JÖRG FLUM

\(L(Q)\)-preservation theorems

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L(Q)-PRESERVATION THEOREMS

JÖRG FLUM

§0. Introduction. Much effort has been spent to prove that the reduced product operation preserves, and sometimes even strengthens the L-equivalence of structures, where L is some infinitary language. A similar result is suggested by the following well-known fact:

Assume D is a nonprincipal ultrafilter on \( \omega \) and, for \( n \in \omega \), \( C_n \) is a set. If the ultra-product \( \prod C_n / D \) is infinite, it has a cardinality \( \geq \aleph_0 \). Hence, by Los’ theorem, (i) if \( \mathcal{A}_n \equiv \mathcal{B}_n \) and \( \varphi(x) \) is a first-order formula, then \( \prod \mathcal{A}_n / D \models Q \varphi(x) \) iff \( \prod \mathcal{B}_n / D \models Q \varphi(x) \), where \( Q \) is the unary quantifier “there are \( \aleph_0 \) many.”

We shall prove some generalizations of (i). In particular, we show

(ii) if \( D \) is a nonprincipal ultrafilter over \( I = \omega \), and \( \mathcal{A}_i \equiv \mathcal{B}_i \), then \( \prod \mathcal{A}_i / D \equiv_{L(Q)} \prod \mathcal{B}_i / D \), where \( L(Q) \) is the language obtained from the first-order language by adding the quantifier \( Q \).

(ii) remains true, if \( D \) is an \( \omega \)-regular or an atomless filter over a set \( I \).

Lipner [7] proved that if \( 2^{\aleph_0} \) is regular, then the \( L(Q) \)-equivalence is preserved under direct products. We show that the assumption “\( 2^{\aleph_0} \) is regular” is necessary. We determine the prefixes for which there is a sentence \( \varphi \) and there are structures \( \mathcal{A}_i \) and \( \mathcal{B}_i \), \( i \in I \), such that \( \mathcal{A}_i \equiv_{L(Q)} \mathcal{B}_i \), \( \prod \mathcal{A}_i \models \varphi \), but non \( \prod \mathcal{B}_i \models \varphi \).

The results of the present paper were announced in [4].

§1. Preliminaries. Let \( L \) be a first-order language. Assume, for simplicity, that \( L \) contains no function symbols and only finitely many predicate and constant symbols. For any cardinal \( \lambda \geq 1 \), let \( L(Q_\lambda) \) be the language obtained from \( L \) by adding the unary quantifier “there exist at least \( \lambda \) many.” \( \mathcal{A} \equiv_{Q_\lambda} \mathcal{B} \) means that \( \mathcal{A} \) and \( \mathcal{B} \) are \( L(Q_\lambda) \)-equivalent. In case \( \lambda = 2^{\aleph_0} \) we omit the subscript \( \lambda \), thus denoting by \( L(Q) \) the logic \( L(2^{\aleph_0}) \).

By \( \prod \mathcal{A}_i / D \) we denote the reduced product of a family \( \langle \mathcal{A}_i \mid i \in I \rangle \) of structures with respect to the filter \( D \) over the set \( I \). For \( f \in \prod I \mathcal{A}_i \) let \( f_D \) be the element of \( \prod I \mathcal{A}_i / D \) containing \( f \). A filter \( D \) over \( I \) is \( \omega \)-regular if there is a family \( \{ X_n \mid n \in \omega \} \) such that \( X_n \in D \) and \( \bigcap_n X_n = \varnothing \). The 2-element Boolean algebra with universe \( \{0, 1\} \) is denoted by \( \mathbb{2} \). Call a filter \( D \) over \( I \) atomless if the reduced power \( \mathbb{2}^I / D \) is an atomless Boolean algebra. For a subset \( X \subseteq I \) let \( X_D \) be the element of \( \mathbb{2}^I / D \) containing the characteristic function of \( X \). For \( X_D \neq 0 \) we denote by \( D \upharpoonright X \) the filter \( \{ X \cap Y \mid Y \in D \} \).

We state two known results for future reference.

1.1. Theorem. For any formula \( \varphi(v_0, \ldots, v_{n-1}) \) of \( L \) there exist finitely many
formulas $\varphi_0(v_0, \ldots, v_{n-1}), \ldots, \varphi_m(v_0, \ldots, v_{n-1})$, and a formula $\psi(x_0, \ldots, x_m)$ of the language of Boolean algebras such that

(a) $\models \varphi_0 \lor \cdots \lor \varphi_m$ and $\models \neg(\varphi_s \land \varphi_t)$ for $0 \leq s < t \leq m$,

(b) for any family $\langle \mathfrak{A}_i | i \in I \rangle$ of structures, any filter $D$ over $I$ and any $f_0, \ldots, f_{n-1} \in \prod_i \mathfrak{A}_i$,

$$\prod_i \mathfrak{A}_i/D \models \varphi[f_0, \ldots, f_{n-1}]$$

iff

$$2^{|D|}/D \models \psi \left[ S \left( \prod_i \mathfrak{A}_i; \varphi_0; f_0, \ldots, f_{n-1} \right), \ldots, S \left( \prod_i \mathfrak{A}_i; \varphi_m; f_0, \ldots, f_{n-1} \right) \right],$$

where $S \left( \prod_i \mathfrak{A}_i; \varphi_0; f_0, \ldots, f_{n-1} \right) = \{ i | \mathfrak{A}_i \models \varphi_0[f_0(i), \ldots, f_{n-1}(i)] \}$ (cf. [3]).

