrections from the total time evolution as well.

Remark 3.3.6
As a consistency check, we note that for \( \kappa = 0 \) the time evolutions of \( q_s \), \( p_s \), and \( H_s \) in Proposition 3.3.4 become independent of the bath coordinate functions and give the expected Hamiltonian time evolution:

\[
A_t q_s|_{t=0} = \cos(\nu t)q_s + \frac{1}{mv} \sin(\nu t)p_s,
\]

\[
A_t p_s|_{t=0} = -mv \sin(\nu t)q_s + \cos(\nu t)p_s,
\]

and

\[
A_t H_s|_{t=0} = \frac{mv^2}{4} \left( \sin(\nu t)^2 + \cos(\nu t)^2 \right) q_s^2 + \frac{1}{4m} \left( \cos(\nu t)^2 + \sin(\nu t)^2 \right) p_s^2 = \frac{1}{4m} p_s^2 + \frac{mv^2}{4} q_s^2.
\]

3.4 Linearly Coupled Harmonic Oscillators II: Examples

Finally, we give the open time evolution for two popular examples of states, the deformed \( \delta \)-functional, classically corresponding to fixed initial values, and the KMS state corresponding to a thermal equilibrium state of the bath.

3.4.1 The Deformed \( \delta \)-Functional

The first example of a state for the bath is a deformation of the \( \delta \)-functional. Thus, fix a point \((q_{00}, p_{00})\) in the bath and consider \( \delta(q_{00}, p_{00}) \). For the Weyl-Moyal star product this will no longer be a positive functional, see e.g. [78, Sect. 7.1.3]. However, for the Wick star product \( \star_{\text{Wick}} \) on the bath, the \( \delta \)-functional will be positive without corrections, which can easily be seen from Equation (3.36). Thus, using the equivalence transformation \( S_B \), we obtain a positive functional \( \delta(q_{00}, p_{00}) \circ S_B \) with respect to the Weyl-Moyal star product, because the equivalence transformation \( S_B \) is precisely a map preserving squares with respect to the Weyl-Moyal star product, which is evident from the explicit formula for \( \star_{\text{Wick}} \). In fact, this was the first example of a map preserving squares which is also massively used in the proofs in [29]. More physically speaking, \( \delta(q_{00}, p_{00}) \circ S_B \) corresponds to a coherent state localized around the point \((q_{00}, p_{00})\), see [9] for further details.

For this particular state, we note that for the observables which are at most linear in \( q \) and \( p \) the operator \( S_B \) does not have a non-trivial effect. Moreover, for the quadratic terms \( \dot{q}_s \), \( p_s^2 \), and \( q_b p_b \) the operator \( S_B \) only gives a constant correction term in first order of \( \hbar \). Explicitly, we obtain

\[
S_B q_b = q_B, \quad S_B p_b = p_B, \quad S_B(q_b p_b) = q_b p_b,
\]

(3.50)
3.4. Linearly Coupled Harmonic Oscillators II: Examples

\[ S_B q_B^2 = q_B^2 + \hbar \frac{1}{2m\nu}, \quad \text{and} \quad S_B p_B^2 = p_B^2 + \hbar \frac{mv}{2}. \]  

(3.51)

From these computations, we see by Equations (3.47), (3.48), and (3.49) that the open time evolutions with respect to \( \delta(q_{0B},p_{0B}) \circ S_B \) are given by

\[ A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B q_B = \left( \Phi_{\delta q_{0B},p_{0B}} \right)^* q_B, \]  

(3.52)

\[ A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B p_B = \left( \Phi_{\delta q_{0B},p_{0B}} \right)^* p_B, \]  

(3.53)

and

\[ A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B H_S = \left( \Phi_{\delta q_{0B},p_{0B}} \right)^* H_S + \frac{\hbar}{16} \left( \nu \sin(vt) - \nu_k \sin(v_k t) \right)^2 + \nu \sin(vt) - \nu_k \sin(v_k t)^2 + 2\nu \cos(vt) - \cos(v_k t). \]  

(3.54)

Remark 3.4.1

The classical open time evolutions of \( q_S, p_S, \) and \( H_S \) in (3.52), (3.53), and (3.54) are obtained by replacing the functions \( q_B, p_B, \) and their powers in the Equations (3.47), (3.48), and (3.49) by their values at \( q_{0B} \) and \( p_{0B}. \)

Remark 3.4.2

Even in this algebraically simplest case of a deformed delta functional, several interesting features of the construction can be seen:

i.) It is easy to see that in Equation (3.54), \( A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B \) is not a \(^*\)-homomorphism, just as expected from our general considerations in Remark 3.2.7.

ii.) The deformation of the \( \delta \)-functional, necessary in order to ensure complete positivity, leads to non-classical components of the open time evolution.

iii.) For \( \hbar \to 0 \), we have \( A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B H_S = \left( \Phi_{\delta q_{0B},p_{0B}} \right)^* H_S \), as expected.

iv.) If \( \kappa = 0 \), that is if system and bath decouple, we have \( \nu_k = \nu \) and hence the non-classical component of \( A_{\delta q_{0B},p_{0B}}^{\delta q_{0B},p_{0B}} S_B H_S \) vanishes.

3.4.2 Formal KMS States

Next, we study quantized states fulfilling a formal KMS condition, corresponding to “thermal equilibrium states” of the bath.

Originally, the KMS condition appeared in the context of thermal Green’s functions in the papers of Kubo [55] and Martin and Schwinger [60]. It was cast into the language of \( C^* \)-algebras by Haag, Hugenholtz, and Winnink [49]. A precise definition is given in the the books by Bratteli and Robinson [18], Haag [48], or Connes [30]. The KMS condition was finally translated to the formalism of star product algebras by Bordemann, Römer, and Waldmann [16].

The example of the KMS state for the coupled harmonic oscillators is of particular interest, because all formal series involved actually end at finite orders of \( \hbar \). Therefore, it leads to a convergent model in terms of deformation quantization. Furthermore, we reach the same results as in the usual quantum mechanics of a harmonic oscillator in a thermal state and even obtain \( 2\pi \hbar \)-times the usual partition function as the normalization factor of the thermal state, avoiding functional-analytic notions like the spectrum.
We first recall that for every symplectic star product \( \star \) for \( C^\infty(M)[[\hbar]] \) there is a unique trace functional
\[
\text{tr} : C^\infty_0(M)[[\hbar]] \rightarrow \mathbb{C}[[\hbar]],
\]
i.e. \( \text{tr}(f \star g) = \text{tr}(g \star f) \). Choosing the normalization of \( \text{tr} \) appropriately one obtains a \textit{positive} trace, see e.g. [78, Sect. 6.3.5] for a detailed discussion and references. For the Weyl-Moyal star product, the trace is known to be
\[
\text{tr}(f) = \int_{\mathbb{R}^{2n}} f(x) \, d^{2n}x,
\]
i.e. the integration with respect to the Liouville volume. In fact, it can be shown that in the symplectic case the lowest order of \( \text{tr} \) is necessarily of this form: It is just the integration over the whole manifold with respect to the Liouville volume.

The second ingredient we need is the \( \star \)-exponential \( \exp \), as introduced in [6]. Instead of defining the exponential function by means of the series, the following approach, favoured in [16], see also [78, Sect. 6.3.1], will be used. For \( H \in C^\infty(M)[[\hbar]] \) one defines \( \exp(\beta H) \in C^\infty(M)[[\hbar]] \) to be the unique solution of the differential equation
\[
\frac{d}{d\beta} \exp(\beta H) = H \star \exp(\beta H)
\]
with the initial condition \( \exp(0) = 1 \). The existence and the uniqueness of the \( \star \)-exponential \( \exp \) follow from a version of Banach’s fixed-point theorem for the \( \hbar \)-adic topology. From the uniqueness, we also get the usual properties as \( H \star \exp(tH) = \exp(tH) \star H, \exp((t+s)H) = \exp(tH) \star \exp(sH) \), and \( \overline{\exp(tH)} = \exp(t\overline{H}) \) for all \( H \in C^\infty(M)[[\hbar]] \) and for all \( s, t \in \mathbb{R} \). The classical limit of \( \exp(H) \) is the ordinary exponential \( \exp(H_0) \) for \( H = H_0 + \sum_{n=1}^{\infty} \hbar^n H_r \).

The KMS condition for the inverse temperature \( \beta \) and the Hamiltonian \( H \) for a \( \mathbb{C}[[\hbar]] \)-linear functional, as formulated in [4, 5] in the context of deformation quantization, leads to the following result: Up to normalization, the KMS functional is uniquely determined and explicitly given by
\[
\mu_{\text{KMS}}(f) = \text{tr}(\exp(-\beta H) \star f) \quad \text{for} \quad f \in C^\infty_0(M)[[\hbar]],
\]
see [16] for the proof and [78, Sect. 7.1.4] for more details on KMS functionals. In particular, we note that (3.58) is a \textit{positive} functional, as, by the properties of \( \exp \) and the cyclicity of \( \text{tr} \),
\[
\mu_{\text{KMS}}(\overline{f} \star f) = \text{tr} \left( \exp \left( -\frac{\beta}{2} H \right) \overline{f} \star f \star \exp \left( -\frac{\beta}{2} H \right) \right) \geq 0,
\]
and its classical limit is a classical KMS functional.

\textbf{Remark 3.4.3}

Depending on the Hamiltonian \( H, \mu_{\text{KMS}} \) may or may not be normalizable. Whenever the Hamiltonian used permits a normalization by rendering the integrations in \( \mu_{\text{KMS}}(1) \) well-defined, \( \frac{1}{\mu_{\text{KMS}}(1)} \mu_{\text{KMS}} \) is denoted by \( \omega_{\text{KMS}} \) and called a \textit{KMS state}.

Before entering our particular example again, we note that in the symplectic case the open quantum time evolution with respect to a KMS functional is necessarily completely positive. This will follow at once from the following proposition.

\textbf{Proposition 3.4.4}

\textit{Let the system} \( S \) \textit{be an arbitrary Poisson manifold and let the bath} \( B \) \textit{be symplectic. Given the KMS functional} \( \mu_{\text{KMS}} \) \textit{with respect to an arbitrary} \( H_0 \in C^\infty(B)[[\hbar]] \) \textit{and inverse temperature} \( \beta > 0 \), \textit{the map} \( \text{id} \otimes \mu_{\text{KMS}} : (C^\infty_0(S \times B)[[\hbar]], \star) \rightarrow (C^\infty_0(S)[[\hbar]], \star) \) \textit{is completely positive}.
Proof: We choose a square preserving map $S$ for the bath whose existence is guaranteed by Theorem 2.3.3. Given a positive $\mathbb{C}[[\hbar]]$-linear functional $\mu : M_n(C^0(\mathbb{S})[[\hbar]]) \to \mathbb{C}[[\hbar]]$, we know from the proof of Theorem 3.2.5, that for every such $\mu$ the combined map

$$
\mu \bar{\otimes} S : (M_n(C^0(\mathbb{S} \times \mathbb{B})[[\hbar]]), \star) \to (C^0(\mathbb{B})[[\hbar]], \cdot)
$$

is positive. It follows that for $F \in M_n(C^0(\mathbb{S} \times \mathbb{B})[[\hbar]])$ the function $(\mu \bar{\otimes} S)(F^* \star F)$ is at every point $x_0 \in \mathbb{B}$ either a formal series with positive lowest order term or zero. To avoid trivialities, assume that $(\mu \bar{\otimes} S)(F^* \star F)$ is not identical to zero. Let $r_0$ be the minimal exponent with $(\mu \bar{\otimes} S)(F^* \star F) = \hbar^r a_r + \cdots$ and $a_0 \geq 0$ not identical to zero. By continuity, there is an open subset $U \subseteq \mathbb{B}$ with $a_0(x_0) > 0$ for $x_0 \in U$. But this implies that $(\mu_{\text{KMS}} \circ S^{-1}) \circ (\mu \bar{\otimes} S)(F^* \star F) = \hbar^r b_r + \cdots$ with $b_r > 0$ since the zeroth order of $S$ is the identity and the zeroth order of $\mu_{\text{KMS}}$ is the integration over all of $\mathbb{B}$. Since $\mu$ is arbitrary and using

$$(\mu_{\text{KMS}} \circ S^{-1}) \circ (\mu \bar{\otimes} S) = \mu \circ (\text{id} \bar{\otimes} \mu_{\text{KMS}}),$$

this shows that $(\text{id} \bar{\otimes} \mu_{\text{KMS}})(F^* \star F)$ is a positive algebra element in $M_n(C^0(\mathbb{S})[[\hbar]])$ with respect to $\star$. Back to our specific example, we consider the harmonic oscillator as the Hamiltonian $H_\hbar \in C^0(\mathbb{R}^2)$ and the Weyl-Moyal star product $\star$, as before. In this case, the star exponential of $H_\hbar$ has been computed explicitly by [6]. One has

$$
\exp(-\beta H_\hbar) = \frac{1}{\cosh\left(\frac{\hbar \beta \nu}{2}\right)} \exp\left(-\frac{2H_\hbar}{\hbar \nu} \tanh\left(\frac{\hbar \beta \nu}{2}\right)\right)
$$

(3.59)

for $\beta > 0$ and $\nu > 0$, which is a well-defined formal power series in $\hbar$. Note that in [6] the exponential $\exp(\frac{\hbar}{\nu} H)$ requires a convergent setting due to the $\hbar$ in the denominator. In our case, the situation is much simpler. In fact, differentiating (3.59) with respect to $\beta$ gives the defining differential equation (3.57) right away. For this, we first note that

$$
\frac{1}{\cosh\left(\frac{\hbar \beta \nu}{2}\right)} \exp\left(-\frac{2H_\hbar}{\hbar \nu} \tanh\left(\frac{\hbar \beta \nu}{2}\right)\right)\bigg|_{\beta=0} = 1.
$$

Then, after setting $A := \frac{1}{\cosh\left(\frac{\beta \nu}{2}\right)} \exp\left(-\frac{2H_\hbar}{\hbar \nu} \tanh\left(\frac{\hbar \beta \nu}{2}\right)\right)$ for convenience, it is a simple, explicit, and straightforward calculation to see that

$$
\frac{d}{d\beta} A = \left(H_\hbar - H_\hbar \tanh\left(\frac{\hbar \beta \nu}{2}\right) - \frac{\hbar \nu}{2} \tanh\left(\frac{\hbar \beta \nu}{2}\right)^2\right) A = -H_\hbar \star A.
$$

As in the textbooks on statistical mechanics, we can now calculate the partition function $Z$ as the normalization factor of the KMS state on the bath by formally calculating Gaussian integrals.

**Proposition 3.4.5**

*The normalization factor $\mu_{\text{KMS}}(1)$ is explicitly given by*

$$
\mu_{\text{KMS}}(1) = 2\pi \hbar \exp\left(-\frac{\hbar \beta \nu}{2}\right) \in \mathbb{R}[[\hbar]].
$$

(3.60)

The partition function is the formal Laurent series

$$
Z = \frac{\exp\left(-\frac{\hbar \beta \nu}{2}\right)}{1 - \exp(-\hbar \beta \nu)} \in \mathbb{R}(\hbar).
$$

(3.61)
The crucial point is that \( \mu_{\text{KMS}}(1) \) has a well-defined classical limit while \( Z \) has a simple pole at \( \hbar = 0 \). Therefore, we can use this normalization factor to obtain the well-defined KMS state

\[
\omega_{\text{KMS}}(f) = \frac{1}{2\pi\hbar Z} \int \exp(-\beta H_0) \ast_f f \, dq_0 \, dp_0
\]

for \( f \in C^\infty(B)[\hbar] \) such that the integral (3.62) is convergent order by order in \( \hbar \). Note that, as \( 2\pi\hbar Z \) begins with a nonzero constant in zeroth order of \( \hbar \), the inverse of \( 2\pi\hbar Z \) is again a well-defined formal power series.

Just as in the case of the \( \delta \)-functional, we shall now compute the quantum open time evolution of the observables \( q_s, p_s, \) and \( H_0 \) with respect to the KMS state \( \omega_{\text{KMS}} \). To this end, we need the expectation values of \( q_0, p_0, q_0^2, p_0^2 \), and \( q_0 p_0 \) in order to evaluate (3.47), (3.48), and (3.49).

**Lemma 3.4.6**

The expectation values of \( q_0, p_0, q_0 p_0, q_0^2, \) and \( p_0^2 \) in the KMS state \( \omega_{\text{KMS}} \) are given by

\[
\omega_{\text{KMS}}(q_0) = \omega_{\text{KMS}}(p_0) = \omega_{\text{KMS}}(q_0 p_0) = 0, \\
\omega_{\text{KMS}}(q_0^2) = \frac{\hbar}{2m} \coth \left( \frac{\hbar \nu}{2} \right), \quad \text{and} \quad \omega_{\text{KMS}}(p_0^2) = \frac{mv}{2} \coth \left( \frac{\hbar \nu}{2} \right).
\]

viewed as formal power series in \( \mathbb{C}[[\hbar]] \), which are convergent for all \( \hbar \).

**Proof:** The results are textbook knowledge. Nevertheless, we sketch the computation in order to illustrate the star product formalism used. The first observation is that the trace functional \( \text{tr} \) for the Weyl-Moyal star product has, by a simple integration by parts, the remarkable feature

\[
\text{tr}(f \ast g) = \text{tr}(fg),
\]

see e.g. [78, Ex. 6.3.33] for instructions. Strictly speaking, one of the functions has to have compact support. However, if one is the Gaussian \( \exp(-\beta H_0) \), the rapid decay allows to perform the integrations by parts also for the other observable being polynomial. Therefore, we can use this feature to simplify \( \omega_{\text{KMS}}(f) \) considerably for the above observables. Since \( \exp(-\beta H_0) \) is just a Gaussian, we are left with the well-known computation of some Gaussian integrals.

