Spin Triplet Effects in the Density of States of Quasi One–Dimensional Wires

Inaugural–Dissertation
zur Erlangung des Doktorgrades
der Fakultät für Mathematik und Physik
der
Albert–Ludwig–Universität Freiburg
im Breisgau

eingereicht von
Wolfgang Körner
aus Ulm

im Juni 2006

Dekan	: Prof. Dr. Josef Hohnerkamp
Leiter der Arbeit	: Prof. Dr. Hermann Grabert
Referent	: Prof. Dr. Hermann Grabert
Koreferent	: Prof. Dr. Hartmann Römer
Datum der mündlichen Prüfung	: 26.7.2006

Contents

Chapter 1

Introduction

This introductory chapter has a threefold aim: explanation of the physical system, historical review and motivation for this work.

The system we are considering in this thesis is a mesoscopic diffusive wire of length L, height b and width a. A sketch is given in figure (1.1). The word mesoscopic indicates that at least the transverse length scales are neither microscopic nor macroscopic. The length L could be macroscopic. Typical transverse extensions in the experiments are of the order of several tens of nanometers, and the length of the wire could reach up to several micrometers (in the experiment of [1]: $L = 29\,\mu\text{m}$, $b = 190\,\text{nm}$ and $a = 22\,\text{nm}$). Due to its transverse extensions our wire is not a strictly one–dimensional system (like for example carbon nanotubes) in the sense that it has only few transmitting channels. The wires we have in mind posses several thousand channels.

The electrons move diffusively through the wire due to the *weak disorder* that is present. A typical piece of a trajectory is depicted in figure (1.1). The expression weak disorder shall indicate that there are more than just a handful of impurities in the wire because in that case the electrons would move ballistically through the wire. Also the expression weak disorder shall indicate that the impurity concentration n_{imp} is not that high that the electrons are captured in bound states (Anderson localization [2]). In the weak disorder regime the electron states are extended over the whole system. There is a characteristic mean free path l along which the electrons move on straight lines. This picture is of course only valid if the Fermi wavelength $\lambda_F = 2\pi\hbar/p_F$ (p_F is the electron momentum at the Fermi level.) is much smaller then l. For metals this condition is typically very well fulfilled.

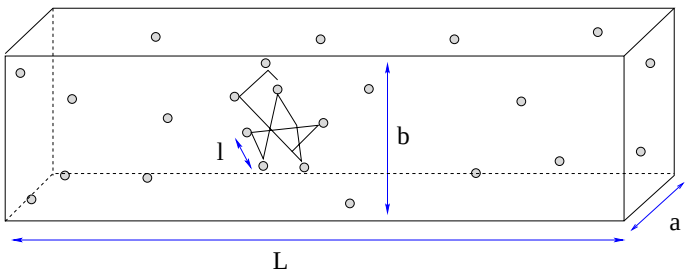

Figure 1.1: Different length scales in a mesoscopic wire

The mean free path is connected to the elastic scattering time τ by $l = v_F\,\tau$ where v_F is the Fermi velocity. Typically l is several tens of nanometers. The exact quantitative definition of

weak disorder is given by the inequality $\epsilon_{\rm F}\,\tau \gg 1$ where $\epsilon_{\rm F}$ is the Fermi energy. In section (2.1) we will show where this condition comes from and how the scattering time τ is connected to the impurity concentration n_{imp}. Concerning the length scales we can characterize our regime by $\lambda_F \ll l, a, b \ll L$. [1]

Let us also consider our system at the level of energy scales. First there is the Fermi energy $\epsilon_{\rm F}$. As we will construct an effective theory for the low energy excitations of the system, the Fermi energy is much bigger than any other energy scale of our problem. Due to the geometry of our wire we get three more characteristic energies. We assume that the diffusion is isotropic and that we have just one diffusion constant D for all directions. The diffusive motion is characterized by $Dt = x^2$. The time $t_a = a^2/D$ is the typical time a particle needs to spread over a distance a. The inverse of t_a is called the Thouless energy $E_a = 1/t_a = D/a^2$. [2] Hence the energies E_a, E_b and E_L are connected to the diffusion of the particle. Further energies are the temperature k_BT and the voltage eV (that one applies in a tunnelling experiment, if one constructs a contact by approaching another piece of metal). We assume that the temperature is very low because otherwise all the effects we want to discuss in this work are destroyed by the thermal fluctuations. Let us look at eV as the energy that is tunable which we use to investigate the different regimes. Assuming that $a < b$ and choosing eV in such a way that we have the following hierarchy of energies

$$k_BT < E_L \ll E_b < E_a < \mathbf{eV} < \epsilon_{\rm F}, \qquad \text{(3D)} \tag{1.1}$$

we are working in the three dimensional regime. Choosing however eV smaller than E_a, one talks of the quasi two–dimensional regime. Translating the hierarchy

$$k_BT < E_L \ll E_b < \mathbf{eV} < E_a < \epsilon_{\rm F} \qquad \text{(quasi 2D)} \tag{1.2}$$

into time scale makes this understandable. When one probes times of order $1/eV$ the particle is already completely spread over the extension a and for the residual effective motion remain only two dimensions. Lowering the energy eV further we cross over to the **quasi one–dimensional regime**:

$$k_BT < E_L \ll \mathbf{eV} < E_b < E_a < \epsilon_{\rm F}. \qquad \text{(quasi 1D)} \tag{1.3}$$

The particle is then completely spread over the cross section of the wire and the time scale $1/eV$ resolves only the movement along the wire. Lowering the energy even below E_L we arrive at a zero dimensional system:

$$k_BT < \mathbf{eV} < E_L \ll E_b < E_a < \epsilon_{\rm F}. \qquad \text{(0D)} \tag{1.4}$$

In this work we are investigating our diffusive wires in the quasi one–dimensional regime.[3] We will explain later why we choose this regime as we need some more ingredients to understand the reasoning. So far, we have outlined what is understood by a diffusive mesoscopic wire and gave the definition of a quasi one–dimensional system, but we neglected completely the interactions between the electrons.[4] However in reality our electrons are coupled by the Coulomb interaction.

[1] In the ballistic regime l is of order L or bigger.

[2] We have set $\hbar = 1$.

[3] In the following we set $a = b$.

[4] Including interactions introduces new energy scales that modify our hierarchy of energies. We will have to introduce two more diffusion constants which measure the propagation of charge and spin. This implies the appearance of six new Thouless energies. We skipped that complication here for the sake of clarity. In section (2.2.5) we will be more precise and explain how the definition of the quasi dimension is modified.

Nevertheless for many phenomena it is sufficient to neglect the Coulomb interaction. The reason is that in metals the screening is very efficient and the effective screened interaction becomes short–range.

For an interacting electron system (= Fermi liquid) without disorder it was shown by Landau ([3], [4] and [5])[5] that one can treat the low lying excitations as non–interacting quasiparticles. The interaction between the electrons can be taken into account by a renormalization of various parameters (for example a renormalized mass). What remains is the quasiparticles, which do not interact, that one could imagine as electrons dragging a cloud of screening charges with them. Up to the nineteen seventies the general feeling was that adding some disorder would not modify essentially the Fermi–liquid theory. Thus it came quite as a surprise when it was shown by Altshuler and Aronov in 1979 ([7], [8]) that interactions in disordered Fermi liquids lead to strong singularities at the Fermi level. For the density of states (DOS) of a quasi one-dimensional systems they predicted a suppression going like $1/\sqrt{eV}$.[6] For the quasi two-dimensional system the singularity should go like $\ln(eV)$, and in three dimension they still predicted a suppression of the DOS according to $\sqrt{eV}$. This reduction of the DOS near the Fermi energy due to the repulsive interactions is usually called *zero bias anomaly*.

In experiments the effect can be seen by a measurement of the tunnelling conductance G.[7] For low temperatures one can derive the following direct connection between the tunnelling conductance G and the DOS ν:

$$\frac{\delta G(V)}{G} = \frac{\delta \nu(V)}{\nu}. \tag{1.5}$$

A derivation of that expression is given in section (9.1) as it is the decisive link to the experiments. In figure (1.2) one can see the decrease of the conductance towards the Fermi energy ($\epsilon = 0$). The intuitive explanation why little disorder can make such a big difference is the following: If we add enough impurities that the electrons travel through the wire no more ballistically but diffusively, it has also an impact on the mutual interactions. Slowed down to the diffusive movement the electrons spend longer time close to each other and thus have more time to interact. In the chapter on perturbation theory we will remind the reader which part of the diffusive theory reflects exactly that crucial feature.

Considering the three different functional dependencies of the suppression of the DOS, we remark that the strongest effect shows up in the quasi one–dimensional system. Also this is understandable in the same spirit. Reducing the dimension and thus the space for the electrons to move increases the effect of the interactions. It is harder for the electrons to avoid each other. As the combined effect of interaction, disorder and spatial restriction is most pronounced in quasi one–dimensional systems we focus in this thesis on this class of systems.

[5] See also Noizieres and Pines [6].

[6] The energy is measured relative to the Fermi energy.

[7] The conductance is defined by $G := \sigma L^{d-2}$ where σ is the conductivity and d the dimension of the conductor. One can define the dimensionless conductance $g = G/(e^2/\hbar)$.

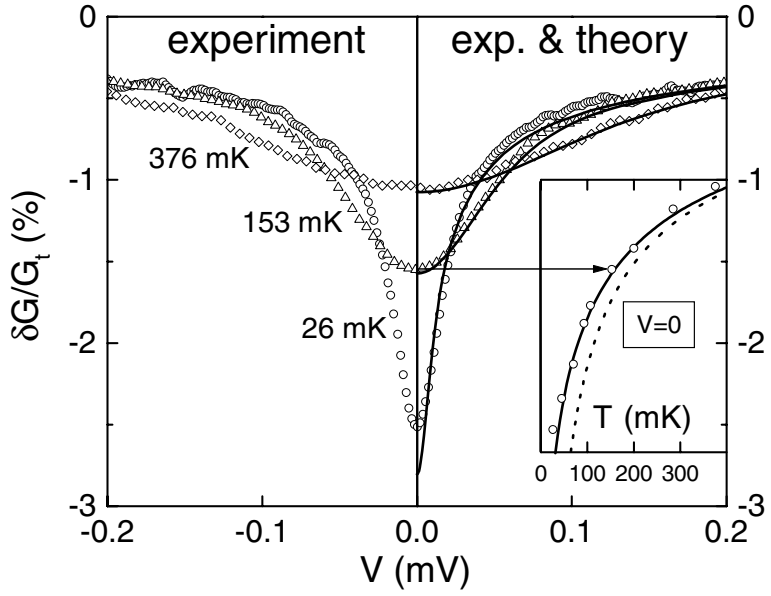

Figure 1.2: Experimental data for the *zero bias anomaly* taken from [1]
The conductance of an aluminum wire was measured for different temperatures.

Before entering the historical review of the field let us discuss the quantity that we are focussing on: The quasiparticle density of states per volume and spin direction (DOS) which we denote for the three dimensional case by $\nu_3(\epsilon)$. It has the unit $\left[\frac{1}{\mathrm{Jm}^3}\right]$. In the remainder we will often just talk of the density of states which is in some books associated with the quantity that has the dimension $\left[\frac{1}{\mathrm{J}}\right]$. Our DOS however will always be related either to a volume or to a length. We denote it either by $\nu_3(\epsilon)$ or $\nu_1(\epsilon)$ $\left[\frac{1}{\mathrm{Jm}}\right]$ indicating the different dimensionality. According to text books[8] the DOS of a free electron gas confined in a box is given by

$$\nu_3(\epsilon) = \frac{\sqrt{2}\, m^{3/2}}{\hbar^3 \pi^2}\, \sqrt{\epsilon} \tag{1.6}$$

where m is the electron mass. The DOS increases with $\sqrt{\epsilon}$. In this work we examine the DOS close to the Fermi energy ϵ_{F}. In order to give a feeling what we mean by close to ϵ_{F}, let us look at some numbers: The energy E_s related to the singlet interaction, that will appear in this work, is of the order of $10^{-9}\epsilon_{\mathrm{F}}$. In the figure (1.3) we plotted $\nu_3(\epsilon)$ for typical values of a metal. The Fermi energy is several eV and the effective mass is close to the free electron mass. For such values the DOS at the Fermi level is of order $10^{47}\frac{1}{\mathrm{Jm}^3}$. We have put an energy strip indicating the regime of our investigation, which is rather disproportionate because even $1000E_s \approx 10^{-5}eV$ is very small.[9] In our calculations we will always approximate the DOS $\nu_3(\epsilon)$ around the Fermi level by a constant because it changes, in the energy strip we are interested in, only very weakly. We set $\nu_3 := \nu_3(\epsilon_{\mathrm{F}})$.

[8] See for example [9].
[9] See also the energy scale in figure (1.2).

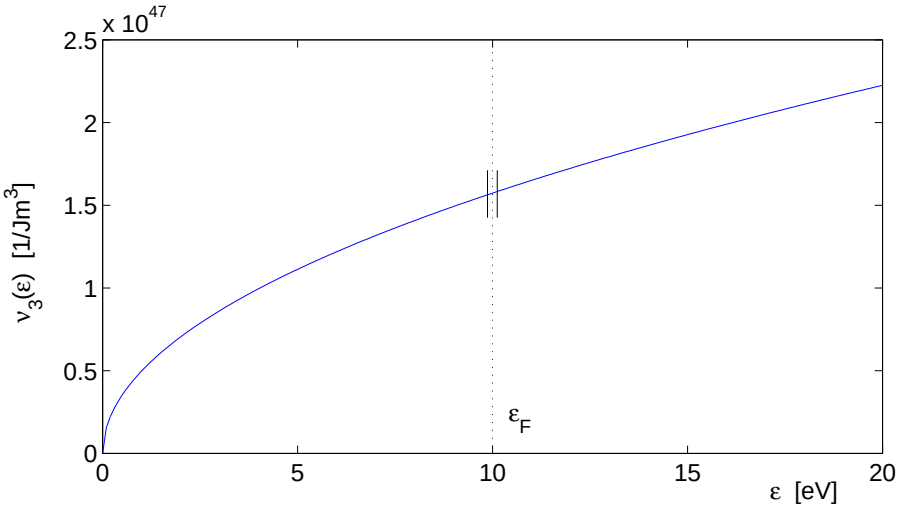

Figure 1.3: Density of states $\nu_3(\epsilon)$ for free electrons in a box.
The sketch shows the small strip we will investigate in real diffusive metals.

There is a wealth of experimental data on the observation of the zero bias anomaly. In the quasi one–dimensional case studies were carried out on thin wires of WRe (Chaudari and Habermeier 1980 [10]), AuPd (Giordano et al. 1979 [11], Giordano 1980 [12], White et al. 1982 [13]), Pt (Masden and Giordano 1981 [14], 1982 [15]), Cu and Ni (White et al. 1982 [13]). The experiments confirmed the $1/\sqrt{eV}$ behavior. Especially we want to mention the beautiful experiments by Imry and Ovadyahu [16] who varied the thickness of an indium oxide film ($a = 16$–260nm) and could observe the transition from the $1/\sqrt{eV}$ to $\ln(eV)$ behavior. The experimental data is shown in figure (1.4).

The curves in figure (1.2) are taken from a recent experiment of Pothier et al. [1]. They stress that the $1/\sqrt{eV}$ behavior arises from the diffusion of the electric potential along the wire and not from the diffusion of the quasiparticles. In section (2.2.5) we will prove the correctness of this statement in the framework of diffusive perturbation theory.

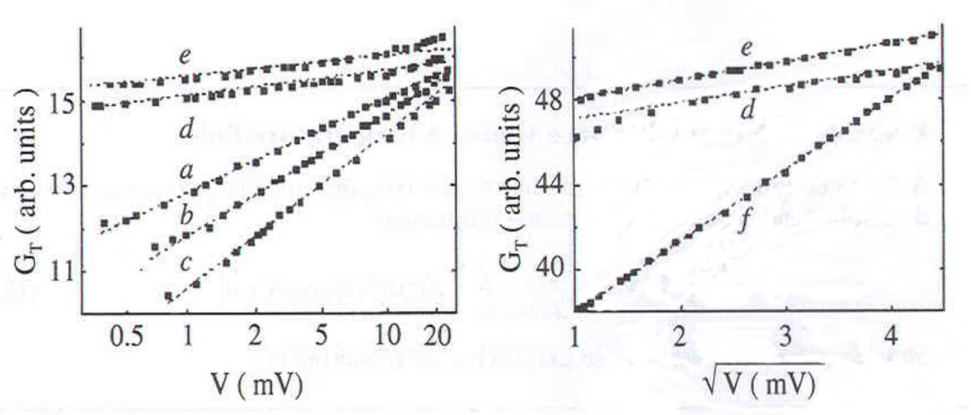

Figure 1.4: Tunneling conductance as function of $\ln V$ (left) and $\sqrt{V}$ (right) for a indium–isolator–lead junction. The different curves are obtained by varying the thickness a of the indium oxide film: a) $a = 16$nm; b) $a = 19$nm; c) $a = 21$nm; d) $a = 31$nm; e) $a = 46$nm; f) $a = 260$nm; see [16]

On the theoretical side we want to mention two recent works that generalized the perturbative results obtained by Altshuler and Aronov. Kamenev and Andreev [17] successfully derived a non–perturbative result for the quasi two–dimensional DOS using a Keldysh σ–model. Inspired by this work, Rollbühler and Grabert ([18], [19]) developed a theory for quasi one–dimensional systems including additionally the inter–electrode interactions. They obtained a non divergent solution for the DOS at low energies, that recovers the $1/\sqrt{eV}$ behavior for higher energies.

In our discussion of the effect of the interaction on the DOS we neglected so far the influence of the spin of the electrons. Also here Altshuler and Aronov were the pioneers who first obtained results using perturbation theory ([20], [21], [22], [23] and [24]). For a quasi one–dimensional system they found two additional singularities in the presence of an external magnetic field. Apart from the $1/\sqrt{eV}$ divergence they predicted poles at $1/\sqrt{|eV + 2\mu_B B_{ex}|}$ and $1/\sqrt{|eV - 2\mu_B B_{ex}|}$.

In the early nineties Raimondi, Castellani and Di Castro [25] reinvestigated the problem using perturbation theory and a renormalization group analysis. They predicted that the divergencies at $1/\sqrt{|eV + 2\mu_B B_{ex}|}$ and $1/\sqrt{|eV - 2\mu_B B_{ex}|}$ do not exist and instead one would see broadened peaks at $\pm 2\mu_B B_{ex}$. Up to now, no experimental papers are known to the author that investigated the dependence of the DOS on the spin. According to private communications Pothier et al. made experiments taking their aluminum wires of the experiment shown above [1]. Up to several Tesla they did not see any signature due to external magnetic field.

Of course, this does not mean too much as one has to analyze carefully under which conditions one can expect a measurable effect. Spin scattering due to magnetic impurities and spin–orbit scattering tend to suppress the effect of the spin. We will discuss this in the chapter on perturbation theory. However one could ask if there are any features at $\pm 2\mu_B B_{ex}$ at all, even without any mechanism of suppression. It could be that the predicted dips or divergencies are an artefact of an insufficient theoretical description. Maybe a theory that takes into account the Coulomb interaction non perturbatively cures the divergencies, like it was shown by Kamenev and Andreev [17] and Rollbühler and Grabert [18] for the spinless case. If there is an effect of the spin on the DOS, it is necessary to estimate its magnitude and seek a suitable regime in which it is most pronounced. In this thesis we will try to answer these questions.

Outlook

We will approach the problem of calculating the DOS near the Fermi level in the presence of disorder and interactions using Green's functions. The Green's functions are a powerful tool in tackling such many–body problems. Therefore we connect the DOS to a Green's function which we then determine approximately taking into account the various effects. Assuming non–interacting electrons confined in a box of volume $\mathcal{V}$ the eigenstates of the Schrödinger equation $H\Psi = E\Psi$ form a discrete set with eigenvalues $\epsilon_p = \frac{p^2}{2m}$. The DOS per volume and spin direction is given by[10]

$$\nu_3(\epsilon) = \frac{1}{\mathcal{V}} \sum_p \delta(\epsilon - \epsilon_p). \tag{1.7}$$

For a macroscopic system the energy levels lie very close. The average level distance $\Delta\epsilon$ can be related to the DOS in the following way: $\Delta\epsilon = \frac{1}{\nu_3 \mathcal{V}}$. In experiments one always has a finite measuring time which leads to the fact that one is averaging over a certain energy interval which is determined by the uncertainty principle. Thus instead of seeing the highly peaked structure of equation (1.7) one obtains a smooth function as depicted in figure (1.3). The Green's function

[10] Note that $\delta(\epsilon - \epsilon_p)$ has the unity $1/J$.

associated with the problem $\big(\epsilon - H_0\big)G(r,r',\epsilon) = \delta(r-r')$ reads in (ϵ,p)–representation[11]

$$G(\epsilon,p) = \frac{1}{\epsilon - \epsilon_p + i\eta\,\mathrm{sign}(\epsilon)} \tag{1.8}$$

where η is a positive infinitesimal quantity. Note that for $\epsilon > 0$ one has the so called retarded Green's function G^R. Writing the real and imaginary parts separately, we get

$$G^R(\epsilon,p) = \frac{\epsilon-\epsilon_p}{(\epsilon-\epsilon_p)^2+\eta^2} - i\frac{\eta}{(\epsilon-\epsilon_p)^2+\eta^2}. \tag{1.9}$$

Hence the imaginary $\mathrm{Im}G^R(\epsilon,p) = -\pi\delta(\epsilon-\epsilon_p)$,[12] and we can express the DOS per spin direction as

$$\boxed{\nu_3(\epsilon) = -\frac{1}{\pi\mathcal{V}}\,\mathrm{Im}\sum_p G^R(p,\epsilon).} \tag{1.11}$$

This relation between ν_3 and G^R holds in general. For a proof consult for example [26]. Here we just wanted to link ν_3 and G^R using an easy and intuitive example.

The thesis is organized as follows: In the next chapter we will extensively use the standard perturbation theory for diffusive metals in order to calculate the DOS for various situations. In particular we will review results in the literature and extend them by accounting for dynamical screening. These results will serve as a benchmark for our non–perturbative results for high energies.[13] Furthermore we will discuss the spin scattering mechanisms due to magnetic impurities and spin–orbit scattering.

Afterwards we will turn to a path integral approach for Fermions. In the chapters 3, 4 and 5 we will explain the construction of a coherent state path integral for Fermions on the Keldysh–contour and show how to implement disorder and spin dependent interactions. In chapter 6 we will combine the effects of disorder and interactions (including spin) writing down an action that models our diffusive quasi one–dimensional wire. Generalizing an idea of Kamenev and Andreev we will be able to determine a non–perturbative expression for the Green's function which will allow us to derive a non–perturbative result for the quasi one–dimensional DOS.

In chapter 7 we implement additionally a constant external magnetic field in the action. We investigate the influence of the external field on the quasi one–dimensional DOS. Using the example of Palladium, which is a promising material as it has a high magnetic susceptibility for low temperatures, we will estimate quantitatively the magnitude of the spin effects on the DOS and discuss their observability.

In the conclusions we will finally summarize our findings and discuss the open questions.

---

[11] See [26], [27] or [28].

[12] In the limit $\eta \to 0$ the real part is the principal value of $\frac{1}{\epsilon-\epsilon_p}$ and the imaginary part is up to a factor of π a representation of the δ–function. One has

$$\frac{1}{x+i\eta} = \mathcal{P}\frac{1}{x} - i\pi\delta(x) \qquad (\eta = 0^+). \tag{1.10}$$

[13] By high energies we mean here several E_s away from the Fermi edge $\epsilon = 0$.

Chapter 2

Perturbation Theory

As mentioned in the introductory part we will tackle the problem of calculating the DOS in the presence of disorder, interactions and external magnetic fields using the Green's function formalism. For the one particle Green's function exists a well established diagram technique. In this chapter we are going to discuss the crucial lowest order diagrams in the Coulomb interaction at zero temperature which already predict the spectacular suppression of the DOS near the Fermi level. However taking only into account the first order diagrams in the interaction leads to a result for the DOS which diverges in the quasi one–dimensional case like $1/\sqrt{\epsilon}$ for small energies. This unphysical divergence can only be cured by taking into account the Coulomb interaction in all orders. However the problem of identifying and summing up the relevant diagrams in the presence of disorder is a quite intricate task. In order to incorporate higher order processes in the Coulomb interaction we later turn to a path integral approach in the following chapters. The effective low energy model we will use is a Keldysh σ–model.

Let us develop the diagrammatic technique step by step including all the different effects like disorder, interactions and external magnetic fields. The chapter is organized as follows: First we describe how the Green's function is changed in the presence of static impurities $U(R_i)$. Then we add an interaction $V(q,\omega)$ and sketch the two basic first order diagrams. We will discuss the so called impurity ladder and explain the concept of quasi one–dimensionality. At that level of complexity we will calculate the correction to the DOS. Afterwards we include an external magnetic field and discuss the effects of magnetic impurities and spin–orbit scattering. In the whole chapter we restrict ourselves to zero temperature. We indicate the retarded and advanced functions by an upper index ^{R} or ^{A}. The propagators without index ^{R} or ^{A} are time–ordered. However in subsequent chapters where all the different time orderings appear in the same equation we will also indicate the time–ordered functions by an upper index ^{T}.

2.1 Green's function in weak disorder

We consider a particle moving through a static disorder potential.[1] The Hamiltonian of the problem is

$$H = \frac{p^2}{2m} + \sum_{i=1}^{N} U(r - R_i)$$

[1] See [26] and [27] for further information.

The second term represents N short range impurities at the positions R_i. It is convenient to rewrite this in second quantization in which we allow many–electron states. However we emphasize that the electrons do not interact with each other except for the constraint on the wave functions imposed by the antisymmetry condition.

$$H = \sum_p \epsilon_p \; c_p^\dagger c_p + \sum_q U(q)\rho_q \sum_p \; c_{p+q}^\dagger c_p$$

where

$$\rho_q = \sum_{i=1}^{N} e^{-iq\cdot R_i}.$$

The energy ϵ_p is measured relative to the Fermi energy. So in fact we are considering $H - \mu N = \sum_p \xi_p \, c_p^\dagger c_p$ where μ is the chemical potential and $\xi_p = \epsilon_p - \mu$. By shifting the energy scale we write however just ϵ_p. The operators c_p and $c_p^\dagger$ are the electron creation and annihilation operators with the anti commutation relation

$$\{c_p \, , c_{p'}^\dagger\} = \delta_{pp'} \qquad\qquad \{c_p \, , c_{p'}\} = \{c_p^\dagger \, , c_{p'}^\dagger\} = 0.$$

The non diagonal time–ordered Green's function is defined as

$$G_{pp'}(t) := -i\langle |\, \mathcal{T}[c_p(t) \; c_{p'}^\dagger(0)] \, |\rangle$$

where $|\rangle$ is the ground state of the noninteracting electron gas. It is filled up to $\epsilon_{\rm F} = \mu(T=0)$. $c_p(t)$ is the Heisenberg operator $c_p(t) = e^{iHt} \; c_p \; e^{-iHt}$ and $\mathcal{T}[c_p(t)c_{p'}^\dagger(0)] := \Theta(t)c_p(t)c_{p'}^\dagger(0) - \Theta(-t)c_{p'}^\dagger(0)c_p(t)$. Calculating the partial derivative with respect to the time t, one gets the equation of motion

$$\big(i\partial_t - \epsilon_p\big)G_{pp'}(t) = \delta_{pp'}\delta(t) + \sum_q U(q)\rho_q G_{p+q,p'}(t).$$

Fourier transforming ($t \to \epsilon$) and iterating, we obtain a series solution to the problem.

$$\begin{aligned} G_{pp'}(\epsilon) = \delta_{pp'}G_p^0(\epsilon) + \frac{1}{\mathcal{V}}G_p^0(\epsilon)U(p-p')\rho_{p-p'}G_{p'}^0(\epsilon) \\ + \frac{1}{\mathcal{V}^2}\sum_q G_p^0(\epsilon)U(q)\rho_q G_{p+q}^0(\epsilon)U(p-q-p')\rho_{p-q-p'}G_{p'}^0(\epsilon) + ... \end{aligned} \tag{2.1}$$

where

$$G_p^0(\epsilon) = \frac{1}{\epsilon - \epsilon_p + i\eta \; \mathrm{sign}(\epsilon)} \tag{2.2}$$

is the unperturbed Green's function for $U = 0$ with $\eta = 0^+$.

We are interested in universal properties of the system that do not depend on the special realization of the disorder. A suitable method is to average over the disorder configurations. If there are N impurities in the system at positions $R_1, R_2, ...R_N$ then G is a functional of this set of vectors $G = G(R_1, R_2, ...R_N)$ and the average is defined as

$$\overline{G} = \frac{1}{\mathcal{V}^N}\int d^3R_1...d^3R_N \; G(R_1, R_2, ...R_N)$$

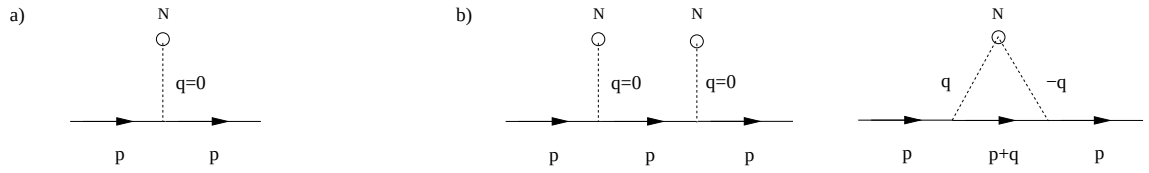

Figure 2.1: a) First and b) second order contribution to the disorder averaged Green's function $G_p(\epsilon)$

where $\mathcal{V}$ is the volume of the system. We assume that each R_i is uncorrelated with all others. Applying this averaging procedure to the series of $G_{pp'}(\epsilon)$, we have averages of the form $\overline{\rho_{q_1}\rho_{q_2}\cdots\rho_{q_N}}$. We find for the first term

$$\overline{\rho_q} = \frac{N}{\mathcal{V}} \int d^3R\, e^{-iq\cdot R} = N\delta_{q,0}.$$

For the second term (which has two contributions $i \neq j$ and $i = j$)

$$\overline{\rho_{q_1}\rho_{q_2}} = N^2\delta_{q_1,0}\ \delta_{q_2,0} + N\delta_{q_1+q_2,0}.$$

One can show in general that the averaging process makes the Green's function diagonal.[2]

$$G_{pp'}(\epsilon) = \delta_{pp'}\ G_p(\epsilon)$$

In figure (2.1) we represented the averaged first and second order terms of the series of equation (2.1) by diagrams. The solid line represents $G_p^0(\epsilon)$. In these diagrams the single dashed lines ending in a circle may be thought of as independent scattering events (in lowest order Born approximation) from different impurities and the pair of dashed lines ending in a single circle corresponds to second order Born scattering from the same impurity.

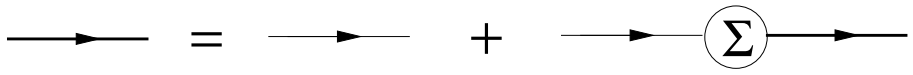

Figure 2.2: Feynman diagram of the Dyson equation of the Green's function

It is not possible to sum all diagrams. However by introducing the concept of irreducible diagrams one can carry out a partial summation. An irreducible diagram is defined as a diagram which cannot be divided into two sub–diagrams joined only by a single $G_p^0(\epsilon)$ line. The sum of the irreducible diagrams (where the incoming and outgoing $G_p^0(\epsilon)$ lines are removed) is called the self energy $\Sigma_p(\epsilon) = \sum_i \Sigma_p^{(i)}(\epsilon)$. This allows us to write the Green's function in the form of a Dyson equation:

$$G_p(\epsilon) = G_p^0(\epsilon) + G_p^0(\epsilon)\Sigma_p(\epsilon)G_p(\epsilon). \tag{2.3}$$

The graphical representation of the Dyson equation is in figure (2.2) and the self–energy is given in figure (2.3). The diagrams in the self–energy can be classified according to their dependence on the impurity density $n_{\rm imp} = \frac{N}{\mathcal{V}}$. We remark that every scattering circle represents a factor n_{imp}. We will restrict ourselves to *low densities* which means keeping only the single scattering circle diagrams (**Assumption 1**).

[2]The averaging process restores the translational invariance. In the remainder we will not write $\overline{G}_{pp'}$ indicating the disorder average but just $G_{pp'}$ in order to keep the notation simple.