We sometimes denote $f_0, \ldots, f_{n-1}$ by $f^n$ and $f^n(i)$ by $f^n(i)$.

1.2. THEOREM. Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless Boolean algebras. Assume that $a_0, \ldots, a_m$ and $b_0, \ldots, b_m$ are partitions of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then $(\mathfrak{A}, a_0, \ldots, a_m)$ and $(\mathfrak{B}, b_0, \ldots, b_m)$ isomorphic if and only if $\{ s | a_s \neq 0 \} = \{ s | b_s \neq 0 \}$ (cf. [8]).

§2.

2.1. LEMMA. Let $D$ be atomless over $I$. Suppose $X \subseteq I$ and $X_D \neq 0$. Then there are subsets $X_\xi \subseteq X$ for $\xi < 2^{n_0}$ such that $X_{\xi_D} \neq 0$ and $X_{\eta_D} \neq X_{\eta_D}$ for $\xi \neq \eta$.

PROOF. Put $Y = I - X$. For $Y_D = 0$ the lemma is obvious. Suppose $Y_D \neq 0$. Then $2^{|D|}/D \cong 2^{|Y|}/D \uparrow Y \times 2^{|X|}/D \uparrow X$. Hence $2^{|D|}/D \uparrow X$ is an atomless Boolean algebra, therefore $|2^{|X|}/D \uparrow X| \geq 2^{n_0}$.

2.2. LEMMA. For any $\varphi(v_0, \ldots, v_n) \in L$ there is a $n_0 \geq 1$ such that for any atomless $D$ over $I$ and any structures $\mathfrak{A}_i, i \in I$, $\prod_i \mathfrak{A}_i/D \models \forall v_0 \cdots \forall v_{n-1} (Q_{n_0}(v_n \varphi \rightarrow Q_{n_0}(v_n))$.

PROOF. Given $\varphi$ choose $\varphi_0, \ldots, \varphi_m$ and $\psi$ such that 1.1 holds. Put $n_0 = 2^{n+1} + 1$.

If, for some $f_0, \ldots, f_{n-1} \in \prod_i \mathfrak{A}_i$, $\prod_i \mathfrak{A}_i/D \models Q_{n_0}(v_n \varphi[f_0, \ldots, f_{n-1}])$, then there are $f, g \in \prod_i \mathfrak{A}_i$, $f \neq g$ such that, for all $s \leq m$,

$$F_s = 0 \text{ iff } G_s = 0,$$

where $F_s = S(\prod_i \mathfrak{A}_i; \varphi_s; f^n, f)$ and $G_s = S(\prod_i \mathfrak{A}_i; \varphi_s; f^n, g)$ (for the definition of $S$ see 1.1).

As $\bigcup_{s=0}^{n_0} F_s = \bigcup_{s=0}^{n_0} G_s = I$, we have $\bigcup_{s=t=0}^{n_0} F_s \cap G_t = I$.

Put $X_{f \neq g} = \{ i | f(i) \neq g(i) \}$. If $f_D \neq g_D$ we get $s, t \leq m$ such that $X_D \neq 0$ holds for $X = F_s \cap G_t \cap X_{f \neq g}$. Choose $X_\xi$, $\xi < 2^{n_0}$, as in Lemma 2.1. Define $h_\xi \in \prod_i \mathfrak{A}_i$ by

$$h_\xi(i) = \begin{cases} f(i) & \text{if } i \notin X_\xi, \\ g(i) & \text{if } i \in X_\xi. \end{cases}$$

Then $\{ i | h_\xi(i) \neq h_\xi(i) \} \supset (X_\xi - X_\eta) \cup (X_\eta - X_\xi)$. Therefore, $h_\xi \neq h_\xi$ holds for $\xi \neq \eta$. For $\xi < 2^{n_0}$ and $r \leq m$ let $H_\xi$ be $S(\prod_i \mathfrak{A}_i; \varphi_r; f^n, h_\xi)$. We defined the $h_\xi$'s such that, for all $r \leq m$,

$$H_{\xi_D} = 0 \text{ iff } F_{\xi_D} = 0.$$

Hence, by 1.1 and 1.2, $\prod_i \mathfrak{A}_i/D \models \varphi[f_0, \ldots, f_{n-1}]$, and therefore $\prod_i \mathfrak{A}_i/D \models Q_{n_0}(v_n \varphi[f_0, \ldots, f_{n-1}])$.

2.3. COROLLARY. Suppose $D$ and $E$ are atomless filters over the sets $I$ and $J$, respectively. If $\prod_i \mathfrak{A}_i/D \equiv \prod_j \mathfrak{B}_j/E$, then $\prod_i \mathfrak{A}_i/D \equiv Q \prod_j \mathfrak{B}_j/E.$
PROOF (cf. [4, Lemma 2.7]). Replace by induction in any sentence $\psi$ of $L(Q)$ each subformula of the form $Q^v_{\vee,\psi}$ by $Q^v_{\vee,\psi}$. Let $\psi^*$ be the resulting formula of $L$. Now, by 2.2, for atomless $D'$ over $I'$ and $\langle e_i|i \in I'\rangle$,

$$\prod_{i \in I'} e_i/D' \models \psi \iff \prod_{i \in I'} e_i/D' \models \psi^*.$$  

Hence, if $\prod_j a_i/D \equiv \prod_j b_j/E$, then $\prod_j a_i/D \equiv_\omega \prod_j b_j/E$.