That the results of Lemma 3.4.6 really are analogous to the usual quantum mechanical results can easily be seen by considering the expectation value of \( H_0 \), which can be computed from Equation (3.64) and is given by

\[
\omega_{\text{KMS}}(H_0) = \frac{1}{2} \hbar \nu \coth \left( \frac{\hbar \nu}{2} \right) = \frac{\hbar \nu}{2} \coth \left( \frac{\hbar \nu}{2} \right).
\]

Using the expectation values (3.63) and (3.64), we can apply the general formulas (3.47), (3.48), and (3.49) and substitute the observables \( q_0, p_0, q_0 p_0, q_0^2, \) and \( p_0^2 \) by their expectation values with respect to \( \omega_{\text{KMS}} \) there. This then gives the open time evolutions of \( q_s, p_s, \) and \( H_s \). Remarkably, many terms disappear, thanks to the vanishing of some terms in (3.63). In detail, we have

\[
\mathcal{A}_{t}^{\omega_{\text{KMS}}} q_s = \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_s t) \right) q_s + \frac{1}{2} \left( \sin(\nu t) + \sin(\nu_s t) \right) p_s, \\
\mathcal{A}_{t}^{\omega_{\text{KMS}}} p_s = -\frac{m}{2} \left( \nu \sin(\nu t) + \nu_s \sin(\nu_s t) \right) q_s + \frac{1}{2} \left( \cos(\nu t) + \cos(\nu_s t) \right) p_s, \\
\mathcal{A}_{t}^{\omega_{\text{KMS}}} H_s = \left( \frac{m}{8} \left( \nu \sin(\nu t) + \nu_s \sin(\nu_s t) \right)^2 + \frac{mv^2}{8} \left( \cos(\nu t) + \cos(\nu_s t) \right) \right) q_s^2 \\
+ \left( \frac{1}{8m} \left( \cos(\nu t) + \cos(\nu_s t) \right)^2 + \frac{mv^2}{8} \left( \sin(\nu t) + \sin(\nu_s t) \right)^2 \right) p_s^2.
\]
and leads to a purely classical open time evolution of \( H \), which is a simple check using the definition of \( \coth \). The classical limits of the observables \( q_k \) as expected. For the classical limit of the quantum open time evolution of \( H \), let us take a closer look at the low and high temperature limits and the classical limit. Let \( \hbar \geq 0 \) and \( \beta \geq 0 \). Then

\[
\begin{align*}
\hbar \coth \left( \frac{\hbar \beta v}{2} \right) & \xrightarrow{\beta \to 0} \infty, \\
\hbar \coth \left( \frac{\hbar \beta v}{2} \right) & \xrightarrow{\beta \to \infty} 0, \\
\hbar \coth \left( \frac{\hbar \beta v}{2} \right) & \xrightarrow{\hbar \to 0} \frac{2}{\beta v},
\end{align*}
\]

which is a simple check using the definition of \( \coth \). The classical limits of the observables \( q_k^2, p_n^2 \), and \( H_b \) in the state \( \omega_{KMS} \) are given by

\[
\begin{align*}
\omega_{KMS}(q_k^2) & \xrightarrow{\hbar \to 0} \frac{1}{m \beta v^2}, \\
\omega_{KMS}(p_n^2) & \xrightarrow{\hbar \to 0} \frac{m}{\beta}, \\
\omega_{KMS}(H_b) & \xrightarrow{\hbar \to 0} \frac{1}{\beta},
\end{align*}
\]

as expected. For the classical limit of the quantum open time evolution of \( H_b \) we get

\[
\begin{align*}
A_t^{KMS} H_b & \xrightarrow{\hbar \to 0} \left( \frac{m}{8} (v \sin(vt) + v_k \sin(v_k t))^2 + \frac{mv^2}{8} (\cos(vt) + \cos(v_k t))^2 \right) q_k^2 \\
& + \left( \frac{1}{8m} (\cos(vt) + \cos(v_k t))^2 + \frac{mv^2}{8} \left( \frac{\sin(vt)}{m v} + \frac{\sin(v_k t)}{mv_k} \right) \right) p_k^2 \\
& + \left( \frac{1}{4} (v \sin(vt) + v_k \sin(v_k t))(\cos(vt) + \cos(v_k t)) \right) q_k p_k \\
& + \left( \frac{1}{4} (v \sin(vt) - v_k \sin(v_k t))^2 + 2v(\cos(vt) - \cos(v_k t))^2 \right) \\
& + \frac{v(\sin(vt) - \frac{v}{v_k} \sin(v_k t))^2}{8 \beta v},
\end{align*}
\]

The low temperature limit \( \beta \to \infty \) follows from Equations (3.67) and (3.68) by

\[
\begin{align*}
A_t^{KMS} H_b & \xrightarrow{\beta \to \infty} \left( \frac{m}{8} (v \sin(vt) + v_k \sin(v_k t))^2 + \frac{mv^2}{8} (\cos(vt) + \cos(v_k t))^2 \right) q_k^2 \\
& + \left( \frac{1}{8m} (\cos(vt) + \cos(v_k t))^2 + \frac{mv^2}{8} \left( \frac{\sin(vt)}{m v} + \frac{\sin(v_k t)}{mv_k} \right) \right) p_k^2 \\
& + \left( \frac{1}{4} (v \sin(vt) + v_k \sin(v_k t))(\cos(vt) + \cos(v_k t)) \right) q_k p_k \\
& + \left( \frac{1}{4} (v \sin(vt) - v_k \sin(v_k t))^2 + 2v(\cos(vt) - \cos(v_k t))^2 \right) \frac{1}{8 \beta v}
\end{align*}
\]

and leads to a purely classical open time evolution of \( H_b \), “freezing” any quantum activity.

**Remark 3.4.7**

i.) For the high temperature limit \( \beta \to 0 \) the time evolution (3.67) diverges by Equations (3.68).
ii.) The decoupling $\kappa = 0$ has the same effect as for the deformed $\delta$-functional, the open time evolution becomes the Hamiltonian time evolution of the system.

We see that for the two common and established examples of states treated in this section, the deformation quantization of open systems leads to established and expected results.

**Remark 3.4.8**
As already mentioned at the beginning of Subsection 3.4.2, we could demonstrate some advantages of the deformation quantizational framework for a harmonic oscillator in a thermal state. The computations were rather simple, the concept of spectrum was not needed, and, of course, the results coincide with the results of the usual quantum mechanical setting.
Chapter 4

Representations, Categories, and Functors Relating to Completely Positive Maps

Stinespring [75] and Kasparov [52] have given interesting generalizations of the GNS construction for \( C^* \)-algebras over \( \mathbb{C} \), relating completely positive maps and representations. The content of this chapter is mostly based on an adaption of their results to unital \( * \)-algebras over quadratic extensions of ordered rings.

The Stinespring-Kasparov representation implied by this construction and the construction itself allow for several representational and categorial constructions which are of independent interest, but which also lead to a connection between the question of existence of completely positive maps and the question of strong Morita equivalence.

In contrast to \( C^* \)-algebras, for general \( * \)-algebras, we have to assume the completely positive maps which we want to represent by the Stinespring-Kasparov construction to be \( * \)-maps. For \( C^* \)-algebras, this assumption is unnecessary, as for them any completely positive map is a \( * \)-map. Also, this assumption is not a restriction when considering the completely positive time evolution maps of Chapter 3. Given the \( \delta \)-functionals, the algebras of smooth complex-valued functions on a smooth manifold have sufficiently many positive linear functionals. Furthermore, the time evolution maps of classical open systems are unital, whence we know by Lemma 1.3.3 that they are \( * \)-maps. By [23, Prop. 4.2], we know that if a \( * \)-algebra has sufficiently many positive linear functionals, then every positive deformation of it has sufficiently many positive linear functionals. Thence, any Hermitian star product algebra has sufficiently many positive linear functionals, whereby the unital time evolution maps of quantum open systems are \( * \)-maps.

In this chapter, we will strongly use the properties of \( * \)-algebras as given in Chapter 1. Again, all algebras are assumed to be unital \( * \)-algebras over \( \mathbb{C} \).

4.1 Stinespring-Kasparov Representations of Completely Positive Maps

Let us give a more or less straightforward adaption of some generalizations of the GNS construction given by Stinespring [75] and Kasparov [52] regarding representations of \( C^* \)-algebras in the adjointable maps on Hilbert modules induced by completely positive maps from \( C^* \)-algebras into adjointable maps on different Hilbert modules, and of some further properties which can, for example, be found in [67].

Definition 4.1.1 (Stinespring-Kasparov representation)

Let \( \mathcal{A}, \mathcal{B} \) be \( * \)-algebras over \( \mathbb{C} \), let \( (\mathcal{E}_B, \langle \cdot, \cdot \rangle^\mathcal{E}_B) \) be a pre-Hilbert module over \( \mathcal{B} \), let \( \rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{E}_B) \) be a completely positive \( * \)-map. A \( * \)-representation \( \pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{F}_B) \) of \( \mathcal{A} \) on a pre-Hilbert module \( (\mathcal{F}_B, \langle \cdot, \cdot \rangle^\mathcal{F}_B) \) over \( \mathcal{B} \) such that \( \rho \) is of the form \( \rho(a) = V^* \pi(a) V \) for all \( a \in \mathcal{A} \), where \( V \in \mathcal{B}(\mathcal{E}_B, \mathcal{F}_B) \), is called a Stinespring-Kasparov representation of \( \rho \). Such a Stinespring-Kasparov representation is called minimal if \( \mathcal{F}_B = \mathbb{C}\text{-span}[\pi(\mathcal{A})V \mathcal{E}_B] \).
Theorem 4.1.2 (Stinespring-Kasparov representation)

Let $A, B$ be $*$-algebras over $C$, let $E_B$ be a pre-Hilbert module over $B$, let $\rho : A \rightarrow \mathcal{B}(E_B)$ be a completely positive $*$-map. Then, there exists a unital $*$-representation $\pi : A \rightarrow \mathcal{B}(\mathcal{F}_B)$ of $A$ on a pre-Hilbert module $(\mathcal{F}_B, \langle \cdot, \cdot \rangle^\mathcal{F}_B)$ over $B$ and $V \in \mathcal{B}(E_B, \mathcal{F}_B)$ such that $\rho$ is of the form $\rho(a) = V^* \pi(a)V$ for all $a \in A$.

Proof: Take the $C$-module $A \otimes C E =: \mathcal{F}$. Then, by the definitions

$$\begin{align*}
\left( \sum_{i} a_i \otimes e_i \right) \cdot b &= \sum_{i} a_i \otimes e_i b, \\
\pi'(a) \left( \sum_{i} a_i \otimes e_i \right) &= \sum_{i} aa_i \otimes e_i
\end{align*}$$

for all $a, a_i \in A, b \in B$, and all $e_i \in E$, the module $\mathcal{F}$ is an $A \otimes B$-bimodule. The definition

$$\left( \sum_{i=1}^{n} a_i \otimes e_i, \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{e}_j \right)_B^\mathcal{F} := \sum_{i=1}^{n} \sum_{j=1}^{m} \langle e_i, \rho(a_i^* \tilde{a}_j) \tilde{e}_j \rangle_B$$

(4.1)

gives an inner product which is completely positive but possibly degenerate for all $a, \tilde{a}_j \in A$ and all $e_i, \tilde{e}_j \in E$. The complete positivity can be seen by considering for all $e \in E$ the map

$$\langle e, - e \rangle_{B}^E : \mathcal{B}(E) \rightarrow B.$$ (4.2)

This map is completely positive as

$$\langle (e, t^*t(e)) \rangle_{B}^E = \sum_{i,j,k=1}^{n} \langle e_i, t_k e_i \rangle_B = \sum_{i,j,k=1}^{n} \langle t_k e_i, e_i \rangle_B$$

is positive for all $t \in M_n(\mathcal{B}(E))$ and all $n$-tupel $(e)$ composed of $e_1, \ldots, e_n \in E$ and all $n \in \mathbb{N}$ because of the complete positivity of the inner product on $E$. As $\rho$ is completely positive, the positivity of (4.1) follows for all $\sum_{i=1}^{n} a_i \otimes e_i \in \mathcal{F}$ from the $n$-positivity of the composition of (4.2) and $\rho$. Analogously, we get $n$-positivity of (4.1) for all $n \in \mathbb{N}$ and thus complete positivity.

Let $(\mathcal{F}^\perp)'$ be the degeneracy submodule of $\mathcal{F}$ with respect to $\langle \cdot, \cdot \rangle_{B}^\mathcal{F}$. Then, the quotient $\mathcal{F} := \mathcal{F} / (\mathcal{F}^\perp)'$ is a $A \otimes \rho \otimes E$ together with the restriction $\langle \cdot, \cdot \rangle_{B}^{\mathcal{F}}$ of the inner product (4.1) defines a pre-Hilbert module because the degeneracy submodule is invariant under the action $\pi'$ of $A$, which therefore also descends to $\mathcal{F}$. By the following simple calculation we get that $\pi'$ is a unital $*$-homomorphism, as $\pi'(1) = \text{id}, \pi'(ab) = \pi(a)\pi'(b)$, and

$$\begin{align*}
\left( \sum_{i=1}^{n} a_i \otimes e_i, \pi'(a^*) \left( \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{e}_j \right) \right)_B^\mathcal{F} &= \sum_{i,j=1}^{n,m} \langle e_i, \rho(a_i^* \tilde{a}_j) \tilde{e}_j \rangle_B^E = \sum_{i,j=1}^{n,m} \langle e_i, \rho((aa_i^*) \tilde{a}_j) \tilde{e}_j \rangle_B^E \\
&= \left( \sum_{i=1}^{n} a_i \otimes e_i, \sum_{j=1}^{m} \tilde{a}_j \otimes \tilde{e}_j \right)_B^\mathcal{F},
\end{align*}$$

whereby we get $\pi'(a^*) = \pi'(a^*)$ for all $a \in A$ and $\pi' : A \rightarrow \mathcal{B}(\mathcal{F})$. Therefore, $\pi'$ descends to a well-defined unital $*$-homomorphism $\pi : A \rightarrow \mathcal{B}(\mathcal{F})$.

Next, we define a right-$B$-linear map $V : E \rightarrow \mathcal{F}$ by $V(e) = 1 \otimes e$ and another right-$B$-linear map $V^* : \mathcal{F} \rightarrow E$ by $V^*(\sum_{i=1}^{n} a_i \otimes e_i) = \sum_{i=1}^{n} \rho(a_i)e_i$. It is easy to see that $VV^*(e) = \rho(1)e$, $VV^*(\sum_{i=1}^{n} a_i \otimes e_i) = \sum_{i=1}^{n} 1 \otimes \rho(a_i)e_i$, and

$$\begin{align*}
\langle e, V^* \sum_{i=1}^{n} a_i \otimes e_i \rangle_B^E &= \sum_{i=1}^{n} \langle e, \rho(a_i)e_i \rangle_B^E = \sum_{i=1}^{n} \langle e, \rho(1a_i)e_i \rangle_B^E
\end{align*}$$
for all $e, e_i \in \mathcal{E}_B$ and all $a_i \in \mathcal{A}$. Therefore, $V^*$ is the adjoint of $V$ and $V^* ((\mathcal{F}')^+) = 0$, whereby $V^*$ descends to the quotient $\mathcal{F}$ as does $V$. This results in $V \in \mathcal{B}(\mathcal{E}, \mathcal{F})$ and $V^* \in \mathcal{B}(\mathcal{F}, \mathcal{E})$. Finally, we have

$$\langle e, \rho(a) \hat{e} \rangle^\mathcal{E}_B = \langle e, V^*(a \otimes \hat{e}) \rangle^\mathcal{E}_B = \langle Ve, a \otimes \hat{e} \rangle^\mathcal{E}_B = \langle Ve, \pi(a)(1 \otimes \hat{e}) \rangle^\mathcal{E}_B = \langle \rho(a)(1 \otimes \hat{e}) \rangle^\mathcal{E}_B,$$

valid for all $a \in \mathcal{A}$ and $e, \hat{e} \in \mathcal{E}$. Again, as $\pi^*$, $V$, and $V^*$ descend to the quotient, this is still valid on equivalence-classes, whence we get the desired form $\rho(a) = V^* \pi(a)V$.

\section*{Remark 4.1.3}
For $\mathcal{C} = \mathbb{C}$ we retain the result of Kasparov [52]. If, additionally, $\mathcal{B} = \mathbb{C}$, we retain the result of Stinespring [75].

In the case $\mathcal{B} = \mathcal{C}$, a statement reverse to the assertion of Theorem 4.1.2 holds:

\section*{Theorem 4.1.4 (Stinespring)}
Let $\mathcal{A}$ be a $^*$-algebra over $\mathcal{C}$, let $\mathcal{K}(\mathcal{C}, \langle \cdot, \cdot \rangle^{\mathcal{C}})$ be a pre-Hilbert module over $\mathcal{C}$, let $\rho : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K}_C)$ be $\mathcal{C}$-linear, and let there be a unital $^*$-representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K}_C)$ of $\mathcal{A}$ on a pre-Hilbert module $(\mathcal{K}_C, \langle \cdot, \cdot \rangle^{\mathcal{K}})$ over $\mathcal{C}$ such that $\rho$ is of the form $\rho(a) = V^* \pi(a)V$ for all $a \in \mathcal{A}$, where $V \in \mathcal{B}(\mathcal{K}_C, \mathcal{K}_C)$. Then $\rho$ is a completely positive $^*$-map.