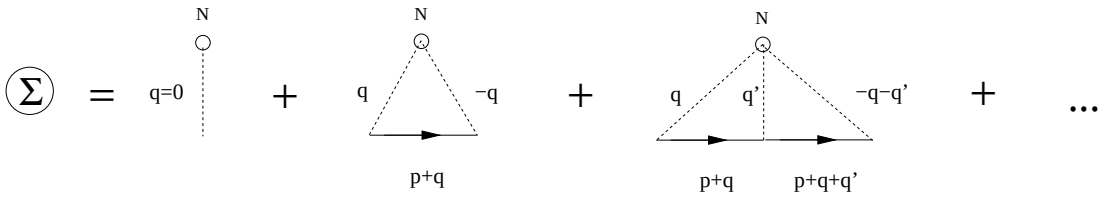

Figure 2.3: The Self–energy

Furthermore we only take into account the two simplest irreducible diagrams that are depicted in figure (2.3) which means that we assume the scattering from a given impurity to be *weak* (**Assumption 2**). [3]

The first term contributes $NU(q=0)$. This simply leads to a shift in the energy.

$$\epsilon_p \to \epsilon_p + \Sigma_p^{(1)}(\epsilon) = \epsilon_p + NU(q=0) = \epsilon_p + n_{\rm imp}\int d^3r\; U(r)$$

We can take care of this term by a redefinition of the energies ϵ_p. The second order Born approximation reads

$$\Sigma_p^{(2)}(\epsilon) = \frac{N}{\mathcal{V}^2}\sum_q U(q)G^0_{p+q}U(-q).$$

As $U(r)$ is real we have $U(q)^* = U(-q)$. Also we assumed $U(q)$ to be short ranged and we can replace in good approximation $U(q)$ by its value at $q=0$ which we denote by U_0. The energy levels in a bulk systems lie very close. As one has a finite observation time t one can not resolve the discrete level spacing $\Delta\epsilon$. If the inequality $\Delta\epsilon \ll \hbar/t$ holds we can replace the q–sum by an energy integral with a suitable smooth density of states $\nu_3(\epsilon)$.

$$\frac{1}{\mathcal{V}}\sum_q \;...\approx \int_{-\epsilon_{\rm F}}^{\infty} d\epsilon\;\nu_3(\epsilon)... \tag{2.4}$$

We have then

$$\Sigma_p^{(2)}(\epsilon) = n_{\rm imp}U_0^2\int_{-\epsilon_{\rm F}}^{\infty} d\epsilon' \frac{\nu_3(\epsilon')}{\epsilon-\epsilon'+i\eta\;{\rm sign}(\epsilon)}. \tag{2.5}$$

As the integrand is peaked around the Fermi energy $\epsilon_{\rm F}$ we can replace $\nu_3(\epsilon_p)$ by its value at $\epsilon_{\rm F}$ which we denote by ν_3. The real part is divergent. This is a consequence of the simple model taken for the scattering potential U. A more realistic model will generally cure the divergence and give rise to a finite contribution that may be absorbed into a redefinition of the energies. The imaginary part (see formula (1.10)) is approximately

$$\Sigma_p^{(2)}(\epsilon) = -i\pi\nu_3 n_{\rm imp}U_0^2\;{\rm sign}(\epsilon).$$

It is convention to define the elastic scattering time

$$\tau = \frac{1}{2\pi\nu_3 n_{\rm imp}U_0^2}. \tag{2.6}$$

[3] Three successive scattering events from one impurity as depicted in the third diagram on the right side of figure (2.3) are then negligible.

Then we obtain for the self–energy

$$\Sigma_p(\epsilon) \approx \Sigma_p^{(1)}(\epsilon) + \Sigma_p^{(2)}(\epsilon) = -\frac{i}{2\tau}\mathrm{sign}(\epsilon)$$

and the damped Green's function reads

$$\boxed{G_\tau(p,\epsilon) = \frac{1}{\epsilon - \epsilon_p + \frac{i}{2\tau}\,\mathrm{sign}(\epsilon)}.} \tag{2.7}$$

Compared to (2.2) η is replaced by $1/2\tau$. Due to the impurities the Green's function has a finite imaginary self–energy part and every state with energy ϵ_p acquires a finite lifetime. In the time domain one has $G_\tau(t,p) \sim e^{-t/2\tau}$. This is understandable as the momentum eigenstates, taken as a basis, are no longer solutions of the problem.

Keep in mind that equation (2.7) is not the general solution of the problem. In neglecting the crossed diagrams (see figure (2.4) diagram d)) we restricted us to the case of weak disorder which is characterized by the inequality $\epsilon_\mathrm{F}\,\tau \gg 1$. In [29] it is explicitly shown how crossed lines lead to an extra factor $\frac{1}{\epsilon_\mathrm{F}\tau}$.

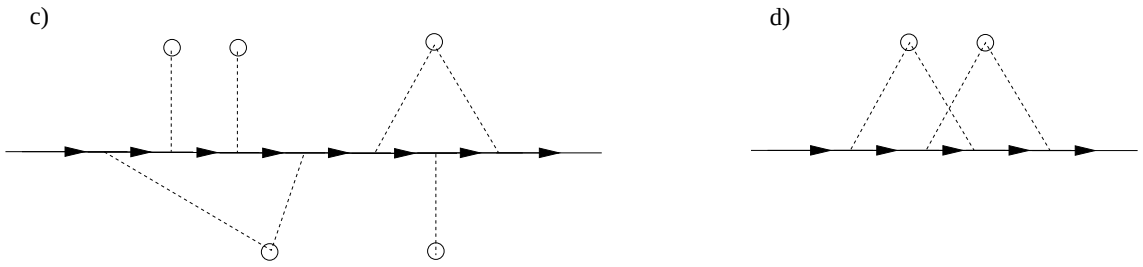

Figure 2.4: c) is included by our self–energy d) is not included in $G_\tau(p,\epsilon)$

Note however that the Green's function G_τ is the self–consistent solution for our regime. Taking the damped Green's function of equation (2.7) as starting point and taking again into account scattering events in the second order Born approximation we will arrive at the same Green's function in the end. This is due to the fact that we only need that the pole lies close to the real axis which amounts to the condition $\epsilon_\mathrm{F}\,\tau \gg 1$.

Having now the basic quantity for our perturbation theory we can start adding a short range interaction between the quasiparticles. Let us denote this potential by $V(q,\omega)$. The terminology *short range* means that the interaction is finite in the limit $q \to 0$ and $\omega \to 0$.

2.2 Calculation of the DOS without external magnetic field

In this part we calculate the DOS of a weakly disordered, interacting wire neglecting the spin dependent interactions. Thus in this section the spin only comes in as a factor of 2. All parameters affecting the spin like external magnetic fields and spin scattering are discussed in the following sections.

Our philosophy in this chapter will be to write down first the expressions to be determined, and then step by step we will explain the meaning of the different objects that appear. However we do not repeat in this work all the rules how to construct Feynman diagrams. For a

introduction see for example [26], [27] or [28]. The correction to the DOS for one spin direction reads

$$\frac{\delta\nu(\epsilon)}{\nu_3} = -\frac{\mathrm{sign}(\epsilon)}{\pi\nu_3}\frac{1}{\mathcal{V}}\sum_p \mathrm{Im}\Big[\delta G(p,\epsilon)\Big] \tag{2.8}$$

where $\mathcal{V}$ is the volume of the system and ν_3 the three–dimensional DOS at the Fermi level of the noninteracting wire per spin direction. The additional $\mathrm{sign}(\epsilon)$ that appears here in comparison to equation (1.11) is due to the fact we use the time–ordered instead of the retarded Green's function. In all our diagrams the solid lines represent the time–ordered Green's function. Note that p is a three–dimensional vector. In principle all vectors in our calculation are three–dimensional. However in the quasi one–dimensional regime one can neglect certain components. This will be discussed in section (2.2.5).

$$\delta G(p,\epsilon) = G(p,\epsilon) - G_\tau(p,\epsilon)$$

is the change of the Green's function $G(p,\epsilon)$ due to the Coulomb interaction. The time–ordered Green's function of the noninteracting wire is given in equation (2.7). In lowest order perturbation theory one has

$$\delta G(p,\epsilon) = G^{(1)}(p,\epsilon) - G_\tau(p,\epsilon) = G_\tau(p,\epsilon)\Sigma^{(1)}(p,\epsilon)G_\tau(p,\epsilon)$$

where the self–energy $\Sigma^{(1)}(p,\epsilon)$ consists of two parts

$$\Sigma^{(1)}(p,\epsilon) = \Sigma^{\mathrm{Exc}}(p,\epsilon) + \Sigma^{\mathrm{Har}}(p,\epsilon)$$

coming from the exchange and the Hartree diagrams depicted in figure (2.5) and (2.6). The indices α and β indicate the spin of the corresponding particle. They can be neglected for our current discussion but will become important when we add an external magnetic field.

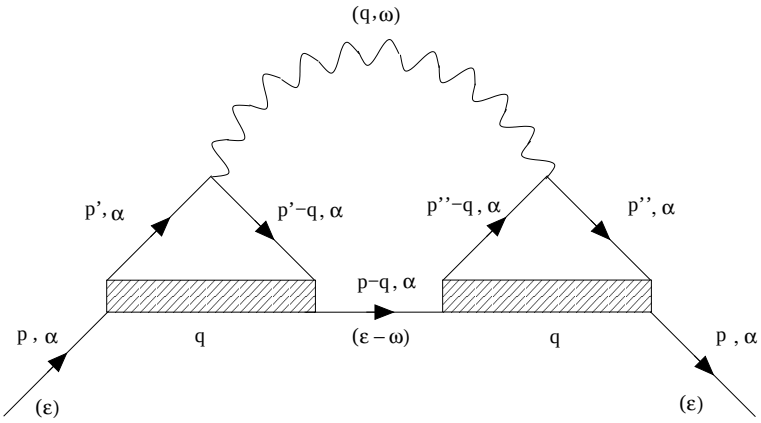

Figure 2.5: Exchange diagram in the presence of disorder.

Let us begin with the exchange diagram. The solid lines represent the damped Green's function G_τ and the wavy line symbolizes the dynamic interaction $V(q,\omega)$. Finally there are the two shaded boxes. These boxes represent the so called diffusion ladder $L(q,\omega)$. A diagrammatic translation of the ladder is given in figure (2.7). One can see that the ladder takes into account

successive scattering events of a particle–hole pair at a sequence of impurities. As it is the key element of the diffusive theory we will derive its form in the next section.[4]

Thus the exchange diagram represents an electron emitting a photon with momentum q and energy ω and reabsorbing it again. The ingoing and outgoing electron lines are connected by scattering events. Let us already stress here that due to the ladder elements before and after the emission of the photon the main contribution comes from **small momenta $\boldsymbol{q}$ and small energies $\boldsymbol{\omega}$**. The functional form of $L(q,\omega)$ will make this obvious. Throughout the whole work we will use the letters q and ω to indicate small quantities whereas p's and ϵ's represent big momenta and energies near the Fermi surface. The correction to the DOS due to the exchange diagram reads

$$\delta\nu^{\text{exc}} = -2\,\frac{\text{sign}(\epsilon)}{\pi\mathcal{V}}\,\text{Im}\sum_p G_\tau(p,\epsilon)\Bigg[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,L(q,\omega)G_\tau(p-q,\epsilon-\omega)L(q,\omega)(iV(q,\omega))$$
$$\frac{1}{\mathcal{V}}\sum_{p'}G_\tau(p',\epsilon)G_\tau(p'-q,\epsilon-\omega)\,\frac{1}{\mathcal{V}}\sum_{p''}G_\tau(p'',\epsilon)G_\tau(p''-q,\epsilon-\omega)\Bigg]G_\tau(p,\epsilon) \tag{2.9}$$

where the additional factor of i in front of $V(q,\omega)$ comes from the internal photon line (see Feynman rules in [28]) and the factor of 2 in front takes into account that the diagram can be drawn for spin $\alpha = 1/2$ and $\alpha = -1/2$.

The Hartree diagram has a different structure. The ingoing electron interacts with a second electron performing a closed loop. The energy ω transferred by the interaction $V(q,\omega)$ has to be 0 as we need to conserve the energy in the upper loop (remember that the impurities are static and therefore can not exchange energy). Note also that the transferred momentum $p' - p''$ is not necessarily small. The second important difference is the additional minus sign coming from the electron loop. As a consequence **the Hartree correction has the opposite sign of the exchange correction**.

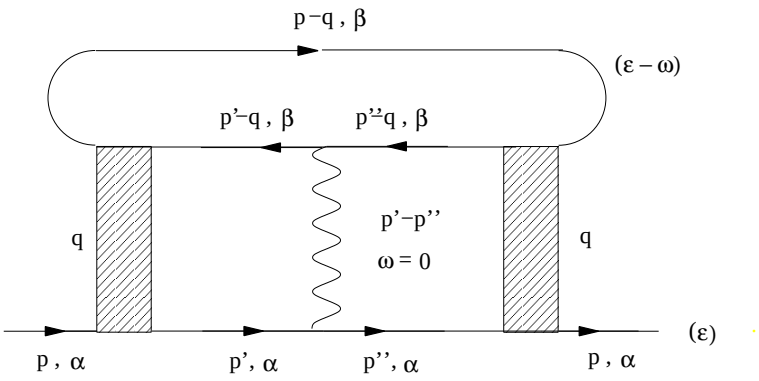

Figure 2.6: Hartree diagram in the presence of disorder.

[4]Inverting the hole line one obtains the so called Cooperon which is the particle–particle ladder. Taking into account the Cooperon one gets the weak localization correction to the conductivity. For further information see [29]

A third point which will become important when we take into account the spin in the presence of an external magnetic field is that we can of course choose the spin of the loop electron independent of the spin of the incoming electron. Here however this freedom just amounts in an additional factor of 2. So the spin degeneracy of the Hartree diagram without external field is 4.

$$\delta\nu^{\mathrm{Har}} = -4\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\mathrm{Im}\sum_p G_\tau(p,\epsilon)\Bigg[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,(-i)\,L(q,\omega)G_\tau(p-q,\epsilon-\omega)L(q,\omega)$$
$$\frac{1}{\mathcal{V}^2}\sum_{p',p''}G_\tau(p',\epsilon)G_\tau(p'-q,\epsilon-\omega)\,V(p'-p'',0)G_\tau(p'',\epsilon)G_\tau(p''-q,\epsilon-\omega)\Bigg]G_\tau(p,\epsilon) \tag{2.10}$$

For the evaluation of (2.9) and (2.10) we need to determine the functions $L(q,\omega)$ and $V(q,\omega)$. This will be done in the next sections.

2.2.1 The ladder approximation

Two particles can scatter at the same static impurity. It can be shown that the leading order term for low energies arises from diagrams without any crossing of lines (see for example [27] or [29]). The relevant processes are shown in figure (2.7). The dashed line with a circle in the middle represents the scattering at a static impurity. Static implies that it can not absorb or emit energy. All the impurity does, is transferring momentum from one particle to the other as can be seen on the right side of the diagram in figure (2.7). The particle moving to the right changes its momentum from p to p' and the hole moving to left changes its momentum from $p'-q$ to $p-q$. Quite general the series of scattering events may be evaluated by solving the integral equation for ladder $L_{p,p',\epsilon}(q,\omega)$.

$$L_{p,p',\epsilon}(q,\omega) = L_0 + L_0\,\frac{1}{\mathcal{V}}\sum_{p''}G_\tau(p'',\epsilon)G_\tau(p''-q,\epsilon-\omega)L_{p'',p',\epsilon}(q,\omega) \tag{2.11}$$

Each impurity line contributes $L_0 = n_{\mathrm{imp}}U_0^2 = \frac{1}{2\pi\tau\nu_3}$ (see equation (2.6)). The ladder generally depends on three momenta and two energies. However in the diffusive limit we will see that the ladder in fact only depends on q and ω and we can drop the subscripts p, p' and ϵ.

$$L(q,\omega) = L_0 + L_0\,\frac{1}{\mathcal{V}}\sum_p G_\tau(p,\epsilon)G_\tau(p-q,\epsilon-\omega)L(q,\omega) \tag{2.12}$$

We define the (spin independent) polarization bubble

$$\Pi_0(q,\omega) = \frac{1}{\mathcal{V}}\sum_p\,G_\tau(p,\epsilon)G_\tau(p-q,\epsilon-\omega). \tag{2.13}$$

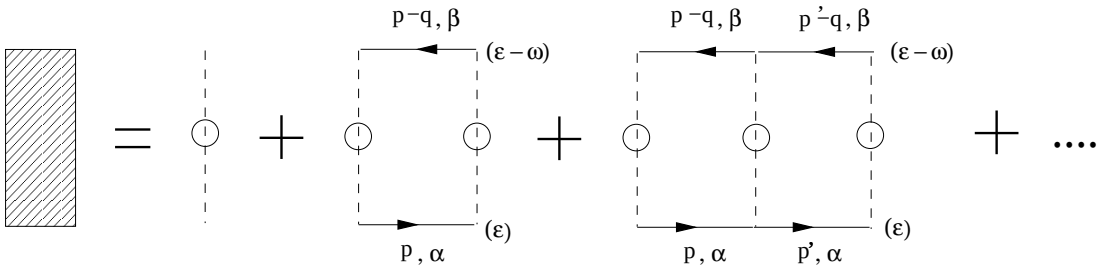

Figure 2.7: Diagrammatic representation of the ladder $L_{\alpha\beta}(q,\omega)$

Usually in the literature one can find diagrams like in figure (2.8) for the polarization bubble. One would say that there are no bubbles in the ladder diagrams in figure (2.7). However bending the Green's function lines straight, one recognizes the pair of Green's functions appearing in the ladder sequences.

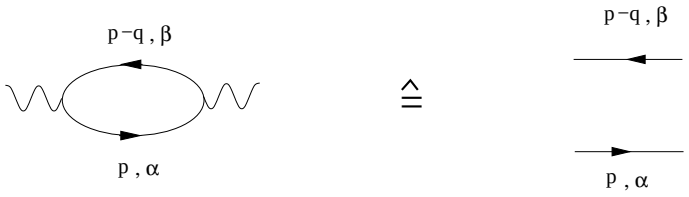

Figure 2.8: Diagrammatic representation of the polarization propagator $\Pi_0^{\alpha\beta}$

We calculate Π_0 in the continuum limit. The standard approximation reads

$$\frac{1}{\mathcal{V}}\sum_p \ldots \approx \int \frac{d^3p}{(2\pi)^3}\ldots = \int d\epsilon_p\, \nu_3(\epsilon_p) \int \frac{d\Omega_p}{4\pi}\ldots \approx \nu_3 \int d\epsilon_p \int \frac{d\Omega_p}{4\pi}\ldots \tag{2.14}$$

where $d\Omega_p = d\phi\, d\theta \sin\theta$ and the boundaries for the ϕ integration are 0 and 2π and the θ integration goes from 0 to π. For the integration over $d\epsilon_p$ the choice of the correct boundaries needs some caution. Usually one finds the following

$$\frac{1}{\mathcal{V}}\sum_p \ldots \approx \nu_3 \int_{-\epsilon_\mathrm{F}}^{\infty} d\epsilon_p \ldots \approx \nu_3 \int_{-\infty}^{\infty} d\epsilon_p \ldots \tag{2.15}$$

The extension of the lower boundary to $-\infty$ is only justified if the contribution of the integrand in the interval $[-\infty, -\epsilon_\mathrm{F}]$ is negligible. The reader can verify that for our integrand the contribution is of order $\frac{\nu_3}{\epsilon+\epsilon_\mathrm{F}+\frac{i}{2\tau}} \sim \nu_3\tau\, \frac{1}{\epsilon_\mathrm{F}\tau} \ll \nu_3\tau$ which is negligible. Thus we calculate Π_0 in the approximation of equation (2.15).

A second comment we have to make before starting the calculation is how the diffusive regime is reflected in our calculations. Diffusion is a continuous process which describes long time developments. Thus the length scale L on which we observe diffusive motion is related to the mean free path l by $L \gg l$. For the k-vectors associated with L we have approximately: $q \approx 1/L$. Thus we have the relation: $\boldsymbol{ql = v_F q\tau \ll 1}$. We do not have only big length scales but also big timescales t. t is much bigger than the elastic scattering time τ: $t \gg \tau$. Relevant energies associated with t are $\omega \approx 1/t$. If we combine these last two statements we obtain a second inequality $\boldsymbol{\omega\tau \ll 1}$. Using the diffusion approximation thus means focussing on small

energies ω and small k–vectors q. This allows some approximations in several integrals. Near the Fermi surface one has

$$\epsilon_{p-q} \approx \epsilon_p - \mathbf{v_F}\cdot\mathbf{q} \qquad \text{where} \qquad \mathbf{v_F} = \nabla_{\mathbf{p}}\,\epsilon_{\mathbf{p}}\big|_{|\mathbf{p}|=\mathbf{p_F}}$$

is the Fermi velocity. We use bold letters for the vectors. Then Π_0 reads

$$\Pi_0(q,\omega) = \nu_3 \int_{-\infty}^{\infty} d\epsilon_p \int \frac{d\Omega_p}{4\pi} \frac{1}{\epsilon - \epsilon_p + \frac{i}{2\tau}\mathrm{sign}(\epsilon)} \; \frac{1}{\epsilon - \omega - \epsilon_p + \mathbf{v_F}\cdot\mathbf{q} + \frac{i}{2\tau}\mathrm{sign}(\epsilon-\omega)}.$$

For the ϵ–integration we can apply the residue theorem. In the complex ϵ_p–plane the poles have to lie on different sides of the real axis in order to get a non vanishing contribution. We obtain

$$\Pi_0(q,\omega) = 2\pi i \nu_3 \int \frac{d\Omega_p}{4\pi} \left(\frac{\Theta(\epsilon)\Theta(\omega-\epsilon)}{\omega - \mathbf{v_F}\cdot\mathbf{q} + \frac{i}{\tau}} - \frac{\Theta(-\epsilon)\Theta(\epsilon-\omega)}{\omega - \mathbf{v_F}\cdot\mathbf{q} - \frac{i}{\tau}} \right).$$

We multiply nominator and denominator by $\pm i\tau$

$$\Pi_0(q,\omega) = \frac{\nu_3\tau}{2} \int d\Omega \left(\frac{\Theta(\epsilon)\Theta(\omega-\epsilon)}{1 - i\omega\tau + i\mathbf{v_F}\cdot\mathbf{q}} + \frac{\Theta(-\epsilon)\Theta(\epsilon-\omega)}{1 + i\omega\tau - i\mathbf{v_F}\cdot\mathbf{q}} \right).$$

Although the angular integral may be evaluated exactly we restrict ourselves to the diffusive limit [5] and expand in frequency and momentum. We find

$$\frac{1}{1 \mp i\omega\tau \mp i\mathbf{v_F}\cdot\mathbf{q}\tau} \approx 1 \pm i\omega\tau \pm i\mathbf{v_F}\cdot\mathbf{q}\tau - (\mathbf{v_F}\cdot\mathbf{q}\tau)^2 + \ldots$$

We neglect higher order terms in $\omega\tau$ and $\mathbf{v_F}\cdot\mathbf{q}\tau$ as this accuracy is high enough to reproduce the results of the macroscopic diffusion theory (see the footnote to equation (2.23)). Now we integrate over the angles noting that $\int_{-1}^{1} d(\cos\theta)\cos\theta = 0$ and $\int_{-1}^{1} d(\cos\theta)\cos^2\theta = \frac{2}{3}$. We get

$$\Pi_0(q,\omega) = 2\pi\nu_3\tau \; \Theta_+(\epsilon,\omega) \; (1 + i|\omega|\tau - Dq^2\tau) \tag{2.16}$$

where $D = \frac{v_F\, l}{3} = \frac{1}{3}v_F^2\tau$ is the classical diffusion constant in three dimensions and we introduced the abbreviation $\Theta_+(\epsilon,\omega) := \Theta(\epsilon)\Theta(\omega-\epsilon) + \Theta(-\epsilon)\Theta(\epsilon-\omega)$. From equation (2.12) we deduce

$$L(q,\omega) = \frac{L_0}{1 - L_0\,\Pi_0(q,\omega)} \tag{2.17}$$

and obtain

$$\boxed{L(q,\omega) = \frac{\Theta_+(\epsilon,\omega)}{2\pi\nu_3\tau^2(Dq^2 - i|\omega|)}.} \tag{2.18}$$

Now we can see explicitly what we announced in the previous section: **The function $L(q,\omega)$ selects small momenta q and small frequencies ω due to its so called diffusive pole**. This diffusive pole will be finally responsible for the correction of the DOS. But before calculating the correction to the DOS we need to discuss the dynamically screened interaction appearing in equations (2.9) and (2.10).

[5] This expansion is sufficient in the low energy (or low temperature) regime when $\epsilon\tau \ll 1$. At higher energies with $\epsilon\tau \geq 1$ one must retain the full frequency and momentum dependence of Π_0. This defines the quasi–ballistic regime. A detailed discussion can be found in [41]

2.2.2 Dynamically screened interaction and polarization function

Let us start with a static bare interaction $V_0(q)$ between our particles. This bare interaction is screened in a many–particle system. Each electron is surrounded by other charge carriers. Thus other electrons do not feel the bare electron charge but an effective screened charge. The reorganization of charges is a dynamical time dependent process which leads to an energy dependent effective interaction $V(q,\omega)$ between the quasiparticles (=electrons + surrounding clouds). $V(q,\omega)$ can be expressed by the bare interaction $V_0(q)$ and the so called polarization propagator $\Pi(q,\omega)$ in the form of a Dyson equation[6]

$$V(q,\omega) = V_0(q) + V_0(q)\Pi(q,\omega)V(q,\omega). \tag{2.19}$$

The graphical representation is shown in figure (2.9). The polarization propagator $\Pi(q,\omega)$ represents all irreducible diagrams with one incoming and one outgoing photon line. Of course, we are not able to take into account all the irreducible diagrams. A common and successful approximation is the so called *random phase approximation* (RPA) which takes into account a certain class of diagrams. For the clean case one can find an excellent description in [28]. In the presence of disorder this approximation has to be modified including ladder diagrams. In the literature this is often called *diffusive RPA*. Although we denoted the polarization propagator in the diffusive approximation by Π_{Dif} in figure (2.9) in order to stress the difference to the exact Π, we will omit the index in the following to keep the notation simple.

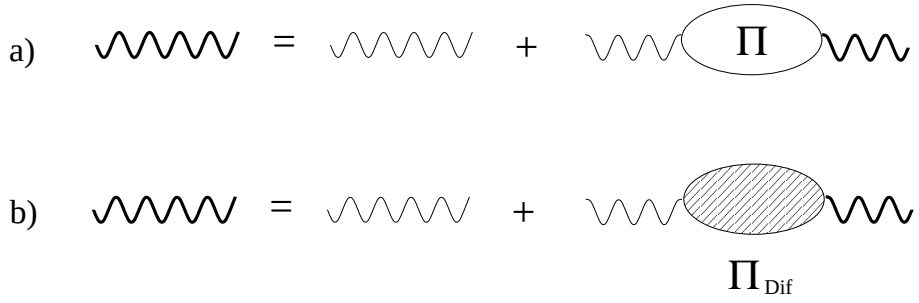

Figure 2.9: Dyson equation: a) exact, b) diffusive RPA–approximation

The first graph on the right side in figure (2.10) shows the first correction to the bare photon line. This diagram is called polarization bubble. It can be interpreted as creation and annihilation of a particle–hole pair.

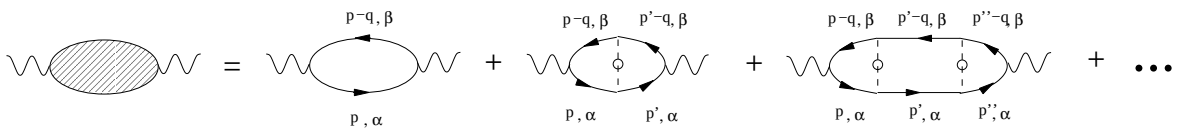

Figure 2.10: Bubble of the diffusive RPA–approximation

In the presence of impurities those two particles can be scattered during their existence (see second and third diagram on the right side in figure (2.10)). Again we include the scattering processes in the ladder approximation.