2.4. COROLLARY. Suppose $D$ is an atomless filter over $I$. If, for $i \in I$, $a_i \equiv b_i$, then $\prod_j a_i/D \equiv_\omega \prod_j b_j/D$.

We shall assume that the reader is familiar with the $r$-move game $G_r$ described by Ehrenfeucht in [2] and which may be played on any pair of structures $\mathfrak{A}$ and $\mathfrak{B}$. We write $\mathfrak{A} \neq \mathfrak{B}$ if the second player has a winning strategy in this game. Then,

2.5. THEOREM. (i) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $L$ with quantifier rank $\leq r$ iff $\mathfrak{A} \neq_\omega \mathfrak{B}$.

(ii) $\mathfrak{A} \equiv \mathfrak{B}$ iff, for all $r \in \omega$, $\mathfrak{A} \neq_\omega \mathfrak{B}$ (cf. [2]).

Lipner ([7], see also Vinner [9]) has introduced a game $G^\lambda_r$ to deal with the language $L(Q_\lambda)$. $G^\lambda_r$ is, as $G_r$, an $r$-move game between two players, I and II, and is played on a pair ($\mathfrak{A}$, $\mathfrak{B}$) of structures. There are two types of moves, $3$-moves and $Q_\lambda$-moves; both end up with a choice of an element from $\mathfrak{A}$ and one from $\mathfrak{B}$. Player I on each move decides which type it is to be. The $3$-moves are the same as moves in the Ehrenfeucht game, i.e. player I picks an element from $\mathfrak{A}$ or from $\mathfrak{B}$ and player II then picks an element from the other structure. $Q_\lambda$-moves take the following form. Player I chooses a subset $X$ of $\lambda$ elements either all from $\mathfrak{A}$ or all from $\mathfrak{B}$; player II then has to choose a set $Y$ of $\lambda$ elements from the other structure. Player I then selects $y$ in $Y$ and symmetrically, II picks $x$ in $X$, thus completing this move. If after $r$ moves the elements $a_0, \ldots, a_{r-1}$ and $b_0, \ldots, b_{r-1}$ have been successively chosen from $\mathfrak{A}$ or all from $\mathfrak{B}$, respectively (the chosen subsets are disregarded), then player II has won if the correspondence $a_i \rightarrow b_i$ for $i < r$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

We write $\mathfrak{A} \neq_\omega \mathfrak{B}$ if player II has a winning strategy in this game. Then

2.6. THEOREM. (i) $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $L(Q_\lambda)$ of quantifier rank $\leq r$ iff $\mathfrak{A} \neq_\omega \mathfrak{B}$.

(ii) $\mathfrak{A} \equiv \mathfrak{B}$ iff, for all $r \in \omega$, $\mathfrak{A} \neq_\omega \mathfrak{B}$ (cf. [7]).

If $a_0, \ldots, a_{n-1} \in A$ and $b_0, \ldots, b_{n-1} \in B$ we write

$$a_0, \ldots, a_{n-1} \sim_\omega b_0, \ldots, b_{n-1}$$

for $(\mathfrak{A}, a_0, \ldots, a_{n-1}) G^\lambda_r (\mathfrak{B}, b_0, \ldots, b_{n-1})$, i.e. by 2.6, if $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ satisfy the same formulas of $L(Q_\lambda)$ with free variables $v_0, \ldots, v_{n-1}$ and quantifier rank $\leq r$. Notice that for $\lambda = 1$ the game $G^\lambda_r$ is essentially Ehrenfeucht's game $G_r$. Therefore, we write $\sim_\lambda$ for $\sim_1$, and in the case $\lambda = 2^\omega$ we denote $G^\lambda_r$ by $G^\omega_r$ and $\sim_\lambda$ by $\sim_\omega$.

2.7. THEOREM. Suppose $D$ is an $\omega$-regular filter over $I$ and, for $i \in I$, $a_i \equiv_\lambda \mathfrak{B}_i$. Then $\prod_i a_i/D \equiv_\lambda \prod_i b_i/D$.

Proof. By $\omega$-regularity there are $X_k \in D$, $k \in \omega$, such that $I = X_0 \supset X_1 \supset \ldots$ and $\bigcap_{k \in \omega} X_k = \emptyset$. For $i \in I$ let $k_i \in \omega$ be such that $i \in X_{k_i-1}$, $i \notin X_{k_i}$. Assume $r \in \omega$. For $f_0, \ldots, f_{n-1} \in \prod_i A_i$, $g_0, \ldots, g_{n-1} \in \prod_i B_i$ and $s \in \omega$, we write

$$f^n R_{s} g^n$$

if, for all $i \in I$, $f^*(i) \sim_{s \cdot k_i} g^*(i)$. 


Thus, if $f^n R g^n$, then the correspondence $f_{mD} \mapsto g_{mD}$, $m < n$, is a partial isomorphism from $\prod A_i/D$ to $\prod B_i/D$.