\textbf{Proof:} Let $\rho(a) = V^* \pi(a)V$ for all $a \in \mathcal{A}$. As $\pi$ and $\langle \psi, \cdot \rangle^{\mathcal{K}} : \mathcal{B}(\mathcal{K}) \rightarrow \mathbb{C}$ are completely positive maps for all $\psi \in \mathcal{K}$, the statement is true. This can easily be seen for $n \in \mathbb{N}$, $\phi_1, \ldots, \phi_n \in \mathcal{K}_C$, $\phi_i = V\phi_i$ for all $i = 1, \ldots, n$, and $A \in M_n(\mathcal{A})^+$:

$$\langle \phi^{(n)}, \rho^{(n)}(A)\phi^{(n)} \rangle^{\mathcal{K}(n)} = \sum_{i,j=1}^n \langle V\phi_i, \pi(A_{ij})V\phi_j \rangle^{\mathcal{K}} = \langle \phi^{(n)}, \pi^{(n)}(A)\phi^{(n)} \rangle^{\mathcal{K}(n)} \geq 0,$$

whereby $\rho^{(n)} : M_n(\mathcal{A})^+ \rightarrow M_n(\mathcal{B}(\mathcal{K}))^+$ for all $n \in \mathbb{N}$. In other words, as $\langle \psi, \cdot \rangle^{\mathcal{K}}$ is a positive linear functional, we get by definition $\rho^{(n)}(A) \in M_n(\mathcal{B}(\mathcal{K}))^+$ for all $A \in M_n(\mathcal{A})^+$. By $\rho(a^*) = V^* \pi(a)^*V = V^* \pi(a)V^* = (V^* \pi(a)V)^* = \rho(a)^*$ for all $a \in \mathcal{A}$, the $^*$-map property follows.

\section*{Remark 4.1.5 (GNS representation)}
In case $\mathcal{B} = \mathcal{C}$, $\mathcal{E}_B = \mathcal{C}$, we know by Example 1.4.6 that the completely positive map $\rho$ in Theorem 4.1.2 actually is a positive linear functional. In this case, the Stinespring-Kasparov construction of Theorem 4.1.2 is just the usual GNS construction for $^*$-algebras, cf. Bordemann and Waldmann [17] or Waldmann [78, Subsec 7.2.2].

\section*{Corollary 4.1.6}
If $\rho$ is unital, the map $V$ is an isometry and $V V^* : \mathcal{F} \rightarrow \mathcal{F}$ is a projection.

\section*{Corollary 4.1.7 (Minimal Stinespring-Kasparov representation)}
The Stinespring-Kasparov representation obtained by the procedure in Theorem 4.1.2 is minimal.

\textbf{Proof:} For $e \in \mathcal{E}$ and $a \in \mathcal{A}$ we have $Ve = 1 \otimes e + (\mathcal{F})^+$ and $(\pi(a)V)e = a \otimes e + (\mathcal{F})^+$, wherefore the $\mathbb{C}$-span of $\pi(A)V(\mathcal{E})$ is given by $\mathbb{C}$-span$[\pi(A)V(\mathcal{E})] = \mathcal{F}$.

\section*{Definition 4.1.8 (Canonical Stinespring-Kasparov representation)}
The minimal Stinespring-Kasparov representation of the construction in Theorem 4.1.2 is called the canonical Stinespring-Kasparov representation of the corresponding completely positive map.
Sometimes, the Stinespring-Kasparov representation of a completely positive map, as constructed in Theorem 4.1.2 in order to show the existence of Stinespring-Kasparov representations, is called dilation of the completely positive map. Given a non-minimal Stinespring-Kasparov representation, it is fairly easy to obtain a minimal Stinespring-Kasparov representation by the following procedure.

**Remark 4.1.9 (Minimalization of Stinespring-Kasparov representations)**

Let $(K_B, \pi, V)$ be a Stinespring-Kasparov representation for a completely positive* map $\rho : A \to B(H_B)$. Let $K'_1 := C\text{-span}\{\pi(A)V\}B$. The induced map $\pi'_1 = \pi|_{K'_1}$ of $\pi$ to $K'_1$ is a well defined *-homomorphism $\pi'_1 : A \to B(K'_1)$, as $\pi(A)K'_1 = K'_1$. But, possibly, the restriction of the inner product to the submodule $K'_1$ is degenerate. We therefore divide by the degeneracy submodule of $K'_1$ and get $K'_1 = K'_1/(K'_1)^\perp$. It is easy to check that, again, $\pi'_1$ leaves the degeneracy submodule invariant and thus induces a *-representation $\pi_1$ onto $K_1$. Analogously, it is equally simple to check on elementary tensors that $\pi, \pi'_1$.

$$(\pi(V^*(\pi(a)Vh')))^{\pi_1} = (V(h)(\pi(a)Vh'))^{\pi_1}$$

for $h, h' \in K_B, a \in A$, and $\pi(a)Vh' \in (K'_1)^\perp$. Therefore $V$ and $V^*$ restrict to $K_1$. Furthermore, as $VK_B \subseteq K_1$, we have $\rho(a) = V^*\pi_1(a)V$ for all $a \in A$. Therefore, $(K_1, \pi_1, V)$ is a minimal Stinespring-Kasparov representation for the completely positive map $\rho$.

**Remark 4.1.10**

In the following, the restriction process described in Remark 4.1.9 in order to obtain a minimal Stinespring-Kasparov representation from a Stinespring-Kasparov representation is called minimalization.

**Lemma 4.1.11 (Uniqueness of Stinespring-Kasparov representations)**

Let $\rho : A \to B(H_B)$ be a completely positive map. Let $(K_i, \pi_i, V_i)$ for $i = 1, 2$ be two minimal Stinespring-Kasparov representations of $\rho$. Then, there exists a unitary map $U \in B(K_1, K_2)$ such that $UV_1 = V_2$ and $U\pi_1U^* = \pi_2$.

**Proof:** If $a_1, \ldots, a_n \in A$ and $h_1, \ldots, h_n \in K_B$, then

$$\sum_{i=1}^n \pi_2(a_i)V_2h_i = \pi_2(\sum_{j=1}^n a_j)V_2h_i \quad (4.3)$$

The map $U : \sum_i \pi_1(a_i)V_1h_i \mapsto \sum_i \pi_2(a_i)V_2h_i$ is surjective by definition. The injectivity follows directly from Equation (4.3) and the non-degeneracy of the inner products (e.g. by proof by contradiction). Hence, $U$ is well-defined and isometric, as is the adjoint map $U^* : \sum_i \pi_2(a_i)V_2h_i \mapsto \sum_i \pi_1(a_i)V_1h_i$. It is easy to see that $U$ is unitary as $U^*U = \text{id} = UU^*$.

$$U\pi_1(a)U^* \sum_i \pi_2(a_i)V_2h_i = \pi_1(a) \sum_i \pi_1(a_i)V_1h_i = \sum_i \pi_2(a_i)V_2h_i = \pi_2(\sum_i \pi_2(a_i)V_2h_i)$$
we have $U\pi_1(a)U^* = \pi_2(a)$ for all $a \in A$. Furthermore, from the definition of $U$, we get

$$UV_1h = U\pi_1(1)V_1h = \pi_2(1)V_2h = V_2h$$

for all $h \in \mathcal{H}_B$. \hfill \blacksquare

Not only are different minimal Stinespring-Kasparov representations of a completely positive map $\rho$ connected by a unitary intertwiner, but also can any $^*$-representation which is connected by a unitary intertwiner to a minimal Stinespring-Kasparov representation of $\rho$ be seen as a minimal Stinespring-Kasparov representation of $\rho$.

**Proposition 4.1.12**

Let $\rho : A \rightarrow \mathfrak{B}(\mathcal{H}_B)$ be a completely positive $^*$-map, let $(\mathcal{K}_B, \pi, V)$ be a minimal Stinespring-Kasparov representation of $\rho$, and let $(\mathcal{F}_B, \pi')$ be a unital $^*$-representation of $A$ on a pre-Hilbert module $\mathcal{F}_B$. If there exists a unitary intertwiner $U \in \mathfrak{B}(\mathcal{K}_B, \mathcal{F}_B)$ such that $U\pi = \pi'U$, then $(\mathcal{F}_B, \pi', UV)$ is a minimal Stinespring-Kasparov representation of $\rho$.

**Proof:** As $U$ is unitary, we have $\pi = U^*\pi'U$. Therefore,

$$\rho = V^*\pi V = V^*U^*\pi'UV = (UV)^*\pi'UV,$$

wherefore $(\mathcal{F}_B, \pi', UV)$ is a Stinespring-Kasparov representation of $\rho$. As $(\mathcal{K}_B, \pi, V)$ is minimal, we have $\mathcal{K}_B = \mathbb{C}\text{-span}(\pi(A)V\mathcal{H}_B)$. $U$ is a unitary map, wherefore it is bijective. Therefore, we get that $\mathcal{F}_B = U\mathcal{K}_B = U\mathbb{C}\text{-span}(\pi(A)V\mathcal{H}_B) = \mathbb{C}\text{-span}(U\pi(A)UV\mathcal{H}_B)$. Thus, we see that $(\mathcal{F}_B, \pi', UV)$ is minimal. \hfill \blacksquare

Two differing completely positive $^*$-maps $\rho : A \rightarrow \mathcal{K}_B$ and $\rho' : A \rightarrow \mathcal{K}_B$ cannot have the same minimal Stinespring-Kasparov representation $(\mathcal{K}_B, \pi, V)$, as then they would be equal as adjointable maps on $\mathcal{H}_B$ for all $a \in A$. Thence, differing completely positive $^*$-maps lead to differing canonical Stinespring-Kasparov representations. As there is exactly one canonical Stinespring-Kasparov representation of any given completely positive $^*$-map, we get the following lemma, assuming any equal maps are identified.

**Lemma 4.1.13**

The “rule” assigning to every completely positive $^*$-map on unital $^*$-algebras its canonical Stinespring-Kasparov representation is injective.

### 4.2 The Internal Tensor Product of Canonical Stinespring-Kasparov Representations

In this section, we take a closer look at the properties of Stinespring-Kasparov representations in the sense of strongly non-degenerate $^*$-representation theory, especially the internal tensor product of Stinespring-Kasparov representation bimodules.

**Remark 4.2.1**

Let $\rho : A \rightarrow \mathfrak{B}(\mathcal{H}_B)$ be a completely positive $^*$-map. Then, any Stinespring-Kasparov representation of $\rho$ is a strongly non-degenerate $^*$-representation. This follows from the unitality of $A$ and the unitality of Stinespring-Kasparov representations.

**Proposition 4.2.2**

Let $\rho : A \rightarrow \mathfrak{B}(\mathcal{H}_B)$ be a completely positive $^*$-map. Then, we get a subcategory $^*$-$\text{SKRep}_B^\rho(\mathcal{A})$ of the category $^*$-$\text{Rep}_B(\mathcal{A})$. 
The objects of $\ast$-SKRep$^\mu_B(A)$ are given by the class of minimal Stinespring-Kasparov representations of $\rho$. The morphisms are given by the unitary intertwiners and the respective identity maps.

By Remark 4.2.1, it is clear that every Stinespring-Kasparov representation for a unital $\ast$-algebra is a strongly non-degenerate $\ast$-representation. Furthermore, by definition, every unitary intertwiner is an intertwiner.

By Lemma 4.1.11, it is clear that, for every pair of objects, there exists a morphism.

**Remark 4.2.3**

We denote by $\ast$-SKRep$^\mu_B(A)$ the subcategory of $\ast$-Rep$^\mu_B(A)$ consisting of the minimal Stinespring-Kasparov representations from $A$ into the adjointable maps on pre-Hilbert $\mathcal{B}$-modules with, not necessarily unitary, intertwiners as morphisms.

**Remark 4.2.4**

Let us take a closer look at possible relations between $\ast$-representations, Stinespring-Kasparov representations, and canonical Stinespring-Kasparov representations:

i.) By setting $V = \text{id}_{\mathcal{H}_B}$, every $\ast$-homomorphism $\pi : A \rightarrow \mathcal{B}(\mathcal{H}_B)$ is a minimal Stinespring-Kasparov representation of itself in its own right. Yet, by the canonical construction in the proof of Theorem 4.1.2 every $\ast$-homomorphism can be dilated to a minimal Stinespring-Kasparov representation of itself which is unitarily equivalent to $(\mathcal{H}_B, \pi, \text{id}_{\mathcal{H}_B})$ by Lemma 4.1.11.

ii.) If $\pi_\rho$ already is a Stinespring-Kasparov representation with regard to a completely positive map $\rho$, the above construction is consistent in the sense that the canonical Stinespring-Kasparov representation of $\pi_\rho$ is also a Stinespring-Kasparov representation of $\rho$ due to the definition of the Stinespring-Kasparov representation $\pi_\rho$ of $\rho$.

iii.) By i.), the identity map $\text{id}_A$ induces a non-canonical Stinespring-Kasparov representation bimodule $\pi_\rho A_{\ast}^\mu \pi_\rho \mathcal{B} \subset A_{\ast}^\mu \mathcal{B}$ with the usual $A$-valued inner product obviously compatible with the left-module structure. On the other hand, the canonical Stinespring-Kasparov representation of $\pi_\rho$ is given by the bimodule $\pi_\rho A_{\ast}^\mu \pi_\rho \mathcal{B}$, which is isomorphic to $A_{\ast}^\mu \mathcal{B} \subset \pi_\rho A_{\ast}^\mu \mathcal{B}$ by Lemma 4.1.11.

Next, we make use of Example 1.4.6. The identification of pre-Hilbert module, adjointable maps, and underlying algebra allows for an interesting construction involving the internal tensor product.

**Theorem 4.2.5**

Let $\rho : A \rightarrow \mathcal{B}$ and $\mu : \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H}_C)$ be completely positive $\ast$-maps of unital $\ast$-algebras over $\mathbb{C}$. Let $A_{\ast}^\mu \mathcal{C}_B \subset A_{\ast}^\mu \mathcal{B} \subset \mathcal{B} \subset A_{\ast}^\mu \mathcal{C}$ be the canonical Stinespring-Kasparov representations of $\rho$ and $\mu$. Then, the internal tensor product

$$A_{\ast}^\mu \mathcal{B} \otimes \mathcal{B} \mathcal{C}_\rho \mathcal{C}$$

(4.4)

is a Stinespring-Kasparov representation of $\mu \circ \rho$, and its minimalization is isomorphic to the canonical Stinespring-Kasparov representation of $\mu \circ \rho$.

**Proof:** As $A_{\ast}^\mu \mathcal{C}_B \subset A_{\ast}^\mu \mathcal{B} \subset \mathcal{B} \subset A_{\ast}^\mu \mathcal{C}$ are strongly non-degenerate $\ast$-representations, it follows that $A_{\ast}^\mu \mathcal{C}_B \otimes \mathcal{C}_\rho \mathcal{C}_B \subset A_{\ast}^\mu \mathcal{B} \subset \mathcal{B}$ is a pre-Hilbert bimodule in $\ast$-Rep$^\mu_\mathcal{C}(A)$, see [28, Thm. 4.7]. It remains to show that $A_{\ast}^\mu \mathcal{B} \otimes \mathcal{B} \mathcal{C}_\rho \mathcal{C}$ is a Stinespring-Kasparov representation of $\mu \circ \rho$ and that its minimalization is isomorphic to the canonical Stinespring-Kasparov representation of $\mu \circ \rho$.

In order to see that it is a Stinespring-Kasparov representation of $\mu \circ \rho$, we follow the construction of Theorem 4.1.2 and turn the $\mathcal{C}$-module $A \otimes \mathcal{B} \otimes \mathcal{C}_\rho \mathcal{C}$ into an $A_{\mathcal{C}}$-bimodule with possibly...
Kasparov representation of is adjointable with adjoint $V$ for all $a$.