[6] $\Pi(q,\omega)$ plays here the same role as $\Sigma_p(\epsilon)$ in equation (2.3)

Writing figure (2.10) as an equation we find

$$\Pi(q,\omega) = -2i \int \frac{d\epsilon}{2\pi} \frac{1}{\mathcal{V}} \sum_p G_\tau(p,\epsilon) G_\tau(p-q,\epsilon-\omega)\, \Gamma_0(p,\epsilon,q,\omega) \tag{2.20}$$

where the factor of 2 in front comes from the spin and $\Gamma_0(p,\epsilon,q,\omega)$ is

$$\Gamma_0(p,\epsilon,q,\omega) = 1 + \frac{1}{\mathcal{V}} \sum_{p'} G_\tau(p,\epsilon) G_\tau(p-q,\epsilon-\omega) L_{p',p,\epsilon}(q,\omega).$$

Putting the expression for $\Gamma_0(p,\epsilon,q,\omega)$ in equation (2.20) one can identify the arising terms with diagrams on the right side of figure (2.10). Using the diffusive results (2.16) and (2.18) we get

$$\Gamma_0(q,\omega) = 1 + \frac{\Theta_+(\epsilon,\omega)}{(Dq^2 - i|\omega|)\tau}. \tag{2.21}$$

With the explicit expression for Γ_0 we are now able to start the calculation of the polarization propagator given in equation (2.20). We begin with the so–called *static part* which is proportional to the 1 in equation (2.21).

$$\Pi_{\text{stat}}(q,\omega) = -2i \int \frac{d\epsilon}{2\pi} \int \frac{d^3p}{(2\pi)^3}\, G_\tau(p,\epsilon) G_\tau(p-q,\epsilon-\omega)$$

As the integrand contains no function with poles for small q and ω we can neglect the q and ω dependence in the appearing Green's function.

$$\Pi_{\text{stat}}(q,\omega) \approx -2i \int \frac{d\epsilon}{2\pi} \int \frac{d^3p}{(2\pi)^3}\, G_\tau(p,\epsilon)^2$$

We start with the integration over ϵ

$$\Pi_{\text{stat}}(q,\omega) \approx -2i\, \frac{1}{2\pi} \int \frac{d^3p}{(2\pi)^3} \left(\left[\frac{-1}{\epsilon - \epsilon_p - \frac{i}{2\tau}} \right]_{-\infty}^{0} + \left[\frac{-1}{\epsilon - \epsilon_p + \frac{i}{2\tau}} \right]_{0}^{\infty} \right)$$

and get

$$\Pi_{\text{stat}}(q,\omega) \approx \frac{-i}{\pi} \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{\epsilon_p + \frac{i}{2\tau}} - \frac{1}{\epsilon_p - \frac{i}{2\tau}} \right) \approx \frac{-i\nu_3}{\pi} \int_{-\infty}^{\infty} d\epsilon_p \left(\frac{-i/\tau}{\epsilon_p^2 + \frac{1}{(2\tau)^2}} \right).$$

As we are in the limit of weak disorder $\epsilon_{\text{F}}\tau \gg 1$ the appearing function is in good approximation $-i\, 2\pi\delta(\epsilon_p)$. Hence

$$\Pi_{\text{stat}}(q,\omega) = -2\nu_3$$

which is nothing else but the total DOS at the Fermi level of the non–interacting system. The *dynamic part* contains a diffusive pole coming from the ladder.

$$\Pi_{\text{dyn}}(q,\omega) = -2i \int \frac{d\epsilon}{2\pi} \int \frac{d^3p}{(2\pi)^3}\, G_\tau(p,\epsilon) G_\tau(p-q,\epsilon-\omega) \frac{\Theta_+(\epsilon,\omega)}{(Dq^2 - i|\omega|)\tau}$$

Using once more (2.16) and keeping only the leading 1 we obtain

$$\Pi_{\rm dyn}(q,\omega) = -4i\pi\nu_3 \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{\Theta_+(\epsilon,\omega)}{(Dq^2 - i|\omega|)}.$$

The ϵ–integral yields $|\omega|$.

$$\Pi_{\rm dyn}(q,\omega) = -2\nu_3 \frac{i|\omega|}{Dq^2 - i|\omega|} \tag{2.22}$$

We find in diffusive ladder approximation

$$\Pi(q,\omega) = \Pi_{\rm stat} + \Pi_{\rm dyn} = -2\nu_3 \frac{Dq^2}{Dq^2 - i|\omega|}$$

and the retarded function is

$$\boxed{\Pi^R(q,\omega) = -2\nu_3 \frac{Dq^2}{Dq^2 - i\omega}.} \tag{2.23}$$

At this point one can often read in the literature, that the form of Π reflects particle conservation and that the pole is a direct consequence of the particle conservation. In order to shed light on this statement we added a footnote and show how to derive the expression for Π starting with the continuity equation.[7]Returning to equation (2.19) we can write down the retarded, dynamically screened interaction in diffusive RPA approximation

$$V^R(q,\omega) = \frac{V_0(q)}{1 + 2e^2\nu_3 V_0(q) \frac{Dq^2}{Dq^2 - i\omega}}. \tag{2.29}$$

[7]The polarization function Π also appears in the context of linear response theory. Studying the response of the density of an electron gas ρ to an applied external field ϕ one can derive a Kubo formula: $\rho(q,t) = \int_{-\infty}^{\infty} dt' \chi(q,t-t')\phi(q,t)$ where the retarded response function $\chi(q,t-t') = -i\Theta(t-t')\langle|\rho(q,t),\rho(-q,t')|\rangle$. Π^R is nothing else than χ (for further information see [26]). That is why Π is also often called density–density response function. Let us derive the expression for the density–density response function from macroscopic equations. On the one hand we have the continuity equation

$$\partial_t\ \rho(x) + \nabla j(x) = 0 \tag{2.24}$$

and on the other hand the current density j is determined by

$$j(x) = -D\nabla\rho(x) + \sigma\ E(x) \tag{2.25}$$

The electric field is connected to the scalar potential by $E = -\nabla\phi$ and the conductivity σ can be connected to the Diffusion constant D via the Einstein relation

$$\sigma = 2e^2\nu_3\ D. \tag{2.26}$$

Hence we have

$$\big(\partial_t - D\Delta\big)\rho(x) = 2e^2\nu_3 D\Delta\phi(x). \tag{2.27}$$

Putting this into the diffusion equation above and Fourier transforming leads to $\big(-i\omega + Dq^2\big)\rho(q,\omega) = -2e^2\nu_3 Dq^2\phi(q,\omega)$. We finally obtain

$$\rho(q,\omega) = \underbrace{-2e^2\nu_3 \frac{Dq^2}{Dq^2 - i\omega}}_{\Pi^R}\ \phi(q,\omega) \tag{2.28}$$

2.2.3 The exchange and the Hartree terms

Having discussed in the last sections the Green's function $G_\tau(p,\epsilon)$, the ladder $L(q,\omega)$ and the dynamically screened interaction $V(q,\omega)$, we are now able to evaluate the expressions given in equations (2.9) and (2.10). In both equations we find the following product of Green's functions[8]

$$\gamma_3 := \frac{1}{\mathcal{V}} \sum_p G_\tau^2(p,\epsilon) G_\tau(p-q,\epsilon-\omega) \tag{2.30}$$

which we calculate using the same approximations as before.

$$\gamma_3 = \nu_3 \int_{-\epsilon_F}^{\infty} d\epsilon_p \int \frac{d\Omega}{\Omega} \frac{1}{[\epsilon - \epsilon_p + \frac{i}{2\tau}\mathrm{sign}(\epsilon)]^2} \frac{1}{\epsilon - \omega - \epsilon_p + \mathbf{v_F}\cdot\mathbf{q} + \frac{i}{2\tau}\mathrm{sign}(\epsilon-\omega)}$$

We extend the lower boundary to $-\infty$ and use the residue theorem. We find

$$\gamma_3 = -2\pi i \nu_3 \tau^2 \int \frac{d\Omega}{\Omega} \left(\frac{\Theta(\epsilon)\Theta(\omega-\epsilon)}{[1-\omega\tau + i\mathbf{v_F}\cdot\mathbf{q}\tau]^2} - \frac{\Theta(-\epsilon)\Theta(\epsilon-\omega)}{[1+i\omega\tau - i\mathbf{v_F}\cdot\mathbf{q}\tau]^2} \right)$$

We keep only the highest order in the diffusive limit

$$\gamma_3 = -2\pi i \nu_3 \tau^2 \Theta_-(\epsilon,\omega) \tag{2.31}$$

where $\Theta_-(\epsilon,\omega) := \Theta(\epsilon)\Theta(\omega-\epsilon) - \Theta(-\epsilon)\Theta(\epsilon-\omega)$. Let us start with the exchange term. Plugging in our results for the different objects

$$\delta\nu^{\mathrm{exc}} = -2\,\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\,\mathrm{Im}\left[\frac{1}{\mathcal{V}} \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[-2\pi i\nu_3\tau^2\Theta_-(\epsilon,\omega)\right] \left(iV(q,\omega)\right) \right.$$
$$\left. \left[\frac{\Theta_+(\epsilon,\omega)}{2\pi\nu_3\tau^2(Dq^2 - i|\omega|)}\right]^2 \left[2\pi\nu_3\tau\ \Theta_+(\epsilon,\omega)\right]^2 \right].$$

One has $\mathrm{sign}(\epsilon)\,\Theta_-(\epsilon,\omega) = \Theta_+(\epsilon,\omega)$ and $\Theta_+^5(\epsilon,\omega) = \Theta_+(\epsilon,\omega)$ and one is left with

$$\delta\nu^{\mathrm{exc}} = -2\,\frac{\nu_3}{\pi\mathcal{V}}\ \sum_q \int_{-\infty}^{\infty} d\omega\ \Theta_+(\epsilon,\omega)\ \mathrm{Im}\left[\frac{V(q,\omega)}{(Dq^2 - i|\omega|)^2}\right].$$

We replace the time–ordered propagators by retarded ones. This is possible because ω is always positive in the integrand and at $T=0$ one has [9]

$$V^R(q,\omega) = \mathrm{Re}V^R(q,\omega) + i\ \mathrm{Im}V^R(q,\omega)\ \mathrm{sign}(\omega).$$

Finally we obtain[10]

$$\delta\nu^{\mathrm{exc}} = -2\,\frac{\nu_3}{\pi\mathcal{V}}\ \sum_q \int_{|\epsilon|}^{\infty} d\omega\ \mathrm{Im}\left[\frac{V^R(q,\omega)}{(Dq^2 - i\omega)^2}\right] \tag{2.32}$$

[8] This is only true if we neglect spin. In the Hartree diagram the spin α can be different from β.

[9] see Appendix 2 in [26]

[10] More precisely the upper integration limit in ω should be $1/\tau$ since the diffusion approximation is only valid in the limit $\omega\tau \ll 1$. The error we commit however is negligible.

The expression (2.32) is still quite general. One can plug in various interaction propagators $V^R(q,\omega)$.

But before evaluating the remaining integrals let us turn to the Hartree term. The Hartree term has a more complicated structure. We can see in the second line of equation (2.10) that the four Green's functions are connected with the interaction $V(p'-p'',0)$ via their momenta. We define like it was done in [39][11] the so called F–factor.

$$F := c_0 \frac{1}{\mathcal{V}^2} \sum_{p',p''} G_\tau(p',\epsilon) G_\tau(p'-q,\epsilon-\omega) V(p'-p'',0) G_\tau(p'',\epsilon) G_\tau(p''-q,\epsilon-\omega) \tag{2.33}$$

where $c_0 = [(2\pi\nu_3\tau)^2 V(0,0)\Theta_+(\epsilon,\omega)]^{-1}$. The Hartree correction then reads

$$\delta\nu^{\mathrm{Har}} = -4\,\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\,\mathrm{Im}\left[\frac{1}{\mathcal{V}}\sum_q \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[-2\pi i\nu_3\tau^2\Theta_-(\epsilon,\omega)\right](-i)\right.$$
$$\left.\left[(2\pi\nu_3\tau)^2 F\, V(0,0)\right]\left[\frac{\Theta_+(\epsilon,\omega)}{2\pi\nu_3\tau^2(Dq^2-i|\omega|)}\right]^2\right].$$

Following the same steps as for the exchange term we finally get

$$\delta\nu^{\mathrm{Har}} = 4\,\frac{\nu_3}{\pi\mathcal{V}}\sum_q \int_{|\epsilon|}^{\infty} d\omega\,\mathrm{Im}\left[\frac{F\,V^R(0,0)}{(Dq^2-i\omega)^2}\right]. \tag{2.34}$$

Remark the two fundamental differences between the exchange and the Hartree expressions. First the Hartree term has the opposite sign and second the interaction $V(p'-p'',0)$ selects processes with different momentum transfer (so called large angle scattering).[12] For the evaluation of (2.32) and (2.34) we follow the lines of [29]. The important contribution comes from the small momentum and energy region due to the diffusive pole $1/(Dq^2-i\omega)^2$. We restrict ourselves to a short range interaction which is equivalent to the assumption that $V^R(0,0)$ is finite. [13] So we can replace in good approximation $V^R(q,\omega)$ by $V^R(0,0)$ in the exchange expression. The total correction to the DOS then reads

$$\frac{\delta\nu_3}{\nu_3} = -\frac{2}{\pi}\,\frac{1}{\mathcal{V}}\sum_q \int_{|\epsilon|}^{\infty} d\omega\,\mathrm{Im}\left[\frac{(1-2F)\,V^R(0,0)}{(Dq^2-i\omega)^2}\right]. \tag{2.36}$$

In fact there is another correction coming from the Cooper channel. Here however we neglect this contribution. Formally one may assume the presence of a small magnetic field which kills the Cooper contribution (see [29] for further information).

[11] Our definition is slightly different from the one in the paper [39] because these authors used retarded and advanced propagators to build up the perturbation theory.

[12] The evaluation of the F–factor is usually done in the following way: Due to the Green's functions that are highly peaked around p_F one sets the absolute value of the appearing momenta in $V(p'-p'',0)$ to p_F and averages the interaction over the angles. We denote this constant by $\overline{V(p'-p'',0)}$. Having replaced the interaction by a constant one can use the result (2.16) for the remaining products of Green's functions and obtains

$$F \approx \frac{\overline{V^R(p'-p'',0)}}{V^R(0,0)}. \tag{2.35}$$

[13] That assumption is also used in the standard Fermi liquid theory.

Simply summing up the contributions of the exchange and of the Hartree diagrams is however dangerous. One is not really summing up two first order contributions concerning the interaction as the dynamically screened interaction $V(q,\omega)$ already is a infinite sum in which $V_0(q)$ appears to all orders (see the discussion in [24] chapter 3.3 and in [30] section III.B.3.C. *Specific heat and tunnelling density of states*). A correct way to handle the problem and at the same time a way to include higher order diagrams in the interaction is the replacement of

$$V_1 := V^R(0,0) \qquad \text{and} \qquad V_2 := F\ V^R(0,0) \tag{2.37}$$

by Fermi liquid scattering amplitudes Γ_1 and Γ_2. We will discuss this procedure in the next section in detail.

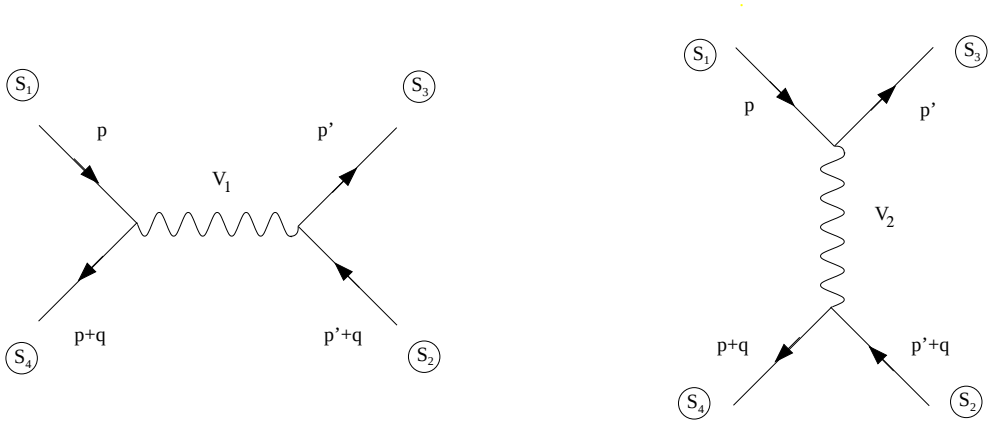

Figure 2.11: Small angle and large angle scattering shown as Feynman diagrams

2.2.4 Effective scattering amplitudes and Landau parameters

According to the discussion of the previous section, the relevant interaction terms are

$$\begin{aligned} H_{\text{int}} = \sum_{s_1,s_2,s_3,s_4} \sum_{p,p'} \sum_{q(\text{small})} \Big(& V_1\ \delta_{s_1,s_4}\delta_{s_2,s_3}\ c^\dagger_{s_1}(p)c^\dagger_{s_2}(p'+q)c_{s_3}(p')c_{s_4}(p+q) \\ & + V_2\ \delta_{s_1,s_3}\delta_{s_2,s_4}\ c^\dagger_{s_1}(p)c^\dagger_{s_2}(p'+q)c_{s_4}(p+q)c_{s_3}(p') \Big) \end{aligned} \tag{2.38}$$

The sum over q is only over small momenta. In figure (2.11) we sketched the two processes that appear in the exchange and the Hartree diagram. One can generalize the upper expressions by replacing

$$V_1 \ \rightarrow\ \Gamma_1 \qquad \text{and} \qquad V_2 \ \rightarrow\ \Gamma_2,$$

which include higher order diagrams in the Coulomb interaction depicted in figure (2.12). Γ_1 and Γ_2 are connected to Landau parameters. Even though the connection to the Landau parameters was derived for the clean case, the general assumption is that the relations hold except close to the impurities. Hence for weak disorder meaning low impurity density the replacement by Γ_1 and Γ_2 should work quite well.

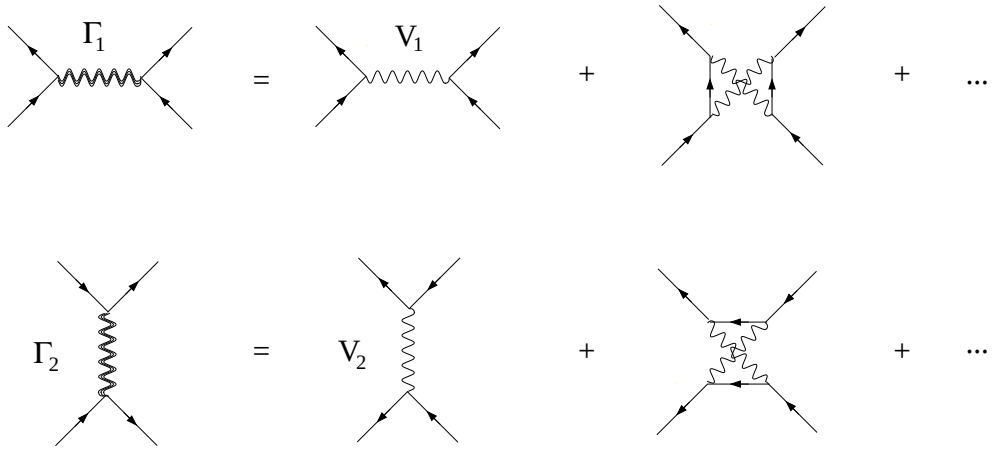

Figure 2.12: Diagrammatic representation of Γ_1 and Γ_2

If there is no mechanism present that could flip the spins one can decompose all diagrams into singlet and triplet diagrams.[14]

$$\Gamma_1\,\delta_{s_1s_4}\delta_{s_2s_3} - \Gamma_2\,\delta_{s_1s_3}\delta_{s_2s_4} = \Big[\Gamma_1 - \frac{\Gamma_2}{2}\Big]\underbrace{\delta_{s_1s_4}\delta_{s_2s_3}}_{\text{singlet}} + \Big[-\frac{\Gamma_2}{2}\Big]\underbrace{\sigma_{s_1s_4}\sigma_{s_2s_3}}_{\text{triplet}} \tag{2.39}$$

We define the singlet and triplet amplitudes

$$\Gamma_s = \Gamma_1 - \Gamma_2/2 \qquad\qquad \Gamma_t = -\Gamma_2/2. \tag{2.40}$$

These effective coupling constants are connected with the Landau parameters F_0^s and F_0^a in the following way:[15]

$$\Gamma_s = \frac{1}{2\nu_3}\underbrace{\frac{F_0^s}{1+F_0^s}}_{A_0^s} \qquad\qquad \Gamma_t = \frac{1}{2\nu_3}\underbrace{\frac{F_0^a}{1+F_0^a}}_{A_0^a}. \tag{2.41}$$

Sometimes one also finds the parameters A_0^s and A_0^a. The Landau parameters can not be calculated for disordered metals from a first principle theory. The Fermi liquid theory however connects the Landau parameters to various response functions which allows to determine them experimentally. In section (9.4) we will look in detail at the Landau parameters in the example of palladium. In summary we made the replacement:

$$V^R(0,0) - 2FV^R(0,0) = V_1 - 2V_2 \qquad \to \qquad \Gamma_1 - 2\Gamma_2 \qquad \to \qquad \Gamma_s + 3\Gamma_t. \tag{2.42}$$

These effective amplitudes are also screened by polarization effects. The difference however to the procedure in section (2.2.2) is that the amplitudes Γ_s and Γ_t already include *all* the static

[14] At this stage the names *singlet* and *triplet* are rather arbitrary and we are just using a mathematical identity in equation (2.39) to rewrite the spin structure. However in the part where we discuss the effect of an external magnetic field we will see how the singlet part remains unaffected, whereas the triplet part splits up into $m = -1, 0, 1$ components.

[15] A derivation of (2.41) can be found in [27] or [31]

diagrams. This means that we only have to take the dynamical part $\Pi_{\rm dyn}$ of the polarization propagator given in equation (2.22) in order to avoid multiple counting of diagrams. The Dyson equation then reads

$$\Gamma_s(q,\omega) = \Gamma_s + \Gamma_s \Pi_{\rm dyn}(q,\omega)\Gamma_s(q,\omega)$$

and one finds

$$\Gamma_s(q,\omega) = \Gamma_s \frac{Dq^2 - i\omega}{Dq^2 - i\omega Z_s} \tag{2.43}$$

where $Z_s = 1 - 2\nu_3\Gamma_s = 1 - A_0^s$. For the triplet interaction one has in the same way

$$\Gamma_t(q,\omega) = \Gamma_t \frac{Dq^2 - i\omega}{Dq^2 - i\omega Z_t} \tag{2.44}$$

where $Z_t = 1 - 2\nu_3\Gamma_t = 1 - A_0^a$. We introduce field diffusion constants $D_s^* = D/Z_s$ and $D_t^* = D/Z_t$ writing the interactions in the form

$$\Gamma_s(q,\omega) = \frac{\Gamma_s}{Z_s}\,\frac{Dq^2 - i\omega}{D_s^* q^2 - i\omega} \qquad\qquad \Gamma_t(q,\omega) = \frac{\Gamma_t}{Z_t}\,\frac{Dq^2 - i\omega}{D_t^* q^2 - i\omega} \tag{2.45}$$

The field diffusion constants measure how fast the fields spread over the conductor. D_s^* determines the propagation of the charge density, whereas D_s^* determines the propagation of the spin density. In general they differ largely from the particle diffusion constant D. Putting these interactions in our formula for the correction to the DOS (2.36) we have

$$\frac{\delta\nu_3}{\nu_3} = -\frac{2}{\pi\mathcal{V}} \sum_q \int_{|\epsilon|}^{\infty} d\omega\ {\rm Im}\left[\frac{\Gamma_s(q,\omega) + 3\Gamma_t(q,\omega)}{(Dq^2 - i\omega)^2}\right] \tag{2.46}$$

which reads explicitly

$$\frac{\delta\nu_3}{\nu_3} = -\frac{2}{\pi\mathcal{V}} \sum_q \int_{|\epsilon|}^{\infty} d\omega\ {\rm Im}\left[\frac{\Gamma_s/Z_s}{(Dq^2 - i\omega)(D_s^* q^2 - i\omega)} + \frac{3\Gamma_t/Z_t}{(Dq^2 - i\omega)(D_t^* q^2 - i\omega)}\right]. \tag{2.47}$$

As a next step we want to evaluate the q–sum. For this purpose we have to discuss the concept of quasi–dimensionality. We will do this in the following section.

2.2.5 Correction to the DOS / Quasi one–dimensional approximation

q is a three–dimensional momentum in equation (2.47). This does not mean necessarily that we convert the sum over q into a three–dimensional integral because we treat a finite size system. The evaluation of the q–sum depends on the energy scale we are looking at. Suppose our energy is the applied voltage eV. As already mentioned in the introduction there exist several characteristic energies for a wire defined by its geometry. In the case of a wire of length L, height a and width a (with $a \ll L$), we have the Thouless energies $E_L = \frac{D}{L^2}$ and $E_a = \frac{D}{a^2}$ that are connected to the particle diffusion. Furthermore one has the Thouless field energies $E_L^i = \frac{D_i^*}{L^2}$ and $E_a^i = \frac{D_i^*}{a^2}$ (where i is either s or t). In order to sort these energies according to their magnitude we need to know how the different diffusion constants are related to each other. From the preceding section we have

$$D_s^* = \frac{D}{1 - A_0^s} \qquad\qquad D_t^* = \frac{D}{1 - A_0^a}. \tag{2.48}$$

Approaching the ferromagnetic state the Fermi liquid parameter $A_0^s \to 1$ and $A_0^a \to -\infty$. For Palladium we found $A_0^s = 0.966$ and $A_0^a = -4$.[16] That implies

$$D_s^* > D > D_t^*. \tag{2.49}$$

We define: A system is quasi one–dimensional concerning its **particle diffusion** if

$$E_a > eV > E_L$$

and a system is quasi one–dimensional concerning the **charge or spin diffusion** if

$$E_a^i > eV > E_L^i \qquad \text{where} \qquad i = \ s \text{ or } t$$

We see that there are various situations possible. For example a system can be quasi one–dimensional concerning the charge diffusion but three–dimensional concerning particle and spin diffusion if the order of energies is

$$E_a^s > eV > E_L^s, E_L, E_a, E_L^t, E_a^t \tag{2.50}$$

In our calculations we will assume that eV is such that the wire is quasi one–dimensional with respect to the particle and singlet field diffusion.[17]

$$E_a^s, E_a > eV > E_L^s, E_L \tag{2.51}$$

Let us consider the q–dependent part of the first term in equation (2.47)

$$\frac{\delta\nu_3^s}{\nu_3} \sim \frac{1}{\mathcal{V}} \sum_{q_x,q_y,q_z} \frac{1}{\left(D(q_x^2+q_y^2+q_z^2) - i\omega\right)\left(D_s^*(q_x^2+q_y^2+q_z^2) - i\omega\right)}.$$

The q–values are quantized due to the finite volume. In transverse direction the smallest momenta q_x and q_y are of order $2\pi/a$ and in the longitudinal direction q_z is of order $2\pi/L$. Let us compare the contributions of the two q–vectors $q_0 = (0, 0, 2\pi/L)$ and $q_1 = (2\pi/a, 0, 2\pi/L)$ in our expression above. The two appearing diffusion constants are D and D_s^* but as D_s^* is much bigger than D we neglect D for a while. We see that the contribution of q_1 is much smaller because

$$\frac{1}{D_s^*/a^2 + D_s^*/L^2} \approx \frac{1}{D_s^*/a^2} \ll \frac{1}{D_s^*/L^2}$$

As we are interested in energies $eV \ll D_s^*/a^2$ we can neglect all terms in our sum where q_x or q_y are different from zero. Thus we sum only over vectors like q_0 with a non vanishing q_z–component in order to determine the main contribution in the energy regime of interest. The sum over q_z we write as an integral assuming an infinitely long wire.

$$\frac{1}{\mathcal{V}} \sum_{q_x,q_y,q_z} \approx \frac{1}{a^2} \sum_{q_x,q_y} \int \frac{dq_z}{2\pi}\, \delta_{q_x,0}\, \delta_{q_y,0} = \frac{1}{a^2} \int \frac{dq_z}{2\pi} \tag{2.52}$$

Thus in summary we replace a **three**–dimensional sum by a **one**–dimensional integral based on the energy inequality (2.51). This justifies the name **quasi one–dimensional system**. For the

[16] See section (9.4) in the appendix.

[17] We do not need to specify the dimensionality concerning triplet field diffusion as the two others already determine the quasi one–dimensional approximation

second term of equation (2.47) the reasoning is the same (according to equation (2.51) we have $eV \ll D/a^2$). Thus in the quasi one–dimensional limit we are left with

$$\frac{\delta\nu_3}{\nu_3} = -\frac{2}{\pi\, a^2}\int\frac{dq_z}{2\pi}\int_{|\epsilon|}^{\infty} d\omega\ \mathrm{Im}\Big[\frac{\Gamma_s/Z_s}{(Dq_z^2-i\omega)(D_s^*q_z^2-i\omega)}+\frac{3\Gamma_t/Z_t}{(Dq_z^2-i\omega)(D_t^*q_z^2-i\omega)}\Big]. \tag{2.53}$$

We replace the interaction amplitudes Γ_s and Γ_t according to equation (2.41) and rename $q_z = q$.

$$\frac{\delta\nu_3}{\nu_3} = -\frac{1}{\pi\, a^2\nu_3}\int\frac{dq}{2\pi}\int_{|\epsilon|}^{\infty} d\omega\ \mathrm{Im}\Big[\frac{A_0^s/(1-A_0^s)}{(Dq^2-i\omega)(D_s^*q^2-i\omega)}+\frac{3A_0^a/(1-A_0^a)}{(Dq^2-i\omega)(D_t^*q^2-i\omega)}\Big] \tag{2.54}$$

Here we introduce the quasi one–dimensional DOS $\boldsymbol{\nu_1 := a^2\nu_3}$. Also we multiply nominator and denominator on the left side of the equation by a^2. According to the residue theorem

$$\int_{|\epsilon|}^{\infty} d\omega\int\frac{dq}{2\pi}\ \mathrm{Im}\ \Big[\frac{1}{(Dq^2-i\omega)(D^*q^2-i\omega)}\Big]=\frac{1}{\sqrt{2}\,(\sqrt{D}+\sqrt{D^*})}\ \frac{1}{\sqrt{|\epsilon|}}. \tag{2.55}$$

Hence we have

$$\frac{\delta\nu_1}{\nu_1} = -\frac{1}{\sqrt{2}\pi\,\nu_1}\Big[\frac{A_0^s/(1-A_0^s)}{(\sqrt{D}+\sqrt{D_s^*})}+\frac{3A_0^a/(1-A_0^a)}{(\sqrt{D}+\sqrt{D_t^*})}\Big]\frac{1}{\sqrt{|\epsilon|}} \tag{2.56}$$

and obtain

$$\boxed{\frac{\delta\nu_1}{\nu_1} = -\frac{1}{\sqrt{2}\pi\,\nu_1\sqrt{D}}\Big[\frac{A_0^s}{(1-A_0^s+\sqrt{1-A_0^s})}+\frac{3A_0^a}{(1-A_0^a+\sqrt{1-A_0^a})}\Big]\frac{1}{\sqrt{|\epsilon|}}} \tag{2.57}$$

Remember that A_0^a is negative and leads instead of a suppression to an augmentation of the DOS. To see the connection with earlier work we note that neglecting the dynamical screening of the interaction and replacing $(1-2F)V^R(0,0)$ by $\Gamma_s+3\Gamma_t$ according to equation (2.42) in equation (2.36), we find that in this case the dimensionality is only defined by D because there appear no other diffusion constants. Using

$$\int_{|\epsilon|}^{\infty} d\omega\int\frac{dq}{2\pi}\ \mathrm{Im}\ \Big[\frac{1}{(Dq^2-i\omega)^2}\Big]=\frac{1}{2\sqrt{2D|\epsilon|}} \tag{2.58}$$

leads us to the known result of Altshuler and Aronov with generalized amplitudes Γ_s and Γ_t.

$$\frac{\delta\nu_1}{\nu_1} = -\frac{1}{\sqrt{2}\,\pi\sqrt{D}\,a^2}\ \frac{\Gamma_s+3\Gamma_t}{\sqrt{|\epsilon|}} \tag{2.59}$$

As above we express everything in terms of the Landau parameters A_0^s and A_0^a using equation (2.41). Then (2.58) becomes

$$\boxed{\frac{\delta\nu_1}{\nu_1} = -\frac{1}{2\sqrt{2}\,\pi\nu_1\sqrt{D}}\ \frac{A_0^s+3A_0^a}{\sqrt{|\epsilon|}}} \tag{2.60}$$

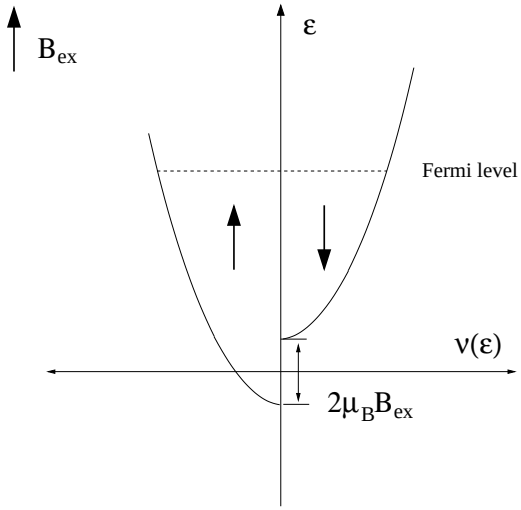

Figure 2.13: Shift of the distribution of the two spin species in an external magnetic field

Looking at equations (2.43) and (2.44) one can see that setting $Z_s = 1 - A_0^s$ and $Z_t = 1 - A_0^a$ to 1 brings us back to the static interactions Γ_s and Γ_t. Performing that limit in equation (2.57) the denominators are replaced by 2 and we obtain (2.60). The result of naive perturbation theory which neglects the dynamical screening of the static amplitudes should be taken with caution because it is only valid for $A_0^s, A_0^a \ll 1$. In ordinary metals the parameter A_0^a is actually small but A_0^s is close to one. Hence one should turn to (2.57). For Palladium, that has an extraordinary high magnetic susceptibility for small temperatures, even the parameter A_0^a becomes large. In section (9.4) we will estimate a value of -4 for A_0^a.

At this stage we can also confirm what Pothier et al. pointed out in their paper [1]. They claimed that in their experiment with aluminum the $1/\sqrt{eV}$ behavior arises from the diffusion of the electric potential. We see from equation (2.56) that for $A_0^a \ll 1$ but A_0^s close to 1, the term containing $D_s^* = D/(1 - A_0^s)$ indeed determines the correction of the DOS.

Just for completeness we note that the energy dependence of the DOS for systems with other dimensions differs from the one in the one–dimensional case. Replacing the q–sum in equation (2.52) not by an one dimensional integration but two or three dimensional integrations one finds in the quasi two–dimensional case

$$\delta\nu_2 \sim \ln(|\epsilon|\tau) \tag{2.61}$$

and in the three dimensional case

$$\delta\nu_3 \sim \sqrt{|\epsilon|} \tag{2.62}$$

For further information see [24].

2.3 Calculation of the DOS in the presence of external field

In this section we redo the calculation adding an external time independent field B_{ex} in z–direction which forces us to take the spin of the quasiparticles into account. The first question, we have to answer, is: How is the Green's function G_τ changed in the presence of a magnetic field? The number of electrons with spin–up and spin–down arrange such that one has a common

Fermi level (see figure (2.3)). The occupation at $T = 0$ is given by[18]

$$\langle |c^\dagger_{\alpha,p} c_{\alpha,p}| \rangle = \Theta(-\xi_p + \alpha\ \omega_s)$$

where $\omega_s = g\mu_B B_{ex}$ and $\alpha = \pm 1/2$. $\mu_B = \frac{e}{2mc}$ is the Bohr magneton and we approximate the gyromagnetic factor g by 2 as we are focussing on metals[19]. Again we rescale our energies $\xi_p \to \epsilon_p$ and obtain for the Green's function

$$\boxed{G_\alpha(\epsilon, p) = \frac{1}{\epsilon - \epsilon_p + \alpha\ \omega_s + \frac{i}{2\tau}\ \mathrm{sign}(\epsilon)}.} \tag{2.63}$$

We see that the external magnetic field causes a Zeeman splitting. Turning off the external field we recover the Green's function of equation (2.7). As next element of the perturbation theory we reinvestigate the ladder $L_{\alpha\beta}(q, \omega)$ which as can be seen in figure (2.7) depends on two spins. The Dyson equation reads

$$L_{\alpha\beta}(q, \omega) = L_0 + L_0 \Pi_0^{\alpha\beta}(q, \omega) L_{\alpha\beta}(q, \omega)$$

where $\Pi_0^{\alpha\beta}$ is depicted in figure (2.8). We have then

$$\Pi_0^{\alpha\beta}(q, \omega) = \int \frac{d^3p}{(2\pi)^3}\ G_\alpha(p, \epsilon) G_\beta(p + q, \epsilon - \omega)$$

where the Green's function is given in equation (2.63). We calculate $\Pi_0^{\alpha\beta}(q, \omega)$ using the approximations we discussed in section (2.2.1). One arrives at

$$\Pi_0^{\alpha\beta}(q, \omega) = \frac{\nu_3 \tau}{2} \int d\Omega \bigg(\frac{\Theta(\epsilon)\Theta(\omega - \epsilon)}{1 - i\omega\tau + i\mathbf{v_F} \cdot \mathbf{q} - i(\alpha - \beta)\ \omega_s \tau} + \frac{\Theta(-\epsilon)\Theta(\epsilon - \omega)}{1 + i\omega\tau - i\mathbf{v_F} \cdot \mathbf{q} + i(\alpha - \beta)\ \omega_s \tau} \bigg).$$

We restrict ourselves to weak magnetic field assuming that $i\omega_s\tau \ll 1$.[20]

$$\frac{1}{1 \mp \omega\tau \mp i\mathbf{v_F} \cdot \mathbf{q}\tau \mp i(\alpha - \beta)\ \omega_s \tau} \approx 1 \pm i\omega\tau \pm i(\alpha - \beta)\ \omega_s \tau \pm i\mathbf{v_F} \cdot \mathbf{q}\tau - (\mathbf{v_F} \cdot \mathbf{q}\tau)^2$$

and we get

$$\Pi_0^{\alpha\beta}(q, \omega) = 2\pi\nu_3\tau\ \Theta_+(\epsilon, \omega)\ (1 + i|\omega|\tau + i(\alpha - \beta)\ |\omega_s|\tau - Dq^2\tau). \tag{2.64}$$

Hence one obtains for the ladder

$$\boxed{L_{\alpha\beta}(q, \omega) = \frac{\Theta_+(\epsilon, \omega)}{2\pi\nu_3\tau^2(Dq^2 - i\omega - i(\alpha - \beta)\ \omega_s)}.} \tag{2.65}$$

[18] without external field we have $\langle |c^\dagger_{\alpha,p} c_{\alpha,p}| \rangle = \Theta(-\xi_p)$

[19] In semiconductors g can differ largely from 2

[20] Note that the inequality allows external magnetic fields of several Tesla. However when will talk of weak external magnetic fields in chapter 7 they are of order 10^{-5} Tesla.