We prove that $\prod A_i/D \cong \prod B_i/D$ by showing that there is a strategy for player II such that after $n$ moves for the chosen elements from $\prod A_i/D$ and $\prod B_i/D$ there are representatives $f_0, \ldots, f_{n-1}$ and $g_0, \ldots, g_{n-1}$ such that $f^n R g^n$. The proof is by induction on $n$. Hence, assume $f^n R g^n$ and let us consider the $(n + 1)$th move: Suppose that player I decides to choose from $A$ (by symmetry, we only need to consider this case).

\textbf{3-move.} I chooses $f_D$ in $\prod A_i/D$. For $i \in I$ pick $g(i) \in B_i$ such that

$$f^n(i), f(i) \sim (r - (n + 1))k_i g^n(i), g(i)$$

In particular, $f^n(i), f(i) \sim (r - (n + 1))k_i g^n(i), g(i)$, hence $f^n, f R g^n$. Q-move. Player I chooses $F_D \subset \prod A_i/D$ with $|F_D| = 2^{\aleph_0}$. Let $F$ contain just one representative in $\prod A_i$ for each element in $F_D$. Thus $F_D = \{f_D | f \in F\}$. Define an equivalence relation $R$ on $F$ by

$$f R h \iff \{i | f^n(i), f(i) \sim (r - (n + 1))k_i f^n(i), h(i)\} \in D.$$ 

Let $(F^c)_{\leq \omega}$ be the $R$-equivalence classes. If $F^c$ is finite, pick $f^c \in F^c$ and choose $g^c$ as in the 3-move. Let $G^c$ be $\{g^c\}$. Assume $F^c$ is infinite and let $h_0, h_1, \ldots$ be different elements from $F^c$. For $k \in \omega$ put

$$Y_k = \{i | f^n(i), h_s(i) \sim (r - (n + 1))k_i f^n(i), h_t(i) \text{ for } s, t \leq k\}.$$ 

Then $Y_k \supseteq Y_{k+1}$ and, by (1), $Y_k \in D$. Define, for $s \in \omega$, $h_s \in \prod A_i$ by

$$h_s(i) = \begin{cases} h_0(i) & \text{if } i \in Y_s, \\ h_t(i) & \text{if } i \notin Y_s. \end{cases}$$

It is easy to check that

(i) $h_{s_0} = h_{s_0}$,

(ii) for all $i \in I$ and all $s, t \in \omega$,

$$f^n(i), h_s(i) \sim (r - (n + 1))k_i f^n(i), h_t(i).$$

By (i) we may assume $h_0, h_1, \ldots \in F$. Let $J = \{i \mid$ there are $s, t \in \omega$ such that $h_s(i) \neq h_t(i)\}$. Obviously, $J_D \neq 0$.

\textbf{Case 1.} $2'/D \upharpoonright J$ is infinite, and hence has cardinality $\geq 2^{\aleph_0}$. Assume $i \in J \cap X_1$. Choose $s, t \in \omega$ such that $h_s(i) \neq h_t(i)$. $i \in X_1$ implies $k_i \geq 2$. Thus, we find $g^s(i), g^t(i) \in B_i$ such that

$$f^n(i), h_s(i), h_t(i) \sim (r - (n + 1))k_i g^n(i), g^s(i), g^t(i).$$

For $i \notin J \cap X_1$ choose $g(i) \in B_i$ with

$$f^n(i), h_0(i) \sim (r - (n + 1))k_i g^n(i), g(i).$$

For $\eta \in 2'$ define $g_\eta \in \prod B_i$ by

$$g_\eta(i) = \begin{cases} g(i) & \text{if } i \notin J \cap X_1, \\ g^\eta(0)(i) & \text{if } i \in J \cap X_1. \end{cases}$$

As $|2'/D \upharpoonright J| \geq 2^{\aleph_0}$, we have $|\{g_\eta | \eta \in 2'\}| \geq 2^{\aleph_0}$. Thus, there is a subset $G^c$ of
\{g_{\eta}\in 2^\mathfrak{I}\}, |G^t| = |F^t|, \text{ and } g_D \neq \bar{g}_D \text{ for } g, \bar{g} \in G^t. \text{ By (2), (3) and (4), for } s \in \omega \text{ and } g \in G^t, f^n, h_s R_{r-(n+1)} g^n, g. \text{ Put } f^c = h_0.

Case 2. $2^I / D \upharpoonright J$ is finite. Thus there is a finite partition $J_0, \ldots, J_k$ of $J$ such that $D_I = D \upharpoonright J_i$ is an ultrafilter over $J_i (1 \leq k)$. For $i \in I$ let $H_i = \{h_\delta(i) | s \in \omega\}$.

\[
\{h_\delta | s \in \omega \} \subseteq \prod_{J_0} H_{J_0} / D \simeq \prod_{J_k} H_{J_k} / D \upharpoonright (I - J) \times \prod_{J_0} H_{J_0} / D_0 \times \cdots \times \prod_{J_k} H_{J_k} / D_k.
\]

As $\{h_\delta | s \in \omega \}$ is infinite and the first factor on the right side contains just one element, $\prod_{J_0} H_{J_0} / D_e$ is infinite for some $e \leq k$. By Los' theorem, for $m \geq 0$,

\[
J^m_e = \{i | |H_i| > m, i \in J_e\} \subseteq D_e.
\]