V descends to $\pi(A \otimes b_1 \otimes [b_2 \otimes h]) = [\mu(a) b_1 b_2] h$.

\[ \langle A \otimes b_1 \otimes [b_2 \otimes h], [a' \otimes b_1'] \otimes [b'_2 \otimes h'] \rangle_c^C = \langle [b_2 \otimes h], \langle a \otimes b_1, [a' \otimes b_1'] \rangle_B, [b'_2 \otimes h'] \rangle_c^{B \otimes B \otimes C} \]  

(4.5)

Next we define a linear map $V : \mathcal{H}_C \rightarrow A \otimes B \otimes B \otimes \mathcal{H}_C$ by $V(h) = [1 \otimes 1] \otimes [1 \otimes h]$ and show that it is adjointable with adjoint $V^*[a \otimes b_1 \otimes [b_2 \otimes h]) = \mu(\rho(a) b_1 b_2) h$.

\[ \langle V(h), [a' \otimes b'_1] \otimes [b'_2 \otimes h'] \rangle_c^C = \langle [1 \otimes 1] \otimes [1 \otimes h], [a' \otimes b'_1] \otimes [b'_2 \otimes h'] \rangle_c^C = \langle [1 \otimes h], [1', \rho(1'^* a') b'_1 \otimes [b'_2 \otimes h'] \rangle_c^{B \otimes B \otimes C} = \langle [1 \otimes h], (1'^* \rho(1'^* a') b'_1) [b'_2 \otimes h'] \rangle_c^{B \otimes C} = \langle h, \mu(\rho(a) b'_1 b'_2) h' \rangle_c^C = \langle h, V^*([a' \otimes b'_1] \otimes [b'_2 \otimes h']) \rangle_c^C \]  

(4.6)

for all $a' \in A$, all $b'_1, b'_2 \in B$, and all $h' \in \mathcal{H}_C$, wherefore $V$ and $V^*$ are adjoint. Furthermore,

\[ VV^*[a \otimes b_1 \otimes [b_2 \otimes h]) = [1 \otimes 1] \otimes [1 \otimes (\mu(\rho(a)b_1 b_2)h)] \]  

(4.7)

and

\[ V^*V(h) = \rho(1) h \]  

(4.8)

for all $a \in A$, all $b_1, b_2 \in B$, and all $h \in \mathcal{H}_C$. Next, we realize that with the unital $\ast$-homomorphism $\pi : A \rightarrow B (A \otimes B \otimes B \otimes \mathcal{H}_C)$ given by $\pi(a')([a \otimes b_1] \otimes [b_2 \otimes h]) = [a' a \otimes b_1 \otimes [b_2 \otimes h]$ we have

\[ \langle h, (\mu \circ \rho)(a) h' \rangle_c^C = \langle h, V^*([a \otimes 1] \otimes [1 \otimes h']) \rangle_c^C = \langle V(h), [a \otimes 1] \otimes [1 \otimes h'] \rangle_c^C = \langle V(h), \pi(a) ([1 \otimes 1] \otimes [1 \otimes h']) \rangle_c^C = \langle h, (V^* \pi(a) V) h' \rangle_c^C \]  

(4.9)

for all $a \in A$ and all $h, h' \in \mathcal{H}_C$. Hereby, we see that $(\mu \circ \rho)(a) = V^* \pi(a) V$ for all $a \in A$ gives a Stinespring-Kasparov representation of $\mu \circ \rho$ on $(A \otimes B) \otimes (B \otimes \mathcal{H}_C)$, because, by calculation (4.6), it is easy to see that $V^*$ maps the degeneracy submodule of $(A \otimes B) \otimes (B \otimes \mathcal{H}_C)$ into zero, wherefore $V, V^*$, and $\pi$ descend to $(A \otimes B) \otimes (B \otimes \mathcal{H}_C)$.

As the $C$-span of $\pi(A)V\mathcal{H}_C = (A \otimes 1) \otimes 1 \otimes (B \otimes (B \otimes \mathcal{H}_C))$ will usually not be equal to the $C$-span of $(A \otimes B) \otimes (B \otimes \mathcal{H}_C)$, $A \otimes B \otimes (B \otimes \mathcal{H}_C)^\mu$ is a Stinespring-Kasparov representation of $\mu \circ \rho$, which usually is not minimal.

Finally, the minimalization of the internal tensor product $A \otimes B \otimes (B \otimes \mathcal{H}_C)^\mu$ is isomorphic to the canonical Stinespring-Kasparov representation, as by Lemma 4.1.11 any two minimal Stinespring-Kasparov representations with respect to the same completely positive map are unitarily equivalent and thus isomorphic.
Remark 4.2.6
In case $B = C$, the minimalization is unnecessary, as $(A \otimes \rho C \otimes C \otimes \mu H C) \cong (A \otimes \rho C \otimes C \otimes H C)$. The latter is given as $(A \otimes C) \otimes (C \otimes H C) \cong A \otimes C \otimes H C$ and as, because of the $C$-linearity of $\rho$ and $\mu$, the degeneracy submodule $(A \otimes \rho C \otimes C \otimes H C)$ isomorphic. The isomorphy of the degeneracy spaces follows from the equality of conditions, as

$$\sum_{i,j} \langle h_i, \mu(a_i \tilde{a}_j) \cdot 1 \cdot 1 \rangle_{H C} = \sum_{i,j} \langle h_i, (\mu \circ \rho)(a_i \tilde{a}_j) \rangle_{H C}$$

(4.10)

for all $a_i, \tilde{a}_j \in A$ and all $h_i, \tilde{h}_j \in H C$.

4.3 Dynamical Systems on $\ast$-Algebras

Dynamical systems in physics are usually defined on particular $\ast$-algebras. Therefore, it is a simple act to generalize the usual notion of a dynamical system to general $\ast$-algebras. Furthermore, we will show an interesting connection between the composition of dynamical maps and tensorial properties of their canonical Stinespring-Kasparov representations.

Definition 4.3.1
Let $A$ be a $\ast$-algebra over $C$. A dynamical system on $A$ is a family $\{\Phi_{t,s}\}_{s \geq t \geq 0}$ of completely positive unital $\ast$-maps $\Phi_{t,s} : A \rightarrow A$, called dynamical maps, with the additional properties

i.) $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$ for all $r \geq s \geq t \geq 0$,

ii.) $\Phi_{t,t} = \text{id}$ for all $t \geq 0$.

In order to illustrate the general applicability of Definition 4.3.1, let us have some examples.

Example 4.3.2
Examples of dynamical systems on $\ast$-algebras are given by:

i.) the classical dynamical systems with possibly time-dependent vector fields, see [1], including but not solely consisting of classical Hamiltonian systems,

ii.) the deformation quantized Hamiltonian systems of Section 2.2,

Combining Theorem 4.2.5 with the time evolution of dynamical systems of Definition 4.3.1, we get the following corollary connecting the time evolution of algebra elements and the internal tensor product of Stinespring-Kasparov representations.

Proposition 4.3.3
Let $\{\Phi_{t,s}\}_{s \geq t \geq 0}$ be a dynamical system on a $\ast$-algebra $A$ over $C$, let $0 \leq t \leq s \leq r$. Then, the minimalized internal tensor product of the canonical Stinespring-Kasparov representations of the dynamical maps $\Phi_s$ and $\Phi_{s,r}$ is isomorphic to the canonical Stinespring-Kasparov representation of the composition $\Phi_r$ of the constituent dynamical maps.

Proof: Considering Example 1.4.6, the assertion is a direct result of Theorem 4.2.5, as the $\Phi_{t,s}$ are completely positive maps and as $\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}$ for all $r \geq s \geq t \geq 0$. ■

Remark 4.3.4
By Proposition 4.3.3, the composition of dynamical maps of a dynamical system, and thus the time evolution of the elements of the $\ast$-algebra, usually interpreted as an algebra of observables, can be encoded in the minimalized internal tensor product of canonical Stinespring-Kasparov representations of the selfsame dynamical maps.
The notion of a dynamical system of Definition 4.3.1 can be specialized to semigroups of time evolution maps resulting in additional algebraic properties such as the notion of a generator.

**Definition 4.3.5**

Let $\mathcal{A}$ be a $^*$-algebra over $\mathbb{C}$. A Markovian dynamical system on $\mathcal{A}$ is a one-parameter semigroup $\{\Phi_t\}_{t \geq 0}$ of completely positive unital $^*$-maps $\Phi_t : \mathcal{A} \to \mathcal{A}$, that is

\begin{align*}
  i.) & \quad \Phi_s \circ \Phi_t = \Phi_{s+t} \text{ for all } s, t \geq 0, \\
  ii.) & \quad \Phi_0 = \text{id}.
\end{align*}

**Remark 4.3.6**

Obviously, Markovian dynamical systems are a special case of dynamical systems in the sense of Definition 4.3.1. Therefore, Corollary 4.3.3 also holds for Markovian dynamical systems.

### 4.4 The External Tensor Product of Stinespring-Kasparov Representations

After having seen a certain compatibility between the internal tensor product and Stinespring-Kasparov representations in Section 4.2, it is natural to check whether a similar compatibility exists with the external tensor product. In this section, we shall discuss in which sense such a compatibility can be given. Furthermore, we shall discuss some conditions regarding the non-degeneracy of the ring-theoretic tensor product of Stinespring-Kasparov representations.

To begin with, let $(\pi_\rho, \mathcal{F}_{\mathcal{B}_1}, V_\rho)$ be a Stinespring-Kasparov representation of a completely positive map $\rho : A_1 \to \mathcal{B}(\mathcal{H}_{\mathcal{B}_1})$ and let $(\pi_\mu, \mathcal{F}_{\mathcal{B}_2}, V_\mu)$ be a Stinespring-Kasparov representation of a completely positive map $\mu : A_2 \to \mathcal{B}(\mathcal{H}_{\mathcal{B}_2})$. Then we know that $\mathcal{F}_{\mathcal{B}_1} \otimes \mathcal{F}_{\mathcal{B}_2}$ and $\mathcal{H}_{\mathcal{B}_1} \otimes \mathcal{H}_{\mathcal{B}_2}$ are modules with possibly degenerate inner products in the sense of external tensor products as in Section 1.4. Furthermore, the tensor product $\rho \otimes_\mathbb{C} \mu : A_1 \otimes \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{H}_{\mathcal{B}_2})$ is a completely positive map that induces a completely positive map into the adjointable maps on the quotient given by the map $\rho \otimes_\mathbb{C} \mu : A_1 \otimes \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_{\mathcal{B}_1} \otimes \mathcal{H}_{\mathcal{B}_2})$. Similarly, the tensor product of two $^*$-representations $\pi_\rho \otimes_\mathbb{C} \pi_\mu$ induces a $^*$-representation $\pi_\rho \otimes_\mathbb{C} \pi_\mu : A_1 \otimes_\mathbb{C} A_2 \to \mathcal{B}(\mathcal{F}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{F}_{\mathcal{B}_2})$, as $\otimes_\mathbb{C}$ is a functor

$\otimes_\mathbb{C} : \text{-Rep}_{\mathcal{B}_1}(A_1) \times \text{-Rep}_{\mathcal{B}_2}(A_2) \to \text{-Rep}_{\mathcal{B}_1 \otimes_\mathbb{C} \mathcal{B}_2}(A_1 \otimes_\mathbb{C} A_2)$.

The results above can, for example, be found in Waldmann [79]. Now, we have almost all ingredients for the following proposition.

**Proposition 4.4.1**

$(\mathcal{F}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{F}_{\mathcal{B}_2}, \pi_\rho \otimes_\mathbb{C} \pi_\mu, V_\rho \otimes_\mathbb{C} V_\mu)$ is a Stinespring-Kasparov representation of the completely positive map $\rho \otimes_\mathbb{C} \mu : A_1 \otimes_\mathbb{C} A_2 \to \mathcal{B}(\mathcal{H}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{H}_{\mathcal{B}_2})$.

**Proof:** What is left to show is that the map $V_\rho \otimes_\mathbb{C} V_\mu$ is well-defined and adjointable on the external tensor product and that the relation $\rho \otimes_\mathbb{C} \mu = (V_\rho \otimes_\mathbb{C} V_\mu)(\pi_\rho \otimes_\mathbb{C} \pi_\mu)(V_\rho \otimes_\mathbb{C} V_\mu)$ holds. Let $h \in \mathcal{H}_{\mathcal{B}_1}$, $h' \in \mathcal{H}_{\mathcal{B}_2}$, $f \in \mathcal{F}_{\mathcal{B}_1}$, and $f' \in \mathcal{F}_{\mathcal{B}_2}$. Then

\[
\langle V_\rho \otimes_\mathbb{C} V_\mu(h \otimes h'), f \otimes f' \rangle_{\mathcal{F}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{F}_{\mathcal{B}_2}} = \langle (V_\rho h \otimes V_\mu h'), f \otimes f' \rangle_{\mathcal{F}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{F}_{\mathcal{B}_2}} = \langle V_\rho h, f \rangle_{\mathcal{B}_1} \otimes \langle V_\mu h', f' \rangle_{\mathcal{B}_2} = \langle h, V_\rho f \rangle_{\mathcal{B}_1} \otimes \langle h', V_\mu f' \rangle_{\mathcal{B}_2} = \langle h \otimes h', (V_\rho \otimes_\mathbb{C} V_\mu)^*(f \otimes f') \rangle_{\mathcal{F}_{\mathcal{B}_1} \otimes_\mathbb{C} \mathcal{F}_{\mathcal{B}_2}}.
\]
wherefore we can see that both $V_{\rho} \otimes_{C} V_{\mu}$ and $(V_{\rho} \otimes_{C} V_{\mu})^*$ are adjointable maps and thence descend to the external tensor products. The relation $\rho \otimes_{\text{ext}} \mu = (V_{\rho} \otimes_{\text{ext}} V_{\mu})^*(\pi_{\rho} \otimes_{\text{ext}} \pi_{\mu})(V_{\rho} \otimes_{\text{ext}} V_{\mu})$ follows by a simple evaluation using the definition of the inner product of the external tensor product similar to the corresponding calculation in the proof of Theorem 4.1.2.

Especially interesting is the case where the ring-theoretic tensor product of pre-Hilbert modules only has a trivial degeneracy submodule. There are counterexamples with non-trivial degeneracy submodule both for the external and for the internal tensor product, as we will see in Example 4.4.11 and in Remark 4.4.12. Nonetheless, for pre-Hilbert modules over $C$ there exist some useful conditions as this question reduces to the question of torsion. For the rest of this section, all pre-Hilbert modules are assumed to be defined over $C$.

As we are interested in ring-theoretic tensor products without degeneracy submodules, we use the construction of the external tensor product without taking the quotient with respect to the degeneracy submodule in the end and give it explicitly as an illustration.

**Definition 4.4.2 (Induced product)**

Let $\mathcal{H}$ and $\mathcal{K}$ be inner product $C$-modules. Let $\langle \cdot , \cdot \rangle_{C}^{\mathcal{H} \otimes \mathcal{K}}$ be the bilinear continuation of the induced mapping

$$
(\phi_{1} \otimes \phi_{2}, \psi_{1} \otimes \psi_{2}) \mapsto \langle \phi_{1}, \psi_{1} \rangle_{C}^{\mathcal{H}} \langle \phi_{2}, \psi_{2} \rangle_{C}^{\mathcal{K}}.
$$

(4.11)

where $\phi_{1}, \psi_{1} \in \mathcal{H}$ and $\phi_{2}, \psi_{2} \in \mathcal{K}$, on the tensor product $\mathcal{H} \otimes_{C} \mathcal{K}$ of inner product modules.

**Remark 4.4.3 (Induced inner product)**

The induced product $\langle \cdot , \cdot \rangle_{C}^{\mathcal{H} \otimes \mathcal{K}}$ fulfills the requirements i) - iii) of Definition 1.4.1 by virtue of its construction and the properties of the respective constituent inner products. We call it induced inner product henceforth without demanding non-degeneracy.

**Lemma 4.4.4 (Induced inner product)**

The induced inner product $\langle \cdot , \cdot \rangle_{C}^{\mathcal{H} \otimes \mathcal{K}}$ on the tensor product $\mathcal{H} \otimes_{C} \mathcal{K}$ of pre-Hilbert modules over $C$ is positive semi-definite.

**Proof:** Let $\Psi \in \mathcal{H} \otimes_{C} \mathcal{K}$ with $\Psi = \sum_{i=1}^{n} \psi_{i} \otimes \phi_{i}$. Then,

$$
\langle \Psi, \Psi \rangle_{C}^{\mathcal{H} \otimes \mathcal{K}} = \sum_{i,j=1}^{n} \langle \psi_{i}, \phi_{i} \rangle_{C}^{\mathcal{H}} \langle \phi_{j}, \phi_{j} \rangle_{C}^{\mathcal{K}}
$$

equals the trace of the product of two positive matrices, for which reason we have $\langle \Psi, \Psi \rangle_{C}^{\mathcal{H} \otimes \mathcal{K}} \geq 0$.

In order to prove Proposition 4.4.6, we need a generalization of the Gram-Schmidt orthogonalization of a minimal generating system of a vector space over a field to the setting of a pre-Hilbert module over $C$. Remember that a pre-Hilbert module over $C$ is torsion-free due to Lemma 1.4.4.

**Remark 4.4.5 (Orthogonalization of pre-Hilbert modules)**

Let $(\mathcal{M}, \langle \cdot , \cdot \rangle_{C}^{\mathcal{M}})$ be a pre-Hilbert $C$-module. Furthermore, let an element $m \in \mathcal{M}$ be written as finite sum $m = \sum_{i=1}^{n} v_{i}e_{i}$ with $e_{i} \in C$ and $v_{i} \in M$. For expediency’s sake take $v_{i} \neq 0 \neq c_{i}$ for all $i = 1, \ldots, n$. Define elements $e_{i} \in \mathcal{M}$ by

$$
e_{1} = v_{1}, \quad e_{i} = v_{i}u_{i-1} - \sum_{k=1}^{i-1} e_{k} \langle e_{k}, v_{i} \rangle_{C}^{\mathcal{M}} u_{i-1}^{1},
$$

(4.12)

where $u_{i-1} = \prod_{k=1}^{i-1} \langle e_{k}, e_{k} \rangle_{C}^{\mathcal{M}}, \quad u_{0} = 1$, and $u_{i-1}^{1} = \prod_{j=1}^{i-1} \prod_{j=1}^{i-1} \langle e_{k}, e_{j} \rangle_{C}^{\mathcal{M}} \langle e_{j}, e_{j} \rangle_{C}^{\mathcal{M}}$. Obviously, we have $u_{i-1} \neq 0 \neq u_{i-1}^{1}$ for all $i = 1, \ldots, n$, as the inner product is non-degenerate. Furthermore, $\langle e_{k}, e_{j} \rangle_{C}^{\mathcal{M}} = 0$ if
4.4. The External Tensor Product of Stinespring-Kasparov Representations

$k \neq j$ as by a simple calculation $0 = \langle e_k U_{k-1}, e_j U_{j-1} \rangle_C^M = \langle e_k, e_j \rangle_C \overline{U_{k-1} U_{j-1}}$ for $k \neq j$. While in general $m$ cannot be expressed in terms of the orthogonal elements $e_i$, by Equation (4.12) there exists a non-zero $c \in C$ such that

\[ mc = \sum_{i=1}^n e_i \tilde{c}_i \]  

(4.13)

with $0 \neq \tilde{c}_i \in C$ for all $i = 1, \ldots, n$, because the inner product is non-degenerate. Actually, $c = \prod_{r=0}^{n-1} U_r$ does as required.