We see that the ladder becomes spin dependent in the presence of an external field. If $\alpha = \beta$ we recover the form (2.18). In order to keep things simple let us calculate the correction to the DOS taking the static amplitudes Γ_1 and Γ_2 for the interaction in the exchange and the Hartree diagrams. The basic effect of the external magnetic field will already be visible at this stage without taking into account the dynamical screening of the interaction amplitudes.

2.3.1 Correction to the DOS using static interactions

In this subsection we calculate the DOS approximating the interaction by the constants Γ_1 and Γ_2 in the exchange and the Hartree diagram respectively. Translating the exchange diagram (2.5) into functions we find

$$\delta\nu_\alpha^{\rm exc} = -\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\,\mathrm{Im}\sum_p G_\alpha(p,\epsilon)\Big[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,L_{\alpha\alpha}(q,\omega)G_\alpha(p-q,\epsilon-\omega)L_{\alpha\alpha}(q,\omega)\,i\Gamma_1$$
$$\frac{1}{\mathcal{V}}\sum_{p'} G_\alpha(p',\epsilon)G_\alpha(p'-q,\epsilon-\omega)\,\frac{1}{\mathcal{V}}\sum_{p''} G_\alpha(p'',\epsilon)G_\alpha(p''-q,\epsilon-\omega)\Big]G_\alpha(p,\epsilon). \tag{2.66}$$

Note that we first examine the correction $\delta\nu_\alpha^{\rm exc}$ for one spin direction α. That is why there is no factor 2 in front like in equation (2.9). Remark also that $L_{\alpha\alpha}(q,\omega) = L(q,\omega)$ and that $\Pi_0^{\alpha\alpha} = \Pi_0$. Finally we need to determine the element

$$\gamma_3^{\alpha\beta} := \frac{1}{\mathcal{V}}\sum_p G_\alpha^2(p,\epsilon)G_\beta(p-q,\epsilon-\omega).$$

Although we have in the exchange term $\alpha = \beta$ we calculate $\gamma_3^{\alpha\beta}$ for two independent spins α and β because this combination will appear in the Hartree expression. Using for the Green's functions equation (2.63) and following the same steps as for γ_3 yields

$$\gamma_3^{\alpha\beta} = -2\pi i\nu_3\tau^2\int\frac{d\Omega}{\Omega}\left(\frac{\Theta(\epsilon)\Theta(\omega-\epsilon)}{[1-\omega\tau-iv_F\cdot q\tau-i(\alpha-\beta)\omega_s\tau]^2}-\frac{\Theta(-\epsilon)\Theta(\epsilon-\omega)}{[1+i\omega\tau+iv_F\cdot q\tau+i(\alpha-\beta)\omega_s\tau]^2}\right).$$

Keeping only the leading order term in the diffusive limit (and neglecting $i(\alpha-\beta)\,\omega_s\tau \ll 1$) one finds

$$\gamma_3^{\alpha\beta} = -2\pi i\nu_3\tau^2\Theta_-(\epsilon,\omega). \tag{2.67}$$

We see that in the limit of weak magnetic fields defined by the condition $\mu_B B_{ex}\tau \ll 1$ the expression $\gamma_3^{\alpha\beta}$ is independent of the external field. For the exchange term where $\alpha = \beta$ anyway we do not need the assumption of weak fields. **We can thus conclude that an external magnetic field has no influence on the exchange term at all.** Multiplying the result by 2 we recover exactly the singlet part given in equation (2.60).

The Hartree term however is sensitive to the external field. According to diagram (2.6) we have

$$\delta\nu_\alpha^{\rm Har} = -\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\,\mathrm{Im}\sum_p\sum_\beta G_\alpha(p,\epsilon)\Big[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,(-i)L_{\alpha\beta}(q,\omega)G_\beta(p-q,\epsilon-\omega)L_{\alpha\beta}(q,\omega)$$

$$\frac{1}{\mathcal{V}^2}\sum_{p',p''}G_\alpha(p',\epsilon)G_\beta(p'-q,\epsilon-\omega)\;\Gamma_2\;G_\alpha(p'',\epsilon)G_\beta(p''-q,\epsilon-\omega)\Big]G_\alpha(p,\epsilon). \tag{2.68}$$

Note that instead of the simple factor of 2 for the internal spin in the loop in equation (2.10) we have to sum over the internal spin β. Using the results for $\Pi_0^{\alpha\beta}$ and the factor $\gamma_3^{\alpha\beta}$ we obtain

$$\frac{\delta\nu_\alpha^{\rm Har}}{\nu_1} = \frac{\Gamma_2}{\pi a^2}\,\mathrm{Im}\sum_\beta\int\frac{dq}{2\pi}\int_{|\epsilon|}^{\infty}d\omega\frac{1}{(Dq^2-i\omega-i(\alpha-\beta)\omega_s)^2}. \tag{2.69}$$

The sum over the spin β generates two terms. Either the spins are parallel ($\beta=\alpha$) or antiparallel ($\beta=-\alpha$).

$$\frac{\delta\nu_\alpha^{\rm Har}}{\nu_1} = \frac{\Gamma_2}{\pi a^2}\,\mathrm{Im}\int\frac{dq}{2\pi}\int_{|\epsilon|}^{\infty}d\omega\Big[\frac{1}{(Dq^2-i\omega)^2}+\frac{1}{(Dq^2-i\omega-i(2\alpha)\omega_s)^2}\Big]$$

In the second integral we shift the integration variable.

$$\frac{\delta\nu_\alpha^{\rm Har}}{\nu_1} = \frac{\Gamma_2}{\pi a^2}\,\mathrm{Im}\int\frac{dq}{2\pi}\Big[\int_{|\epsilon|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}+\int_{|\epsilon+2\alpha\omega_s|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}\Big]$$

For $\alpha=\frac{1}{2}=\uparrow$

$$\frac{\delta\nu_\uparrow^{\rm Har}}{\nu_1} = \frac{\Gamma_2}{\pi a^2}\,\mathrm{Im}\int\frac{dq}{2\pi}\Big[\int_{|\epsilon|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}+\int_{|\epsilon+\omega_s|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}\Big]$$

and for $\alpha=-\frac{1}{2}=\downarrow$

$$\frac{\delta\nu_\downarrow^{\rm Har}}{\nu_1} = \frac{\Gamma_2}{\pi a^2}\,\mathrm{Im}\int\frac{dq}{2\pi}\Big[\int_{|\epsilon|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}+\int_{|\epsilon-\omega_s|}^{\infty}\frac{d\omega}{(Dq^2-i\omega)^2}\Big].$$

We evaluate the integrals using equation (2.58). The total change of the DOS (both spin directions) is

$$\frac{\delta\nu_1}{\nu_1} = 2\,\frac{\delta\nu^{\rm exc}}{\nu_1}+\frac{\delta\nu_\uparrow^{\rm Har}}{\nu_1}+\frac{\delta\nu_\downarrow^{\rm Har}}{\nu_1}.$$

We find

$$\frac{\delta\nu_1}{\nu_1} = -\frac{1}{2\sqrt{2}\pi\sqrt{D}a^2}\Big(\frac{2\Gamma_1-2\Gamma_2}{\sqrt{|\epsilon|}}-\frac{\Gamma_2}{\sqrt{|\epsilon-\omega_s|}}-\frac{\Gamma_2}{\sqrt{|\epsilon+\omega_s|}}\Big).$$

Using that $2\Gamma_1-2\Gamma_2=2\Gamma_s+2\Gamma_t$ and $-\Gamma_2=2\Gamma_t$ and replacing the Γ_i's by the Landau parameters we obtain

$$\boxed{\frac{\delta\nu_1}{\nu_1} = -\frac{1}{2\sqrt{2}\pi\sqrt{D\nu_1}}\left(\frac{A_0^s + A_0^a}{\sqrt{|\epsilon|}} + \frac{A_0^a}{\sqrt{|\epsilon - \omega_s|}} + \frac{A_0^a}{\sqrt{|\epsilon + \omega_s|}}\right).} \tag{2.70}$$

The external magnetic field creates two new poles at the shifted positions $-A_0^a/\sqrt{|\epsilon - \omega_s|}$ and $-A_0^a/\sqrt{|\epsilon + \omega_s|}$. At his level the name triplet amplitude for A_0^a becomes understandable. The $3A_0^a$ split up in the presence of a magnetic field like the triplet levels in an atom. Sending $\omega_s \to 0$ in the equation (2.70) we recover the result of equation (2.60).[21]

2.3.2 Dynamical screening of the static interactions in the presence of an external magnetic field

Determining the dynamically screened interaction in the presence of an external magnetic field is more complicated than in the zero field case as the screening process represented by the polarization function $\Pi^{\alpha\beta}(q,\omega)$ (depicted in figure (2.10)) is affected by the external field. Going through the steps of the calculation in section (2.2.2) and replacing G_τ by G_α and G_β according to the depicted diagrams one arrives in diffusive ladder approximation at

$$\Pi^{\alpha\beta}(q,\omega) = -\nu_3 \frac{Dq^2 - i(\alpha - \beta)\,\omega_s}{Dq^2 - i|\omega| - i(\alpha - \beta)\,\omega_s} \tag{2.73}$$

and especially

$$\Pi^{\alpha\beta}_{\text{dyn}}(q,\omega) = -\nu_3 \frac{i|\omega|}{Dq^2 - i|\omega| - i(\alpha - \beta)\,\omega_s}. \tag{2.74}$$

Remember that we only need the dynamic part of $\Pi^{\alpha\beta}(q,\omega)$ for the screening as all the static diagrams are already included in the effective amplitudes Γ_s and Γ_t.

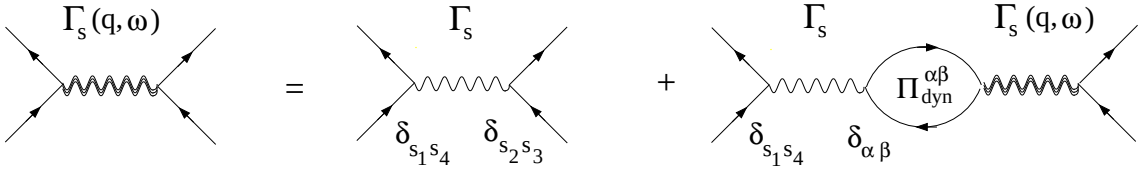

Figure 2.14: Diagrammatic Dyson series for the singlet interaction

[21] The result (2.70) was first derived by Altshuler and Aronov. In [24] one finds in chapter 6.

$$\delta\nu = -\frac{1}{2\sqrt{2}\pi\sqrt{D}}\left(\frac{\lambda_\nu^{(j=0)} + \frac{1}{2}\lambda_\nu^{(j=1)}}{\sqrt{\epsilon}} + \frac{\frac{1}{2}\lambda_\nu^{(j=1)}}{\sqrt{\epsilon - \omega_s}} + \frac{\frac{1}{2}\lambda_\nu^{(j=1)}}{\sqrt{\epsilon + \omega_s}}\right) \tag{2.71}$$

where

$$\lambda_\nu = \lambda_\nu^{(j=0)} + \frac{3}{2}\lambda_\nu^{(j=1)} \tag{2.72}$$

and

$$\lambda_\nu = \nu_1\big[V^R(0,0) - 2\overline{V^R(p'-p'',0)}\big].$$

The additional factors of 1/2 in front of $\lambda_\nu^{(j=1)}$ in equation (2.71) compared to (2.70) are due to the different definition of the triplet part in equation (2.72).

It is convenient to introduce the quantum number m which takes the values $-1, 0, 1$. Additional to (2.74) we will use

$$\Pi^m_{\rm dyn}(q,\omega) = -\nu_3 \, \frac{i|\omega|}{Dq^2 - i|\omega| - im\,\omega_s}. \tag{2.75}$$

Let us begin to set up a Dyson equation like (2.19) for the singlet amplitude $\Gamma_s(q,\omega)$. According to figure (2.14) we have

$$\Gamma^s_{s_1 s_2 s_3 s_4}(q,\omega) = \Gamma_s \, \delta_{s_1 s_4}\delta_{s_2 s_3} + \Gamma_s \, \delta_{s_1 s_4} \sum_{\alpha\beta} \delta_{\alpha\beta} \, \Pi^{\alpha\beta}_{\rm dyn}(q,\omega) \, \Gamma^s_{\beta s_2 s_3 \alpha}(q,\omega)$$

which is

$$\Gamma^s_{s_1 s_2 s_3 s_4}(q,\omega) = \Gamma_s \, \delta_{s_1 s_4}\delta_{s_2 s_3} + \Gamma_s \, \delta_{s_1 s_4} \sum_{\alpha} \Pi^{\alpha\alpha}_{\rm dyn}(q,\omega) \, \Gamma^s_{\alpha s_2 s_3 \alpha}(q,\omega)$$

We can replace $\Pi^{\alpha\alpha}_{\rm dyn}(q,\omega)$ by

$$\Pi^{(m=0)}_{\rm dyn}(q,\omega) = -\nu_3 \frac{i|\omega|}{Dq^2 - i|\omega|} \tag{2.76}$$

that is independent of the spin. We discuss only the nonzero components of $\Gamma^s_{s_1 s_2 s_3 s_4}$. We find for example the two coupled equations[22]

$$\Gamma^s_{1111}(q,\omega) = \Gamma_s + \Gamma_s \, \Pi^{(m=0)}_{\rm dyn}(q,\omega) \, \left[\Gamma^s_{1111}(q,\omega) + \Gamma^s_{1221}(q,\omega)\right] \tag{2.77}$$

and

$$\Gamma^s_{1221}(q,\omega) = \Gamma_s + \Gamma_s \, \Pi^{(m=0)}_{\rm dyn}(q,\omega) \, \left[\Gamma^s_{1111}(q,\omega) + \Gamma^s_{1221}(q,\omega)\right].$$

Subtracting the equations we find that $\Gamma^s_{1111}(q,\omega) = \Gamma^s_{1221}(q,\omega)$. Plugging this into equation (2.77) we get

$$\Gamma^s_{1111}(q,\omega) = \frac{\Gamma_s}{1 - 2\,\Pi^{(m=0)}_{\rm dyn}(q,\omega)\,\Gamma_s}.$$

The same result we obtain for $\Gamma^s_{2112}(q,\omega)$ and $\Gamma^s_{2222}(q,\omega)$. So we can summarize that the dynamically screened singlet interaction does not depend on the spins and therefore is not affected by the external magnetic field. Thus we recover the equation (2.43).

$$\Gamma^s_{s_1 s_2 s_3 s_4}(q,\omega) = \Gamma_s(q,\omega) \, \delta_{s_1 s_4}\delta_{s_2 s_3}. \tag{2.78}$$

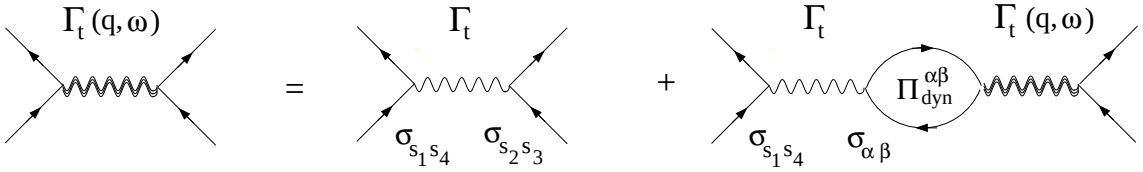

Figure 2.15: Diagrammatic Dyson series for the triplet interaction

[22]Only in this subsection we denote $\frac{1}{2} \equiv \uparrow \equiv 1$ and $-\frac{1}{2} \equiv \downarrow \equiv 2$. Thus instead of writing $\Gamma^s_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}$ we just write Γ^s_{1122}

The Dyson equation for the triplet interaction has a different spin structure which is depicted in figure (2.15). [23] At each vertex we have σ matrices instead of δ matrices.

$$\Gamma^t_{s_1s_2s_3s_4}(q,\omega) = \Gamma_t\ \sigma_{s_1s_4}\sigma_{s_2s_3} + \Gamma_t\ \sigma_{s_1,s_4}\sum_{\alpha,\beta}\sigma_{\alpha\beta}\ \Pi^{\alpha\beta}_{\rm dyn}(q,\omega)\ \Gamma^t_{\beta s_2s_3\alpha}(q,\omega)$$

We discuss only the nonzero components of $\Gamma^t_{s_1s_2s_3s_4}$. Starting for example with $\Gamma^t_{s_1s_2s_2s_1}(q,\omega)$, we obtain the pair of equations

$$\Gamma^t_{1111}(q,\omega) = \Gamma_t\big[1 + \ \Pi^{11}_{\rm dyn}(q,\omega)\ \Gamma^t_{1111}(q,\omega) - \Pi^{22}_{\rm dyn}(q,\omega)\ \Gamma^t_{2112}(q,\omega)\big] \tag{2.79}$$

$$\Gamma^t_{2112}(q,\omega) = -\Gamma_t\big[1 + \ \Pi^{11}_{\rm dyn}(q,\omega)\ \Gamma^t_{1111}(q,\omega) - \Pi^{22}_{\rm dyn}(q,\omega)\ \Gamma^t_{2112}(q,\omega)\big].$$

It follows $\Gamma^t_{1111}(q,\omega) = -\Gamma^t_{2112}(q,\omega)$. Putting this into equation (2.79) yields

$$\Gamma^t_{1111}(q,\omega) = \frac{\Gamma_t}{1 - 2\ \Pi^{(m=0)}_{\rm dyn}(q,\omega)\ \Gamma_t}.$$

In the same way we get $\Gamma^t_{2222}(q,\omega) = -\Gamma^t_{1221}(q,\omega)$, and $\Gamma^t_{2222}(q,\omega)$ is identical to $\Gamma^t_{1111}(q,\omega)$. So for the $m = 0$ components we recover the result of equation (2.44). Turning to the combination $\Gamma^t_{1212}(q,\omega)$ we find the Dyson equation

$$\Gamma^t_{1212}(q,\omega) = 2\Gamma_t - 2\Gamma_t\ \Pi^{21}_{\rm dyn}(q,\omega)\ \Gamma^t_{1212}$$

which leads to

$$\Gamma^t_{1212}(q,\omega) = \frac{2\Gamma_t}{1 + 2\ \Pi^{(m=1)}_{\rm dyn}(q,\omega)\ \Gamma^t}. \tag{2.80}$$

And finally for $\Gamma^t_{2121}(q,\omega)$ we get

$$\Gamma^t_{2121}(q,\omega) = \frac{2\Gamma_t}{1 + 2\ \Pi^{(m=-1)}_{\rm dyn}(q,\omega)\ \Gamma^t} \tag{2.81}$$

where we used $\Pi^{21}_{\rm dyn}(q,\omega) = \Pi^{(m=1)}_{\rm dyn}(q,\omega)$ and $\Pi^{12}_{\rm dyn}(q,\omega) = \Pi^{(m=-1)}_{\rm dyn}(q,\omega)$. We can summarize all the partial results by

$$\Gamma^t_{s_1s_2s_3s_4}(q,\omega) = \tilde{\Gamma}^t_{s_1s_2s_3s_4}(q,\omega)\ \sigma_{s_1s_4}\sigma_{s_2s_3} \tag{2.82}$$

where $\tilde{\Gamma}^t_{1212}(q,\omega) = \frac{1}{2}\Gamma^t_{1212}(q,\omega)$ and $\tilde{\Gamma}^t_{2121}(q,\omega) = \frac{1}{2}\Gamma^t_{2121}(q,\omega)$ as the factor of 2 is generated by the combination of σ's. Turning off the dynamical screening, which means practically sending $\Pi^m_{\rm dyn}(q,\omega) \to 0$, brings us back to the static result $\Gamma^t_{s_1s_2s_3s_4} = \Gamma^t\ \sigma_{s_1s_4}\sigma_{s_2s_3}$. Written with the quantum number m, the dynamically screened interaction reads

$$\Gamma^m_t(q,\omega) = \Gamma_t\ \frac{Dq^2 - i\omega - im\omega_s}{Dq^2 - i\omega Z - im\omega_s}. \tag{2.83}$$

[23] see also equation (2.39) for the spin structure

2.3.3 Correction to the DOS including a dynamically screened interaction

With the results (2.78) and (2.83) we are now able to calculate the correction to the DOS taking into account the modified dynamically screened interaction in the presence of an external magnetic field. The diagrams that have to be calculated are depicted in figure (2.16). The replacements for the triplet diagram are given in the brackets. Looking at figure (2.16) one remarks that the singlet–triplet formulation is more symmetric than the exchange–Hartree formulation we started with. All one has to do to go from the singlet to the triplet diagram is to change the spin matrices at the interaction vertices from $\delta_{ij} \to \sigma_{ij}$ and change the interaction propagator $\Gamma_s(q,\omega) \to \Gamma_t(q,\omega)$. Let us begin with the calculation of the singlet correction to the DOS per spin direction.

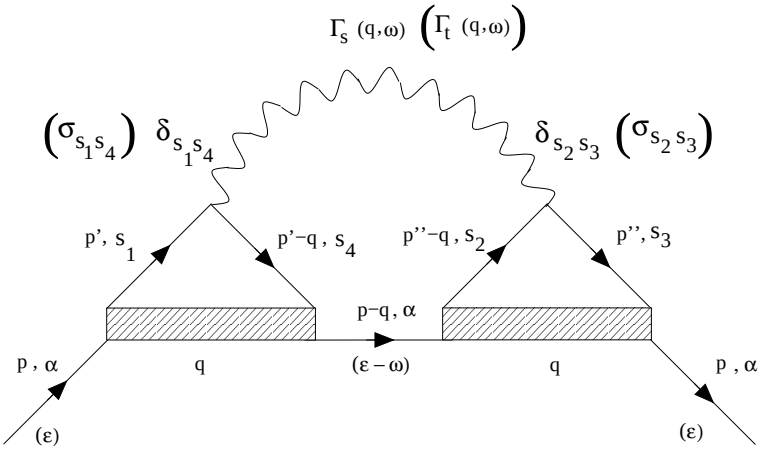

Figure 2.16: Diagrammatic representation of singlet (triplet) correction to the DOS

$$\delta\nu_\alpha^{\rm sing} = -\frac{\mathrm{sign}(\epsilon)}{\pi\mathcal{V}}\,\mathrm{Im}\sum_{\substack{s_1,s_2,\\ s_3,s_4}}\sum_p G_\alpha(p,\epsilon)\Big[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;\delta_{s_1\alpha}\;L_{s_1s_4}(q,\omega)G_\alpha(p-q,\epsilon-\omega)$$

$$i\,\Gamma_s(q,\omega)\;\delta_{s_1s_4}\,\delta_{s_2s_3}\;\frac{1}{\mathcal{V}}\sum_{p'}G_{s_1}(p',\epsilon)G_{s_4}(p'-q,\epsilon-\omega)$$

$$L_{s_3s_2}(q,\omega)\;\frac{1}{\mathcal{V}}\sum_{p''}G_{s_3}(p'',\epsilon)G_{s_2}(p''-q,\epsilon-\omega)\;\delta_{s_3\alpha}\Big]G_\alpha(p,\epsilon).$$

Evaluating all the sums in the diffusive approximation in the quasi one–dimensional limit, we arrive at

$$\frac{\delta\nu_\alpha^{\rm sing}}{\nu_3} = -\frac{1}{\pi a^2}\int\frac{dq}{2\pi}\int_{|\epsilon|}^{\infty}d\omega\;\mathrm{Im}\Big[\frac{\Gamma_s(q,\omega)}{(Dq^2-i\omega)^2}\Big]. \tag{2.84}$$

which multiplied by 2 for the spin degeneracy reproduces exactly the singlet part given in equation (2.57). This is no surprise as all the ingredients forming the singlet part where not affected by the external field.

The situation however changes for the triplet part. In our calculation in section (2.3.1) with static interaction amplitudes we observed a Zeeman splitting of the three triplet terms. For the triplet contribution per spin direction we have according to the diagram (2.16)

$$\delta\nu_\alpha^{\rm trip} = -\frac{{\rm sign}(\epsilon)}{\pi\mathcal{V}}\,{\rm Im}\sum_{\substack{s_1,s_2,\\ s_3,s_4}}\sum_p G_\alpha(p,\epsilon)\Big[\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\delta_{s_1\alpha}\,L_{s_1s_4}(q,\omega)G_\alpha(p-q,\epsilon-\omega)$$

$$i\,\tilde{\Gamma}_t(q,\omega)\,\sigma_{s_1s_4}\,\sigma_{s_2s_3}\,\frac{1}{\mathcal{V}}\sum_{p'}G_{s_1}(p',\epsilon)G_{s_4}(p'-q,\epsilon-\omega)$$

$$L_{s_3s_2}(q,\omega)\,\frac{1}{\mathcal{V}}\sum_{p''}G_{s_3}(p'',\epsilon)G_{s_2}(p''-q,\epsilon-\omega)\,\delta_{s_3\alpha}\Big]G_\alpha(p,\epsilon).$$

For the spin–up part we get

$$\delta\nu_\uparrow^{\rm trip} = -\frac{{\rm sign}(\epsilon)}{\pi\mathcal{V}}\,{\rm Im}\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\Pi_0^2(q,\omega)\gamma_3(q,\omega)\Big[L^2(q,\omega)\Gamma_{1111}(q,\omega)+L_{12}^2(q,\omega)\Gamma_{1212}(q,\omega)\Big]$$

and for the spin–down part we obtain

$$\delta\nu_\downarrow^{\rm trip} = -\frac{{\rm sign}(\epsilon)}{\pi\mathcal{V}}\,{\rm Im}\frac{1}{\mathcal{V}}\sum_q\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\Pi_0^2(q,\omega)\gamma_3(q,\omega)\Big[L^2(q,\omega)\Gamma_{2222}(q,\omega)+L_{21}^2(q,\omega)\Gamma_{2121}(q,\omega)\Big]$$

Summing up these two contributions yields

$$\frac{\delta\nu_3^{\rm trip}}{\nu_3} = -\frac{2}{\pi a^2}\sum_m\int\frac{dq}{2\pi}\int_{|\epsilon|}^{\infty}d\omega\,{\rm Im}\Big[\frac{\Gamma_t}{(Dq^2-i\omega Z_t-im\omega_s)(Dq^2-i\omega-im\omega_s)}\Big] \tag{2.85}$$

where m takes the values $-1,0,1$. Note that in comparison to equation (2.69) there is a additional Z_t coming from the dynamically screened interaction. The $m=0$ term is treated like the singlet part. For the $m=1$ and $m=-1$ terms however we need to redo the integrations.

We define $w_1=\omega Z+m\,\omega_s$ and $w_2=\omega+m\,\omega_s$. Assuming that w_1 and $w_2>0$ we get using the residue theorem

$$\int\frac{dq}{2\pi}\frac{1}{(Dq^2-iw_1)(Dq^2-iw_2)} = \frac{i^{3/2}}{2(w_1-w_2)}\Big(\frac{1}{\sqrt{w_2}}-\frac{1}{\sqrt{w_1}}\Big). \tag{2.86}$$

The energy integral for the $m=1$ component is then

$$\int_{|\epsilon|}^{\infty}d\omega\,\frac{1}{\omega}\Big(\frac{1}{\sqrt{\omega Z_t+\omega_s}}-\frac{1}{\sqrt{\omega+\omega_s}}\Big)=\frac{2}{\sqrt{\omega_s}}\,\Big[{\rm arctanh}\Big(\sqrt{\frac{|\epsilon Z_t+\omega_s|}{\omega_s}}\Big)-{\rm arctanh}\Big(\sqrt{\frac{|\epsilon+\omega_s|}{\omega_s}}\Big)\Big] \tag{2.87}$$

For the $m=-1$ part one obtains a similar result where ${\rm arctanh}(x)$ is replaced by $\arctan(x)$ and there is a multiplicative factor of -1. Taking into account the effect of dynamical screening in the presence of an external field we thus obtain (remember that $\Gamma_t=A_0^a/2\nu_3$ and $Z_t=1-A_0^a$)

$$\begin{aligned}\frac{\delta\nu_1}{\nu_1} =& -\frac{1}{\sqrt{2}\pi\,\nu_1\sqrt{D}}\left[\frac{A_0^s}{(1-A_0^s+\sqrt{1-A_0^s})}+\frac{A_0^a}{(1-A_0^a+\sqrt{1-A_0^a})}\right]\frac{1}{\sqrt{|\epsilon|}}\\ &-\frac{1}{\sqrt{2}\pi\nu_1\sqrt{D}\sqrt{\omega_s}}\left[\operatorname{arctanh}\left(\sqrt{\frac{|(1-A_0^a)\,\epsilon+\omega_s|}{\omega_s}}\right)-\operatorname{arctanh}\left(\sqrt{\frac{|\epsilon+\omega_s|}{\omega_s}}\right)\right]\\ &+\frac{1}{\sqrt{2}\pi\nu_1\sqrt{D}\sqrt{\omega_s}}\left[\arctan\left(\sqrt{\frac{|(1-A_0^a)\,\epsilon-\omega_s|}{\omega_s}}\right)-\arctan\left(\sqrt{\frac{|\epsilon-\omega_s|}{\omega_s}}\right)\right]\end{aligned} \tag{2.88}$$

This is a first central result of our work. It is the generalization of the result in equation (2.70), which was only valid for A_0^s and A_0^a smaller than 1. We check that for $m = 1$ if we can reproduce our known results (the calculation for the $m = -1$ is completely analogous). Let us first send $\omega_s \to 0$. Then

$$\frac{1}{\sqrt{\omega_s}}\left[\operatorname{arctanh}\left(\sqrt{\frac{|(1-A_0^a)\,\epsilon+\omega_s|}{\omega_s}}\right)-\operatorname{arctanh}\left(\sqrt{\frac{|\epsilon+\omega_s|}{\omega_s}}\right)\right]=\frac{1}{\sqrt{|\epsilon|}}\left(\frac{1}{\sqrt{(1-A_0^a)}}-1\right)+\mathcal{O}(\omega_s) \tag{2.89}$$

Putting all together we obtain

$$\frac{\delta\nu_3^{m=1}}{\nu_3}\approx-\frac{1}{\sqrt{2}\pi\nu_1\sqrt{D}}\frac{A_0^a}{(1-A_0^a+\sqrt{1-A_0^a})}\frac{1}{\sqrt{|\epsilon|}}+\mathcal{O}(\omega_s) \tag{2.90}$$

where the next order term is proportional to ω_s. The leading order term is one of the three triplets in equation (2.57). The other limit we can verify is the case of static interaction. In the limit $Z_t \to 1$ we should recover the result (2.70). Using

$$\frac{1}{\sqrt{\omega_s}}\left[\operatorname{arctanh}\left(\sqrt{\frac{|(1-A_0^a)\,\epsilon+\omega_s|}{\omega_s}}\right)-\operatorname{arctanh}\left(\sqrt{\frac{|\epsilon+\omega_s|}{\omega_s}}\right)\right]=\frac{A_0^a}{2\sqrt{|\epsilon+\omega_s|}}+\mathcal{O}\left((A_0^a)^2\right) \tag{2.91}$$

we obtain indeed

$$\frac{\delta\nu_3^{m=1}}{\nu_3}=-\frac{A_0^a}{2\sqrt{2}\pi\nu_1\sqrt{D}}\frac{1}{\sqrt{|\epsilon+\omega_s|}}+\mathcal{O}\big((1-Z_t)\big). \tag{2.92}$$

2.4 Spin scattering – Suppression mechanisms for the triplets

In the preceding sections we have examined the effect of the triplet for various situations. However all this took place under idealized conditions. Of course, we assumed the temperature to be low enough because otherwise thermal fluctuations would already smear out the effects. But there are other mechanisms that could suppress the effect of the triplets.