For $i \in J_e$ let $m_i$ be such that $i \in J_{e, m_i} - 1 \cap X_{m_i-1}, i \notin J_{e, m_i} \cap X_{m_i}$. Pick $m_i$ elements from $H_i$, say $h_{\delta_0}(i), \ldots, h_{\delta_{m_i-1}}(i)$. As $m_i \leq k_i$ we find $g_0(i), \ldots, g_{m_i-1}(i) \in B_i$ such that

\[
f^n(i), h_0(i), \ldots, h_{\delta_{m_i-1}(i)} \sim (r-(n+1))_{r \in \omega} f^n(i), g_0(i), \ldots, g_{m_i-1}(i).
\]

For $i \notin J_e$ choose $g(i) \in B_i$ such that

\[
f^n(i), h_0(i), \ldots, h_{\delta_{m_i-1}(i)} \sim (r-(n+1))_{r \in \omega} f^n(i), g(i).
\]

$D_e$ is $\omega$-regular and, for $t \in \omega, \{i | m_i \geq t\} \cap X_{\alpha_e} \subseteq X_e \subseteq D_e$ holds. Thus, $|\prod_{\alpha_e} m_i / D| \geq 2^{\aleph_0}$. Therefore, $|\{\eta | \eta \in 2^\mathfrak{I}\} \geq 2^{\aleph_0}$. Hence, we find a subset $G^t$ of $\{g_{\eta} | \eta \in \prod_{\alpha_e} m_i\}$, $|G^t| = |F^t|$ such that $g_D \neq \bar{g}_D$ for $g, \bar{g} \in G^t$.

By (2), (5) and (6), for all $s \in \omega$ and $g \in G^t$,

\[
f^n, h_s R_{r-(n+1)} g^n, g.
\]

Put $f^c = h_0$. Let $G = \bigcup_{\alpha_e} G^t$. Player II chooses the subset $G_D = \{g_D | g \in G\} \subseteq \prod_{\alpha_e} B_i / D$. If now player I chooses $g_D \in G_D$, say $g \in G^t$, player II responds by choosing $f^c$.

We needed the $\omega$-regularity of $D$ only in Case 2, i.e. only if $2^I / D$ has an atom. Thus the above proof contains a proof of Corollary 2.4, but does not yield the stronger version 2.3.

In case $2^{\aleph_0} = \aleph_1$, 2.7 is contained in [1] and [6]. It should be clear to the reader familiar with [1] how to change the above proof to obtain the following stronger version of 2.7.

2.8. **THEOREM.** Let $L$ be a countable language. Suppose $D$ is an $\omega$-regular filter over $I$ and, for $i \in I$, $\mathfrak{A}_i \equiv \mathfrak{A}_i$. Then $\prod \mathfrak{A}_i / D \equiv_{\mathfrak{A}_1(\mathfrak{Q})} \prod \mathfrak{A}_i / D$.

One consequence of Lipner's results in [7] is

\[
\text{(7) if } |I| \leq \aleph_0 \text{ and for } i \in I, \mathfrak{A}_i \equiv_{\mathfrak{A}_1} \mathfrak{A}_i, \text{ then } \prod \mathfrak{A}_i \equiv_{\mathfrak{A}_1} \prod \mathfrak{A}_i.
\]

2.9. **THEOREM.** Assume $|I| \leq \aleph_0$ and let $D$ be a filter over $I$. If, for $i \in I$, $\mathfrak{A}_i \equiv_{\mathfrak{A}_1}$ then $\prod \mathfrak{A}_i / D \equiv_{\mathfrak{A}_1} \prod \mathfrak{A}_i / D$.

**Proof.** By (7) we may assume that $I = \omega$. For $n \in \omega$ choose $X_n \in D$ with $n \notin X_n$; if there is such an $X_n$; otherwise put $X_n = \emptyset$. Let $X = \bigcap_n X_n$. Suppose $X \notin D$ and put $J = I - X$. Then $D \upharpoonright J$ is an $\omega$-regular filter. Thus, by 2.7 and (7),

\[
\prod \mathfrak{A}_i / D \simeq \prod \mathfrak{A}_i \times \prod \mathfrak{A}_i / D \upharpoonright J \equiv_{\mathfrak{A}_1} \prod \mathfrak{A}_i \times \prod \mathfrak{A}_i / D \upharpoonright J \simeq \prod \mathfrak{A}_i / D.
\]
Use (7) in case $X \in D$.

2.10. Theorem. (a) Let $D$ be an ultrafilter over $I$. If, for $i \in I$, $\mathcal{A}_i \equiv_{\mathcal{A}_i} \mathcal{B}_i$, then $\Pi_i \mathcal{A}_i / D \equiv_{\mathcal{A}_i} \Pi_i \mathcal{B}_i / D$. 

(b) Assume $|I|$ is less than the first measurable cardinal and $D$ is any nonprincipal ultrafilter over $I$. If, for $i \in I$, $\mathcal{A}_i \equiv_{\mathcal{A}_i} \mathcal{B}_i$, then $\Pi_i \mathcal{A}_i / D \equiv_{\mathcal{A}_i} \Pi_i \mathcal{B}_i / D$.

Proof. If $D$ is a nonprincipal, $\omega$-complete ultrafilter, then $D$ is $\kappa$-complete and $|I| \geq \kappa$, where $\kappa$ is the first measurable cardinal. Then $D$ preserves $L(\kappa)$-equivalence, in particular $L(Q)$-equivalence.