**Proposition 4.4.6**

Let $\mathcal{H}$ and $\mathcal{K}$ be pre-Hilbert modules over $C$ such that the module $\mathcal{H} \otimes \mathcal{K}$ is torsion-free. Then the induced inner product $\langle \cdot, \cdot \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$ is positive definite.

**Proof:** We have to show that the induced product $\langle \cdot, \cdot \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$ is positive definite. Therefore, let $\Psi \in \mathcal{H} \otimes \mathcal{K}$ be given by $\Psi = \sum_{i=1}^n \phi_i \otimes \psi_i$, where $\phi_i \in \mathcal{H}$ and $\psi_i \in \mathcal{K}$ for all $i = 1, \ldots, n$. Let a non-zero $z \in \mathbb{C}$ be such that $\phi_i z = \sum_{i=1}^n e_i c_i$ and $\psi_i z = \sum_{i=1}^n e_i d_i$ can be expanded in terms of orthogonalized elements $e_i \in \mathcal{H}$ and $f_j \in \mathcal{K}$ with non-zero coefficients $c_i, d_j \in C$. According to Remark 4.4.5, such a $z \in \mathbb{C}$ exists. Then, we have

\[ \langle \Psi, \Psi \rangle_C^{\mathcal{H} \otimes \mathcal{K}} = 0 \iff \langle \Psi, \psi_i \rangle_C^{\mathcal{H} \otimes \mathcal{K}} (\overline{z})^2 = 0, \quad \text{and} \quad \langle \Psi z^2, \Psi z \rangle_C^{\mathcal{H} \otimes \mathcal{K}} = \langle \Psi, \psi_i \rangle_C^{\mathcal{H} \otimes \mathcal{K}} (\overline{z})^2 \]  

(4.14)

as $\mathbb{C}$ has no zero divisors and as $\mathcal{H} \otimes \mathcal{K}$ is torsion-free. Thus, we can check the positive definiteness by evaluating $\langle \Psi z^2, \Psi z \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$ instead of $\langle \Psi, \Psi \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$. By evaluating

\[ \langle \Psi z^2, \Psi z \rangle_C^{\mathcal{H} \otimes \mathcal{K}} = \left( \sum_{i=1}^n (\phi_i z) \otimes (\psi_i z), \sum_{i=1}^n (\phi_i z) \otimes (\psi_i z) \right)_C^{\mathcal{H} \otimes \mathcal{K}} \]

\[ = \sum_{i,j=1}^n \overline{c}_i d_j \langle e_i, e_j \rangle_C^{\mathcal{H} \otimes \mathcal{K}} \langle f_i, f_j \rangle_C^{\mathcal{K}} \]

\[ = \sum_{i=1}^n c_i d_i \langle e_i, e_i \rangle_C^{\mathcal{H} \otimes \mathcal{K}} \langle f_i, f_i \rangle_C^{\mathcal{K}} \]

we see that $\langle \Psi z^2, \Psi z \rangle_C^{\mathcal{H} \otimes \mathcal{K}} = 0$ and thereby $\langle \Psi, \psi_i \rangle_C^{\mathcal{H} \otimes \mathcal{K}} = 0$ if and only if $c_i d_i = 0$ for all $i = 1, \ldots, n$, which means $c_i = 0$ or $d_i = 0$ for all $i = 1, \ldots, n$ as any quadratic extensions of ordered rings are always integral domains, see Appendix A.1. This gives $\Psi^2 = 0$ and by the torsion-freeness $\Psi = 0$, whereby $\langle \cdot, \cdot \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$ is positive definite. 

**Remark 4.4.7**

By Remark 1.4.2, we know that for positive semi-definite inner products non-degeneracy and positive definiteness concur. Furthermore, by Lemma 1.4.4, Lemma 4.4.4, and Proposition 4.4.6, on a tensor product $\mathcal{H} \otimes \mathcal{K}$ of pre-Hilbert modules over $C$ with the induced inner product $\langle \cdot, \cdot \rangle_C^{\mathcal{H} \otimes \mathcal{K}}$ positive definiteness and torsion freeness are equivalent notions, wherefor we have the equality of the submodules

\[ \text{tor} (\mathcal{H} \otimes \mathcal{K}) = (\mathcal{H} \otimes \mathcal{K})^- \]  

(4.15)

From Proposition 4.4.6, we immediately get the following Corollary.
Corollary 4.4.8 (Condition of compatibility of tensor product and pre-Hilbert property)
Let $\mathcal{H}$ and $\mathcal{K}$ be pre-Hilbert modules over $\mathbb{C}$ such that the module $\mathcal{H} \otimes \mathcal{K}$ is torsion-free. Let $\langle \cdot, \cdot \rangle_{\mathcal{C}}^{\mathcal{H} \otimes \mathcal{K}}$ be the induced inner product. Then, $\left( \mathcal{H} \otimes \mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{C}}^{\mathcal{H} \otimes \mathcal{K}} \right)$ is a pre-Hilbert module over $\mathbb{C}$.

The following lemma follows directly from Lemma A.3.12 and Lemma A.3.16.

Lemma 4.4.9
Let $\mathcal{H}$ and $\mathcal{K}$ be modules over a commutative ring $\mathbb{C}$. Then, the following statements hold:

i.) The tensor product $\mathcal{H} \otimes \mathcal{K}$ of flat modules over an integral domain $\mathbb{C}$ is torsion-free.

ii.) The tensor product $\mathcal{H} \otimes \mathcal{K}$ of torsion-free modules over a principal ideal domain $\mathbb{C}$ is torsion-free.

The quadratic extensions of ordered rings are always integral domains. On the other hand, in the case of formal deformation quantization, the underlying ring $\mathbb{C}[[\hbar]]$ is a principal ideal domain, see Appendix A.1 for further details. Therefore, the following corollary is of special interest.

Corollary 4.4.10
Let $\mathcal{C}$ be a quadratic extension of an ordered ring. Let $\mathcal{C}$ additionally be an integral domain. Then, the ring-theoretic tensor product of pre-Hilbert modules over $\mathcal{C}$ with the induced inner product is a pre-Hilbert module over $\mathcal{C}$ if the pre-Hilbert modules involved are flat as modules over $\mathcal{C}$. Alternatively, let $\mathcal{C}$ additionally be a principal ideal domain. Then, the ring-theoretic tensor product of pre-Hilbert modules over $\mathcal{C}$ with the induced inner product is a pre-Hilbert module over $\mathcal{C}$, as torsion-free modules over a principal ideal domain are flat and pre-Hilbert modules over $\mathcal{C}$ are always torsion-free by Lemma 1.4.4.

An interesting problem is the relation of flat modules and pre-Hilbert modules. Obviously, flat modules and pre-Hilbert modules are both torsion-free, and the notions are not mutually exclusive as we can have flat pre-Hilbert modules (the obvious example being free pre-Hilbert modules). Thus, the interesting question is whether there exist non-flat pre-Hilbert modules. Example 4.4.11 gives an answer to that question, using an external tensor product of pre-Hilbert modules over $\mathcal{C}$ resulting in a degenerate inner product module.

Example 4.4.11 (Non-flat pre-Hilbert module)
Let $\mathbb{K}$ be an ordered field. Let $\mathbb{R} = \mathbb{K}[x, y]$ be polynomials in two variables over $\mathbb{K}$ and let $\mathcal{C} = \mathbb{R}(i)$. The ring $\mathbb{R}$ is ordered if an additional ordering prescription $\mathbb{R} = (\mathbb{K}[x])[y]^{\mathbb{R}}$ is introduced. Take the maximal ideal $\mathcal{H}_{\mathbb{C}} = (x\mathbb{C} + y\mathbb{C})$ of $\mathbb{C}$. Together with the inner product $\langle w, z \rangle = \overline{wz}$ the ideal $\mathcal{H}_{\mathbb{C}}$ is a pre-Hilbert module over $\mathcal{C}$ and, thus, torsion-free as a module.

Obviously, there is no $z \in \mathcal{C}$ such that $x = yz$. Therefore, $x \otimes y - y \otimes x$ is a non-zero element in $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$. Thence, $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ is not flat, as $x \otimes y - y \otimes x \in \mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ is a torsion element with regard to $xy \in \mathcal{C}$.

Another way of describing this effect is via the non-definiteness of the induced inner product:

$$\langle x \otimes y - y \otimes x, x \otimes y - y \otimes x \rangle = \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle \langle y, x \rangle - \langle x, y \rangle \langle y, x \rangle + \langle y, y \rangle \langle x, x \rangle = 0.$$  

As the tensor product $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{H}_{\mathbb{C}}$ is not torsion-free, it is not flat, whereby $\mathcal{H}_{\mathbb{C}}$ could not have been flat.

Remark 4.4.12

i.) Example 4.4.11 shows that flatness and the property of being pre-Hilbert do not necessarily coincide. In order to ensure the ring-theoretic tensor product of two pre-Hilbert modules over $\mathcal{C}$ with the induced inner product to be a pre-Hilbert module over $\mathbb{C}$ (in other words, torsion-free or non-degenerate), one should consider flat pre-Hilbert modules.

ii.) Interpreting $\mathcal{C}$ as an algebra over itself, Example 4.4.11 also gives an example of a degenerate algebraic tensor product of pre-Hilbert modules as

$$\langle x \otimes y - y \otimes x, f \otimes g \rangle = \langle y, \langle x, f \rangle g - \langle x, f \rangle g \rangle = yxf - xyf = 0$$

for all $f, g \in \mathcal{H}_{\mathbb{C}}$. Therefore, in general, it is necessary to divide by the degeneracy submodule and thus use the internal tensor product in order to achieve non-degeneracy.
4.5 The Category $\ast$-$\text{CP}_B(A)$, the Functor $\otimes$, and the Functor $\text{cSK}$

As a generalization of the category of strongly non-degenerate $\ast$-representations and pre-Hilbert bimodules introduced in Section 1.4, we no longer assume to have an algebra homomorphism, but at least a completely positive, possibly unital, $\ast$-map into the adjointable maps on a pre-Hilbert module.

Take the completely positive $\ast$-maps $\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)$. This is not a very unusual requirement, as, by Lemma 1.3.3, if $\mathcal{B}(\mathcal{H}_B)$ has sufficiently many positive linear functionals and if $\rho$ is unital, this is given. Furthermore, by Proposition 1.4.8, $\mathcal{B}(\mathcal{H}_B)$ has sufficiently many positive linear functionals if $\mathcal{B}$ has sufficiently many positive linear functionals and if $C$ is such that $2 \in \mathbb{R}$ is invertible.

As we do not treat representations but completely positive $\ast$-maps, we cannot speak of representation pre-Hilbert modules. Instead, we will call the pre-Hilbert modules, upon which the images of the completely positive maps act, target pre-Hilbert modules.

Let us define the $\ast$-category $\ast$-$\text{cp}_B(A)$ over $C$ of completely positive $\ast$-maps. The objects of $\ast$-$\text{cp}_B(A)$ are given by the completely positive $\ast$-maps $\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)$. Given two completely positive $\ast$-maps $\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)$ and $\rho' : A \rightarrow \mathcal{B}(\mathcal{H}_B')$, we define CP-intertwiners as adjointable module morphisms $T \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}_B')$ fulfilling $T \rho(a) = \rho'(a') T$ for all $a \in A$. As $\rho$ and $\rho'$ are $\ast$-maps, by

$$(T \rho(a))^* = (\rho'(a) T)^* \Leftrightarrow \rho(a^*) T^* = T^* \rho'(a^*)$$

for all $a \in A$, it is easy to see that $T^*$ is a CP-intertwiner if and only if $T$ is a CP-intertwiner. By setting the CP-intertwiners to be the morphisms, we get a $\ast$-category, as being a CP-intertwiner is a $C$-linear condition. If, in addition, we require the objects to be unital, we denote the category by $\ast$-$\text{CP}_B(A)$ and call it the category of unital completely positive $\ast$-maps.

Remark 4.5.1

We are only considering unital $\ast$-algebras. Therefore, $\ast$-$\text{Rep}_B(A)$ is a subcategory of $\ast$-$\text{CP}_B(A)$ as every $\ast$-homomorphism is a unital completely positive $\ast$-map.

Inspired by the design of the representation pre-Hilbert bimodule in the course of the construction of the Stinespring-Kasparov representation in Theorem 4.1.2, we give the following tensorial construction, leading to a functor on the $\ast$-$\text{CP}_B(A)$. This construction is called composition.

Given two completely positive unital $\ast$-maps $\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)$ and $\mu : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H}_C)$, we define a right $\mathcal{C}$-module by

$$\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C,$$  \hspace{1cm} (4.16)

where an inner product is defined on elementary tensors $\phi \otimes x \in \mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C$ and $\psi \otimes y \in \mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C$ by

$$\langle \phi \otimes x, \psi \otimes y \rangle_{\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C} = \langle x, \mu(\langle \phi, \psi \rangle_{\mathcal{H}_B}) \rangle_{\mathcal{H}_C}.\hspace{1cm} (4.17)$$

This inner product is by its construction C- and $\mathcal{C}$-sesquilinear and completely positive by an argument similar to that in the proof of Theorem 4.1.2. It is possibly degenerate on $\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C$, wherefore we will be interested in the quotient

$$\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C = (\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C) / (\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C)^\perp,$$  \hspace{1cm} (4.18)

on which it is non-degenerate by definition. Thence, $\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C$ together with the inner product (4.17) is a pre-Hilbert module after sesquilinear extension of the inner product. Next, define the composition of objects on $\mathcal{H}_B \otimes_{\mathcal{C}} \mathcal{H}_C$ by

$$\rho \otimes_{\mathcal{C}} \mu : a \mapsto (\phi \otimes x \mapsto (\rho(a)\phi) \otimes x).\hspace{1cm} (4.19)$$
In other words,

$$(\rho \otimes_B \mu)(a) = \rho(a) \otimes_C \text{id}_{\mathcal{E}_C}, \quad (4.20)$$

wherefore it is easy to see that $\rho \otimes_B \mu$ is a unital map. Then, we check that the composition of objects gives an adjointable map.

$$(\phi \otimes x, (\rho \otimes_B \mu)(a) \psi \otimes y)^{\mathcal{E}_B \otimes_C \mathcal{E}_C} = (\phi \otimes x, (\rho(a) \psi) \otimes y)^{\mathcal{E}_B \otimes_C \mathcal{E}_C} = \left( x, \mu\left( \langle \phi, \rho(a) \psi \rangle_B^2 \right) y \right)^{\mathcal{E}_C}$$

$$= \left( x, \mu\left( \langle \rho(a^*) \phi, \psi \rangle_B^2 \right) y \right)^{\mathcal{E}_C} = \langle \rho(a^*) \phi \otimes x, \psi \otimes y \rangle^{\mathcal{E}_B \otimes_C \mathcal{E}_C}$$

$$= \langle \rho(b) \phi \otimes x, \psi \otimes y \rangle^{\mathcal{E}_B \otimes_C \mathcal{E}_C}, \quad (4.21)$$

for all $a \in A$, all $\phi, \psi \in \mathcal{H}_B$, and all $x, y \in \mathcal{H}_C$, whereby $(\rho \otimes_B \mu)(a)^* = (\rho \otimes_B \mu)(a^*)$ for all $a \in A$. Therefore, the composition of objects induces a map $\rho \otimes_B \mu : A \rightarrow \mathcal{B}((\mathcal{H}_B \otimes_B \mathcal{H}_C)$ on the quotient, which we denote by the same symbol.

In order to prove the complete positivity of $\rho \otimes_B \mu : A \rightarrow \mathcal{B}(\mathcal{H}_B \otimes_B \mathcal{H}_C)$, we first consider the map $\Phi$ given for all $T \in \mathcal{B}(\mathcal{H}_B)$ and all $[\phi \otimes x] \in \mathcal{H}_B \otimes_B \mathcal{H}_C$ by

$$\Phi(T) : [\phi \otimes x] \mapsto \left[ T\phi \otimes x \right].$$

This map is well-defined, which can be seen by checking that it is adjointable.

$$\langle (T \phi) \otimes x, \psi \otimes y \rangle^{\mathcal{H}_B \otimes_C \mathcal{H}_C} = \left( x, \mu\left( \langle T\phi, \psi \rangle_B^2 \right) y \right)^{\mathcal{H}_C}$$

$$= \left( x, \mu\left( \langle \phi, T^* \psi \rangle_B^2 \right) y \right)^{\mathcal{H}_C}$$

$$= \langle \phi \otimes x, (T^* \psi) \otimes y \rangle^{\mathcal{H}_B \otimes_C \mathcal{H}_C},$$

for all $\phi \otimes x, \psi \otimes y \in \mathcal{H}_B \otimes_C \mathcal{H}_C$, whereby we see that $\Phi(T)^* = \Phi(T^*)$. Thus, $\Phi$ is also a $^*$-map. Furthermore, as $\Phi$ is adjointable, it passes to the quotient and we get $\Phi(T) \in \mathcal{B}(\mathcal{H}_B \otimes_B \mathcal{H}_C)$. Additionally, $\Phi$ is a homomorphism as

$$\Phi(T \circ S) ([\phi \otimes x]) = [(T \circ S)\phi] \otimes x = [\Phi(T)(S\phi)] \otimes x = \Phi(T)(S([\phi \otimes x])) = \Phi(T) \circ \Phi(S) ([\phi \otimes x])$$

for all $T, S \in \mathcal{B}(\mathcal{H}_B)$ and all $[\phi \otimes x] \in \mathcal{H}_B \otimes_B \mathcal{H}_C$. Hence, $\Phi$ is a $^*$-homomorphism and thus completely positive. Now, $\rho \otimes_B \mu$ is a completely positive map because, for all $a \in A$, it is equal to the composition of the completely positive maps $\rho$ and $\mu$ by considering

$$(\rho \otimes_B \mu)(a)([\phi \otimes x]) = \rho(a)(\phi)(\otimes x) = \Phi(\rho(a))(\phi \otimes x) \quad (4.22)$$

for all $[\phi \otimes x] \in \mathcal{H}_B \otimes_B \mathcal{H}_C$.