Let us discuss very shortly the effect of magnetic impurities and spin–orbit interactions. Magnetic impurities are modelled by the Hamiltonian

$$H_s=\sum_{s_1,s_2}\sum_{p,p'}U_s^{s_1s_2}(p,p')\;c_{p,s_1}^\dagger\;c_{p',s_2} \tag{2.93}$$

with

$$U_s^{s_1 s_2}(p,p') = \sum_{i=1}^{N_s} \int dr\; e^{-irp}\; U_s(r-R_i)\; \vec{S}\cdot\vec{\sigma}_{s_1 s_2}\; e^{irp'} \tag{2.94}$$

where $\vec{S}$ is the spin of the magnetic impurity located at R_i with the scattering amplitude $U_s(r-R_i)$. In the presence of spin–orbit interaction one adds the term

$$H_{\rm so} = \sum_{s_1,s_2}\sum_{p,p'} U_{\rm so}^{s_1 s_2}(p,p')\; c^\dagger_{p,s_1}\; c_{p',s_2} \tag{2.95}$$

with

$$U_{\rm so}^{s_1 s_2}(p,p') = \frac{1}{4m^2c^2}\sum_{i=1}^{N_{so}} \sigma_{s_1 s_2} \int dr\; e^{-irp}\big[\nabla\times U_{\rm so}(r-R_i)\big]\; e^{irp'} \tag{2.96}$$

where R_i indicates an ion site and the corresponding potential is $U_{\rm so}(r-R_i)$. In second order Born–approximation one obtains the spin–spin and spin–orbit scattering times τ_s and $\tau_{\rm so}$ that can be written as (This is obtained along similar lines as our treatment of static impurities in section (2.1).)

$$\frac{1}{\tau_s} = \pi\nu_3 n_s U_s^2\; S(S+1) \tag{2.97}$$

and[24]

$$\frac{1}{\tau_{\rm so}} = \pi\nu_3 n_{\rm tot} U_{\rm so}^2\; \overline{(p\times p')^2}. \tag{2.98}$$

where n_s is the concentration of magnetic impurities and $n_{\rm tot}$ is the total impurity concentration.

Taking into account these additional effects one finds a modified scattering time in the damped single particle Green's function which leads to a modified form of the diffusive ladder

$$\boxed{L^{(j,m)}(q,\omega) = \frac{1}{Dq^2 - i\omega + j/t_s - im\omega_s}} \tag{2.99}$$

where the total spin relaxation time is given by

$$\frac{1}{t_s} = \frac{4}{3}\left(\frac{1}{\tau_s} + \frac{1}{\tau_{\rm so}}\right) \tag{2.100}$$

Note that for the singlet component ($j=0$) the term j/t_s vanishes and we still have the diffusive pole. The triplets ($j=1$) are however suppressed as we have no longer a divergence for small q and ω due to the $1/t_s$ in the denominator. For more details see [29].

2.5 Summary

In this chapter we discussed extensively the calculation of the DOS using the techniques of perturbation theory. First we reproduced the results by Altshuler and Aronov ([7], [8]) and extended them using the effective coupling amplitudes Γ_s and Γ_t, which are connected with the Landau parameters. Second we investigated the impact of an external magnetic field on the

[24] The line over $p\times p'$ indicates the angle average.

correction to the DOS. Also here we first rederived the result of Altshuler and Aronov ([24]) for static interactions. In a second step we included dynamical screening of the static interaction amplitudes which lead to our generalized result (2.88).

The results obtained are in good agreement with the experiments for higher energies, as mentioned in the introductory chapter. However approaching the Fermi edge the results diverge. This is due the fact that for small energies the interactions in the singlet and triplet channel can no longer be treated as small perturbations. Defining the singlet and triplet energies

$$E_s = \frac{(A_0^s)^2}{32\pi\hbar D(\nu_3 \mathrm{a}^2)^2(1 - A_0^s + \sqrt{1 - A_0^s})^2} \qquad E_t = \frac{(A_0^a)^2}{32\pi\hbar D(\nu_3 \mathrm{a}^2)^2(1 - A_0^a + \sqrt{1 - A_0^a})^2} \tag{2.101}$$

we can write equation (2.57) in the form[25]

$$\frac{\delta\nu_1}{\nu_1} = -\frac{4}{\sqrt{\pi}}\left(\sqrt{\frac{E_s}{\epsilon}} - 3\sqrt{\frac{E_t}{\epsilon}}\right). \tag{2.103}$$

Estimations of E_s and E_t for Palladium can be found in appendix (9.4). We see that if our probing energy $\epsilon = eV$ approaches E_s, we enter in the region where perturbation theory starts to fail.

In order to solve this problem one needs to take into account higher order processes in $\Gamma_s(q,\omega)$ and $\Gamma_t(q,\omega)$. Rollbühler and Grabert could successfully derive a non–perturbative result for the quasi one-dimensional DOS using a path integral approach for Fermions ([18], [19]). They achieved to sum up the exchange diagram to all orders which would correspond to a summation of the singlet diagrams in our language. In this work we will try to generalize this approach to electrons with spin in order to take into account the triplet diagrams and investigate the impact of the spin on the DOS. Therefore we will explain in the next three chapters the formalism of fermionic path integrals.

[25] The square roots of theses energies are

$$\sqrt{E_s} = \frac{A_0^s}{4\nu_1\sqrt{2\pi D}(1 - A_0^s + \sqrt{1 - A_0^s})} \qquad \text{and} \qquad \sqrt{E_t} = -\frac{A_0^a}{4\nu_1\sqrt{2\pi D}(1 - A_0^a + \sqrt{1 - A_0^a})}. \tag{2.102}$$

We put an additional minus sign in the definition of $\sqrt{E_t}$ because the Landau parameter A_0^a is negative. We need these expressions in section (6.6.1).

Chapter 3

Path Integrals for Fermions

In this chapter we will present a short overview of fermionic path integrals. First we will introduce the mathematical concept of Grassmann variables which is necessary for the definition of Fermion coherent states. Using these Fermion coherent states we will then formulate the fermionic many–body problem in terms of a fermionic coherent state path integral. In this formulation we will later calculate correlation functions. For a more detailed discussion we refer the reader to [19], [31], [32], [33] and [34].

With regard to the treatment of disorder we will use the Keldysh technique. Therefore our fermionic fields will live on the Keldysh contour.[1]

3.1 Grassmann variables

Before starting with the definition of Grassmann variables let us motivate the need of such peculiar objects. Path integrals for Bosons can be expressed as integrals over complex valued functions. Several complex valued functions simply commute like Bosons do. For Fermions however we need anti–commuting objects in order to implement the correct symmetry. Thus if we want to formulate the many–fermion problem with the help of path integrals we need numbers that posses that anti–commutation property. The conventional Feynman path integral is set up using the position eigenstates. In this work, however, we will use the so called coherent states for the construction of our path integral. Coherent states for Bosons are rather well known objects. They already appear in the context of the harmonic oscillator. The **coherent states are** defined as **the eigenstates of the annihilation operator**. For Bosons

$$a_\alpha|\zeta\rangle = \zeta_\alpha|\zeta\rangle \qquad\qquad \zeta_\alpha \in \mathbb{C} \tag{3.1}$$

where a_α is a bosonic annihilation operator, $\alpha \in \mathcal{I}$ is a (multi–)index representing a set of quantum numbers and $|\zeta\rangle$ is a coherent state.

Suppose we have constructed a fermionic coherent state $|\psi\rangle$. Operating with a fermionic annihilation operator c_α on the state $|\psi\rangle$ we obtain

$$c_\alpha|\psi\rangle = \psi_\alpha|\psi\rangle \qquad\qquad \psi_\alpha \in \mathcal{G} \tag{3.2}$$

where ψ_α is an element of the Grassmann algebra $\mathcal{G}$ (which we will shortly define). ψ_α is necessarily an anti–commuting number. This necessity becomes clear if we look at two different annihilation operators acting on a coherent state:

[1] There are other field theoretical approaches based on the replica trick [35] or on the supersymmetry [36].

$$\psi_\alpha\psi_\beta|\psi\rangle = c_\alpha c_\beta|\psi\rangle = -c_\beta c_\alpha|\psi\rangle = -\psi_\beta\psi_\alpha|\psi\rangle. \tag{3.3}$$

The anticommutation relation for the operators imposes

$$\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha \tag{3.4}$$

for the Grassmann variables. This relation can be seen as the definition of Grassmann variables. Additionally we demand that these variables anti–commute with fermionic operators and state vectors with odd particle number.

Definition of the Grassmann algebra

A Grassmann algebra $\mathcal{G}$ is defined by a set generators $\{\psi_\alpha|\alpha = 1, ..., n\}$ where each pair of elements anti–commute

$$\psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0 \tag{3.5}$$

so that in particular

$$\psi_\alpha^2 = 0. \tag{3.6}$$

The basis of the Grassmann algebra is made of all distinct products of generators. Up to a sign factor there are 2^n distinct products of generators $\{1, \psi_{\alpha_1}, \psi_{\alpha_1}\psi_{\alpha_2}, ..., \psi_{\alpha_1}..\psi_{\alpha_n}\}$ where by convention $\alpha_1 < \alpha_2 < ... < \alpha_n$. For Grassmann variables one cannot set up an order relation like $>$ or $<$. Grassmann numbers cannot be big or small. However one can define a conjugation operation $\psi \to \psi^*$. Similar to the complex conjugation one defines

$$(\psi^*)^* = \psi \qquad (\lambda\psi)^* = \lambda^*\psi^* \quad \lambda \in \mathbb{C} \qquad (\psi_1\psi_2)^* = \psi_2^*\psi_1^*. \tag{3.7}$$

Having the formal definition we proceed to the rules of calculus that will be applied in the path integral calculations for fermions. Since $\psi^2 = 0$ each function can easily be written in a Taylor series:

$$f(\psi) = a + b\,\psi \tag{3.8}$$

where a and b can be ordinary or Grassmann numbers. An important example is $e^\psi = 1 + \psi$. More general one could have

$$g(\psi, \psi^*) = a + b\,\psi^* + c\,\psi + d\,\psi^*\psi. \tag{3.9}$$

Generalization to more degrees of freedom is straightforward.

Integration of Grassmann functions

It is possible to define a differentiation for Grassmann functions which we will not explain here as we do not need it for our calculations. However we need to know how to integrate a Grassmann function. As there is no analog of the familiar sum motivating the Riemann integral for ordinary variables, the integration over Grassmann variables is defined as a linear mapping. One only needs two definitions

$$\int d\psi\, 1 = 0 \qquad \int d\psi\, \psi = 1. \tag{3.10}$$

However this is not enough to calculate the integral of the function $f(\psi)$ given in equation (3.8). In (3.10) the 1 and ψ are adjacent to $d\psi$. We define that $d\psi$ behaves like a Grassmann variable

and demand that 1 and ψ have to be adjacent to $d\psi$ in order to apply (3.10). Practically that means we have to commute or anti–commute the different terms under the integral until 1 and ψ are directly to the right of $d\psi$ and then have the permission to apply the integration rules. For $f(\psi)$ we obtain for example:

$$\int d\psi\, f(\psi) = \pm a \int d\psi\, 1 \pm b \int d\psi\, \psi = \pm b \tag{3.11}$$

where the upper sign is for the case that the coefficients are complex numbers or consist of an even number of Grassmann variables and the minus sign appears if the coefficients are an odd number of Grassmann variables.

In order to integrate the function $g(\psi, \psi^*)$ of equation (3.9) we also need to define the integration rules for ψ^*:

$$\int d\psi^*\, 1 = 0 \qquad\qquad \int d\psi^*\, \psi^* = 1. \tag{3.12}$$

We obtain

$$\int d\psi^*\, d\psi\, g(\psi, \psi^*) = -d \tag{3.13}$$

and especially for Gaussian integrals we have

$$\int d\psi^*\, d\psi\, e^{a\psi^*\psi} = \int d\psi^*\, d\psi\, (1 - a\psi^*\psi) = a. \tag{3.14}$$

A motivation for this definition of integration is that many results look similar to those of complex integration. For instance the definition of the δ–function:

$$\delta(\psi_\alpha - \psi_\beta) = \int d\psi_\gamma\, e^{\psi_\gamma(\psi_\alpha - \psi_\beta)} = (\psi_\alpha - \psi_\beta). \tag{3.15}$$

Taking the function $f(\psi)$ one can easily verify that $\delta(\psi_\alpha - \psi_\beta)$ has the desired property $\int d\psi_\alpha\, \delta(\psi_\alpha - \psi_\beta) f(\psi_\beta) = f(\psi_\beta)$.

A major difference towards ordinary integration is however the transformation law under a change of variables. We examine it doing the calculation for one degree of freedom (restricting ourselves to complex coefficients a,b and λ):

$$\int d\psi\, f(\lambda\psi) = b\lambda. \tag{3.16}$$

On the other hand with $\chi = \lambda\psi$

$$\int d\psi\, f(\lambda\psi) = \int d\chi\, J\, f(\chi) = b\, J. \tag{3.17}$$

We see that $J = \lambda$. For ordinary integrals however one would get $J = \partial\psi/\partial\chi = 1/\lambda$. So in conclusion we get the inverse Jacoby determinant. The corresponding is true for higher dimensions.

Gaussian Integrals

Setting up the path integral formalism for our problem will lead to integrals of exponential functions which are polynomials in complex variables or Grassmann variables. In the case of

quadratic forms these are straightforward generalizations of the familiar Gaussian integral. For brevity we just give the formulas for the case of real and complex variables (Bosonic case) and refer the reader to [31] for the proofs.

We begin with the multidimensional integral over real variables.

$$\boxed{\int \frac{dx_1...dx_n}{(2\pi)^{n/2}}\, e^{-\frac{1}{2}\,x_iA_{ij}x_j+x_iJ_i} = [\mathrm{det}A]^{-1/2}\, e^{\frac{1}{2}\,J_iA^{-1}_{ij}J_j}} \tag{3.18}$$

where A is a real symmetric positive definite matrix and the summation is over repeated indices. This identity is established by changing variables to reduce it to diagonal form and using the Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{ax^2} = \sqrt{\frac{\pi}{a}}$.

For pairs of complex conjugate variables a similar identity is valid:

$$\int \prod_{i=1}^{n} \frac{dx_i^*\, dx_i}{2\pi i}\, e^{-x_i^*H_{ij}x_j+J_i^*x_i+J_ix_i^*} = [\mathrm{det}H]^{-1}\, e^{J_i^*H^{-1}_{ij}J_j} \tag{3.19}$$

for any matrix H with positive Hermitian part. Finally let us establish an analogous identity for Grassmann variables. We assume that A is a hermitian but not necessarily postive definite $\mathbb{C}$–valued $n \times n$ matrix and ξ_α, ξ_α^* $(\alpha = 1, ..., n)$ are Grassmann variables. Then

$$\boxed{\int \prod_{\alpha=1}^{n} d\psi_\alpha^* d\psi_\alpha\, e^{-\psi_\alpha^*A_{\alpha\beta}\psi_\beta+\xi_\alpha^*\psi_\alpha+\psi_\alpha^*\xi_\alpha} = [\mathrm{det}A]\, e^{\xi_\alpha^*A^{-1}_{\alpha\beta}\xi_\beta}} \tag{3.20}$$

Note that in comparison with equation (3.19) we also find on the right hand side the inverse matrix in the exponential but for the prefactor we have the inverse result. For the proof let us use vector notation and abbreviate the conjugated and transposed vector $(\psi^*)^T = \psi^\dagger$. The strategy to evaluate the left hand side of equation (3.20) is the same as in the bosonic case. We diagonalize the matrix A

$$A = M^{-1}DM \tag{3.21}$$

where D is a diagonal matrix. Furthermore we introduce the vectors

$$\chi = M\psi \qquad\qquad \chi^\dagger = \psi^\dagger M^{-1}$$

and

$$\eta = M\xi \qquad\qquad \eta^\dagger = \xi^\dagger M^{-1}.$$

The change of variables from ψ_α and ψ_α^* to the χ_α and χ_α^* brings us a factor $\mathrm{det}(M)\mathrm{det}(M^{-1}) = \mathrm{det}(MM^{-1}) = 1$.

$$\int \prod_{\alpha=1}^{n} d\psi_\alpha^* d\psi_\alpha\, e^{-\psi^\dagger A\psi+\xi^\dagger\psi+\psi^\dagger\xi} = \int \prod_{\alpha=1}^{n} d\chi_\alpha^* d\chi_\alpha\, e^{-\chi^\dagger D\chi+\eta^\dagger\chi+\chi^\dagger\eta}$$

Let us look at the exponential

$$\begin{aligned}\exp(-\chi_\alpha^*D_\alpha\chi_\alpha + \eta_\alpha\chi_\alpha + \chi_\alpha^*\eta_\alpha) &= 1 - \chi_\alpha^*D_\alpha\chi_\alpha + \eta_\alpha^*\chi_\alpha + \chi_\alpha^*\eta_\alpha + \chi_\alpha^*\eta_\alpha\eta_\alpha^*\chi_\alpha \\ &= 1 + \eta_\alpha^*\chi_\alpha + \chi_\alpha^*\eta_\alpha + \chi_\alpha^*(D_\alpha + \eta_\alpha\eta_\alpha^*)\chi_\alpha\end{aligned}$$

So we find according to equation (3.13)

$$\int d\chi_\alpha^* d\chi_\alpha \, \exp(-\chi_\alpha^* D_\alpha \chi_\alpha + \eta_\alpha \chi_\alpha + \chi_\alpha^* \eta_\alpha) = D_\alpha + \eta_\alpha^* \eta_\alpha$$

which we can rewrite as an exponential function

$$D_\alpha + \eta_\alpha^* \eta_\alpha = D_\alpha (1 + D_\alpha^{-1} \eta_\alpha^* \eta_\alpha) = D_\alpha \exp(D_\alpha^{-1} \eta_\alpha^* \eta_\alpha).$$

Furthermore

$$\prod_{\alpha=1}^{n} D_\alpha = \det(D) = \det(MAM^{-1}) = \det(A).$$

Hence we finally obtain

$$\int \prod_{\alpha=1}^{n} d\psi_\alpha^* d\psi_\alpha \, e^{-\psi^\dagger A \psi + \xi^\dagger \psi + \psi^\dagger \xi} = \prod_{\alpha=1}^{n} D_\alpha \exp(D_\alpha^{-1} \eta_\alpha^* \eta_\alpha) = \det(A) \, e^{\xi^\dagger A^{-1} \xi}$$

which proves equation (3.20). These Gaussian integration formulas are also valid for operators ($n \to \infty$) if the boundary conditions allow to determine a unique inverse operator.

3.2 Fermion coherent states

The Fermion coherent states can be defined by

$$|\psi\rangle = e^{-\sum_\alpha \psi_\alpha c_\alpha^\dagger} |0\rangle = \prod_\alpha \Big(1 - \psi_\alpha c_\alpha^\dagger\Big) |0\rangle \tag{3.22}$$

where $c_\alpha^\dagger$ is the Fermion creation operator (that creates a Fermion in state α: $c_\alpha^\dagger |0\rangle_\alpha = |1\rangle_\alpha$) and $|0\rangle = |0\rangle_{\alpha_1} \otimes ... \otimes |0\rangle_{\alpha_n}$ is the vacuum state. Note that the product on the right hand side is well defined since the factors $(1 - \psi_\alpha c_\alpha^\dagger)$ commute with each other. The state $|\psi\rangle$ is indeed an eigenstate of c_α (which means that it fulfills equation (3.2)). One has

$$\begin{aligned} c_\alpha |\psi\rangle &= c_\alpha \Big(|0\rangle_\alpha - \psi_\alpha |1\rangle_\alpha\Big) \prod_{\beta \neq \alpha} \Big(|0\rangle_\beta - \psi_\beta |1\rangle_\beta\Big) \\ &= \Big(0 + \psi_\alpha |0\rangle_\alpha\Big) \prod_{\beta \neq \alpha} \Big(|0\rangle_\beta - \psi_\beta |1\rangle_\beta\Big) \end{aligned}$$

Now we use the property $\psi_\alpha^2 = 0$:

$$c_\alpha |\psi\rangle = \psi_\alpha \Big(|0\rangle_\alpha - \psi_\alpha |1\rangle_\alpha\Big) \prod_{\beta \neq \alpha} \Big(|0\rangle_\beta - \psi_\beta |1\rangle_\beta\Big) = \psi_\alpha |\psi\rangle.$$

That's it. We have constructed coherent states for Fermions which we will use in the following to build up the path integral formulation for the Fermion many–body problem. The adjoint of a coherent state $|\psi\rangle$ is denoted by the bra vector $\langle\psi|$ and is given by

$$\langle\psi| = \langle 0| e^{-\sum_\alpha c_\alpha \psi_\alpha^*} = \langle 0| \prod_\alpha \Big(1 - c_\alpha \psi_\alpha^*\Big). \tag{3.23}$$

It is straightforward to prove that $\langle\psi|$ is a left eigenvector for the creation operator $c_\alpha^\dagger$ with eigenvalue ψ_α^*.

Scalar product of two coherent states

Since $(1+\psi_\alpha^* c_\alpha)$ and $(1-\psi'_\beta c_\beta^\dagger)$ commute for $\alpha \neq \beta$ we get

$$\langle\psi|\psi'\rangle = \langle 0|\prod_\alpha \Big(1+\psi_\alpha^* c_\alpha\Big)\Big(1-\psi'_\beta c_\beta^\dagger\Big)|0\rangle = \prod_\alpha \Big(1+\psi_\alpha^*\psi'_\alpha\Big) = e^{\psi_\alpha^*\psi'_\alpha}. \tag{3.24}$$

Closure relation for coherent states

An important property of the coherent states is that every vector in the Fock space $\mathcal{F}$ can be expressed as a linear combination of coherent states and the unity operator $\mathbb{I}$ in $\mathcal{F}$ can be represented by the closure relation

$$\mathbb{I} = \int \Big(\prod_{\alpha=1}^n d\psi_\alpha^* d\psi_\alpha\Big)\, e^{-\sum_{\alpha=1}^n \psi_\alpha^*\psi_\alpha}\, |\psi\rangle\langle\psi|. \tag{3.25}$$

For simplicity let us prove this relation just for one degree of freedom (for the general prove see [31] or [34]):

$$\begin{aligned}
\int d\psi^* d\psi\, e^{-\psi^*\psi}\, |\psi\rangle\langle\psi| &= \int d\psi^* d\psi\, (1-\psi^*\psi)\Big(|0\rangle - \psi|1\rangle\Big)\Big(\langle 0| - \langle 1|\psi^*\Big) \\
&= \int d\psi^* d\psi\, (1-\psi^*\psi)\Big(|0\rangle\langle 0| + \psi|1\rangle\langle 1|\psi^*\Big) \\
&= \int d\psi^* d\psi\, \Big(-\psi^*\psi|0\rangle\langle 0| + \psi^*\psi|1\rangle\langle 1|\Big) \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{I}.
\end{aligned}$$

Representation of the Trace by Coherent States

In order to calculate the partition function or the generating functional we need a representation of the trace of an operator A in terms of Fermion coherent states. We start writing the trace using a complete orthonormal system $\{\phi_i\}$ and then introduce the unity operator given in equation (3.25)

$$\begin{aligned}
\mathrm{Tr}A &= \sum_i \langle\phi_i|A|\phi_i\rangle \\
&= \int \Big(\prod_{\alpha=1}^n d\psi_\alpha^* d\psi_\alpha\Big)\, e^{-\sum_{\alpha=1}^n \psi_\alpha^*\psi_\alpha}\, \sum_i \langle\phi_i|\psi\rangle\langle\psi|A|\phi_i\rangle.
\end{aligned}$$

Next we use

$$\langle\theta_i|\psi\rangle\langle\psi|\theta_j\rangle = \langle -\psi|\theta_j\rangle\langle\theta_i|\psi\rangle \tag{3.26}$$

where θ_i and θ_j are Fock states. We need the relation only for $i = j$:

$$\mathrm{Tr}A = \int \left(\prod_{\alpha=1}^{n} d\psi_\alpha^* d\psi_\alpha \right) e^{-\sum_{\alpha=1}^{n} \psi_\alpha^* \psi_\alpha} \sum_i \langle -\psi|A|\phi_i\rangle\langle\phi_i|\psi\rangle.$$

Replacing finally $\sum_i |\phi_i\rangle\langle\phi_i|$ by $\mathbb{I}$ we obtain

$$\mathrm{Tr}A = \int \left(\prod_{\alpha=1}^{n} d\psi_\alpha^* d\psi_\alpha \right) e^{-\sum_{\alpha=1}^{n} \psi_\alpha^* \psi_\alpha} \langle -\psi|A|\psi\rangle. \tag{3.27}$$

Thus the trace of an operator is expressed by integrals over Grassmann variables. Now that we have all the mathematical tools, we can construct the path integral for Fermions in terms of coherent states.

3.3 Coherent state path integral for Fermions

In this section we apply the formalism of the Grassmann variables and coherent states to the general expression for the partition function. The partition function reads

$$Z = \mathrm{Tr}\ e^{-\beta H}.$$

where H is a second quantization Hamilton operator in normal ordered form (all the creation operators $c_\alpha^\dagger$ are to the left of any annihilation operator c_α) and $\beta = 1/k_B T$ is the inverse temperature. With equation (3.27) we can write Z using coherent states. We restrict ourselves to one degree of freedom and therefore drop the index α.

$$Z = \int d\psi^* d\psi\ e^{-\psi^*\psi}\ \langle -\psi|e^{-\beta H}|\psi\rangle$$

The idea is to split $e^{-\beta H}$ along the complex contour $\mathcal{C}$ from $z = 0$ to $z = -i\beta$ (see figure (3.1)) into small steps Δ_j , $(j = 1, ..., P)$ with

$$\sum_{j=1}^{P} \Delta_j = -i\beta \qquad \text{such that} \qquad e^{-\beta H} = \prod_{j=1}^{P} e^{-i\Delta_j H}.$$

We have[2]

$$Z = \int d\psi^* d\psi\ e^{-\psi^*\psi}\ \langle -\psi| \prod_{j=1}^{P} e^{-i\Delta_j H}|\psi\rangle$$

$$= \int d\psi^* d\psi\ e^{-\psi^*\psi}\ \langle -\psi| \prod_{j=1}^{P} \Big(1 - i\Delta_j H + \mathcal{O}(\Delta_j^2)\Big)|\psi\rangle.$$

Making the pieces Δ_j smaller and smaller such that $\lim_{P\to\infty}\ \max_j|\Delta_j| = 0$ we can in good approximation neglect the quadratic corrections. We rename $\psi \to \psi_0$ and introduce $P-1$ unity operators along the contour.

$$\mathbb{I} = \int d\psi_j^* d\psi_j\ e^{-\sum_j \psi_j^*\psi_j} |\psi_j\rangle\langle\psi_j|\,, \qquad (j = 1, ..., P-1) \tag{3.28}$$

[2]In the literature this procedure is often called *Trotter Breakup*

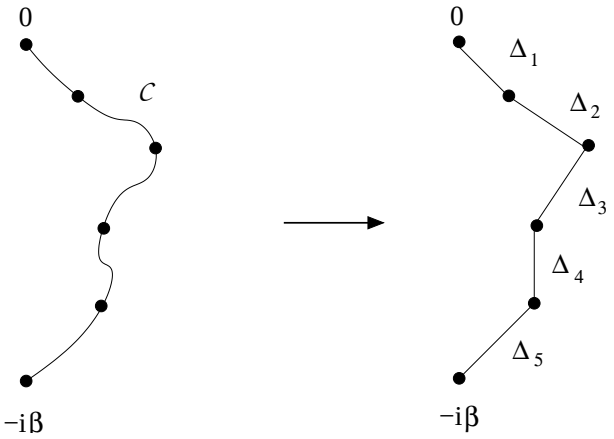

Figure 3.1: Trotter decomposition of the contour $\mathcal{C}$

Thus we get

$$Z = \int d\psi^* d\psi_0 \Big(\prod_{j=1}^{P-1} d\psi_j^* d\psi_j \Big)\; e^{-\psi^* \psi_0}\; \langle -\psi | \Big(1 - i\Delta_P H \Big) | \psi_{P-1} \rangle \times$$

$$e^{-\psi_{P-1}^* \psi_{P-1}}\; \langle \psi_{P-1} | \Big(1 - i\Delta_{P-1} H \Big) | \psi_{P-2} \rangle \;\times ... \times\; e^{-\psi_2^* \psi_2}\; \langle \psi_2 | \Big(1 - i\Delta_2 H \Big) | \psi_1 \rangle$$

$$\times\; e^{-\psi_1^* \psi_1}\; \langle \psi_1 | \Big(1 - i\Delta_1 H \Big) | \psi_0 \rangle .$$

As H is normal ordered we can now easily evaluate the matrix elements

$$\langle \psi_j | \Big(1 - i\Delta_j H(c^\dagger, c) \Big) | \psi_{j-1} \rangle = \langle \psi_j | \psi_{j-1} \rangle \; \Big(1 - i\Delta_j H(\psi_j^*, \psi_{j-1}) \Big).$$

With the scalar product (3.24) we finally get

$$\langle \psi_j | \Big(1 - i\Delta_j H(c^\dagger, c) \Big) | \psi_{j-1} \rangle = e^{\psi_j^* \psi_{j-1}}\; e^{-i\Delta_j H(\psi_j^*, \psi_{j-1})}.$$

We substitute $-\psi^* \to \psi_P^*$, $d\psi^* d\psi_0 = -d\psi_P^* d\psi_0 = d\psi_0 d\psi_P^*$ and obtain

$$Z = \int d\psi_0\; d\psi_P^* \Big(\prod_{j=1}^{P-1} d\psi_j^* d\psi_j \Big)\; e^{\psi_P^* \psi_0}\; \Big(e^{\psi_P^* \psi_{P-1}}\; e^{-i\Delta_P H(\psi_P^*, \psi_{P-1})} \Big) \times$$

$$e^{-\psi_{P-1}^* \psi_{P-1}} \Big(e^{\psi_{P-1}^* \psi_{P-2}}\; e^{-i\Delta_{P-1} H(\psi_{P-1}^*, \psi_{P-2})} \Big) \times ... \times\; e^{-\psi_1^* \psi_1} \Big(e^{\psi_1^* \psi_0}\; e^{-i\Delta_1 H(\psi_1^*, \psi_0)} \Big).$$

With the help of the δ–function of equation (3.15) we can write

$$e^{\psi_P^* \psi_0} = \int d\psi_P\; \delta(\psi_P + \psi_0)\; e^{-\psi_P^* \psi_P}.$$

In that way we created a product of P similar terms

$$\prod_{j=1}^{P} e^{-\psi_j^*\psi_j}\big(e^{\psi_j^*\psi_{j-1}}e^{-i\Delta_j H(\psi_j^*,\psi_{j-1})}\big)$$

$$= \prod_{j=1}^{P} \exp\bigg(i\Delta_j\bigg[i\,\psi_j^*\bigg(\frac{\psi_j-\psi_{j-1}}{\Delta_j}\bigg)\bigg] - H(\psi_j^*,\psi_{j-1})\bigg)$$

$$= \exp\bigg(i\sum_{i=1}^{P}\Delta_j\bigg[i\,\psi_j^*\bigg(\frac{\psi_j-\psi_{j-1}}{\Delta_j}\bigg)\bigg] - H(\psi_j^*,\psi_{j-1})\bigg).$$

For the transformation we used the property $e^{-\psi_j^*\psi_j}e^{\psi_j^*\psi_{j-1}} = e^{-\psi_j^*\psi_j+\psi_j^*\psi_{j-1}}$ which is true because $(\psi_j^*)^2 = 0$. So let us write down our final result

$$Z = \int d\psi_0\bigg(\prod_{j=1}^{P} d\psi_j^* d\psi_j\bigg)\delta(\psi_P+\psi_0)\exp\bigg(i\sum_{i=1}^{P}\Delta_j\bigg[\psi_j^*\bigg(\frac{\psi_j-\psi_{j-1}}{\Delta_j}\bigg)\bigg] - H(\psi_j^*,\psi_{j-1})\bigg).$$

We consider the limit $P\to\infty$ and $\Delta_j\to 0$. The expressions for the Riemann sum and the differential quotient suggest to use the following notation

$$\int d\psi_0\bigg(\prod_{j=1}^{P} d\psi_j^* d\psi_j\bigg)\delta(\psi_P+\psi_0) \qquad\to\qquad \int \mathcal{D}\psi^*\mathcal{D}\psi$$

$$\sum_{i=1}^{P}\Delta_j\big[...\big] \qquad\to\qquad \int_{\mathcal{C}} dz[...]$$

$$\frac{\psi_j-\psi_{j-1}}{\Delta_j} \qquad\to\qquad \frac{\partial\psi}{\partial z} = \partial_z\psi$$

and

$$H(\psi_j^*,\psi_{j-1}) \qquad\to\qquad H(\psi^*,\psi).$$

This allows us to write the partition function in the following elegant form

$$\boxed{Z = \int \mathcal{D}\psi^*\mathcal{D}\psi\exp\Big(iS[\psi^*,\psi]\Big)} \tag{3.29}$$

with the action S given by

$$\boxed{S = \int_{\mathcal{C}} dz\big[\psi^* i\partial_z\psi - H(\psi^*,\psi)\big].} \tag{3.30}$$

The contour $\mathcal{C}$ is here an arbitrary contour from 0 to $-i\beta$. For our purposes we will need the Keldysh contour that will be explained in the next section.