Is 2.10(a) true for filters? We prove in the next section that we get a negative answer, if $2^{\aleph_0}$ is singular. We do not know if 2.10(a) is true for filters in case $2^{\aleph_0}$ is regular.

§3. Throughout this section let $\lambda$ be an infinite cardinal. Define the cardinals $j(\lambda)$ and $k(\lambda)$ by:

(a) $j(\lambda)$ is the least $\mu$ such that $2^\mu \geq \lambda$.

(b) $k(\lambda)$ is the least $\mu$ such that, for some family $\langle \lambda_\xi \mid \xi < \mu \rangle$ of cardinals $\lambda_\xi < \lambda$, $\prod_{\xi < \mu} \lambda_\xi \geq \lambda$.

For any $\lambda$, we have $k(\lambda) \leq j(\lambda) \leq \lambda$. We say that $E(\lambda)$ holds, if $k(\lambda) = j(\lambda)$. If $E(\lambda)$ holds, then $2^{cf(\lambda)} \geq \lambda$. As $k(2^{\aleph_0}) = j(2^{\aleph_0}) = \aleph_1$, $E(2^{\aleph_0})$ holds.

Lipner proves that the $L(Q, \lambda)$-equivalence is preserved by direct products, if $\lambda$ is regular and $E(\lambda)$ holds. He shows that the assumption "$E(\lambda)$ holds" is necessary but leaves open the question if the regularity of $\lambda$ is necessary. Lipner’s proof and the proof in [10] are "global" as will be the proof of 3.4 below. We sketch a local proof of Lipner’s theorem, i.e. player II obtains a winning strategy for the product structure by playing in each component according to a winning strategy.

3.1. Theorem. Assume $E(\lambda)$ holds and $\lambda$ is regular. If, for $i \in I$, $\mathcal{A}_i \equiv_{\mathcal{A}_i} \mathcal{B}_i$, then $\Pi_i \mathcal{A}_i \equiv_{\mathcal{A}_i} \Pi_i \mathcal{B}_i$.

Proof (Sketch). For $f_0, \ldots, f_{n-1} \in \Pi_i \mathcal{A}_i$, $g_0, \ldots, g_{n-1} \in \Pi_i \mathcal{B}_i$ and $s \in \omega$ we write

$f^n R_s g^n$ iff, for all $i \in I, f^n(i) \sim_{\mathcal{A}_i} g^n(i)$.

For $r \in \omega$ we prove $\Pi_i \mathcal{A}_i \equiv_{\mathcal{A}_i} \Pi_i \mathcal{B}_i$ by showing that there is a strategy for player II such that if after $n$ moves $f_0, \ldots, f_{n-1}$ and $g_0, \ldots, g_{n-1}$ have been chosen, then $f^n R_r g^n$. The proof is by induction on $n$ and the $3$-move is handled similar to the $3$-move in 2.7. Assume that $f^n R_r g^n$ and that in the $(n + 1)$th move player I chooses $F \subseteq \Pi_i \mathcal{A}_i$ with $|F| = \lambda$. If there is an $i_0 \in I$ such that $|\{f(i_0) \mid f \in F\}| = \lambda$ we do a $Q$-move in the $i_0$-component and an $3$-move for $i \neq i_0$. Otherwise, for all $i \in I$,

(1) $|\{f(i) \mid f \in F\}| < \lambda$.

Then, define an equivalence relation $R$ on $F$ by

$f R h$ iff $f^n, f R_{r-(n+1)} f^n, h$.

If $R$ has $\lambda$ many equivalence classes, choose, for each $f \in F$, a $g_f \in \Pi_i \mathcal{B}_i$ as in the $3$-move. Then, $|\{g_f \mid f \in F\}| = \lambda$ and player II may choose this subset. If $R$ has less than $\lambda$ equivalence classes, by regularity of $\lambda$ there is a class $F_0$ with $\lambda$ many ele-
ments. Let $J$ be $\{i \mid \text{there are } f, \bar{f} \in F_{0} \text{ with } f(i) \neq \bar{f}(i)\}$. By (1) and $E(\lambda)$, $|2'| \geq \lambda$.

Now we handle this situation similar to Case 1 of the $Q$-move in 2.7.

Lipner's global proof yields a stronger result, which we state in the terminology of this paper. Fix $n$: As there are only finitely many nonequivalent formulas of $L(Q_{\lambda})$ with free variables $v_{0}, \ldots, v_{n-1}$ and quantifier rank $\leq r$, the relation $\sim_{r}^{\lambda}$ has only finitely many equivalence classes, say $E_{n}^{0}, \ldots, E_{n}^{k}$, where $k = k(n, \lambda)$.