**Proposition 4.5.2**

Let $\rho : A \rightarrow \mathcal{B}(\mathcal{E}_B), \rho' : A \rightarrow \mathcal{B}(\mathcal{E}_B')$, $\mu : B \rightarrow \mathcal{B}(\mathcal{F}_C)$, and $\mu' : B \rightarrow \mathcal{B}(\mathcal{F}_C')$ be unital completely positive $^*$-maps. Furthermore, let $T \in \mathcal{B}(\mathcal{E}_B, \mathcal{E}_B')$ and $S \in \mathcal{B}(\mathcal{F}_C, \mathcal{F}_C')$ be CP-intertwiners of $\rho, \rho'$ and $\mu, \mu'$ respectively. Then, the ring-theoretic tensor product

$$T \otimes_C S : \mathcal{E}_B \otimes_C \mathcal{F}_C \rightarrow \mathcal{E}_B' \otimes_C \mathcal{F}_C' \quad (4.23)$$

induces a CP-intertwiner

$$T \otimes_B S : \mathcal{E}_B \otimes_B \mathcal{F}_C \rightarrow \mathcal{E}_B' \otimes_B \mathcal{F}_C', \quad (4.24)$$

whose adjoint is given by $T^* \otimes_B S^*$.
The composition $\otimes$ yields a functor
\[
\otimes_B : \ast\text{-CP}_{B}(A) \times \ast\text{-CP}_{C}(B) \to \ast\text{-CP}_{C}(A),
\] (4.27)
where the unital completely positive $\ast$-maps are composed as defined above and the morphisms are composed using Proposition 4.5.2.

**Proof:** The only things left to show are the preservation of the identity morphism and of the actual composition of morphisms, which are easy to see. ■

**Remark 4.5.4**
Theorem 4.5.3 allows for Rieffel induction-like constructions or changes of the base algebra. See Rieffel [71] for the Rieffel induction in the setting of $C^\ast$-algebras, see Bursztyn and Waldmann [25] for the setting of $\ast$-algebras.

The composition of objects is associative up to an isometric isomorphism, which we will check both on the level of the mappings and on the level of the underlying target pre-Hilbert modules.

**Proposition 4.5.5**
For any three unital completely positive $\ast$-maps $\rho : A \to B(\mathcal{E}_B)$, $\mu : B \to B(\mathcal{F}_C)$, $\nu : C \to B(\mathcal{S}_D)$ there exists an isometric isomorphism
\[
a(\rho, \mu, \nu) : (\mathcal{E}_B \otimes_B \mathcal{F}_C) \otimes_C \mathcal{S}_D \to \mathcal{E}_B \otimes_B (\mathcal{F}_C \otimes_C \mathcal{S}_D),
\] (4.28)
called associativity.
Proof: First, we check that there exists a well-defined isometric isomorphism

\[
(\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D / ((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D) \to (\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D
\]

(4.29)

\[
(z \otimes y) \mapsto [z \otimes y] \otimes x
\]

(4.30)

In order to do this, consider \( \phi \in ((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D)^{\perp} \) and show that \( \phi \) becomes an element in the degeneracy submodule of \((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D\). Let us denote the image of \( \phi \) in \((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D\) by \([\phi]\). A general element in \((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D\) is a finite linear combination of elements of the form \((z \otimes y) \otimes x\). Thence, for \( \phi = \sum_i [z_i \otimes y_i] \otimes x_i \) one computes

\[
\langle [\phi], [z \otimes y] \otimes x \rangle_{\mathcal{G}_D}^{\mathcal{E}_B \otimes \mathcal{F}_C \otimes \mathcal{G}_D} = \sum_i \langle [z_i \otimes y_i] \otimes x_i, [z \otimes y] \otimes x \rangle_{\mathcal{G}_D}^{\mathcal{E}_B \otimes \mathcal{F}_C \otimes \mathcal{G}_D}
\]

\[
= \sum_i \langle x_i, \nu \left((z_i \otimes y_i), [z \otimes y]\right)_{\mathcal{E}_B \otimes \mathcal{F}_C} \rangle_{\mathcal{G}_D}
\]

\[
= \sum_i \langle x_i, \nu \left(z_i \otimes y_i, z \otimes y\right)_{\mathcal{E}_B \otimes \mathcal{F}_C} \rangle_{\mathcal{G}_D}
\]

\[
= \sum_i \langle z_i \otimes y_i \otimes x_i, (z \otimes y) \otimes x \rangle_{\mathcal{G}_D}^{\mathcal{E}_B \otimes \mathcal{F}_C \otimes \mathcal{G}_D}
\]

\[
= 0
\]

(4.31)

as \( \phi \) is orthogonal to all elements of \((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D\). Thus, (4.29) is well-defined and obviously right \( \mathcal{D} \)-linear. Furthermore, by (4.31) it is clear that (4.29) is isometric. An isometric map between inner product modules with non-degenerate inner products is injective. Therefore, (4.29) is injective. It is surjective by definition, yielding an isometric bijection. Thence, we get an isometric isomorphism because the inverse equals the adjoint, as can be seen by a calculation completely analogous to (4.31).

A completely analogous statement holds for the map

\[
\mathcal{E}_B \otimes \mathcal{F}_C (\mathcal{G}_C \otimes \mathcal{G}_D) / \left(\mathcal{E}_B \otimes \mathcal{F}_C (\mathcal{G}_C \otimes \mathcal{G}_D)\right)^{\perp} \to \mathcal{E}_B \otimes \mathcal{F}_C (\mathcal{G}_C \otimes \mathcal{G}_D)
\]

(4.32)

\[
z \otimes (y \otimes x) \mapsto [z \otimes [y \otimes x]]
\]

(4.33)

Next, we show that the canonical isomorphism of right \( \mathcal{D} \)-modules

\[
(\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D \ni (z \otimes y) \otimes x \mapsto z \otimes (y \otimes x) \in \mathcal{E}_B \otimes \mathcal{G}_C (\mathcal{F}_C \otimes \mathcal{G}_D)
\]

(4.34)

induces a well-defined isometric isomorphism

\[
\mathcal{E}(\rho, \mu, \nu) : \mathcal{E}_B \otimes \mathcal{F}_C (\mathcal{G}_C \otimes \mathcal{G}_D) \to (\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{G}_D
\]

(4.35)

of inner product modules over \( \mathcal{D} \), called associativity. On elementary tensors this reads as

\[
\mathcal{A}(\rho, \mu, \nu) ([z \otimes y] \otimes x) = [z \otimes [y \otimes x]].
\]

(4.36)

Let us check on the level of representatives that (4.34) is isometric. For this, we calculate

\[
\left\langle (z \otimes y) \otimes x, (z' \otimes y') \otimes x' \right\rangle_{\mathcal{E}_B \otimes \mathcal{F}_C \otimes \mathcal{G}_D} = \left\langle x, \nu \left((z \otimes y, z' \otimes y')_{\mathcal{E}_B \otimes \mathcal{F}_C} \right) x' \right\rangle_{\mathcal{G}_D}
\]

\[
= \left\langle x, \nu \left((y, \mu (z, z')_{\mathcal{F}_C} \right) x' \right\rangle_{\mathcal{G}_D}
\]

\[
= \left\langle y \otimes x, \mu (z, z')_{\mathcal{F}_C} \right\rangle_{\mathcal{G}_D}
\]

\[
= \left\langle (z \otimes (y \otimes x), (z' \otimes (y' \otimes x'))_{\mathcal{E}_B \otimes \mathcal{F}_C \otimes \mathcal{G}_D}
\]

(4.37)
for all \( x, x' \in \mathcal{E}_B \), all \( y, y' \in \mathcal{F}_C \), and all \( z, z' \in \mathcal{S}_D \). The map (4.34) clearly is a right \( \mathcal{D} \)-linear isomorphism with the obvious inverse. Therefore, we obtain an isometric isomorphism in the quotient

\[
(\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{S}_D / ((\mathcal{E}_B \otimes \mathcal{F}_C) \otimes \mathcal{S}_D)^+ \rightarrow \mathcal{E}_B \otimes (\mathcal{F}_C \otimes \mathcal{S}_D) / (\mathcal{E}_B \otimes (\mathcal{F}_C \otimes \mathcal{S}_D))^+.
\] (4.38)

Together with the isomorphisms (4.29) and (4.32) we obtain the isometric isomorphism \( \mathfrak{a} \) that encodes the associativity of \( \mathfrak{a} \). Finally, as

\[
\begin{align*}
\rho \otimes_B \mu : & A \rightarrow \mathfrak{B} (\mathcal{E}_B \otimes_B \mathcal{F}_C), \\
\rho \otimes_B \mu : & A \rightarrow [\phi \otimes x] \mapsto [(\rho(a)\phi) \otimes x],
\end{align*}
\] (4.39)

we have

\[
(\rho \otimes_B \mu) \otimes_C \nu : a \mapsto \left( [\phi \otimes x] \otimes \chi \right) \mapsto \left( [(\rho(a)\phi) \otimes x] \otimes \chi \right)
\] (4.40)

and

\[
\rho \otimes_B (\mu \otimes_C \nu) : a \mapsto \left( [\phi \otimes (x \otimes \chi)] \mapsto [(\rho(a)\phi) \otimes (x \otimes \chi)] \right)
\] (4.41)

whereby \( \mathfrak{a} \) is a CP-intertwiner of the compositions (4.40) and (4.41) of completely positive maps, due to the associativity of the composition of target pre-Hilbert modules up to isometric isomorphisms.

The composition \( \mathfrak{a} \) is not only associative up to an isometric isomorphism but, actually, up to a natural isometric isomorphism.

**Proposition 4.5.6**

Let \( \rho, \rho' \in \ast\text{-CP}_B(A) \), \( \mu, \mu' \in \ast\text{-CP}_B(\mathfrak{B}) \), and \( \nu, \nu' \in \ast\text{-CP}_D(\mathfrak{C}) \) such that \( \rho : A \rightarrow \mathfrak{B} (\mathcal{E}_B) \), \( \rho' : A \rightarrow \mathfrak{B} (\mathcal{E}_B') \), \( \mu : \mathcal{B} \rightarrow \mathfrak{B} (\mathcal{F}_C) \), \( \mu' : \mathcal{B} \rightarrow \mathfrak{B} (\mathcal{F}_C') \), \( \nu : \mathfrak{C} \rightarrow \mathfrak{B} (\mathcal{S}_D) \), and \( \nu' : \mathfrak{C} \rightarrow \mathfrak{B} (\mathcal{S}_D') \) be CP-intertwiners of \( \rho, \rho' \), \( \nu, \nu' \), and \( \mu, \mu' \) respectively. Then, we get

\[
\mathfrak{a} (\rho', \mu', \nu') \circ ((U \otimes_B T) \otimes_C S) = (U \otimes_B (T \otimes_C S)) \circ \mathfrak{a} (\rho, \mu, \nu),
\] (4.42)

whence \( \mathfrak{a} \) is a natural transformation.

**Proof:** Let \( x \in \mathcal{E}_B \), \( y \in \mathcal{F}_C \), and \( z \in \mathcal{S}_D \) and consider \( [[x \otimes y] \otimes z] \in (\mathcal{E}_B \otimes_B \mathcal{F}_C) \otimes \mathcal{S}_D \). Then, we compute

\[
\begin{align*}
\mathfrak{a} (\rho', \mu', \nu') \circ ((U \otimes_B T) \otimes_C S)([[x \otimes y] \otimes z]) &= \mathfrak{a} (\rho', \mu', \nu') (\rho'(Ux \otimes_B T(y)) \otimes_C S(z)) \\
&= [U(x) \otimes_B [T(y) \otimes_C S(z)]] \\
&= (U \otimes_B (T \otimes_C S)) ([x \otimes [y \otimes z]]) \\
&= (U \otimes_B (T \otimes_C S)) \circ \mathfrak{a} (\rho, \mu, \nu) ([x \otimes y] \otimes z).
\end{align*}
\]

As the composition of pre-Hilbert modules is spanned by the classes of elementary tensors, the assertion is proved.

The above composition is not the only functor acting on \( \ast\text{-CP}_B(A) \). An interesting feature of the canonical Stinespring-Kasparov representation of a completely positive map is that it can be viewed as a functor in the following sense.

**Theorem 4.5.7**

The canonical Stinespring-Kasparov representations of completely positive maps induce a functor

\[
\text{cSK} : \ast\text{-CP}_B(A) \rightarrow \ast\text{-Rep}_B(A).
\] (4.43)
Proof: The functor \( \text{cSK} \) is constructed by assigning to each object \( \rho : A \to \mathcal{B}(\mathcal{H}_B) \) in \( \text{*-CP}_B(A) \) its canonical Stinespring-Kasparov representation \( \pi_\rho : A \to \mathcal{B}(A \otimes_\rho \mathcal{H}_B) \) and to each CP-intertwiner \( T \in \mathcal{B}\left(\mathcal{H}_B, \mathcal{H}_B'\right) \) the map \( \widetilde{T} = \text{id}_A \otimes \rho T \in \mathcal{B}\left(A \otimes_\rho \mathcal{H}_B, A \otimes_\rho \mathcal{H}_B'\right) \). It is then easy to check that \( \widetilde{T}\pi_\rho = \pi_\rho \widetilde{T} \), as

\[
\widetilde{T}\pi_\rho(a)([a' \otimes h]) = \widetilde{T}([aa' \otimes h]) = [aa' \otimes Th] = \pi_\rho(a)([a' \otimes Th]) = \pi_\rho(a)([a' \otimes h])
\]

for all \( a, a' \in A \) and all \( h \in \mathcal{H}_B \). First, we show that \( \text{id}_A \otimes_\mathbb{C} T \) is an adjointable map in \( \mathcal{B}(A \otimes_\mathbb{C} \mathcal{H}, A \otimes_\mathbb{C} \mathcal{H}') \) with adjoint \( (\text{id}_A \otimes_\mathbb{C} T)^* = \text{id}_A \otimes_\mathbb{C} T^* \) by checking

\[
\left( \sum_j b_j \otimes h_j', \widetilde{T} \sum_i a_i \otimes h_i \right)_{\mathcal{B}}^{A \otimes_\mathbb{C} \mathcal{H}'} = \sum_{i,j} \left( b_j \otimes h_j', a_i \otimes Th_i \right)_{\mathcal{B}}^{A \otimes_\mathbb{C} \mathcal{H}'} = \sum_{i,j} \left( h_j', \rho(b_j^* a_i)Th_i \right)_{\mathcal{B}}^{\mathcal{H}'} = \sum_{i,j} \left( h_j', T\rho(b_j^* a_i)h_i \right)_{\mathcal{B}}^{\mathcal{H}'} = \sum_{i,j} \left( T^* h_j', \rho(b_j^* a_i)h_i \right)_{\mathcal{B}}^{\mathcal{H}'} = \left( \sum_j b_j \otimes T^* h_j', \sum_i a_i \otimes h_i \right)_{\mathcal{B}}^{A \otimes_\mathbb{C} \mathcal{H}'} ,
\]

whereby \( (\text{id}_A \otimes_\mathbb{C} T)^* \) and \( \text{id}_A \otimes_\mathbb{C} T \) descend to the quotient. Therefore, \( \widetilde{T} \in \mathcal{B}\left(A \otimes_\rho \mathcal{H}, A \otimes_\rho \mathcal{H}'\right) \) and \( \widetilde{T}^* \in \mathcal{B}\left(A \otimes_\rho \mathcal{H}', A \otimes_\rho \mathcal{H}\right) \) are well-defined intertwiners. By the construction of intertwiners from CP-intertwiners, it is obvious that \( \text{cSK} \) maps the composition of CP-intertwiners to the composition of intertwiners and identities to identities.

Furthermore, the functor \( \text{cSK} \) inherits certain uniqueness properties of the canonical Stinespring-Kasparov construction.

**Proposition 4.5.8**

The functor

\[
\text{cSK} : \text{*-CP}_B(A) \to \text{*-Rep}_B(A).
\]

is injective.

Proof: By Lemma 4.1.13, the function assigning to every completely positive unital \( \text{*-} \)-map between unital \( \text{*-} \)-algebras its canonical Stinespring-Kasparov representations is injective. Furthermore, differing CP-intertwiners lead to differing intertwiners by an argument analogous to the one for completely positive maps. Therefore, the functor \( \text{cSK} \) is injective.