A standard basic example for the application of coherent state path integrals is the partition function for a fermionic particle. The Hamiltonian of this system is

$$H = \epsilon\, c^\dagger c.$$

Applying the rules of Grassmann calculus one finally finds

$$Z = 1 + e^{-\beta\epsilon}$$

which is the known result that can also be deduced in the conventional operator formalism.

3.3.1 The contour–ordered Green's function

A key object of interest in our calculations will be Green's function of the system which is defined as expectation value of an annihilation and creation operator. However we have to be more precise as there exist several Green's functions. For the calculation of the density of states for example we can take the retarded Green's function (see equation (1.11)). Of course, the retarded Green's function can be expressed in terms of other Green's function possessing a different time structure. As we want to use the coherent state path integral method to treat our problem we first need to examine which kind of time structure we obtain calculating a fermionic coherent state path integral. So let us look at[3]

$$\begin{aligned} G_c(z(\tau), z(\tau')) &= -i\langle\psi(z(\tau)\psi^*(z(\tau'))\rangle_{\text{path integral}} \\ &= -\frac{i}{Z}\int \mathcal{D}\psi^*\mathcal{D}\psi\, \Big(\psi(z(\tau))\psi^*(z(\tau'))\Big)\, e^{iS[\psi^*,\psi]} \end{aligned} \tag{3.31}$$

where $z(\tau)$ and $z(\tau')$ are two points of the discretized path and τ is the parametrization variable of the contour. The explicit calculation for a free particle described by the Hamiltonian H given in equation (3.3) can be found in [19] (or for many interacting particles in [34]). One finds the following result

$$G_c(z(\tau), z(\tau')) = \Theta(\tau - \tau')G^>(z(\tau), z(\tau')) - \Theta(\tau' - \tau)G^<(z(\tau), z(\tau')) \tag{3.32}$$

where

$$G^>(z(\tau), z(\tau')) = -i(1 - n_F(\epsilon))\, e^{-i\epsilon\left(z(\tau)-z(\tau')\right)} \tag{3.33}$$

and

$$G^<(z(\tau), z(\tau')) \quad = \quad i\, n_F(\epsilon)\, e^{-i\epsilon\left(z(\tau)-z(\tau')\right)} \tag{3.34}$$

where $n_F(\epsilon) = 1/(e^{\beta\epsilon}+1)$. We call this type of order **contour–order** and the Green's function defined by equation (3.31) we call the contour–ordered Green's function G_c. In order to understand the expression contour–ordered let us look at the definition of $G^>$ and $G^<$ which can be found for example in [28][4]:

$$G^>(z(\tau), z(\tau')) = -i\big\langle\, \psi(z(\tau))\psi^\dagger(z(\tau'))\,\big\rangle \qquad \text{where} \quad \tau > \tau' \tag{3.35}$$

[3] We put the index c because the result will be the contour ordered Green's function.

[4] These definitions are made for ordinary field operators.

$$G^{<}(z(\tau), z(\tau')) = \; i\big\langle\, \psi^{\dagger}(z(\tau'))\psi(z(\tau))\,\big\rangle \qquad \text{where} \quad \tau' > \tau \tag{3.36}$$

where $\langle ... \rangle = \langle ... \rangle_{\text{path integral}}$.[5] In the upper two equations the Grassmann variable with the earlier time is to the right and the Grassmann variable with the later time is to the left. We see that in equation (3.31) the two contributions are ordered in that sense. One can introduce the contour–ordering operator $T_{\mathcal{C}}$ that already arranges the different ψ's according to their position on the contour:

$$\begin{aligned} G_c(z(\tau), z'(\tau')) &= -i\langle T_{\mathcal{C}}\big[\psi(z(\tau))\psi^*(z(\tau'))\big]\rangle \\ &= -i\Theta(\tau - \tau')\langle\psi(z(\tau))\psi^*(z(\tau'))\rangle + i\Theta(\tau' - \tau)\langle\psi^*(z(\tau'))\psi(z(\tau))\rangle. \end{aligned} \tag{3.37}$$

3.4 Keldysh formalism

In this section we will describe the Keldysh formalism that allows to treat problems involving disorder. Originally the Keldysh formalism was invented to calculate Green's functions in non–equilibrium situations ([37], [38]).

The idea is to start in a well defined state and propagate it to the non–equilibrium regime where all the complicated interactions act on the state and then propagate it back. All effects except for the interactions cancel as they appear twice during the propagation but with opposite sign. This idea can be used for a system containing disorder. One starts with a system without disorder and fixes the normalization constants of the quantities one wants to calculate. Afterwards one adiabatically switches on the disorder potential. The normalization then gets lost but is recovered after the back propagation of the system.

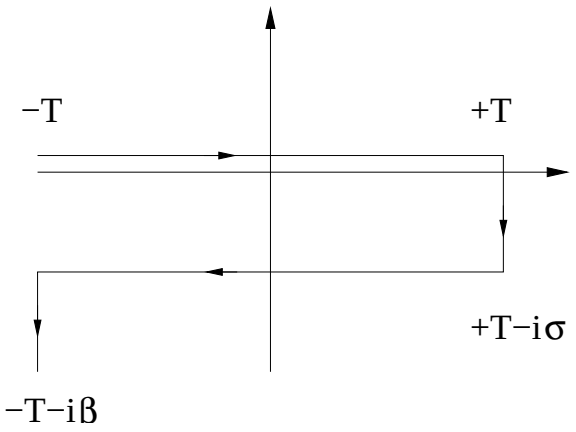

Figure 3.2: A possible contour

A contour $\mathcal{C}$ that realizes the idea described above is shown in figure (3.2). In the preceding section (3.3) we constructed a coherent state path integral for an arbitrary contour from 0 to $-i\beta$. Of course, we can translate the contour in the complex plane to $-T$ and $-T - i\beta$ where T is a real constant. Sending $T \to \infty$ and $\sigma \to 0$ we obtain the classical Keldysh contour which is depicted in figure (3.3). This is the contour we will use in this work. The part which is parallel to the imaginary axis is moved to $-\infty$. It is common to divide the Keldysh contour in an upper and a lower branch.

[5] In the future we will usually omit the index $_{\text{path integral}}$.

$$\mathcal{C} = \mathcal{C}^+ + \mathcal{C}^- \tag{3.38}$$

The starting point for the upper branch is $-\infty + i\eta$ with an infinitesimal $\eta > 0$. The contour $\mathcal{C}^+$ runs parallel to the real axis to $\infty + i\eta$ which is the starting point for the lower branch $\mathcal{C}^-$. $\mathcal{C}^-$ returns shifted by the infinitesimal η below the real axis, to $-\infty - i\eta$.

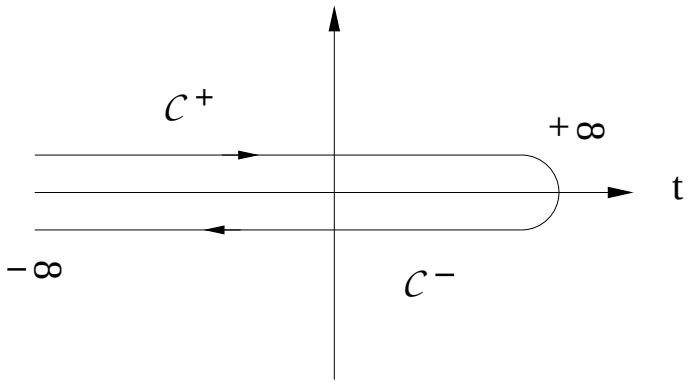

Figure 3.3: The Keldysh–contour

3.5 Matrix representation on the Keldysh contour

Before we discuss the matrix representation of the action S defined in equation (3.30) let us repeat what we did so far. In section (3.3) we constructed a coherent state path integral for Fermions based on the Grassmann variables. In subsection (3.3.1) we discussed which kind of time–ordering the path integral produces. Finally in section (3.4) we chose the Keldysh contour and explained why it is advantageous for our problem. The fact that the Keldysh contour has two branches suggests the use of a 2×2 matrix structure for the action. We show the construction for the example of free electrons. The Hamilton operator for free electrons is $H_0 = \Delta/2m$. [6]

We put the Hamiltonian in equation (3.30) for the action. Furthermore we introduce two so called source fields J and J^* which will allow us to express correlation functions as partial derivatives with respect to these fields. We have then

$$S[\psi, \psi^*, J, J^*] = \int_{\mathcal{C}} dt \int dr\ \psi^*(r,t) G_0^{-1}(r,t)\psi(r,t) + \int_{\mathcal{C}} dt \int dr J^*(r,t)\psi(r,t) + \psi^*(r,t)J(r,t) \tag{3.39}$$

where $t = \mathrm{Re}\, z(\tau)$ and $t' = \mathrm{Re}\, z(\tau')$ and

$$G_0^{-1}(r,t) = \left(i\partial_t + \frac{\Delta}{2m}\right). \tag{3.40}$$

[6] We neglect the spin in this discussion.

We define the generating functional

$$Z[J, J^*] = \int \mathcal{D}\psi\mathcal{D}\psi^* \; e^{iS[\psi,\psi^*,J,J^*]}. \tag{3.41}$$

The name generating functional will be explained at the end of the section where we can show the usefulness of this functional. We split the Keldysh contour in the upper and lower branch and label the variables living on the different branches with index 1 for $\mathcal{C}^+$ and 2 for $\mathcal{C}^-$. We write the action as $S = S_1 - S_2$

$$S_1[\psi_1, \psi_1^*, J_1, J_1^*] = \int_{-\infty}^{\infty} dt \int dr \; \psi_1^*(r,t) G_0^{-1}(r,t)\psi_1(r,t) + \int_{-\infty}^{\infty} dt \int dr \; J_1^*(r,t)\psi_1(r,t) + \psi_1^*(r,t)J_1(r,t)$$

$$S_2[\psi_2, \psi_2^*, J_2, J_2^*] = \int_{-\infty}^{\infty} dt \int dr \; \psi_2^*(r,t) G_0^{-1}(r,t)\psi_2(r,t) + \int_{-\infty}^{\infty} dt \int dr \; J_2^*(r,t)\psi_2(r,t) + \psi_2^*(r,t)J_2(r,t)$$

The minus sign in front of S_2 comes from the reversed integration direction in the t–domain. We set $x = (r, t)$.

$$S_1[\psi_1, \psi_1^*, J_1, J_1^*] = \int dx \; \psi_1^*(x) G_0^{-1}(x)\psi_1(x) + \int dx J_1^*(x)\psi_1(x) + \psi_1^*(x)J_1(x) \tag{3.42}$$

$$S_2[\psi_2, \psi_2^*, J_2, J_2^*] = \int dx \; \psi_2^*(x) G_0^{-1}(x)\psi_2(x) + \int dx J_2^*(x)\psi_2(x) + \psi_2^*(x)J_2(x) \tag{3.43}$$

We introduce the vector notation

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \Psi^* = \begin{pmatrix} \psi_1^*, \psi_2^* \end{pmatrix} \tag{3.44}$$

and

$$J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \qquad J^* = \begin{pmatrix} J_1^*, J_2^* \end{pmatrix}. \tag{3.45}$$

Then the actions reads

$$S[\Psi, \Psi^*, J, J^*] = \int dx \Psi^*(x)\sigma_z \; \breve{G}_0^{-1}\Psi(x) + \int dx J^*(x)\sigma_z \Psi(x) + \Psi^*(x)\sigma_z \; J(x) \tag{3.46}$$

where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \breve{G}_0^{-1} = \begin{pmatrix} G_0^{-1} & 0 \\ 0 & G_0^{-1} \end{pmatrix}. \tag{3.47}$$

We have then for the generating functional

$$Z[J, J^*] = \int \mathcal{D}\Psi\mathcal{D}\Psi^* \; e^{iS[\Psi,\Psi^*,J,J^*]}. \tag{3.48}$$

All we did in comparison to equation (3.41) is to introduce the matrix notation which will turn out to be very convenient for the calculations.

The action is a quadratic form in ψ and ψ^* (or Ψ and Ψ^*) hence we use equation (3.20) for the integration over $\mathcal{D}\Psi$ and $\mathcal{D}\Psi^*$. We obtain

$$Z[J,J^*] = \det\left(-i\breve{G}_0^{-1}\right)\, e^{-i\int dxdx' J^*(x)\sigma_z \left(\breve{G}_0^{-1}\right)^{-1}(x,x')\,\sigma_z J(x')}. \tag{3.49}$$

From our earlier considerations we know that the entries of the 2×2 matrix $\left(\breve{G}_0^{-1}\right)^{-1}(x,x')$ are contour ordered so we call them

$$\hat{G}_0 \;=\; \begin{pmatrix} G_0^T(x,x') & G_0^<(x,x') \\ G_0^>(x,x') & G_0^{\bar{T}}(x,x') \end{pmatrix}. \tag{3.50}$$

Before we explain the usefulness of this notation and why the $(1,1)$–element G_0^T is for example just the ordinary time–ordered free electron Green's function let us comment on the σ_z's on both sides of $(\breve{G}_0^{-1})^{-1}$. Writing down the exponent explicitly (neglecting the position arguments r and r' for clarity) we have

$$\begin{aligned} S[J,J^*] = -i\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\Big(& J_1^*(t)G_0^T(t,t')J_1(t') - J_1^*(t)G_0^<(t,t')J_2(t') \\ & - J_2^*(t)G_0^>(t,t')J_1(t') + J_2^*(t)G_0^{\bar{T}}(t,t')J_2(t')\Big) \end{aligned} \tag{3.51}$$

We see that the two minus signs appearing in front of the second and third term are due to the Keldysh structure (Inversion of the integration direction). The fourth term has a plus sign since both fields are on the lower branch. In order to understand what is behind the entries of $\hat{G}$ let us look at the normalized functional

$$Z[J,J^*] = e^{-i\int dxdx' J^*(x)\sigma_z \hat{G}_0(x,x')\sigma_z J(x')}. \tag{3.52}$$

We call it normalized because $Z[0,0]=1$. We can obtain a certain entry $(\hat{G}_0)_{m,n}$ (for $n,m=1$ or 2) by calculating the following functional derivatives:

$$\left(\hat{G}_0\right)_{m,n} = i\,(-1)^{n+m}\,\frac{\delta}{\delta J_n}\frac{\delta}{\delta J_m^*}Z[J,J^*]\bigg|_{J=0,J^*=0}. \tag{3.53}$$

The prefactor $(-1)^{n+m}$ takes into account the signs coming from the σ_z matrices. On the other hand we can calculate these functional derivatives with respect to the source fields **before** we have integrated out the Fermions. The normalized functional before the integrations reads

$$Z[J,J^*] = \frac{1}{\det\left(-i\breve{G}_0^{-1}\right)}\int \mathcal{D}\Psi\mathcal{D}\Psi^*\; e^{i\int dx\Psi^*(x)\sigma_z\,\breve{G}_0^{-1}\Psi(x)+\int dxJ^*(x)\sigma_3\Psi(x)+\Psi^*(x)\sigma_z\,J(x)}. \tag{3.54}$$

Calculating the functional derivatives of $Z[J,J^*]$ like in equation (3.53) and setting $J=J^*=0$, we obtain the following expectation values

$$\hat{G}_0 = -i\begin{pmatrix} \langle\psi_1(x)\psi_1^*(x')\rangle & \langle\psi_1(x)\psi_2^*(x')\rangle \\ \langle\psi_2(x)\psi_1^*(x')\rangle & \langle\psi_2(x)\psi_2^*(x')\rangle \end{pmatrix} \tag{3.55}$$

where

$$\langle ...\rangle = \frac{1}{\det\left(-i\breve{G}_0^{-1}\right)}\int \mathcal{D}\Psi\mathcal{D}\Psi^*\; ...\; e^{i\int dx\Psi^*(x)\sigma_z\,\breve{G}_0^{-1}\Psi(x)}. \tag{3.56}$$

From equation (3.55) we see immediately that the $(1,2)$– and $(2,1)$–matrix elements coincide with the definitions given in equations (3.35) and (3.36).

$$G_0^<(x,x') \;=\; i\big\langle \psi_2^*(x')\psi_1(x)\big\rangle \tag{3.57}$$

$$G_0^>(x,x') \;=\; -i\big\langle \psi_2(x)\psi_1^*(x')\big\rangle \tag{3.58}$$

The time ordering for these two components is obvious since one Grassmann field lives on the upper branch and the other one on the lower branch. The field on the lower branch is always *later* in the sense of contour ordering. For the the $(1,1)$– and $(2,2)$–elements we have to distinguish two cases as both Grassmann fields are on the same branch.

If $z(\tau)$ and $z(\tau')$ are both on $\mathcal{C}^+$ we have

$$T_{\mathcal{C}}[\psi(r,t)\psi^*(r',t')] = \begin{cases} \psi(r,t)\psi^*(r',t') & \text{for} \quad t > t' \\ -\psi^*(r',t')\psi(r,t) & \text{for} \quad t < t' \end{cases} \tag{3.59}$$

So this is just the normal time–ordering (see [28]). On the upper branch $T_{\mathcal{C}}$ coincides with time–ordering operator T_T and we obtain the usual time–ordered Green's function as the $(1,1)$–matrix element.

$$G_0^T(x,x') \;=\; \Theta(t-t')G_0^>(x,x') + \Theta(t'-t)G_0^<(x,x') \tag{3.60}$$

If $z(\tau)$ and $z(\tau')$ are both on $\mathcal{C}^-$ we have

$$T_{\mathcal{C}}[\psi(r,t)\psi^*(r',t')] = \begin{cases} \psi(r,t)\psi^*(r',t') & \text{for} \quad t < t' \\ -\psi^*(r',t')\psi(r,t) & \text{for} \quad t > t' \end{cases} \tag{3.61}$$

This time order is called anti–time order because the field with the *smaller time* is to the left and not to the right like in the time–ordered case. However on the lower branch a smaller value for $t = \operatorname{Re} z(\tau)$ signifies a later point on the contour. Hence one obtains as the $(2,2)$–matrix element the anti–time–ordered Green's function

$$G_0^{\bar{T}}(x,x') \;=\; \Theta(t-t')G_0^<(x,x') + \Theta(t'-t)G_0^>(x,x'). \tag{3.62}$$

The explicit form of the different Green's functions for the case of free electrons are given in subsection (3.5.2) in the (ϵ,p)–representation.

In summary we explained how to invert $\breve{G}_0^{-1}$ and showed what kind of Green's functions appear in the different entries of $\hat{G}_0$. They are all contour–ordered. Of course, all these Green's functions are not independent. They fulfill several identities (see (3.69), (3.70) and (3.71)). This interdependence becomes most transparent if one changes to a **rotated representation**. Before explaining the special rotation we introduce three more definitions which are not necessary but allow a more condensed notation.

$$\bar{J} = \big(J_1^*, -J_2^*\big) \qquad \breve{G}_0 \;=\; \begin{pmatrix} G_0^T(x,x') & -G_0^<(x,x') \\ G_0^>(x,x') & -G_0^{\bar{T}}(x,x') \end{pmatrix} \qquad \bar{\Psi} = \big(\psi_1^*, -\psi_2^*\big) \tag{3.63}$$

The first two definitions allow to absorb the σ_z's to the left and to the right of $\hat{G}_0(x,x')$ in equation (3.52) as $\bar{J} = J^*\sigma_z$ and $\breve{G} = \hat{G}\sigma_z$. Now the generating functional simply reads

$$Z[J,\bar{J}] = e^{-i\int dx dx' \bar{J}(x)\breve{G}_0(x,x')J(x')}. \tag{3.64}$$

We could have introduced these definitions together with the definition of $\bar{\Psi}$ already at the beginning of this section. However the author believes that the presentation made above is much more transparent concerning the signs that appear due to the splitting of the Keldysh contour in two branches. Once one has understood the origin of all the signs one can go over to the more condensed notation.

3.5.1 Keldysh rotation

There exists a convenient representation for the 2×2 matrix of Green's function in which one entry is 0 and on the diagonal one finds the retarded and the advanced Green's function G^R and G^A. Usually the Keldysh rotation is defined by the orthogonal matrix

$$L_u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{3.65}$$

In this work however we will take the matrix[7]

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.66}$$

which is also orthogonal and has the convenient property that it is its own inverse $L^{-1} = L$. Let us define the **rotated version of the Green's function**:

$$\begin{aligned} \tilde{G} = L\breve{G}L^{-1} &= L \begin{pmatrix} G^T & -G^< \\ G^> & -G^{\bar{T}} \end{pmatrix} L^{-1} \\ &= \frac{1}{2} \begin{pmatrix} G^T - G^< + (G^> - G^{\bar{T}}) & G^T + G^< + (G^> + G^{\bar{T}}) \\ G^T - G^< - (G^> - G^{\bar{T}}) & G^T + G^< - (G^> + G^{\bar{T}}) \end{pmatrix} \end{aligned} \tag{3.67}$$

which reads

$$\tilde{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \tag{3.68}$$

using the following standard definitions for the retarded

$$G^R(x,x') = G^T(x,x') - G^<(x,x') = G^>(x,x') - G^{\bar{T}}(x,x') \tag{3.69}$$

advanced

$$G^A(x,x') = G^T(x,x') - G^>(x,x') = G^<(x,x') - G^{\bar{T}}(x,x') \tag{3.70}$$

and Keldysh

$$G^K(x,x') = G^T(x,x') + G^{\bar{T}}(x,x') = G^>(x,x') + G^<(x,x') \tag{3.71}$$

Green's functions. In equilibrium, in (ϵ, p)–representation the Keldysh Green's function G^K is connected to the retarded and advanced Green's functions by

$$G^K(\epsilon,p) = F(\epsilon)(G^R(\epsilon,p) - G^A(\epsilon,p)) \tag{3.72}$$

[7] Our notation is close to the work [19] and differs slightly from [17].

where $F(\epsilon) = \tanh(\frac{\epsilon}{2\beta}) = 1 - 2n_F(\epsilon)$ with the Fermi function $n_F(\epsilon) = 1/(e^{\beta\epsilon} + 1)$.

We refer to the representation in which G^R, G^A and G^K appear as the **rotated representation** while the **unrotated representation** refers to the original version with $\breve{G}$. Matrices and vectors that are given in the rotated representation can be identified by their tilde (like $\tilde{G}$).

3.5.2 The Green's functions for free electrons

In this subsection we write down the explicit form of all the free electron Green's functions ($H_0 = \Delta/2m$) in the (ϵ, p)–representation as we will need them in further calculation. In the following the we will use the abbreviation $\epsilon_p = p^2/2m$, $n_F(\epsilon) = 1/(e^{\beta\epsilon}+1)$ and $F(\epsilon) = 1-2n_F(\epsilon)$. We have then

$$G^{>}(\epsilon, p) = -2\pi i(1 - n_F(\epsilon))\ \delta(\epsilon - \epsilon_p) \qquad \text{and} \qquad G^{<}(\epsilon, p) = 2\pi i\, n_F(\epsilon)\, \delta(\epsilon - \epsilon_p) \tag{3.73}$$

$$G^R(\epsilon, p) = \frac{1}{\epsilon - \epsilon_p + i\eta} \qquad \text{and} \qquad G^A(\epsilon, p) = \frac{1}{\epsilon - \epsilon_p - i\eta} \tag{3.74}$$

where $\eta = 0^+$.

$$G^T(\epsilon, p) = (1 - n_F(\epsilon))G^R(\epsilon, p) + n_F(\epsilon)G^A(\epsilon, p) \tag{3.75}$$

$$G^{\bar{T}}(\epsilon, p) = -(1 - n_F(\epsilon))G^A(\epsilon, p) - n_F(\epsilon)G^R(\epsilon, p) \tag{3.76}$$

and finally

$$G^K(\epsilon, p) = 2\pi i F(\epsilon)\ \delta(\epsilon - \epsilon_p). \tag{3.77}$$

For more details we recommend [28].

Chapter 4

Electrons in a Disorder Potential

In this chapter we consider noninteracting electrons moving in a static disorder potential $U(r)$.[1] A static disorder corresponds to a situation in which the diffusive motion of the electrons is elastic that means without changing their energy. Such a model does not describe the process of inelastic diffusion which is responsible for the loss of phase coherence.[2] The short range disorder potential $U(r)$ is assumed to be Gaussian that means it is determined by its first and second moment.

$$\langle U(r)\rangle = 0 \qquad\qquad \langle U(r)U(r')\rangle = \frac{1}{2\pi\nu_3\tau}\,\delta(r-r') \tag{4.1}$$

ν_3 is the bulk DOS at the Fermi level and τ the elastic scattering time which determines the strength of disorder potential. The whole prefactor is chosen such that τ coincides with the lifetime obtained in second order Born approximation by perturbation theory (see equation (2.6)). If one wants to learn more about the introduction of disorder in a system and what kind of different approaches exist one can consult chapter 2.2 of [40].[3]

Given the two correlators, how do we calculate disorder dependent quantities $A[U]$? The answer is we have to average the quantity over the disorder potential. The standard Gaussian distribution is given by

$$g_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2} \tag{4.2}$$

where μ is the expectation value and σ the standard deviation. This allows us to set up the distribution function for our case.

$$\langle A[U]\rangle = \frac{1}{\mathcal{N}_U}\int \mathcal{D}U\; A[U]\; e^{-\pi\nu_3\tau\int dr\; U^2(r)} \tag{4.3}$$

where $\mathcal{N}_U$ is given by

$$\mathcal{N}_U = \int \mathcal{D}U\; e^{-\pi\nu_3\tau\int dr\; U^2(r)}. \tag{4.4}$$

Note that the disorder averaging does not yield a simple integral but a path integral as U depends on the continuous variable r.

[1] Also here we neglect the spin degree of freedom of the electrons.
[2] For more information see for example [24].
[3] Unfortunately the book is in French.

4.1 Disorder averaging and introduction of the Q–fields

So let us take our free electrons which we already discussed is the previous chapter and add a disorder potential $U(r)$. Note that $U(r)$ has no time dependence and hence takes the same values on the upper and lower branch of the Keldysh contour. We define $\breve{U}(x) = U(x)\,\mathbb{I} \equiv U(r)\,\mathbb{I}$. Writing $\breve{U}(x)$ we always mean the time independent potential $U(r)\,\mathbb{I}$. All we have to do is to replace H_0 by $H_0 + U$ in equation (3.30) in order to write down the action for the problem. So the additional part is $\int dx\ \bar{\Psi}(x)\breve{U}(x)\Psi(x)$:

$$S[\bar{\Psi},\Psi,\bar{J},J] = \int dx\ \bar{\Psi}(x)[\breve{G}_0^{-1} + \breve{U}(x)]\Psi(x) + \int dx\ \left(\bar{J}(x)\Psi(x) + J(x)\bar{\Psi}(x)\right). \tag{4.5}$$

In comparison to equation (3.46) we also applied the notation defined in equation (3.63) which allows us to absorb the signs coming the σ_z's.

$$\langle Z[\bar{J},J]\rangle = \int \mathcal{D}U \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{iS[\bar{\Psi},\Psi,\bar{J},J,U]} \tag{4.6}$$

In summary we have to perform three path integrals over $\mathcal{D}\Psi$, $\mathcal{D}\bar{\Psi}$ and $\mathcal{D}U$ in order to determine the generating functional $Z[J,\bar{J}]$. Starting with the integration over the fermionic fields Ψ and $\bar{\Psi}$ is impossible because one would have to invert $[\breve{G}_0^{-1} + \breve{U}(x)]$. Hence we have to average first over all disorder configurations.

$$\begin{aligned}\langle Z[\bar{J},J]\rangle &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ \langle e^{iS[\bar{\Psi},\Psi,\bar{J},J,U]}\rangle \\ &= \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int dx\ \bar{\Psi}(x)\breve{G}_0^{-1}\Psi(x)}\langle e^{i\int dx\ \bar{\Psi}(x)\breve{U}(x)\Psi(x)}\rangle\ e^{i\int dx\ \left(\bar{J}(x)\Psi(x)+J(x)\bar{\Psi}(x)\right)}\end{aligned} \tag{4.7}$$

Now

$$\begin{aligned}\langle e^{i\int dx\ \bar{\Psi}(x)\breve{U}(x)\Psi(x)}\rangle &= \frac{1}{\mathcal{N}_U}\int \mathcal{D}U\ e^{i\int dr\int dt\ \bar{\Psi}(r,t)\,\breve{U}(r)\Psi(r,t)} e^{-\pi\nu_3\tau\int dr\ U^2(r)} \\ &= \frac{1}{\mathcal{N}_U}\int \mathcal{D}U\ e^{-\frac{1}{2}\int dr\ U(r)[2\pi\nu_3\tau]U(r)}\ e^{\int dr\left[i\int dt\ \bar{\Psi}(r,t)\Psi(r,t)\right]U(r)}.\end{aligned} \tag{4.8}$$

The path integral has a Gaussian form and we obtain

$$\begin{aligned}\langle e^{i\int dx\ \bar{\Psi}(x)\breve{U}(x)\Psi(x)}\rangle &= e^{\frac{1}{2}\int dr\left[i\int dt\ \bar{\Psi}(r,t)\Psi(r,t)\right]\frac{1}{2\pi\nu_3\tau}\left[i\int dt'\ \bar{\Psi}(r,t')\Psi(r,t')\right]} \\ &= e^{-\frac{1}{4\pi\nu_3\tau}\int dr\left[\int dt\ \bar{\Psi}(r,t)\Psi(r,t)\right]^2}.\end{aligned} \tag{4.9}$$

We see that the disorder averaging yields a term in the exponential with 4 Ψ's. This is rather inconvenient because the integration over the fermionic fields is no longer Gaussian and can not be done exactly.

We can get rid of this quartic term introducing an additional matrix field Q by a Hubbard–Stratonovich–Transformation. Of course we will not choose Q identical to U because then we

would just walk back. The matrix field Q is chosen such that it is suitable to describe the low energy behavior of our system (that means the deviations from the homogeneous particle distribution). Further motivation and information for the decoupling can be found in [36].

The connection of Q to physical quantities becomes clearer when we discuss the space independent saddle point solution for Q. We will see the connection to the quasiclassical Green's function g.