Then 3.1 is a special case of

3.2. THEOREM. Assume $\lambda$ is regular and $E(\lambda)$ holds. Suppose $f_{0}, \ldots, f_{n-1} \in \prod_{j} A_{j}$, and $g_{0}, \ldots, g_{n-1} \in \prod_{j} B_{j}$, and $r \in \omega$. For $s \leq k(n, 2r)$ put

$$Y_{s} = \{ i \in I \mid (f^{n}(i)) \in E_{n, 2r}^{s}\} \quad \text{and} \quad Z_{s} = \{ j \in J \mid (g^{n}(j)) \in E_{n, 2r}^{s}\}.$$

Then $f^{n} \sim_{s}^{\lambda} g^{n}$, if the following conditions are satisfied:

(a) $|Y_{s}| = |Z_{s}|$, if $|Y_{s}|$ is finite,
(b) $K_{0} \leq |Y_{s}| < j(\lambda)$ iff $K_{0} \leq |Z_{s}| < j(\lambda)$,
(c) $j(\lambda) \leq |Y_{s}| \leq \lambda$ iff $j(\lambda) \leq |Z_{s}| \leq \lambda$,
(d) $\lambda \leq |Y_{s}|$ iff $\lambda \leq |Z_{s}|$.

The proof is similar to that of 3.4 below.

3.3. EXAMPLES. (a) Let $\lambda$ be singular, say $\lambda = \bigcup_{\xi \in A_{\mu}}$ where $K_{0} \leq \mu, \mu_{c} < \lambda$. Assume $L$ has a binary predicate $\leq$ and an individual constant $c$. Let $\psi$ be

$$\forall x c \leq x \land \forall x(x \leq x \land \forall y(x \leq y \rightarrow (x = c \lor x = y))).$$

Assume that, for $\xi \in A_{\mu}$, $A_{\xi} = A_{\mu}$ and $B_{\xi} = K_{0}$. Then $A_{t} = A_{\lambda} B_{\xi}$, and $f \in \prod_{\mu} A_{\xi}$. If there is a $\xi_{0}$ such that $f(\xi) = c_{\xi}$ for $\xi \neq \xi_{0}$ and $f(\xi_{0}) \neq c_{\xi_{0}}$, then

$$\bigwedge_{\mu} A_{\xi} \models \forall y((y \leq x \land \neg y = x) \rightarrow y = c)[f].$$

Thus (recall $\lambda = \bigcup_{\xi \in A_{\mu}}$), $\bigwedge_{\mu} A_{\xi} \models \varphi$ where

$$\varphi = Q_{\lambda} x \forall y((y \leq x \land \neg y = x) \rightarrow y = c).$$

Let $g \in \prod_{\mu} B_{\xi}$. If $\prod_{\mu} B_{\xi} \models \forall y((y \leq x \land \neg y = x) \rightarrow y = c)[g]$, then there is a $\xi_{0}$ such that $g(\xi) = c_{\xi}$ for $\xi \neq \xi_{0}$ and $g(\xi_{0}) \neq c_{\xi_{0}}$. But there are only finitely $\mu_{c}$, hence less than $\lambda$, such $g$'s. Thus $\bigwedge_{\mu} B_{\xi} \models \neg \varphi$. In particular, non $\bigwedge_{\mu} A_{\xi} \equiv_{\lambda} \prod_{\mu} B_{\xi}$. (b) Assume $\lambda_{n}$ is singular. Let $c, d$ be two individual constants and $\leq$ a binary predicate symbol. It is not difficult to construct (similar to $a$) for $n \in \omega$ structures $A_{n}$ and $B_{n}$ such that $A_{n} = B_{n}$, $\prod_{n} A_{n} = Q \rightarrow Q y y \leq x$, and non $\prod_{n} B_{n} \models Q x \rightarrow Q y y \leq x$.

Thus not every formula whose prefix contains $Q$ and $\forall$, or $Q$ and $\neg Q$ is "product-stable". If $E(\lambda)$ holds, every formula only containing $Q$ and $\exists$ is product-stable. More precisely:

Call $\varphi$ a $(Q_{\lambda}, \exists)$-formula if $\varphi$ belongs to the smallest subset of $L(Q_{\lambda})$ which contains all atomic formulas and negations of atomic formulas and is closed under $\land, \lor, Q_{\lambda}$ and $\exists$.

3.4. THEOREM. Suppose $E(\lambda)$ holds. Let $\varphi$ be any $(Q_{\lambda}, \exists)$-sentence. If for $i \in I$ $A_{i} \equiv_{\lambda} B_{i}$, then

$$\bigwedge_{i} A_{i} \models \varphi \iff \bigwedge_{i} B_{i} \models \varphi.$$
PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two structures. Let $\mathcal{G}_\lambda^\mathfrak{A}$ be the game obtained from $G_\lambda^\mathfrak{A}$ by the following modification: In each move player I has to pick from the structure $\mathfrak{A}$ an element in a $\exists$-move and a subset of power $\lambda$ in a $\forall$-move. An analysis of the proof of 2.6 shows that

(2) every $(\forall \lambda, \exists)$-sentence of quantifier rank $\leq r$ holding in $\mathfrak{A}$ holds in $\mathfrak{B}$.

By symmetry, it suffices to prove that, for all $r \in \omega$, $\prod \mathfrak{A}_1 \mathcal{G}_\lambda^\mathfrak{A} \prod \mathfrak{B}_1$. We do this by showing there is a strategy for the second player with the following property:

Assume after $n$ moves elements $f_0, \ldots, f_{n-1} \in \prod \mathfrak{A}_1$ and $g_0, \ldots, g_{n-1} \in \prod \mathfrak{B}_1$ have been chosen. For $s \leq k(n, 2r)$ (for notation see the remarks preceding 3.2) put

$$Y_s = \{ i \in I | (f^n(i)) \in E^n_{n,2(r-n)} \} \quad \text{and} \quad Z_s = \{ i \in I | (g^n(i)) \in E^n_{n,2(r-n)} \}.$$ 

Then

(a) $|Y_s| \leq |Z_s|$, 
(b) $Y_s = \emptyset$ iff $Z_s = \emptyset$.