**Remark 4.5.9**

The functor \( \text{cSK} \) is certainly not surjective as can be seen by the non-canonical Stinespring-Kasparov representation bimodule \( \text{A} \otimes_\text{A} \text{id}_A \in \text{*-SKRep}_A(A) \) of Remark 4.2.4, induced by the identity map on a unital \( \text{*-} \)-algebra \( A \), which certainly is not a canonical Stinespring-Kasparov representation of any completely positive map.
As $^*\text{Rep}_B(A)$ is a subcategory of $^*\text{CP}_B(A)$, the functor $cSK$ restricts to a functor

$$cSK : ^*\text{Rep}_B(A) \to ^*\text{CP}_B(A).$$

(4.46)

Actually, the functor $cSK$ maps into the subcategory $^*\text{SKRep}_B(A)$. Considering a $^*$-homomorphism

$$\sigma : A \to \mathcal{B}(\mathcal{H}_B),$$

(4.47)

we know by Remark 4.2.4 that the choice $V = id_{\mathcal{H}_B}$ leads to a minimal Stinespring-Kasparov representation of $\sigma$ given by $(\mathcal{H}_B, \omega, id_{\mathcal{H}_B})$, whereby we can identify every element of $^*\text{Rep}_B(A)$ given by a $^*$-homomorphism $\sigma$ with that particular element of $^*\text{SKRep}_B(A)$. Let us call this minimal Stinespring-Kasparov representation of $\sigma$ the identical Stinespring-Kasparov representation. Actually, the identification of a $^*$-representation with its identical Stinespring-Kasparov representation gives another functor

$$iSK : ^*\text{Rep}_B(A) \to ^*\text{SKRep}_B(A).$$

(4.48)

By Lemma 4.1.11, we know that two minimal Stinespring-Kasparov representations with respect to the same completely positive map are unitarily equivalent, i.e. the resulting representation pre-Hilbert bimodules are isometrically isomorphic. Thence, the identical Stinespring-Kasparov representation $(\mathcal{H}_B, \sigma, id_{\mathcal{H}_B})$ of $\sigma$ is unitarily equivalent to the canonical Stinespring-Kasparov representation of $\sigma$ given by $(\mathcal{K}_B, \pi_\sigma, V_\sigma)$. Then, we show that the isometric isomorphism of the representation pre-Hilbert bimodules of the identical and the canonical Stinespring-Kasparov representations of any $^*$-representation $\sigma$,

$$\tau : ^*\text{SKRep}_B(A) \ni A^*\mathcal{H}_B \to A^*\mathcal{K}_B \in ^*\text{SKRep}_B(A),$$

(4.49)

is a natural isomorphism between the functors $iSK$ and the functor $cSK$. As we already know that $\tau$ is an isometric isomorphism of the representation pre-Hilbert bimodules, we only have to show the compatibility of $\tau$ with $iSK$ and $cSK$ in order to show the naturality.

**Proposition 4.5.10**

Let $\sigma : A \to \mathcal{B}(\mathcal{H}_B)$ be a $^*$-homomorphism with the identical Stinespring-Kasparov representation $(\mathcal{H}_B, \sigma, id_{\mathcal{H}_B})$ and let $\pi_\sigma : A \to \mathcal{B}(\mathcal{K}_B)$ be the canonical Stinespring-Kasparov representation of $\sigma$ given by $(\mathcal{K}_B, \pi_\sigma, V_\sigma)$. Let $\sigma' : A \to \mathcal{B}(\mathcal{H}_B')$ be a $^*$-homomorphism with the identical Stinespring-Kasparov representation $(\mathcal{H}_B', \sigma', id_{\mathcal{H}_B'})$ and let $\pi_{\sigma'} : A \to \mathcal{B}(\mathcal{K}_B')$ be the canonical Stinespring-Kasparov representation of $\sigma'$ given by $(\mathcal{K}_B', \pi_{\sigma'}, V_{\sigma'})$. Let $T \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_B')$ be an intertwiner of $\sigma$ and $\sigma'$. Then

$$cSK(T) \circ \tau = \tau \circ iSK(T),$$

(4.50)

whereby $\tau$ is a natural isomorphism.

**Proof:** Let $h \in \mathcal{H}_B$. Then,

$$cSK(T) \circ \tau(id_{\mathcal{H}_B} h) = cSK(T)(V_\sigma h) = cSK(T)\left([1 \otimes h]_{\pi_\sigma}\right) = [1 \otimes Th]_{\pi_{\sigma'}},$$

and

$$\tau \circ iSK(T)(id_{\mathcal{H}_B} h) = \tau(id_{\mathcal{H}_B} Th) = V_{\sigma'}(Th) = [1 \otimes Th]_{\pi_{\sigma'}},$$

whereby the assertion follows.

$\blacksquare$
4.6 Bicategorical Results

The bicategory $\text{Bimod}^{\text{str}}$ encoding the strong Morita equivalence of its underlying $^*$-algebras is closely related to $^*$-representations. As we are only considering unital $^*$-algebras, in our case, the objects of $\text{Bimod}^{\text{str}}$ are the unital $^*$-algebras. The 1-morphisms and the 2-morphisms are given by the categories $^*$-$\text{Rep}_{\mathcal{B}}(A)$ of strongly non-degenerate $^*$-representations with respect to pairs of unital $^*$-algebras. The composition is given by the internal tensor product and the identity 1-morphism with respect to an object $\mathcal{A}$ is given by the bimodule $A_A \in {^*\text{Rep}_{\mathcal{A}}(A)}$. A ring-theoretic version has been given in Benabou [11]. Good references for the strong version on $^*$-algebras are given by Waldmann [79] and Bursztyn and Waldmann [28].

In the following, we construct a right-unital $^*$-bicategory from completely positive unital $^*$-maps as an extension of $\text{Bimod}^{\text{str}}$, thereby relating the existence of completely positive $^*$-maps between $^*$-algebras and the question of strong Morita equivalence of those $^*$-algebras.

Remember the category $^*$-$\text{CP}_{\mathcal{B}}(A)$ and the functor $\otimes$ from Section 4.5. We induce the non-unital bicategory $\text{CP}^*$ from the category $^*$-$\text{CP}_{\mathcal{B}}(A)$ and the functor $\otimes$.

Let the class $\text{CP}_0$ of objects of $\text{CP}^*$ consist of all unital $^*$-algebras $A$ over the quadratic extension $\mathbb{C}$ of an ordered ring $\mathbb{R}$. Then, take for each two objects $A, B \in \text{CP}_0$ the category $^*$-$\text{CP}_{\mathcal{B}}(A)$ and set the objects $\text{Ob}(^*$-$\text{CP}_{\mathcal{B}}(A))$ of $^*$-$\text{CP}_{\mathcal{B}}(A)$ as the 1-morphisms $^*$-$\text{CP}^1(\mathcal{A}, \mathcal{B})$ from $A$ to $B$. The morphisms $\text{Morph}(^*$-$\text{CP}_{\mathcal{B}}(A))$ of the category $^*$-$\text{CP}_{\mathcal{B}}(A)$, given by the CP-intertwiners $T_\rho = \mu T$ are the 2-morphisms $^*$-$\text{CP}^2(\rho, \mu)$ from $\rho$ to $\mu$. For each three objects $A, B, C \in \text{CP}_0$, we take the composition functor $\otimes$ for the 1-morphisms and the 2-morphisms as in Section 4.5. By Proposition 4.5.5 and Proposition 4.5.6, we know that the composition $\otimes$ is associative up to a natural isometric isomorphism.

Finally, we have to check the associativity coherence condition. From the definition and the properties of the composition $\otimes$ in Section 4.5 both on the level of target pre-Hilbert modules and on the level of completely positive maps, the associativity coherence condition is fulfilled.

As $\mathfrak{a}$ is unitary, the conditions of being a non-unital $^*$-bicategory of Section 1.4 are all fulfilled as we constructed $\text{CP}^*$ from a $^*$-category basing the composition on ring-theoretic tensor products. Thus, we get the following theorem.

Theorem 4.6.1

$\text{CP}^*$ is a non-unital $^*$-bicategory.

Furthermore, $\text{CP}^*$ is a non-unital extension of $\text{Bimod}^{\text{str}}$ in the sense that $\text{Bimod}^{\text{str}}$ is embedded in $\text{CP}^*$.

Theorem 4.6.2

$\text{Bimod}^{\text{str}}$ is embedded in $\text{CP}^*$ as a non-unital $^*$-bicategory.

Proof: In both cases, the objects are given by the unital $^*$-algebras. The 1-morphisms of $\text{Bimod}^{\text{str}}$ are given by pre-Hilbert bimodules. As a pre-Hilbert bimodule consists of a pre-Hilbert module with a compatible $^*$-representation, we can interpret the 1-morphisms of $\text{Bimod}^{\text{str}}$ as 1-morphisms of $\text{CP}^*$ because every $^*$-homomorphism is a unital completely positive $^*$-map. The 2-morphisms of $\text{Bimod}^{\text{str}}$ are the intertwiners of the $^*$-representations included in the 1-morphisms. These intertwiners obviously are CP-intertwiners if we interpret the $^*$-representations as completely positive maps.

Let $\pi_1 : A \rightarrow \mathcal{B}(\mathcal{H}_B)$ and let $\pi_2 : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H}_C)$ be compatible $^*$-representations on pre-Hilbert modules. Then

$$\langle [\phi \otimes x], [\psi \otimes y] \rangle_{\mathcal{H}_C} \otimes_{\mathcal{B}} \mathcal{H}_C = \langle x, \pi_2 (\langle \phi, \psi \rangle_{\mathcal{B}}) y \rangle_{\mathcal{H}_C}$$

for all $[\phi \otimes x], [\psi \otimes y] \in \mathcal{H}_B \otimes_{\mathcal{B}} \mathcal{H}_C$ and

$$\langle [\phi \otimes x], [\psi \otimes y] \rangle_{\mathcal{H}_C} \otimes_{\mathcal{B}} \mathcal{H}_C = \langle x, \pi_2 (\langle \phi, \psi \rangle_{\mathcal{B}}) y \rangle_{\mathcal{H}_C}$$

(4.51)
4.6. Bicategorical Results

for all \([\phi \otimes x], [\psi \otimes y] \in \mathcal{H}_B \otimes_B \mathcal{H}_C\). Thus, for \(*\)-homomorphisms as completely positive unital \(*\)-maps, the definitions of the internal tensor product \(\otimes_B\) and of the composition \(\circ_B\) concur.

As \(\text{Bimod}^\text{str}\) is embedded in \(\mathbf{CP}^\text{str}\), for \(*\)-homomorphisms, the different concepts of an identity element would have to concur. Indeed, interpreting \(\mathcal{A}_A \in \mathbf{\text{Rep}}_A(A)\) as a \(*\)-representation \(\text{id}_A : A \rightarrow A \cong \mathcal{B}(A, A)\), the only candidate for an identity 1-morphism \(\text{id}_A \in \mathbf{CP}^\text{str}_1(A, A)\) that exists for each object \(A \in \mathbf{CP}^\text{str}_0\) is given by \(\text{id}_A : A \rightarrow A \cong \mathcal{B}(A, A)\) with the identification from Example 1.4.6.

Using this induced identity, \(\mathbf{CP}^\text{str}\) is a right-unital bicategory. To illustrate this, we consider the relevant construction. For each two objects \(A, B \in \mathbf{CP}^\text{str}_0\) and for \(\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)\), we define on elementary tensors

\[
\text{right}(\rho) : \mathcal{H}_B \otimes_B \mathcal{B}_B \rightarrow \mathcal{H}_B
\]

\[
[h \otimes b] \mapsto hb
\]

(4.53)

for all \(h \in \mathcal{H}_B\) and all \(b \in \mathcal{B}_B\). Obviously, \text{right} is surjective. It is isometric on representatives

\[
\langle \phi \otimes b, (\rho \otimes_B \text{id}_B) (a)(\phi' \otimes b') \rangle_{\mathcal{H}_B \otimes_C \mathcal{B}_B} = \langle \phi \otimes b, (\rho(a)\phi') \otimes b' \rangle_{\mathcal{H}_B \otimes_C \mathcal{B}_B}
\]

\[
= \langle b, \text{id}_B (\langle \phi, \rho(a)\phi' \rangle_{\mathcal{H}_B}) b' \rangle_{\mathcal{B}_B}
\]

\[
= \langle \phi, \rho(a)\phi' \rangle_{\mathcal{B}_B} b'
\]

\[
= \langle \phi b, \rho(a)\phi' \rangle_{\mathcal{B}_B}
\]

(4.54)

for all \(a \in \mathcal{A}\), all \(b, b' \in \mathcal{B}\), and all \(\phi, \phi' \in \mathcal{H}_B\) as \(\mathcal{B}_B \cong \mathcal{B}\) and hence \(\mathcal{H}_B \mathcal{B}_B = \mathcal{H}_B\). As it is isometric, it passes to the quotient. As right is surjective and isometric between pre-Hilbert modules, it is unitary and an isometric isomorphism. Furthermore, \text{right}(\rho)\) is a morphism between \(\rho(a)\) and \((\rho \otimes_B \text{id}_B) (a)\), which follows from

\[
\rho(a) \circ \text{right}(\rho)((h \otimes b)) = \rho(a)hb = \text{right}(\rho)(([\rho(a)h \otimes b])) = \text{right}(\rho) \circ (\rho \otimes_B \text{id}_B)(a)([h \otimes b])
\]

(4.55)

for all \(h \in \mathcal{H}_B\), all \(a \in \mathcal{A}\), and all \(b \in \mathcal{B}_B\) by the right-\(\mathcal{B}\)-linearity of \(\mathcal{B}(\mathcal{H}_B)\) and as

\[
\rho \otimes_B \text{id}_B : a \mapsto ([h \otimes b] \mapsto ([\rho(a)h] \otimes b))
\]

Finally, \text{right} is a natural isomorphism.

Proposition 4.6.3

Let \(\rho : A \rightarrow \mathcal{B}(\mathcal{H}_B)\) and \(\rho' : A \rightarrow \mathcal{B}(\mathcal{H}'_B)\) be 1-morphisms from \(A\) to \(\mathcal{B}\). Let \(T \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}'_B)\) be a \(\mathbf{CP}\)-intertwiner, that is \(T \rho(a) = \rho'(a)T\) for all \(a \in \mathcal{A}\). Then,

\[
T \circ \text{right}(\rho) = \text{right}(\rho') \circ (T \otimes_B \text{id}_B).
\]

(4.56)

Proof: For elementary tensors we obtain

\[
T \circ \text{right}(\rho)(a)([\phi \otimes b]) = T(\phi b) = (T\phi)b
\]

\[
= \text{right}(\rho')(a)((T\phi) \otimes b) = \text{right}(\rho') (a) \circ (T \otimes_B \text{id}_B)([\phi \otimes b])
\]

(4.57)

for all \(a, a' \in \mathcal{A}\), all \(b \in \mathcal{B}\), and all \(\phi \in \mathcal{H}_B\), wherefore the assertion holds.

Therefore, we get the following theorem.

Theorem 4.6.4

\(\mathbf{CP}^\text{str}\) is a right-unital \(*\)-bicategory.

\(\mathbf{CP}^\text{str}\) is a right-unital extension of \(\text{Bimod}^\text{str}\) as the inner products concur on \(*\)-homomorphisms and as the identity on \(\mathbf{CP}^\text{str}\) is induced from \(\text{Bimod}^\text{str}\).
Theorem 4.6.5
\textbf{Bimod}^{str} \textit{is embedded in } \textbf{CP}^{*} \textit{as a right-unital } \ast\textit{-bicategory.}

Remark 4.6.6
There is a defect keeping \textbf{CP}^{*} \textit{from being a left-unital bicategory. It shows in the attempted construction of } \textit{left}, \textit{where an algebraic obstruction appears, which only vanishes for homomorphisms. In the following, we give details on that defect.}

Define \textit{left} on elementary tensors by
\[
\textit{left}(\rho) : \mathcal{A} \otimes \mathcal{H}_{B} \to \mathcal{H}_{B}
\]
\[
[a \otimes h] \mapsto \rho(a) h.
\]
(4.58)

As \textit{left} is not isometric it is not well-defined. Actually, the lack of isometry stems from the same obstruction as we show below, but it is easier to see in what follows. By
\[
\rho(a') \circ \textit{left}(\rho) ([a \otimes h]) = \rho(a') \rho(a) h
\]
\[
\textit{left}(\rho) \circ (\text{id}_{\mathcal{A}} \otimes \rho(a')) (a') ([a \otimes h]) = \textit{left}(\rho) ([a' a \otimes h]) = \rho(a') h
\]
(4.59)
for all \(a, a' \in \mathcal{A}\), we see that \textit{left} could only be a natural isomorphism if \(\rho\) was a \ast\text{-homomorphism.}

Remark 4.6.7
The obstruction for the existence of a left identity shows that \textbf{CP}^{*} \textit{is a real generalization of } \textbf{Bimod}^{str}.

Remark 4.6.8
In the Sections 4.5 and 4.6, we could see that the slight generalization from \ast\text{-homomorphisms to unital completely positive } \ast\text{-maps leads to generalized categorial structures, which are still compatible with and include the categorial structures concerning } \ast\text{-homomorphisms. Considering the deliberations in the introduction to Chapter 4, the time evolution maps of interesting, non-conservative physical systems generally are unital completely positive } \ast\text{-maps and not } \ast\text{-homomorphisms. Thus, from a physical point of view, a more exhaustive investigation of the categorial properties of unital completely positive } \ast\text{-maps might result in interesting new mathematical tools for the description and treatment of non-conservative systems.}
Appendix A

Rings, Algebras, and Modules

In this chapter, some introductory definitions, relations, and results of the theory of rings, algebras, and modules are given in order to facilitate the understanding of Chapter 1. Most of the content of this chapter can be found in introductory textbooks, e.g. the books of Lam [56], Lang [57], Eisenbud [37], and Atiyah and MacDonald [3].

A.1 Rings

Definition A.1.1 (Ring)
A ring \( R \) is a set, together with two laws of composition (multiplication and addition), satisfying the following conditions:

\begin{enumerate}
\item \( R \) is a commutative group with respect to addition.
\item The multiplication is associative and has a unit element.
\item For all \( r, s, t \in R \) we have distributivity:
\[ (r + s)t = rt + st \quad \text{and} \quad r(s + t) = rs + rt. \] (A.1)
\end{enumerate}

As usual, the unit element of addition is denoted by 0, the unit element of multiplication by 1, where, in general, we do not assume 0 \( \neq \) 1. Furthermore, \( rs = sr \) for all \( r, s \in R \), whereby we always assume a ring \( R \) to be commutative.