The quartic term is explicitly

$$\begin{aligned}
&-\frac{1}{4\pi\tau\nu_3}\int dr\int dt\int dt'\big[\bar\Psi(r,t)\Psi(r,t)\big]\big[\bar\Psi(r,t')\Psi(r,t')\big]\\
&=-\frac{1}{4\pi\tau\nu_3}\int dr\int dt\int dt'\big[\psi_1^*(r,t)\psi_1(r,t)-\psi_2^*(r,t)\psi_2(r,t)\big]\big[\psi_1^*(r,t')\psi_1(r,t')-\psi_2^*(r,t')\psi_2(r,t')\big]\\
&=-\frac{1}{4\pi\tau\nu_3}\int dr\int dt\int dt'\big[\psi_1^*(r,t)\psi_1(r,t)\psi_1^*(r,t')\psi_1(r,t')+\psi_2^*(r,t)\psi_2(r,t)\psi_2^*(r,t')\psi_2(r,t')\\
&\qquad\qquad-\psi_1^*(r,t)\psi_1(r,t)\psi_2^*(r,t')\psi_2(r,t')-\psi_2^*(r,t)\psi_2(r,t)\psi_1^*(r,t')\psi_1(r,t')\big].
\end{aligned}\tag{4.10}$$

Now we regroup the Grassmann fields pairing always a field ψ_i with a conjugated field ψ_j^* where one of them has the argument t and the other one the argument t'.

$$\begin{aligned}
&=+\frac{1}{4\pi\tau\nu_3}\int dr\int dt\int dt'\Big[\big[\psi_1(r,t)\psi_1^*(r,t')\big]\,\big[\psi_1(r,t')\psi_1^*(r,t)\big]+\big[\psi_2(r,t)\psi_2^*(r,t')\big]\,\big[\psi_2(r,t')\psi_2^*(r,t)\big]\\
&\qquad-\big[\psi_1(r,t)\psi_2^*(r,t')\big]\,\big[\psi_2(r,t')\psi_1^*(r,t)\big]-\big[\psi_2(r,t)\psi_1^*(r,t')\big]\big[\psi_1(r,t')\psi_2^*(r,t)\big]\Big]
\end{aligned}\tag{4.11}$$

In order to decouple this expression we consider the following path integral

$$I[A]=\frac{1}{\mathcal{N}}\int\mathcal{D}Q\;e^{\frac{\pi\nu_3}{4\tau}\int drdtdt'\mathrm{tr}\left\{Q(r,t,t')Q(r,t',t)\right\}+\int drdtdt'\mathrm{tr}\left\{Q(r,t,t')A(r,t',t)\right\}}\tag{4.12}$$

where $Q(r,t,t')$ is a real, diagonalizable 2×2 matrix field. For given r,t,t' the eigenvalues should have a larger real than imaginary part. A is an arbitrary 2×2 matrix field and 'tr' denotes the trace over the 2×2 matrix. The normalization constant is

$$\mathcal{N}=\int\mathcal{D}Q\;e^{-\frac{\pi\nu_3}{4\tau}\int drdtdt'\mathrm{tr}\left\{Q(r,t,t')Q(r,t',t)\right\}}.\tag{4.13}$$

One calculates $I[A]$ using the standard method of diagonalization and finds

$$\begin{aligned}
I[A]&=e^{\frac{\tau}{\pi\nu_3}\int drdtdt'\sum_{i,j=1}^2A_{ij}A_{ji}(r,t',t)}\\
&=\exp\bigg(\frac{\tau}{\pi\nu_3}\int drdtdt'\big[A_{11}(r,t,t')A_{11}(r,t',t)+A_{22}(r,t,t')A_{22}(r,t',t)\\
&\qquad A_{12}(r,t,t')A_{21}(r,t',t)+A_{21}(r,t,t')A_{12}(r,t',t)\big]\bigg).
\end{aligned}\tag{4.14}$$

We define

$$\mathrm{Tr} = \int dr \int dt \int dt' \ \mathrm{tr}. \tag{4.15}$$

That allows to write our result in the following compact form

$$I[A] = \frac{1}{\mathcal{N}} \int \mathcal{D}Q \ e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\}+\mathrm{Tr}\{QA\}} = e^{\frac{\tau}{\pi\nu_3}\mathrm{Tr}\{A^2\}}. \tag{4.16}$$

The idea is now to take the right hand side of the upper equation and identify a product $\psi_i(r,t)\psi_j(r,t')$ with a matrix element of A.

$$e^{\frac{\tau}{\pi\nu_3}\mathrm{Tr}\{A^2\}} \stackrel{!}{=} e^{-\frac{1}{4\pi\nu_3\tau}\int dr\left[\int dt \ \bar{\Psi}(r,t)\Psi(r,t)\right]^2} \tag{4.17}$$

From equation (4.11) and (4.14) we get[4]

$$\begin{pmatrix} A_{11}(r,t,t') & A_{12}(r,t,t') \\ A_{21}(r,t,t') & A_{22}(r,t,t') \end{pmatrix} = \frac{1}{2\tau} \begin{pmatrix} \psi_1(r,t)\psi_1^*(r,t') & -\psi_1(r,t)\psi_2^*(r,t') \\ \psi_2(r,t)\psi_1^*(r,t') & -\psi_2(r,t)\psi_2^*(r,t') \end{pmatrix} \tag{4.18}$$

Using identity (4.16) we can write down our final result.

$$\boxed{e^{\frac{1}{4\pi\nu_3\tau}\int dr\left[\int dt\bar{\Psi}(r,t)\Psi(r,t)\right]^2} = \frac{1}{\mathcal{N}} \int \mathcal{D}Q \ e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\}-\mathrm{Tr}\{\bar{\Psi}Q\Psi\}}} \tag{4.19}$$

Let us repeat in words what happened. Performing the disorder average we obtained a quartic term in ψ. In order to recover an action that is quadratic in ψ we introduced an auxiliary matrix field $Q(r,t,t')$. Thus the price that we have to pay for a Gaussian action in ψ is according to equation (4.19) a path integral over $\mathcal{D}Q$.

For our purposes it will be enough to solve the path integral over $\mathcal{D}Q$ in saddle point approximation neglecting even the quadratic corrections (or fluctuations) of Q. The generating functional now reads

$$\langle Z[\bar{J},J]\rangle = \int \mathcal{D}Q \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \ e^{iS[\bar{\Psi},\Psi,\bar{J},J,Q]} \tag{4.20}$$

where

$$\begin{aligned} iS[\bar{\Psi},\Psi,\bar{J},J,Q] = &-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\} + i\int dx \ \left(\bar{J}(x)\Psi(x) + J(x)\bar{\Psi}(x)\right) \\ &+ i\int dx \int dx' \bar{\Psi}(x)\left[\breve{G}_0^{-1}(x)\delta(x-x') + \frac{i}{2\tau}Q(x,x')\delta(r-r')\right]\Psi(x') \end{aligned} \tag{4.21}$$

where $\delta(x-x') = \delta(r-r')\delta(t-t')$. Note that the matrix field $Q(r,t,t')$ is local in space but depends on two different times. Setting

$$\breve{G}^{-1}[Q](x,x') = \breve{G}_0^{-1}(x)\delta(x-x') + \frac{i}{2\tau}Q(x,x')\delta(r-r') \tag{4.22}$$

[4]This choice is not unique. See [19] for further information.

we can write the action in the compact form

$$iS[\bar{\Psi},\Psi,\bar{J},J,Q] = -\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\} + i\mathrm{Tr}\big\{\bar{J}\Psi + \bar{\Psi}J\big\} + i\mathrm{Tr}\big\{\bar{\Psi}\breve{G}^{-1}[Q]\Psi\big\}. \tag{4.23}$$

Here we generalized the meaning of Tr as the number of integrations is not the same for all terms. The trace Tr should be understood in the following as the abbreviation for the integration over the continuous variables plus the trace over the matrix structure.

Integrating out the Fermions

After the introduction of Q we can calculate the path integral over the Fermions exactly as the action is Gaussian in Ψ and $\bar{\Psi}$.

$$\begin{aligned}\langle Z[\bar{J},J]\rangle &= \int \mathcal{D}Q\; e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\}} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\; e^{i\mathrm{Tr}\left\{\bar{J}\Psi+\bar{\Psi}J\right\}+i\mathrm{Tr}\left\{\bar{\Psi}\breve{G}^{-1}[Q]\Psi\right\}} \\ &= \int \mathcal{D}Q\; e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\}}\; \mathrm{Det}\left[\breve{G}_0^{-1} + \frac{i}{2\tau}Q\right]\; e^{-i\mathrm{Tr}\{\bar{J}\breve{G}[Q]J\}}\end{aligned} \tag{4.24}$$

where

$$\breve{G}[Q](x,x') = \left[\breve{G}^{-1}[Q]\right]^{-1} = \left[\breve{G}_0^{-1} + \frac{i}{2\tau}Q\right]^{-1} \tag{4.25}$$

and Det denotes the determinant of the operator with respect to the Keldysh components as well as with respect of the space and time variables. Thanks to the identity

$$\mathrm{Det}(A) = \exp(\mathrm{Tr}\ln(A)) \tag{4.26}$$

we can write the determinant as an exponential.

$$\langle Z[\bar{J},J]\rangle = \int \mathcal{D}Q\; e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{Q^2\}}\; e^{\mathrm{Tr}\ln[\breve{G}_0^{-1}+\frac{i}{2\tau}Q]}\; e^{-i\mathrm{Tr}\{\bar{J}\breve{G}[Q]J\}} \tag{4.27}$$

This representation allows further treatment as we will show in the next section where we determine the saddle point equation. But before determining $Z[\bar{J},J]$ in saddle point approximation let us switch to the Keldysh rotated version.

Keldysh Rotation

We introduce several unities $\mathbb{I} = L\,L^{-1} = L\,L$ and use the possibility of cyclic permutations under the trace.[5] Thanks to the spectral theorem we can also transform the logarithm: $L\ln(A)L^{-1} = \ln(LAL^{-1})$. We define

$$\tilde{Q} = LQL^{-1} \tag{4.28}$$

and

$$\tilde{G}_0^{-1} = L\breve{G}_0^{-1}L^{-1} = LG_0^{-1}\,\mathbb{I}\,L^{-1} = G_0^{-1}\,\mathbb{I}. \tag{4.29}$$

Then we have

$$\tilde{G}_0^{-1}[Q] = L\breve{G}_0^{-1}[Q]L^{-1} = \tilde{G}_0^{-1} + \frac{i}{2\tau}\tilde{Q} \tag{4.30}$$

[5] where L is given in (3.66).

and

$$\tilde{G}[\tilde{Q}] = L\breve{G}[LQL^{-1}]L^{-1}. \tag{4.31}$$

We finally define

$$\tilde{J} = LJ \qquad \text{and} \qquad \tilde{\bar{J}} = \bar{J}L^{-1}. \tag{4.32}$$

Putting altogether yields

$$\boxed{\langle Z[\tilde{\bar{J}}, \tilde{J}]\rangle = \int \mathcal{D}\tilde{Q}\; e^{-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\tilde{Q}^2\}}\; e^{\mathrm{Tr}\ln[\tilde{G}_0^{-1}+\frac{i}{2\tau}\tilde{Q}]}\; e^{-i\mathrm{Tr}\{\tilde{\bar{J}}\tilde{G}[\tilde{Q}]\tilde{J}\}}.} \tag{4.33}$$

4.2 Determination of the saddle point equation

In order to determine $Z[\bar{J}, J]$ in saddle point approximation we have to derive the **saddle point equation**. At the saddle point the variational derivative with respect to $\tilde{Q}$ vanishes.

$$\frac{\delta}{\delta\tilde{Q}} S[\tilde{Q}, J = 0] = 0 \tag{4.34}$$

In order to calculate the variational derivative we write

$$\tilde{Q} = \tilde{Q}^{SP} + \delta\tilde{Q} \tag{4.35}$$

and expand $S[\tilde{Q}, J = 0]$ in orders of $\delta\tilde{Q}$. Let us start with the quadratic term in $\tilde{Q}$ of the exponential in equation (4.33).

$$-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\tilde{Q}^2\} = -\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{(\tilde{Q}^{SP} + \delta\tilde{Q})^2\} = -\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{(\tilde{Q}^{SP})^2\} - \frac{\pi\nu_3}{2\tau}\mathrm{Tr}\{\tilde{Q}^{SP}\delta\tilde{Q}\} + \mathcal{O}(\delta\tilde{Q}^2) \tag{4.36}$$

The second term in which $\tilde{Q}$ appears is $\mathrm{Tr}\ln G[\tilde{Q}]^{-1}$.[6]

$$\begin{aligned}
\mathrm{Tr}\ln[\tilde{G}_0^{-1} + \frac{i}{2\tau}\tilde{Q}] &= \mathrm{Tr}\ln\big[\tilde{G}_0^{-1} + \frac{i}{2\tau}\tilde{Q}^{SP} + \delta\tilde{Q}\big] \\
&= \mathrm{Tr}\ln\left[\tilde{G}[\tilde{Q}^{SP}]^{-1}\left(\mathbb{I} + \tilde{G}[\tilde{Q}^{SP}]\frac{i}{2\tau}\delta\tilde{Q} + \mathcal{O}(\delta\tilde{Q}^2)\right)\right] \\
&= \mathrm{Tr}\ln\left[\tilde{G}[\tilde{Q}^{SP}]^{-1}\right] + \mathrm{Tr}\left[\tilde{G}[\tilde{Q}^{SP}]\frac{i}{2\tau}\delta\tilde{Q}\right] + \mathcal{O}(\delta\tilde{Q}^2)
\end{aligned} \tag{4.37}$$

In summary the linear terms in $\delta\tilde{Q}$ writing explicitly all the arguments read

$$\int dr \int dr' \int dt \int dt' \left(-\frac{\pi\nu_3}{2\tau}\tilde{Q}^{SP} + \frac{i}{2\tau}\tilde{G}[\tilde{Q}^{SP}]\right)(r, t, r, t')\; \delta\tilde{Q}(r', t', t)\delta(r - r'). \tag{4.38}$$

This expression has to vanish for an arbitrary variation $\delta\tilde{Q}$. This condition implies that

[6] Note that we do not have to consider the third term in equation (4.33) because it is 0 when to $J = 0$.

$$\left(-\frac{\pi\nu_3}{2\tau}\tilde{Q}^{SP}+\frac{i}{2\tau}\tilde{G}[\tilde{Q}^{SP}]\right)(r,t,r,t')=0 \tag{4.39}$$

which allows us to express the saddle point for the matrix field $\tilde{Q}^{SP}$ in terms of the Green's function $\tilde{G}[\tilde{Q}^{SP}]$ taken at $r=r'$.

$$\tilde{Q}^{SP}(r,t,t')=\frac{i}{\pi\nu_3}\tilde{G}[\tilde{Q}^{SP}](r,t,r,t') \tag{4.40}$$

Using the definition of $\tilde{G}$ the self–consistent saddle point equation reads

$$\boxed{\tilde{Q}^{SP}(r,t,t')=\frac{i}{\pi\nu_3}\left[\tilde{G}_0^{-1}+\frac{i}{2\tau}\tilde{Q}^{SP}\right]^{-1}_{r=r'}} \tag{4.41}$$

There are different approaches to solve the saddle point equation. Some of them are discussed in [19]. We will restrict ourselves to the case of a space–independent solution of equation (4.41) that we will discuss in the next section.

4.3 Solution of the saddle point equation

Suppose we consider a translational invariant system which corresponds to a homogenous density distribution. Then the saddle point solution $\tilde{Q}^{SP}$ does not depend on the spatial coordinate.

We choose here the shortest way to present the space–independent saddle point solution simply by starting with the solution which we denote by $\tilde{\Lambda}$ and verifying that it solves the saddle point equation (4.41). [7] The solution is

$$\boxed{\tilde{\Lambda}(\epsilon)=\begin{pmatrix}1 & 2F(\epsilon)\\ 0 & -1\end{pmatrix}\qquad\qquad \tilde{\Lambda}(t)=\begin{pmatrix}\delta(t) & 2F(t)\\ 0 & -\delta(t)\end{pmatrix}} \tag{4.42}$$

where $F(\epsilon)=\tanh(\frac{\epsilon}{2\beta})=1-2n_F(\epsilon)$. We write equation (4.41) in (ϵ,p) space and consider the limit of a bulk systems which allows us to go over to the continuum limit for the summation over p.[8]

$$\tilde{\Lambda}(\epsilon)=\frac{i}{\pi\nu_3}\frac{1}{\mathcal{V}}\sum_p\tilde{G}(p,\epsilon)=\frac{i}{\pi}\int_{-\epsilon_F}^{\infty}d\epsilon_p\,\tilde{G}(\epsilon_p,\epsilon) \tag{4.43}$$

We have to determine $\tilde{G}$ in order to calculate the right side of the equation.

$$\begin{aligned}\tilde{G}(p,\epsilon)&=\left[\begin{pmatrix}G_0^{-1}(p,\epsilon) & 0\\ 0 & G_0^{-1}(p,\epsilon)\end{pmatrix}+\frac{i}{2\tau}\begin{pmatrix}1 & 2F(\epsilon)\\ 0 & -1\end{pmatrix}\right]^{-1}\\ &=\left[\begin{pmatrix}\epsilon-\epsilon_p & 0\\ 0 & \epsilon-\epsilon_p\end{pmatrix}+\frac{i}{2\tau}\begin{pmatrix}1 & 2F(\epsilon)\\ 0 & -1\end{pmatrix}\right]^{-1}\end{aligned} \tag{4.44}$$

[7] Another approach can be found in [19].

[8] We use the standard approximation for the p–sum that was already explained in the chapter on Perturbation theory.

where $\epsilon_p = p^2/2m$. Inverting the matrix we find

$$\tilde{G}(p,\epsilon) = \begin{pmatrix} \frac{1}{\epsilon-\epsilon_p+\frac{i}{2\tau}} & \frac{iF(\epsilon)/\tau}{(\epsilon-\epsilon_p)^2+1/4\tau^2} \\ 0 & \frac{1}{\epsilon-\epsilon_p-\frac{i}{2\tau}} \end{pmatrix}. \tag{4.45}$$

The result is not surprising. Remember the general structure of $\tilde{G}$

$$\tilde{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}. \tag{4.46}$$

On the diagonal we recover the standard noninteracting retarded and advanced Green's functions damped by weak disorder that can be derived using perturbation theory (see section (2.1)). They can be obtained from equation (2.7) by setting $\mathrm{sign}(\epsilon) = 1$ for the retarded one and to -1 for the advanced one. It is also easy to verify that the Keldysh entry of equation (4.45) fulfills the relation $G^K(\epsilon,p) = F(\epsilon)(G^R(\epsilon,p) - G^A(\epsilon,p))$. There remains the integration over ϵ_p. For the retarded component we find[9]

$$\frac{i}{\pi}\int_{-\epsilon_F}^{\infty} d\epsilon_p \, \frac{1}{\epsilon - \epsilon_p + \frac{i}{2\tau}} = 1 \tag{4.47}$$

and for the advanced component we get in the same way -1. For the Keldysh component we note that $1/\tau$ is very small compared to ϵ_F (remember the assumption $\epsilon_F\tau \gg 1$) which justifies

$$\frac{i}{\pi}\int_{-\epsilon_F}^{\infty} d\epsilon_p \, \frac{iF(\epsilon)/\tau}{(\epsilon-\epsilon_p)^2+1/4\tau^2} = \frac{i}{\pi}\int_{-\epsilon_F}^{\infty} d\epsilon_p \, 2\pi F(\epsilon)\delta(\epsilon-\epsilon_p) = 2F(\epsilon) \tag{4.48}$$

Thus we proved that the right hand side and the left hand side of equation (4.43) are identical and that the space independent saddle point solution for Q is given by the matrices in equation (4.42).

Note that Λ solves equation (4.43) for any function $F(\epsilon)$. This is understandable since any distribution function is allowed for noninteracting electrons. In [17] it is discussed how interaction effects drive the system towards the equilibrium distribution.

The entries of the matrices in equation (4.42) might look familiar to persons who already worked with the quasiclassical Green's function g. Λ is nothing else but the quasiclassical Green's function for non–interacting electrons in a weakly disordered system. A possible derivation of the quasiclassical Green's function is given in [42], [43] and [44].

As already mentioned earlier it suffices for our purposes to solve the path integral over $\tilde{Q}$ in saddle point approximation. However let us shortly comment on the corrections to the saddle point solution.

[9] The real part vanishes if one interprets the integral as principal value integral.

Fluctuations of the Q-fields

At the saddle point the next order correction for the action is quadratic.

$$iS[\tilde{Q}] = iS[\tilde{Q}^{SP}] + iS_2[\tilde{Q}, \delta\tilde{Q}] + \mathcal{O}\big(\delta\tilde{Q}^3\big) \tag{4.49}$$

The terms proportional to $\delta\tilde{Q}^2$ are

$$iS_2[\tilde{Q}, \delta\tilde{Q}] = -\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\big\{\delta\tilde{Q}\big\} + \frac{1}{8\tau^2}\mathrm{Tr}\big[\tilde{G}(x,y)\delta\tilde{Q}(y,y')\tilde{G}(y',y'')\delta\tilde{Q}(y'',x')\big]. \tag{4.50}$$

These fluctuations lead to a renormalization of the diffusion constant (see chapter 5 of [36])

$$D = D_0\left(1 - \frac{1}{\pi\nu_d}\int \frac{d^dq}{(2\pi)^d}\frac{1}{D_0q^2 - i\omega}\right) \tag{4.51}$$

where d indicates the quasi dimension. The correction term leads to a reduction of the diffusion constant and hence of the conductivity. This is the so called **weak localization**.[10] Additionally taking into account a magnetic field one can suppress this effect as one destroys the interference of time reversed closed paths that lead to the localization of the wavefunction. Experimentally one finds a decrease in the magneto resistance while increasing the magnetic field. Already very low magnetic fields suppress the effect of weak localization. More details about weak localization can be found in [46], [47] and [48].

[10] We are interested here in good metals far from localization where the weak localization correction is small.

Chapter 5

Interacting Electrons

In this chapter we want to discuss the treatment of electron–electron interactions in the coherent state path integral formalism.[1] We take into account the spin of the electrons but neglect possible disorder. The combined problem of interactions and disorder will be discussed in the next chapter.

Let us start quite generally with

$$S_{\text{int}}[\psi,\psi^*] = -\frac{1}{2}\sum_{s_1,s_2}\int dx_1 ... \int dx_4\; V(x_1-x_2, x_3-x_2, x_4-x_1)\; \psi^*_{s_1}(x_1)\psi^*_{s_2}(x_2)\psi_{s_2}(x_3)\psi_{s_1}(x_4). \tag{5.1}$$

S_{int} describes a general two–particle interaction that is translationally invariant in space and time [2] where V is the interaction potential and the indices s_1 and s_2 at the Grassmann fields denote the spin variables. For our purposes it is sufficient to select two areas in phase space in order to describe the low energy physics (close to the Fermi surface) of our problem. These two different kinds of scattering events which are sometimes referred to as small angle and large angle scattering of a particle hole pair are depicted in figure (5.1). The white spots symbolize the holes whereas the black spots represent the electrons.

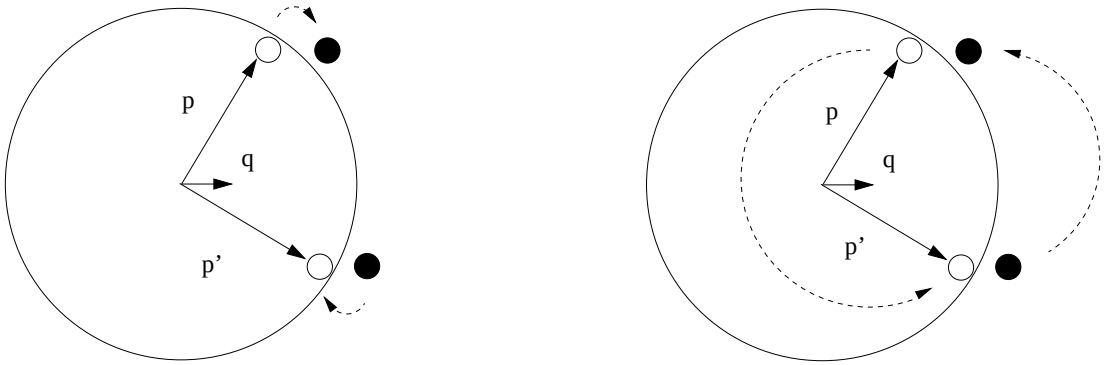

Figure 5.1: Small angle and large angle scattering shown in k-space
(the big circle symbolizes the Fermi sphere)

[1] See [49], [50].

[2] Remember $x = (r, t)$.

There exists a third low energy process which is called $2k_F$–scattering of a particle–hole pair. It is common to assume the presence of a small magnetic field which introduces a 'mass term' [3] that kills the contribution of the so called Cooper channel. Hence we just assume the presence of a small magnetic field and restrict ourselves to small and large angle scattering. In momentum space one has

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\sum_{s_1,s_2,s_3,s_4}\sum_{p,p'}\sum_{q({\rm small})}\Big(V_1\ \delta_{s_1s_4}\ \delta_{s_2s_3}\ \psi^*_{s_1}(p)\psi^*_{s_2}(p'+q)\psi_{s_3}(p')\psi_{s_4}(p+q) + V_2\ \delta_{s_1s_3}\ \delta_{s_2s_4}\ \psi^*_{s_1}(p)\psi^*_{s_2}(p'+q)\psi_{s_4}(p+q)\psi_{s_3}(p')\Big). \tag{5.2}$$

One can assume that for the range of momenta, we are considering, the interaction potential V can be approximated by two different constants V_1 and V_2. The sum over $\mathbf{q}$ is only over **small** momenta. This ensures that the two processes cover different regions in phase space. The processes are drawn as Feynman diagrams in figure (5.2).

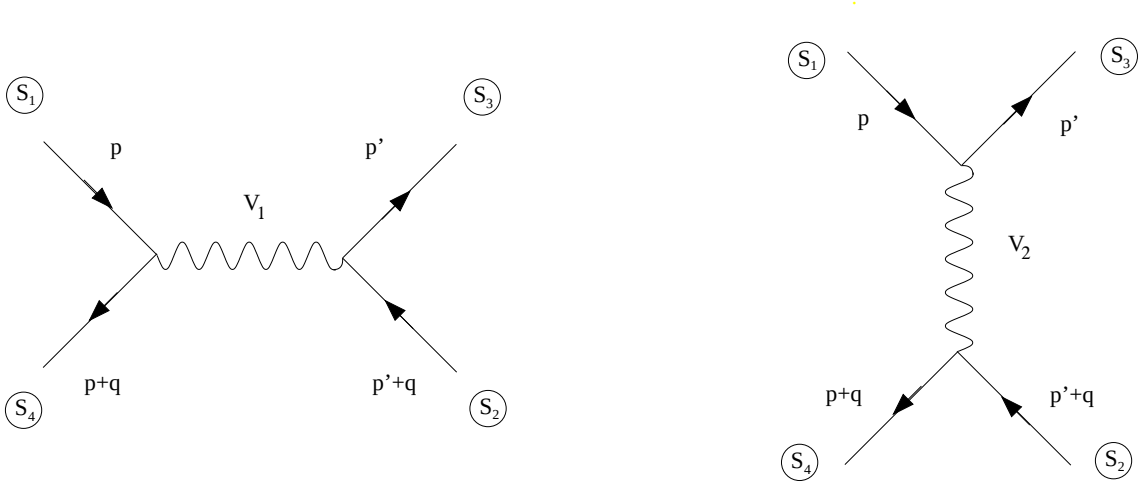

Figure 5.2: Small angle and large angle scattering shown as Feynman diagrams

This picture is identical to the one we have drawn in section (2.2.3) where we discussed the two lowest order diagrams that appear in perturbation theory. Hence we know what is behind V_1 and V_2 (see equation (2.37)). Setting up an action S with these elementary processes which appear in the exponential of the generating function $Z[J,\bar{J}]$ is thus an attempt to take into account these processes in infinite order.

As in the case of disorder (see previous chapter) we encounter four Fermion terms which are disturbing as the action is no longer quadratic in the Grassmann fields. Also here we introduce auxiliary fields in order to cure the problem. Let us turn to coordinate space as we want to determine the form of $S_{\rm int}$ on the Keldysh contour.[4]

[3] A discussion can be found in [29].

[4] Constant interactions in momentum space turn into point like interactions in coordinate space.

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\sum_{s_1..s_4}\int dx\int dx'\Big(V_1\ \delta_{s_1s_4}\ \delta_{s_2s_3}\delta(x-x')\ \psi^*_{s_1}(x)\psi^*_{s_2}(x')\psi_{s_3}(x')\psi_{s_4}(x)$$
$$+ V_2\ \delta_{s_1s_3}\ \delta_{s_2s_4}\delta(x-x')\ \psi^*_{s_1}(x)\psi^*_{s_2}(x')\psi_{s_4}(x)\psi_{s_3}(x')\Big) \tag{5.3}$$

which reads after interchanging the last two Grassmann variables in the second term (remember $\psi_a\psi_b = -\psi_b\psi_a$) and performing the x' integral:

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\sum_{s_1...s_4}\int dx\Big(V_1\ \delta_{s_1s_4}\ \delta_{s_2s_3} - V_2\ \delta_{s_1s_3}\ \delta_{s_2s_4}\Big)\ \psi^*_{s_1}(x)\psi^*_{s_2}(x)\psi_{s_3}(x)\psi_{s_4}(x) \tag{5.4}$$

Next we use the identity

$$\delta_{s_1s_3}\ \delta_{s_2s_4} = \frac{1}{2}\ \delta_{s_1s_4}\ \delta_{s_2s_3} + \frac{1}{2}\ \sigma_{s_1s_4}\ \sigma_{s_2s_3} \tag{5.5}$$

which allows us to write the interaction in the form

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\sum_{s_1..s_4}\int dx\Big(\Big[V_1 - \frac{V_2}{2}\Big]\delta_{s_1s_4}\delta_{s_2s_3} + \Big[-\frac{V_2}{2}\Big]\sigma_{s_1s_4}\sigma_{s_2s_3}\Big)\ \psi^*_{s_1}(x)\psi^*_{s_2}(x)\psi_{s_3}(x)\psi_{s_4}(x). \tag{5.6}$$

We introduce the singlet and triplet amplitudes[5]

$$V_s = V_1 - \frac{V_2}{2} \qquad \text{and} \qquad V_t = -\frac{V_2}{2} \tag{5.7}$$

and rearrange the ψ's forming **charge density** $\rho(x)$ and **spin density** $\mathbf{s}^n(x)$

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\int dx\Big(\Big[\sum_{s_1}\psi^*_{s_1}(x)\psi_{s_1}(x)\Big]\ V_s\ \Big[\sum_{s_2}\psi^*_{s_2}(x)\psi_{s_2}(x)\Big]$$
$$+\Big[\sum_{s_1,s_4}\psi^*_{s1}(x)\sigma_{s_1s_4}\psi_{s_4}(x)\Big]\ V_t\ \Big[\sum_{s_2,s_3}\psi^*_{s_2}(x)\sigma_{s_2s_3}\psi_{s_3}(x)\Big]\Big) \tag{5.8}$$

where we used $\psi^*_a\psi_b = -\psi_b\psi^*_a$. We define

$$\rho(x) = \sum_s \psi^*_s(x)\psi_s(x) \qquad \mathbf{s}^n(x) = \sum_{s_1,s_2}\psi^*_{s_1}(x)\sigma^n_{s_1s_2}\psi_{s_2}(x) \qquad \text{for } n = x,y,z. \tag{5.9}$$

Then the interaction term reads

[5] In the end we will go over to the more general amplitudes Γ_s and Γ_t which we introduced in section (2.2.4). Here however we stick to the notation with V's as it is more common.

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\int dx\Big(\rho(x)\ V_s\ \rho(x) + \vec{\mathbf{s}}(x)\ V_t\ \vec{\mathbf{s}}(x)\Big). \tag{5.10}$$

So far nothing has happened apart from writing the action using densities. As next step we rewrite the four Fermion(=two density) terms in the equation above using Hubbard–Stratonovich–Transformations. This brings the fermionic problem to a Gaussian form at the price of the complication caused by the appearance of some new fields.