The proof is by induction on $n$. Thus, suppose $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ satisfy (a) and (b), and in the $(n + 1)$th move player I does an $\exists$-move and picks $f \in \prod \mathfrak{A}_1$. Suppose $s \leq k(n, 2(r - n))$ and $Y_s \neq \emptyset$; let $j: Y_s \to Z_s$ be injective. If $i \in Y_s$ and $(f^n(i), f(i)) \in E^n_{i+1,2(r-(n+1))}$ choose $g(j(i)) \in B_{j(i)}$ such that $(g^n(j(i)), g(j(i))) \in E^n_{i+1,2(r-(n+1))}$. Fix $i_0 \in Y_s$. If $i \in Z_s$, $i \notin j(Y_s) \choose g(i) \in B_i$ with $(g^n(i), g(i)) \in E^n_{i+1,2(r-(n+1))}$. It is easy to see that $f^n, f$ and $g^n, g$ satisfy (a) and (b).

Next we consider the case when player I makes a $\forall$-move and chooses a subset $F \subseteq \prod \mathfrak{A}_1$ of power $\lambda$.

Case 1. There is an $i_0 \in I$ such that $|\{ f \in F \}| = \lambda$. Then, for some $e < k(n + 1, 2(r - (n + 1)))$ there are $\lambda$ many $f \in F$ such that

$$(f^n(i_0), f(i_0)) \in E^n_{i+1,2(r-(n+1))}.$$ 

If $i_0 \in Y_s$ pick any $j \in Z_s$ and choose $\lambda$ many elements $g^\xi(j) \in B_j (\xi < \lambda)$ with

$$(g^n(j), g^\xi(j)) \in E^n_{i+1,2(r-(n+1))}.$$ 

Now this case is finished similarly to the $\exists$-move.

Case 2. For all $i \in I$, $|\{ f(i) \}| \leq \lambda$. Then, as $E(\lambda)$ holds, $2^{\lambda+1} \geq \lambda$, where $J = \{ i | |\{ f(i) \}| \leq 2 \}$. Thus, $|J| \geq j(\lambda)$.

Case 2a. For some $s \leq k(n, 2(r - n))$, some $f \in F$ there are $i, j \in Y_s$ and $e, m \leq k(n + 1, 2(r - (n + 1)))$, $e \neq m$, such that

$$|J \cap Y_s| \geq j(\lambda), \quad (f^n(i), f(i)) \in E^n_{i+1,2(r-(n+1))}$$

and

$$(f^n(j), f(j)) \in E^n_{m+1,2(r-(n+1))}.$$ 

Choose a subset $Z \subseteq Z_s$ such that $|Z| \geq j(\lambda)$ and $|Z_s - Z| = |Z_s|$. For $j \in Z$ choose $g^\eta(j), g^\xi(j) \in B_j$ with $(g^n(j), g^\eta(j)) \in E^n_{i+1,2(r-(n+1))}, (g^n(j), g^\xi(j)) \in E^n_{m+1,2(r-(n+1))}$. Now this case is finished by extending, similar to the $\exists$-move, for $\eta < \lambda$. Choose $g^{\eta}(j) | j \in Z$ to an element $g^\eta \in \prod \mathfrak{B}_1$.

Case 2b. Assume otherwise that for all $s \in \omega$ such that $|J \cap Y_s| \geq j(\lambda)$, all $f \in F$ there is an $e \leq k(n + 1, 2(r - (n + 1)))$ such that, for all $i \in Y_s$, $(f^n(i), f(i)) \in$
Then, by a cardinality argument, for some $s$ there are $i_0 \in Y_s$ and $f, h \in F$ such that $|J \cap Y_s| \geq j(\lambda), f(i_0) \neq h(i_0)$ and

$$f^n(i_0), f(i_0) \sim \lambda_{2(\tau - (n + 1))} f^n(i_0), h(i_0).$$

For $j \in Z_\tau$ choose $g^0(j), g^1(j) \in B_j$ with

$$g^n(j), g^0(j), g^1(j) \sim \lambda_{2(\tau - (n + 1))} f^n(i_0), f(i_0), h(i_0).$$

Now finish this case by extending each $\langle g^n(j) | j \in Z_\tau \rangle, \eta \in 2^{\Omega_\tau}, \text{ to an element } g^n \in \Pi_1 B_\tau.$

As the proof shows, the hypothesis of 3.4 can be weakened considerably. The proof is global; thus we need only global assumptions as in 3.2. Secondly, whenever in components $(\mathcal{A}_i, f^n(i))$ and $(\mathcal{B}_i, g^n(j))$ was played, player I chose from $(\mathcal{A}_i, f^n(i))$. Thus we do not need the full $L(Q_\tau)$-equivalence of $\mathcal{A}_i$ and $\mathcal{B}_i$.

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MATHEMATISCHES INSTITUT DER ALBERT-LUDWIGS-UNIVERSITÄT
78 FREIBURG, GERMANY