Remark A.1.2 (Zero ring)
A ring \( R \) where 1 = 0 is just the zero ring, as for every \( r \in R \) the equation \( r = 1r = 0r = 0 \) holds. Therefore, in the following, we will only consider rings with 0 \( \neq \) 1, obviously without loss of generality.

A ring homomorphism is a mapping \( \phi \) of a ring \( R \) into a ring \( S \) such that

\begin{enumerate}
\item \( \phi(r + r') = \phi(r) + \phi(r') \), so that \( \phi \) is a homomorphism of commutative groups,
\item \( \phi(rr') = \phi(r)\phi(r') \),
\item \( \phi(1) = 1 \).
\end{enumerate}

In other words, \( \phi \) respects addition, multiplication, and the identity element.

A subset \( S \) of a ring \( R \) is a subring of \( R \) if \( S \) is closed under addition and multiplication and contains the identity element of \( R \). The identity mapping of \( S \) into \( R \) then is a ring homomorphism. Obviously, the composition of ring homomorphisms is a ring homomorphism.
A non-zero element $a$ of a commutative ring is called a zero divisor if there exists a non-zero element $b$ such that $ab = 0$. An integral domain is a commutative ring which has no zero divisors with an additive identity 0 and a multiplicative identity 1 such that $0 \neq 1$.

An ideal $I$ of a ring $R$ is a subset of $R$ which is an additive subgroup and for which the statement $RI \subseteq I$ holds. The quotient group $R/I$ inherits a uniquely defined multiplication from $R$ which makes it into a ring, the quotient ring. The elements of $R/I$ are cosets of $I$ in $R$, and the mapping $\phi : R \rightarrow R/I$ which maps each $r \in R$ to its coset $r + I$ is a surjective ring homomorphism.

An ideal $I \subseteq R$ is called a principal ideal if it is generated by a single element of $R$. The multiples $Rr$ of a ring element $r$ obviously form a principal ideal. A ring $R$ is called a principal ideal ring if every ideal is a principal ideal. A principal ideal domain is an integral domain in which every ideal is principal.

Example A.1.3 (Principal ideal domains)
Examples of principal ideal domains relevant for deformation quantization are:

i.) the ring of integers $\mathbb{Z}$,

ii.) any field $\mathbb{K}$,

iii.) rings of polynomials $\mathbb{K}[\lambda]$ in one variable with coefficients in a field,

iv.) rings of formal power series $\mathbb{K}[[\lambda]]$ in one variable with coefficients in a field.

Later on, we will get interesting examples of pre-Hilbert modules by considering finitely generated ideals. Therefore, some knowledge about the relations between elements of a ring generating either the same ideal or disjoint ideals will prove to be helpful.

Definition A.1.4 (Unit elements)
An element $r \in R$, such that $rs = 1$ for some $s \in R$, is called unit or unit element in $R$.

Lemma A.1.5 (Unit elements)
$r \in R$ is a unit $\iff$ $Rr = R = R1$

Proof: Let $Rr = R$. Then there exists an element $s \in R$ such that $sr = 1$. Conversely, let $s \in R$ be such that $sr = 1$. Then $Rr$ contains the 1. As $R(Rr) \subseteq Rr$ by the definition of an ideal, we therefore have $Rr = R$. ■

Example A.1.6 (Unit elements)
A field is a ring in which $1 \neq 0$ and every non-zero element is a unit.

Lemma A.1.7 (Generating identical ideals)
Let $R$ be an integral domain, and let $r, s$ be non-zero elements of $R$. Then $r, s$ generate the same ideal if and only if there exists a unit element $u \in R$ such that $r = us$.

Proof: Let $u \in R$ be such a unit. We then have $Rr = Rru = Rs$. Conversely, let $Rr = Rs$. Then, we can write $r = sa$ and $s = rb$ with $a, b \in R$. Therefore, $r = rba$, whereby $r(1 - ba) = 0$ and, therefore, $ab = 1$ and $a$ is a unit. ■

Lemma A.1.8 (Generating subideals)
Let $R$ be an integral domain, and let $r, s$ be non-zero elements of $R$. Then $Rr \subseteq Rs$ if and only if there exists an element $t \in R$ such that $r = ts$

Proof: Let $t \in R$ such that $r = ts$. We then have $Rr = Rts \subseteq Rs$. Conversely, let $Rr \subseteq Rs$. Then, we can write $r = ts$ with $t \in R$, because $r \in Rs$. ■
Corollary A.1.9 (Generating non-enclosed ideals)
Let $\mathbb{R}$ be an integral domain, and let $r, s$ be non-zero elements of $\mathbb{R}$ such that $\mathbb{R}r \not\subseteq \mathbb{R}s$ and $\mathbb{R}s \not\subseteq \mathbb{R}r$. Then, there are no elements $t, t' \in \mathbb{R}$ such that $r = ts$ and $t'r = s$.

Proof: The statement is obvious by Lemma A.1.8.

A counter-example to principle ideal domains is indicated by the preceding Lemmata.

Example A.1.10 (Integral domain with non-principal ideal)
Let $\mathbb{R} = \mathbb{K}[x, y]$ be the ring of polynomials in two variables over a field $\mathbb{K}$. Take the maximal ideal $I = x\mathbb{R} + y\mathbb{R}$. As there is no $r \in \mathbb{R}$ such that $x = ky$ or $y = kx$, the ideal $I$ obviously is not generated by a single element of $\mathbb{R}$, and therefore $\mathbb{R}$ is not a principal ideal domain.

Definition A.1.11 (Ordered ring)
A commutative ring with unit $1 \neq 0$ is called ordered if it is the disjoint union $\mathbb{R} = \mathbb{P} \cup \{0\} \cup \mathbb{P}$ of positive elements $\mathbb{P}$, negative elements $-\mathbb{P}$, and $0$, where the requirements $\mathbb{P} \cdot \mathbb{P} \subseteq \mathbb{P}$ and $\mathbb{P} + \mathbb{P} \subseteq \mathbb{P}$ are fulfilled.

On such a disjoint union we can define an ordering relation $< \mathbb{R}$ by setting $r < s$ if $r - s \in \mathbb{P}$. Analogously, the symbols $\geq$, and $\leq$ are used, where $r = s$ means $r - s = 0$. Finally, one defines the absolute value function by $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$.

Lemma A.1.12
Every commutative ordered ring is an integral domain.

Obviously, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are ordered rings, as is any ordered field. Furthermore, if $\mathbb{R}$ is an ordered ring, then the ring of polynomials $\mathbb{R}[x]$ and the ring of formal power series $\mathbb{R}[[x]]$ are ordered rings by calling a polynomial or formal series positive if the lowest non-trivial order is positive.

In an ordered ring $\mathbb{R}$, all squares of non-zero elements are positive, and therefore especially $1 = \mathbb{I} \in \mathbb{P}$, where $\mathbb{R}$ has characteristic $0$ as $1 + \cdots + 1 \in \mathbb{P}$.

Now take the quadratic ring extension $\mathbb{C} = \mathbb{R}(i)$ by a square root of $-1$ with the usual addition and multiplication observing $i^2 = -1$. For $z \in \mathbb{C}$ we write $z = a + ib$, as usual. Then, the complex conjugation is an involutive $\mathbb{R}$-linear ring automorphism of $\mathbb{C}$. Furthermore, $z \in \mathbb{C}$ is real if $\mathbb{z} = z$, and we have $\mathbb{a}^2 + b^2 = \mathbb{z}^2 \in \mathbb{R}$ for all $z \in \mathbb{C}$.

A.2 Algebras

There is a close relationship between rings and unital associative algebras, and at least in the case of commutative rings and commutative unital associative algebras this relationship can be made precise in the following way.

Let $\phi : \mathbb{R} \to \mathbb{S}$ be a ring homomorphism of commutative rings. Let $r \in \mathbb{R}$ and $s \in \mathbb{S}$, and define a product $r \cdot s = \phi(r)s$. Interpreting $\mathbb{S}$ as an algebra over the ring $\mathbb{R}$ by viewing $\cdot$ as scalar multiplication, a commutative unital associative algebra is given by two rings, a ring homomorphism, and a natural definition of scalar multiplication. The same line of thought gives a close relation between rings, associative algebras, and modules.

Definition A.2.1 (Algebra)
An algebra $A$ over a ring $\mathbb{R}$ is a set, together with two laws of composition (addition and multiplication), satisfying the following conditions:

i.) $A$ is a commutative group with respect to addition.

ii.) The multiplication is associative.
iii.) There exists a mapping \( \mu : R \times A \to A \) such that, if we write \( ra \) for \( \mu(r, a) \) (\( r \in R, a \in A \)), the following axioms are satisfied:

\[
(r(a + b)) = ra + rb, \\
((r + s)a) = ra + sa, \\
((rs)a) = r(sa), \\
1a = a
\]

for all \( r, s \in R, a, b \in A \).

We call an algebra unital if it contains the unit element 1 with respect to multiplication.

Again, as in the case of rings, a distinction between 0 and 1 comes naturally for unital algebras.

**Remark A.2.2 (Zero algebra)**

A unital algebra \( A \) where \( 1 = 0 \) is just the zero algebra, as for every \( a \in A \) the equation \( a = a1 = a0 = 0 \) holds.

### A.3 Modules

In the following, we only consider commutative rings.

**Definition A.3.1 (Right modules over rings)**

Let \( R \) be a ring. A right module over \( R \), or a right \( R \)-module \( M \) is an Abelian group together with an operation of \( R \) on \( M \) such that for all \( r, s \in R \) and \( m, n \in M \) the relations

\[
(r(sm)) = (rs)m, \quad (r + s)m = rm + sm, \quad \text{and} \quad r(m + n) = rm + rn
\]

(A.2)

are valid.

**Remark A.3.2 (Modules over rings)**

In a similar way, one defines left \( R \)-modules and \( (R, R) \)-bimodules. In the following sections, we shall deal only with right \( R \)-modules, unless specified differently. Therefore, we call these simply \( R \)-modules \( M \) or even modules \( M \) if the reference is clear.

Further on, we will need the ring-theoretic tensor product \( M \otimes R N \) of two \( R \)-modules \( M \) and \( N \). This product is generated by formal products \( m \otimes n \) where \( m \in M \) and \( n \in N \). Thus, the elements are of the form \( \sum_{i=1}^{s} m_i \otimes n_i \), where \( m_i \in M \), \( n_i \in N \), and \( s \in N \) is finite. This product should behave similarly to the point-wise product, but not be actually realized, that is the tensor product of \( m \) and \( n \) just gives the abstract “value” \( m \otimes n \). Therefore, we demand that

\[
(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, \\
m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, \\
(m \otimes n)r = (mr) \otimes n = m \otimes (nr),
\]

where \( m, m_1, m_2 \in M \), \( n, n_1, n_2 \in N \), and \( r \in R \).

A submodule of an \( R \)-module \( M \) is an Abelian subgroup of \( M \) closed under the action of \( R \).

**Definition A.3.3 (Factor module)**

Let \( R \) be a commutative ring. For an \( R \)-module \( M \) and a submodule \( N \subseteq M \), the factor module \( M/N \) is given by the set of equivalence classes of elements of \( M \) by the equivalence relation

\[
m_1 \equiv m_2 \mod N \iff m_1 - m_2 \in N
\]
with the uniquely determined module structure, for which the canonical surjective map $M \mapsto M/N$ is a homomorphism:

$$a \cdot (m + N) = am + N.$$ 

**Definition A.3.4 (Tensor product I)**

Let $R$ be a commutative ring, let $M, N$ be $R$-modules. An $R$-module $\mathcal{P}$ together with an $R$-bilinear mapping $h : M \times N \rightarrow \mathcal{P}$ is called tensor product if for any $R$-bilinear mapping $g : M \times N \rightarrow \Omega$, where $\Omega$ is an arbitrary $R$-module, there exists exactly one other $R$-linear mapping $f : \mathcal{P} \rightarrow \Omega$ such that $g = f \circ h$ up to isomorphy.

**Theorem A.3.5 (Tensor product)**

For all $R$-modules $M$ and $N$ there exists a tensor product of $M$ and $N$ over $R$. Any two such tensor products are isomorphic.

**Definition A.3.6 (Tensor product II)**

For two $R$-modules $M$ and $N$ the expression $M \otimes_R N$ denotes a choice of a tensor product of $M$ and $N$ over $R$ with the bilinear mapping $\cdot \otimes \cdot : M \times N \rightarrow M \otimes_R N$, $(x, y) \mapsto x \otimes y$.

**Remark A.3.7 (Tensor product)**

The map $\cdot \otimes \cdot$ is the “universal” bilinear map on $M \times N$, as $\text{Hom}_R(M \otimes_R N, \mathcal{P}) \cong \text{Hom}_R(M, \text{Hom}_R(N, \mathcal{P}))$ are naturally isomorphic as $R$-modules.

**Remark A.3.8 (Tensor product of homomorphisms)**

Let $f_i \in \text{Hom}_R(M_i, N_i)$ for $R$-modules $M_i$ and $N_i$ and $i = 1, 2$. The tensor product $(f_1 \otimes f_2)(m \otimes n) = f_1(m) \otimes f_2(n)$ defines a linear mapping $\text{Hom}_R(M_1, N_1) \otimes_R \text{Hom}_R(M_2, N_2) \rightarrow \text{Hom}_R(M_1 \otimes_R M_2, N_1 \otimes_R N_2)$. Thus, it is easy to see that the composition of homomorphisms is compatible with the tensor product.

**Lemma A.3.9**

Let $M, N, \mathcal{P}$ be $R$-modules and let $M_i$ be a family of $R$-modules.

i.) $(M \otimes_R N) \otimes_R \mathcal{P} \cong M \otimes_R (N \otimes_R \mathcal{P})$ by $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$.

ii.) $M \otimes_R N \cong N \otimes_R M$ by $m \otimes n \mapsto n \otimes m$.

iii.) $(\bigoplus_i M_i) \otimes_R N \cong \bigoplus_i (M_i \otimes_R N)$ by $(\sum_i m_i) \otimes n \mapsto \sum_i (m_i \otimes n)$.

**Definition A.3.10 (Flat module)**

Let $R$ be a commutative ring and let $A$ be a module. A module $M$ over $R$ is called flat if its left tensor product maps short exact sequences into short exact sequences, that is if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad \text{(A.3)}$$

be a short exact sequence of $R$-modules. A module $M$ over $R$ is called flat if its left tensor product maps short exact sequences into short exact sequences, that is if

$$0 \rightarrow M \otimes_R A \rightarrow M \otimes_R B \rightarrow M \otimes_R C \rightarrow 0 \quad \text{(A.4)}$$

is a short exact sequence.

**Example A.3.11**

There are some simple examples to Definition A.3.10:

i.) A direct sum of modules is flat if and only if the summands are flat.

ii.) Every ring is flat as a module over itself.
Every free module is flat.

Lemma A.3.12 (Tensor product of flat modules)
The tensor product of two flat modules is flat.

\[ \text{Proof: This follows from Definition A.3.10 and the fact that the associativity of tensor products is an isomorphism of modules.} \]

\[ \blacksquare \]

Lemma A.3.13
Let \( M \) be an \( R \)-module. Then the following statements are equivalent:

\( i. \) \( M \) is flat.

\( ii. \) For every ideal \( I \) in \( R \) the mapping \( M \otimes_R I \to M, \ m \otimes r \mapsto rm \) is injective.

Definition A.3.14 (Torsion element)
Let \( R \) be an integral domain, and let \( M \) be an \( R \)-module. We call an element \( m \in M \) a torsion element if there exists a non-zero element \( r \in R \) such that \( rm = 0 \). The set of torsion elements is denoted by \( \text{tor}(M) \).

Remark A.3.15 (Torsion)
\( \text{tor}(M) \) is a submodule of \( M \), the torsion submodule. The module \( M \) is called a torsion module if \( \text{tor}(M) = M \) and torsion-free if \( \text{tor}(M) = 0 \). The factor module \( M/\text{tor}(M) \) is obviously torsion-free.

\[ \text{Proof: \( \text{tor}(M) \) is not empty since \( 0 \in \text{tor}(M) \). Let \( m, n \in \text{tor}(M) \) so that there exist \( r, s \neq 0 \in R \) such that \( 0 = rm = sn \). Since \( rs(m + n) = rsm + rsn = 0, rs \neq 0 \), this implies that \( m + n \in \text{tor}(M) \). So \( \text{tor}(M) \) is a subgroup of \( M \). Clearly \( tm \in \text{tor}(M) \) for any non-zero \( t \in R \). This shows that \( \text{tor}(M) \) is a submodule of \( M \).} \]

\[ \blacksquare \]

Lemma A.3.16
Every flat module over an integral domain is torsion-free. On the converse, a torsion-free module over a principal ideal domain is flat.

Last, but not least, we give a definition of modules over algebras.

Definition A.3.17 (Modules over algebras)
Let \( R \) be a commutative ring, and let \( A \) be an associative algebra over \( R \). Then a right \( A \)-module \( M_A \) is an \( R \)-module together with an \( R \)-module homomorphism

\[ M_A \otimes_R A \to M, \ m \otimes a \mapsto ma, \]

such that

\[ (ma_1)a_2 = m(a_1a_2) \]

is valid for all \( a_1, a_2 \in A \) and all \( m \in M \).
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