5.1 Introduction of the fields ϕ and $\vec{B}$

Let us begin with the introduction of a scalar field $\phi(x)$ which couples to the charge density $\rho(x)$.

$$\exp\left[-\frac{1}{2}\rho\ V_s\ \rho\right] = \frac{1}{N_\phi}\int \mathcal{D}\phi\ \exp\left[-\frac{e^2}{2}\phi\ V_s^{-1}\ \phi + ie\phi\ \rho\right] \tag{5.11}$$

where e is the elementary charge and the normalization constant

$$N_\phi = \int \mathcal{D}\phi\ \exp\left[-\frac{e^2}{2}\phi\ V_s^{-1}\ \phi\right]. \tag{5.12}$$

We used the symbolic expressions

$$\phi\ V_s^{-1}\ \phi = \int dx \int dx' \phi(x')\ V_s^{-1}(x,x')\ \phi(x) \tag{5.13}$$

$$\phi\ \rho = \int dx\ \phi(x)\ \rho(x) \tag{5.14}$$

In this notation the inverse function is understood as an integral kernel inverse to the corresponding bare singlet interaction.

$$\int dx_1 V_s^{-1}(x,x_1)V_s(x_1,x') = \delta(x-x') \tag{5.15}$$

In our case $V_s(x_1,x')$ was just $V_s\ \delta(x_1-x')$. In the following we will write $V_s(x,x')$. On the Keldysh contour equation (5.10) with the generalized interactions reads:

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\int_{\mathcal{C}} dt \int dr \int_{\mathcal{C}} dt' \int dr' \Big(\rho(r,t)\ V_s(r,t,r',t')\ \rho(r',t') + \vec{\mathbf{s}}(r,t)\ V_t(r,t,r',t')\ \vec{\mathbf{s}}(r',t')\Big) \tag{5.16}$$

Like for ψ's we introduce a doublet notation for the densities:

$$\rho = \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} \qquad \text{and} \qquad \mathbf{s}^n = \begin{pmatrix}\mathbf{s}_1^n\\ \mathbf{s}_2^n\end{pmatrix} \tag{5.17}$$

for $n = x,y,z$. ρ_1 and $\mathbf{s}_1^n$ are on the upper branch and ρ_2 and $\mathbf{s}_2^n$ are on the lower branch.

$$S_{\rm int}[\psi,\psi^*] = -\frac{1}{2}\int dx \int dx' \Big(\rho^T(x)\ \left[\sigma_z \breve{V}_s(x,x')\sigma_z\right]\ \rho(x') + \sum_n (\mathbf{s}^n)^T(x)\ \left[\sigma_z \breve{V}_t(x,x')\sigma_z\right]\ \mathbf{s}^n(x')\Big) \tag{5.18}$$

The σ_z matrices on both sides of $\breve{V}_s(x,x')$ and $\breve{V}_t(x,x')$ are due to the signs that we acquire by splitting up the Keldysh contour. $\breve{V}_s$ is the matrix

$$\breve{V}_s \;=\; \begin{pmatrix} V^T(x,x') & V^<(x,x') \\ V^>(x,x') & V^{\bar{T}}(x,x') \end{pmatrix} \tag{5.19}$$

where entries are defined by the following expectation values

$$V^>(x,x') \;=\; -ie^2\big\langle \phi_1(x)\phi_2(x')\big\rangle_\phi \tag{5.20}$$

$$V^<(x,x') \;=\; -ie^2\big\langle \phi_2(x')\phi_1(x)\big\rangle_\phi \tag{5.21}$$

$$V^T(x,x') \;=\; \Theta(t-t')V^>(x,x') + \Theta(t'-t)V^<(x,x') \tag{5.22}$$

$$V^{\bar{T}}(x,x') \;=\; \Theta(t-t')V^<(x,x') + \Theta(t'-t)V^>(x,x') \tag{5.23}$$

Compare (5.19) to equation (3.50) for the electron Green's function. The reasoning is analogous to the one presented in section (3.5) hence we will not repeat it here. Of course we encounter the same time structure for the triplet interaction

$$\breve{V}_t \;=\; \begin{pmatrix} V_t^T(x,x') & V_t^<(x,x') \\ V_t^>(x,x') & V_t^{\bar{T}}(x,x') \end{pmatrix} \tag{5.24}$$

with

$$V_t^>(x,x') \;=\; -i\mu_B^2\big\langle B_1^n(x)B_2^n(x')\big\rangle_{B^n} \tag{5.25}$$

$$V_t^<(x,x') \;=\; -i\mu_B^2\big\langle B_2^n(x')B_1^n(x)\big\rangle_{B^n} \tag{5.26}$$

$$V_t^T(x,x') \;=\; \Theta(t-t')V_t^>(x,x') + \Theta(t'-t)V_t^<(x,x') \tag{5.27}$$

$$V_t^{\bar{T}}(x,x') \;=\; \Theta(t-t')V_t^<(x,x') + \Theta(t'-t)V_t^>(x,x') \tag{5.28}$$

for $n = x, y, z$. μ_B is the Bohr magneton.

Remark that we have four different angled brackets here: $\langle ... \rangle_\phi$ and $\langle ... \rangle_{B^n}$ where $n = x, y, z$. They belong to the four new path integrals that we obtain by introducing the auxiliary fields ϕ and $\vec{B} = (B^x, B^y, B^z)^T$ (compare to equation (4.3) of the disorder chapter). We denote these auxiliary fields by

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \text{and} \qquad B^n = \begin{pmatrix} B_1^n \\ B_2^n \end{pmatrix} \tag{5.29}$$

The fields ϕ_α and B_α^n are real. For the singlet part we have then

$$e^{iS_{\rm int}[\Phi]} = \exp\Big[-\frac{i}{2}\int dx \int dx' \big[\rho^T(x)\sigma_z\big]\ \breve{V}_s(x,x')\ \big[\sigma_z\rho(x')\big]\Big]$$

$$= \frac{1}{\mathcal{N}_\phi}\int \mathcal{D}\Phi\ \exp\Big[\frac{ie^2}{2}\int dx\ \Phi^T(x)\big[\sigma_z V_s^{-1}(x)\big]\Phi(x) + ie\ \big[\rho^T(x)\sigma_z\big]\Phi(x)\Big] \tag{5.30}$$

with a normalization constant $\mathcal{N}_\phi = \mathrm{Det}[i\sigma_z V_s^{-1}]^{1/2}$. $\big[\sigma_z V_s^{-1}(x)\big]$ is the inverse operator of $\breve{V}_s$. For the triplet part we have

$$e^{iS_{\rm int}[\vec{B}]} = \exp\Big[-\frac{i}{2}\sum_n\int dx \int dx' \big[(\mathbf{s}^n)^T(x)\sigma_z\big]\ \breve{V}_t(x,x')\ \big[\sigma_z\mathbf{s}^n(x')\big]\Big]$$

$$= \frac{1}{\mathcal{N}_B}\int \mathcal{D}\vec{B}\ \exp\Big[\frac{i\mu_B^2}{2}\sum_n\int dx\ (B^n)^T(x)\big[\sigma_z V_t^{-1}(x)\big]B^n(x) + i\mu_B\ \big[(\mathbf{s}^n)^T(x)\sigma_z\big]B^n(x)\Big] \tag{5.31}$$

where $\mathcal{D}\vec{B} = \int \mathcal{D}B^x\ \mathcal{D}B^y\ \mathcal{D}B^z$ and $\mathcal{N}_B = \mathrm{Det}[i\sigma_z V_t^{-1}]^{3/2}$. It is convenient to translate back the charge and spin density operators ρ and $\mathbf{s}^n$ according to equation (5.9) into products of ψ's and ψ^*'s since we want to add the action for the free electrons in order to have the complete action describing an interacting electron gas. Therefore we write

$$\begin{aligned}\rho^T(x)\sigma_z\Phi(x) &= \psi_1(x)^*\phi_1(x)\psi_1(x) - \psi_2(x)^*\phi_2(x)\psi_2(x)\\ &= \bar{\Psi}(x)\begin{pmatrix}\phi_1(x) & 0\\ 0 & \phi_2(x)\end{pmatrix}\Psi(x)\\ &= \bar{\Psi}(x)\ \phi_\alpha(x)\gamma^\alpha\ \Psi(x)\end{aligned} \tag{5.32}$$

where [6]

$$\gamma_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \text{and} \qquad \gamma_2 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{5.33}$$

The minus sign coming from σ_z is absorbed in $\bar{\Psi}$. In the same way

$$(\mathbf{s}^n)^T(x)\sigma_z B^n(x) = \sum_{s_1,s_2}\bar{\Psi}_{s_1}(x)\ \sigma^n_{s_1s_2}\ B^n_\alpha\gamma^\alpha\ \Psi_{s_2}(x). \tag{5.34}$$

Let us now finally add the action of the free electrons plus the source field terms (see equation (3.46)).

[6] We use Einstein sum convention $\phi_\alpha\gamma^\alpha = \sum_\alpha \phi_\alpha\gamma^\alpha$.

5.2 Complete action and generating functional

In equation (3.46) we have the action of spinless electrons. The generalization in order to include the spin degree of freedom is straightforward. We simply double the space from $\psi \to \psi_\uparrow, \psi_\downarrow$ and $\psi^* \to \psi^*_\uparrow, \psi^*_\downarrow$.

$$\Psi = \begin{pmatrix} \psi_{1,\uparrow} \\ \psi_{2,\uparrow} \\ \psi_{1,\downarrow} \\ \psi_{2,\downarrow} \end{pmatrix} \qquad \text{and} \qquad \hat{\Psi} = \begin{pmatrix} \psi^*_{1,\uparrow} \\ -\psi^*_{2,\uparrow} \\ \psi^*_{1,\downarrow} \\ -\psi^*_{2,\downarrow} \end{pmatrix}^T \tag{5.35}$$

and the sources

$$J = \begin{pmatrix} J_{1,\uparrow} \\ J_{2,\uparrow} \\ J_{1,\downarrow} \\ J_{2,\downarrow} \end{pmatrix} \qquad \text{and} \qquad \hat{J} = \begin{pmatrix} J^*_{1,\uparrow} \\ -J^*_{2,\uparrow} \\ J^*_{1,\downarrow} \\ -J^*_{2,\downarrow} \end{pmatrix}^T \tag{5.36}$$

The action for free electrons then reads

$$S[\Psi, \hat{\Psi}, J, \hat{J}] = \int dx\ \hat{\Psi}(x)\ \hat{G}_0^{-1}\Psi(x) + \int dx\ \hat{J}(x)\Psi(x) + \hat{\Psi}(x)\ J(x) \tag{5.37}$$

where $\hat{G}_0^{-1} = G_0^{-1}\mathbb{I}_4$ and $\mathbb{I}_4$ is the unity matrix in the four dimensional vector space. From now on we will always indicate which unity matrix is meant by either writing $\mathbb{I}_2$ or $\mathbb{I}_4$. The lower index indicates the dimensionality of the vector space. A quantity O in the spin $\times$ Keldysh–space (dimension $2 \times 2 = 4$) is marked in general by a hat: $\hat{O}$. Although we already used the symbols Ψ and J in equation (3.44) and (3.45) we believe that they can not be confused with ones in equation (5.35) and (5.36), as Ψ and J appear always combined with quantities having a 'hat' (including spin) or a 'bar' (neglecting spin)[7].

Combining (5.30), (5.31) and (5.37), we obtain the complete action for our interacting electron gas at low energies.

$$\begin{aligned} S[\Psi, \hat{\Psi}, \Phi, \vec{B}] = &\int dx\ \hat{\Psi}(x)\left[\left(i\frac{\partial}{\partial t} + \frac{\Delta}{2m}\right)\ \mathbb{I}_4 + \mathbb{I}_2 \otimes e\phi_\alpha(x)\gamma^\alpha + \vec{\sigma} \otimes \mu_B \vec{B}_\alpha(x)\gamma^\alpha\right]\Psi(x) \\ &+ \frac{e^2}{2}\int dx\ \Phi^T(x)\ \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{\mu_B^2}{2}\int dx \sum_n (B^n)^T(x)\ \sigma_z V_{0,t}^{-1}(x) B^n(x) \\ &+ \int dx\ \left(\hat{J}(x)\Psi(x) + \hat{\Psi} J(x)\right) \end{aligned} \tag{5.38}$$

We changed the notation for V_s^{-1} and V_t^{-1} to $V_{0,s}^{-1}$ and $V_{0,t}^{-1}$ meaning that these operators are the inverse of the respective static short range interactions V_s and V_t. Later, when we will take

[7] $\hat{J}$ or $\bar{J}$ for example

into account dynamical screening we will use the notation V_s^{-1} and V_t^{-1} for the inverse of the dynamically screened interactions $V_s(q,\omega)$ and $V_t(q,\omega)$.

The first 2×2 matrix gives the structure in spin space whereas the second 2×2 matrix determines the structure in Keldysh space and $\otimes$ indicates the direct product. The linear term in the fields explicitly reads

$$F_{\phi,\vec{B}} = \begin{pmatrix} \phi_1 + B_1^z & 0 & B_1^x - iB_1^y & 0 \\ 0 & \phi_2 + B_2^z & 0 & B_2^x - iB_2^y \\ B_1^x + iB_1^y & 0 & \phi_1 - B_1^z & 0 \\ 0 & B_2^x + iB_2^y & & \phi_2 - B_2^z \end{pmatrix} \tag{5.39}$$

where we set $\boldsymbol{e = \mu_B = 1}$. We will use this convention throughout the whole work and only explicitly write e or μ_B in final results. The generating functional is

$$Z[J,\hat{J}] = \int \mathcal{D}\Psi\mathcal{D}\hat{\Psi} \int \mathcal{D}\Phi \int \mathcal{D}\vec{B}\ \exp\Big(iS[\Psi,\hat{\Psi},J,\hat{J},\Phi,\vec{B}]\Big). \tag{5.40}$$

The action S is quadratic in the Fermion fields ψ and ψ^* at the price of eight new path integrals over the auxiliary fields ϕ_i, B_i^x, B_i^y and B_i^z (i=1, 2). Before integrating out the electrons exactly we apply the Keldysh rotation L.

Keldysh rotation

The Keldysh rotation matrix L was a 2×2 matrix. In our treatment including spin we encounter however also 4×4 matrices. As we only want to rotate in Keldysh space we add a unity matrix $\mathbb{I}_2$ in spin space

$$L_s = \mathbb{I}_2 \otimes L. \tag{5.41}$$

We begin with the rotation of the coupling term

$$\int dx\ \hat{\Psi}(x)\Big[\mathbb{I}_2 \otimes \phi_\alpha(x)\gamma^\alpha + \vec{\sigma}\otimes\vec{B}_\alpha(x)\gamma^\alpha\Big]\Psi(x). \tag{5.42}$$

Introducing $L_s\ L_s^{-1} = \mathbb{I}_4$ and setting $L_s\Psi = \tilde{\Psi}$ and $\hat{\Psi}L_s^{-1} = \tilde{\hat{\Psi}}$ we obtain[8]

$$\int dx\ \tilde{\hat{\Psi}}(x)\Big[\mathbb{I}_2 \otimes \phi_\alpha(x)L\gamma^\alpha L^{-1} + \vec{\sigma}\otimes\vec{B}_\alpha(x)L\gamma^\alpha L^{-1}\Big]\tilde{\Psi}(x) \tag{5.43}$$

where we used

$$L_s(M\otimes\gamma^\alpha)L_s^{-1} = \mathbb{I}_2\otimes L\ (M\otimes\gamma^\alpha)\ \mathbb{I}_2\otimes L^{-1} = M\otimes L\gamma^\alpha L^{-1}. \tag{5.44}$$

M could be an arbitrary 2×2 matrix. By construction the Keldysh rotation has no effect on the spin space structure. Thus we can focus on the Keldysh structure. Now

$$L\gamma^1 L^{-1} \ = \ \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ = \frac{1}{2}\Big[\mathbb{I}_2 + \sigma_x\Big] \tag{5.45}$$

and

$$L\gamma^2 L^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\Big[\mathbb{I}_2 - \sigma_x\Big]. \tag{5.46}$$

[8]The convention is again that quantities having a tilde are in the rotated representation.

Thus we have

$$\phi_\alpha \, L\gamma^\alpha L^{-1} = \frac{1}{2}\begin{pmatrix}\phi_1+\phi_2 & \phi_1-\phi_2\\ \phi_1-\phi_2 & \phi_1+\phi_2\end{pmatrix}. \tag{5.47}$$

We define the **rotated fields**[9]

$$\tilde{\Phi} = \begin{pmatrix}\tilde{\phi}_1\\ \tilde{\phi}_2\end{pmatrix} = L\Phi = \frac{1}{2}\begin{pmatrix}\phi_1+\phi_2\\ \phi_1-\phi_2\end{pmatrix}. \tag{5.48}$$

and the **rotated matrices**

$$\tilde{\gamma}_1 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad \tilde{\gamma}_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{5.49}$$

For the components of $\vec{B}$ we can do the same

$$\tilde{B}^n = \begin{pmatrix}\tilde{B}^n_1\\ \tilde{B}^n_2\end{pmatrix} = LB^n = \frac{1}{2}\begin{pmatrix}B^n_1+B^n_2\\ B^n_1-B^n_2\end{pmatrix}. \tag{5.50}$$

In this way the coupling term keeps its form

$$\int dx \; \tilde{\bar{\Psi}}(x)\Big[\mathbb{I}_2 \otimes \tilde{\phi}_\alpha(x)\tilde{\gamma}^\alpha + \vec{\sigma}\otimes\vec{\tilde{B}}_\alpha(x)\tilde{\gamma}^\alpha\Big]\tilde{\Psi}(x). \tag{5.51}$$

The interaction matrices $\hat{V}_s$ and $\hat{V}_t$ in the rotated representation read

$$\tilde{V} = L\,\hat{V}_i\,L^{-1} = \begin{pmatrix}V_i^K & V_i^R\\ V_i^A & 0\end{pmatrix} \qquad \text{for} \qquad i = s,t \tag{5.52}$$

where

$$V^R(x,x') = V^T(x,x') - V^<(x,x') = V^>(x,x') - V^{\bar{T}}(x,x') \tag{5.53}$$

$$V^A(x,x') = V^T(x,x') - V^>(x,x') = V^<(x,x') - V^{\bar{T}}(x,x') \tag{5.54}$$

$$V^K(x,x') = V^T(x,x') + V^{\bar{T}}(x,x') = V^>(x,x') + V^<(x,x') \tag{5.55}$$

and $V^K(q,\omega) = B(\omega)[V^R(q,\omega) - V^A(q,\omega)]$. For the expressions that are quadratic in the fields we remark that

$$L\,\sigma_z V^{-1}(x)\,L^{-1} = \sigma_x V^{-1}(x). \tag{5.56}$$

In summary we obtain for the action in the rotated representation:

$$\begin{aligned} S[\tilde{\Psi},\tilde{\bar{\Psi}},\tilde{\Phi},\vec{\tilde{B}}] = &\int dx\; \tilde{\bar{\Psi}}(x)\,\tilde{\hat{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}]\;\tilde{\Psi}(x) + \int dx\;\Big(\tilde{\bar{J}}(x)\tilde{\Psi}(x) + \tilde{\bar{\Psi}}\tilde{J}(x)\Big)\\ &+ \int dx\;\tilde{\Phi}^T(x)\;\sigma_x V^{-1}_{0,s}(x)\tilde{\Phi}(x) + \int dx \sum_n \tilde{B}^n(x)\;\sigma_x V^{-1}_{0,t}(x)\tilde{B}^n(x). \end{aligned} \tag{5.57}$$

[9] Note that our definition is different from [19] and follows the convention of [17].

where

$$\hat{\tilde{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}] = \left[\left(i\frac{\partial}{\partial t}+\frac{\Delta}{2m}\right)\ \mathbb{I}_4 + \mathbb{I}_2\otimes\tilde{\phi}_\alpha(x)\tilde{\gamma}^\alpha + \vec{\sigma}\otimes\vec{\tilde{B}}_\alpha(x)\tilde{\gamma}^\alpha\right] \tag{5.58}$$

Remember our convention for the notation. The tilde indicates that we are in the Keldysh rotated representation and the additional hat indicates that we are dealing with a quantity of the four dimensional vector space.

Integrating out the Fermions

After the introduction of Φ and $\vec{B}$ we can calculate the path integral over the Fermions exactly because the action is Gaussian in Ψ and $\hat{\Psi}$.

The generating functional in the rotated representation reads

$$Z[\tilde{J},\hat{\tilde{J}}] = \int\mathcal{D}\tilde{\Psi}\mathcal{D}\hat{\tilde{\Psi}}\int\mathcal{D}\tilde{\Phi}\int\mathcal{D}\vec{\tilde{B}}\ \exp\left(iS[\tilde{\Psi},\hat{\tilde{\Psi}},\tilde{J},\hat{\tilde{J}},\tilde{\Phi},\vec{\tilde{B}}]\right) \tag{5.59}$$

where the action is given in equation (5.57). Integrating out the Fermions yields

$$\begin{aligned} Z[\tilde{J},\hat{\tilde{J}}] = \int\mathcal{D}\tilde{\Phi}\int\mathcal{D}\vec{\tilde{B}}\ \mathrm{Det}\big[\hat{\tilde{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}]\big]\ \exp\Big(i\int dx\int dx'\hat{\tilde{J}}(x)\ \hat{\tilde{G}}[\tilde{\Phi},\vec{\tilde{B}}](x,x')\tilde{J}(x') \\ + i\int dx\ \tilde{\Phi}^T(x)\ \sigma_x V_{0,s}^{-1}(x)\tilde{\Phi}(x) + i\int dx\sum_n(\tilde{B}^n)^T(x)\ \sigma_x V_{0,t}^{-1}(x)\tilde{B}^n(x)\Big) \end{aligned} \tag{5.60}$$

where

$$\hat{\tilde{G}}[\tilde{\Phi},\vec{\tilde{B}}] = \left[\left(i\frac{\partial}{\partial t}+\frac{\Delta}{2m}\right)\ \mathbb{I}_4 + \mathbb{I}_2\otimes\tilde{\phi}_\alpha(x)\tilde{\gamma}^\alpha + \vec{\sigma}\otimes\vec{\tilde{B}}_\alpha(x)\tilde{\gamma}^\alpha\right]^{-1}. \tag{5.61}$$

Using the identity $\mathrm{Det}(A) = \exp(\mathrm{Tr}\ln(A))$ we can write the function under the path integrals as pure exponential

$$Z[\tilde{J},\hat{\tilde{J}}] = \int\mathcal{D}\tilde{\Phi}\int\mathcal{D}\vec{\tilde{B}}\ \exp\left(iS[\tilde{J},\hat{\tilde{J}},\tilde{\Phi},\vec{\tilde{B}}]\right). \tag{5.62}$$

where the effective action is

$$\begin{aligned} S[\tilde{J},\hat{\tilde{J}},\tilde{\Phi},\vec{\tilde{B}}] = -i\mathrm{Tr}\ln\left[\hat{\tilde{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}]\right] + \int dx\int dx'\hat{\tilde{J}}(x)\ \hat{\tilde{G}}[\tilde{\Phi},\vec{\tilde{B}}](x,x')\tilde{J}(x') \\ + \int dx\ \tilde{\Phi}^T(x)\ \sigma_x V_{0,s}^{-1}(x)\tilde{\Phi}(x) + \int dx\sum_n(\tilde{B}^n)^T(x)\ \sigma_x V_{0,t}^{-1}(x)\tilde{B}^n(x). \end{aligned} \tag{5.63}$$

Again like in the previous chapter where we discussed the treatment of disorder we determine the saddle point of the action. However in order to describe Coulomb Blockade effects it does not suffice to evaluate the path integrals over the auxiliary fields Φ and $\vec{B}$ by taking their value at the saddle point. In addition we have to take into account the field fluctuations (quadratic corrections) around the saddle point.

5.3 Determination of the saddle point

We will determine the saddle point in the **rotated representation**. The saddle point is defined by

$$\frac{\delta}{\delta\tilde{\phi}_\alpha(x)}S[\tilde{\Phi},\vec{\tilde{B}},\tilde{J}=0]=0 \qquad \text{and} \qquad \frac{\delta}{\delta\tilde{B}^n_\alpha(x)}S[\tilde{\Phi},\vec{\tilde{B}},\tilde{J}=0]=0. \tag{5.64}$$

In order to calculate the derivatives we write

$$\tilde{\phi}_\alpha=\tilde{\phi}^{SP}_\alpha+\delta\tilde{\phi}_\alpha \qquad\qquad \tilde{B}^n_\alpha=\big(\tilde{B}^{SP}_\beta\big)^n+\delta\tilde{B}^n_\alpha \tag{5.65}$$

$\big(\tilde{B}^{SP}_\beta\big)^n$ is the n–th component of the saddle point field $\tilde{B}^{SP}_\beta$. n is either x, y or z. The quadratic term in the field ϕ is

$$S_\phi=\frac{1}{2}\int dx\ (\tilde{\Phi}^{SP}+\delta\tilde{\Phi})(x)\big[\sigma_x\ V_0^{-1}(x)\big](\tilde{\Phi}^{SP}+\delta\tilde{\Phi})(x). \tag{5.66}$$

So the term proportional to $\delta\phi_\alpha(x)$ is

$$\frac{\delta S_\phi}{\delta\tilde{\phi}_\alpha(x)}=\Big[\big[\ \sigma_x\ V^{-1}_{0,s}(x)\big]\tilde{\Phi}^{SP}(x)\Big]^\alpha. \tag{5.67}$$

In the same way we obtain from the quadratic part S_B of the action in $\vec{\tilde{B}}$ for the different components

$$\frac{\delta S_B}{\delta\tilde{B}^n_\alpha(x)}=\Big[\big[\ \sigma_x\ V^{-1}_{0,t}(x)\big]\big(\tilde{B}^{SP}\big)^n(x)\Big]^\alpha. \tag{5.68}$$

The other term we have to look at is

$$-\,i\mathrm{Tr}\ln\big[\hat{\tilde{G}}^{-1}\big[\tilde{\Phi},\vec{\tilde{B}}\big]\big]. \tag{5.69}$$

The inverse Green's function reads:

$$\begin{aligned}\hat{\tilde{G}}[\tilde{\Phi},\vec{\tilde{B}}]^{-1}&=\Big[G_0^{-1}\ \mathbb{I}_4+\mathbb{I}_2\otimes\tilde{\phi}_\alpha(x)\tilde{\gamma}^\alpha+\vec{\sigma}\otimes\vec{\tilde{B}}_\alpha(x)\tilde{\gamma}^\alpha\Big]\\&=\Big[G_0^{-1}\ \mathbb{I}_4+\mathbb{I}_2\otimes\tilde{\phi}^{SP}_\alpha(x)\tilde{\gamma}^\alpha+\vec{\sigma}\otimes\vec{\tilde{B}}^{SP}_\alpha(x)\tilde{\gamma}^\alpha\Big]+\mathbb{I}_2\otimes\delta\tilde{\phi}_\alpha\tilde{\gamma}^\alpha+\vec{\sigma}\otimes\delta\tilde{B}_\alpha\tilde{\gamma}^\alpha\\&=\hat{\tilde{G}}[\tilde{\Phi}^{SP},\tilde{B}^{SP}]^{-1}(x)+\mathbb{I}_2\otimes\delta\tilde{\phi}_\alpha\tilde{\gamma}^\alpha+\vec{\sigma}\otimes\delta\tilde{B}_\alpha\tilde{\gamma}^\alpha.\end{aligned} \tag{5.70}$$

We get to first order in the fields:

$$\hat{\tilde{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}]=\hat{\tilde{G}}[\tilde{\Phi}^{SP},\tilde{B}^{SP}]^{-1}(x)\Big[\mathbb{I}_4+\int dx'\hat{\tilde{G}}^{SP}[\tilde{\Phi}^{SP},\tilde{B}^{SP}](x,x')\big[\mathbb{I}_2\otimes\delta\tilde{\phi}_\alpha(x')\tilde{\gamma}^\alpha+\vec{\sigma}\otimes\delta\vec{\tilde{B}}_\alpha(x')\tilde{\gamma}^\alpha\big]\Big]. \tag{5.71}$$

In the following let us abbreviate

$$\hat{\tilde{G}}[\tilde{\Phi}^{SP},\tilde{B}^{SP}]^{-1}(x)=(\hat{\tilde{G}}^{SP})^{-1}(x) \qquad \text{and} \qquad \hat{\tilde{G}}^{SP}[\tilde{\Phi}^{SP},\tilde{B}^{SP}](x,x')=\hat{\tilde{G}}^{SP}(x,x'). \tag{5.72}$$

Then we have

$$\ln \tilde{\hat{G}}^{-1}[\tilde{\Phi}, \vec{\tilde{B}}] \approx \left[\ln(\tilde{\hat{G}}^{SP})^{-1} + \int dx' \; \tilde{\hat{G}}^{SP}(x,x')\big[\mathbb{I}_2 \otimes \delta\tilde{\phi}_\alpha(x')\tilde{\gamma}^\alpha + \vec{\sigma} \otimes \delta\vec{\tilde{B}}_\alpha(x')\tilde{\gamma}^\alpha\big]\right] \tag{5.73}$$

where we used $\ln(AB) = \ln(A) + \ln(B)$ and $\ln(1+A) = A + \mathcal{O}\big(A^2\big)$. Hence the partial derivatives with respect to the fields are

$$\frac{\delta}{\delta\tilde{\phi}_\alpha(x)}\left(-i\mathrm{Tr}\ln\left[G[\tilde{\Phi}, \vec{\tilde{B}}]^{-1}\right]\right) = \mathrm{tr}\left[-i\tilde{G}^{SP}(x,x)\;\big[\mathbb{I}_2 \otimes \tilde{\gamma}^\alpha\big]\right] \tag{5.74}$$

and

$$\frac{\delta}{\delta\tilde{B}^n_\alpha(x)}\left(-i\mathrm{Tr}\ln\left[\tilde{\hat{G}}[\tilde{\Phi}, \vec{\tilde{B}}]^{-1}\right]\right) = \mathrm{tr}\left[-i\tilde{\hat{G}}^{SP}(x,x)\;\big[\sigma_n \otimes \tilde{\gamma}^\alpha\big]\right] \tag{5.75}$$

where $n = x, y, z$. We do not have to consider the term $\int dx \int dx' \tilde{\hat{J}}(x)\; \tilde{G}[\tilde{\Phi}, \vec{\tilde{B}}](x,x')\tilde{J}(x')$ of equation (5.63) as the functional derivatives are taken at $\tilde{J} = 0$. So we can already write down the saddle point equations for the fields

$$\boxed{\left[\big[\,\sigma_1\; V^{-1}_{0,s}(x)\big]\tilde{\Phi}^{SP}(x)\right]^\alpha = \frac{1}{e}\;\mathrm{tr}\left[i\tilde{\hat{G}}^{SP}(x,x)\;\big[\mathbb{I}_2 \otimes \gamma^\alpha\big]\right]} \tag{5.76}$$

and for the different components of the magnetic field

$$\boxed{\left[\big[\,\sigma_1\; V^{-1}_{0,t}(x)\big]\big(\tilde{B}^{SP}\big)^n(x)\right]^\alpha = \frac{1}{\mu_B}\;\mathrm{tr}\left[i\tilde{\hat{G}}^{SP}(x,x)\;\big[\sigma_n \otimes \gamma^\alpha\big]\right]} \tag{5.77}$$

We added the constants e and μ_B in theses final results. For the scalar field $\tilde{\phi}$ equation (5.76) is nothing else but the well known Poisson equation from electrodynamics. On the right side we have a sum of Green's functions taken at equal points in space which corresponds to a charge density.[10] In the ordinary Poisson equation one finds on the left side $\Delta/4\pi$ where Δ is the Laplace operator. Here however we assumed a special form of short range interaction $V^{-1}_{0,s}(x) = \frac{1}{V_s}$.

The charge density on the right side only appears because we neglected the positive charge background. In the Keldysh formalism one does not have to take it into account as the contributions on the upper and lower branch cancel each other.

For the components of the magnetic field we find a corresponding equation. Here we find on the right side a spin density and on the left side the inverse operator of the triplet interaction.

We will show in the following section that $\tilde{\Phi}^{SP} = 0$ and $\vec{\tilde{B}}^{SP} = 0$ is a self consistent solution of the saddle point equations.

[10] One might wonder if the unit is correct because we have $1/e$ instead of e in front of the density (see [19]). The answer is: The equation is correct due to the different unit of $V^{-1}_{0,s}$. Our saddle point equations can be directly compared to the ones in [42].

5.4 Solution of the saddle point equations

Like for the saddle point equation of Q we consider a translationally invariant system. We will show that

$$\boxed{\tilde{\Phi}^{SP} = 0 \qquad \text{and} \qquad \vec{\tilde{B}}^{SP} = 0} \tag{5.78}$$

is a solution of the saddle point equations. Let us begin with the right hand side of the equations (5.76) and (5.77). We denote the entries of the Green's function by

$$\tilde{\hat{G}}^{SP}(x,x') = \begin{pmatrix} G^R_{\uparrow\uparrow} & G^K_{\uparrow\uparrow} & G^R_{\uparrow\downarrow} & G^K_{\uparrow\downarrow} \\ 0 & G^A_{\uparrow\uparrow} & 0 & G^A_{\uparrow\downarrow} \\ G^R_{\downarrow\uparrow} & G^K_{\downarrow\uparrow} & G^R_{\downarrow\downarrow} & G^K_{\downarrow\downarrow} \\ 0 & G^A_{\downarrow\uparrow} & 0 & G^A_{\downarrow\downarrow} \end{pmatrix}. \tag{5.79}$$

We skip the ^{SP} for the entries otherwise the notation would be overloaded. One obtains for the four different traces that contain $\tilde{\gamma}^1$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\mathbb{I}_2 \otimes \tilde{\gamma}^1\right]\right] \ = \ \left[G^R_{\uparrow\uparrow} + G^A_{\uparrow\uparrow} + G^R_{\downarrow\downarrow} + G^A_{\downarrow\downarrow}\right] \tag{5.80}$$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_x \otimes \tilde{\gamma}^1\right]\right] \ = \ \left[G^R_{\uparrow\downarrow} + G^A_{\uparrow\downarrow} + G^R_{\downarrow\uparrow} + G^A_{\downarrow\uparrow}\right] \tag{5.81}$$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_y \otimes \tilde{\gamma}^1\right]\right] = i\left[(G^R_{\uparrow\downarrow} + G^A_{\uparrow\downarrow}) - (G^R_{\downarrow\uparrow} + G^A_{\downarrow\uparrow})\right] \tag{5.82}$$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_z \otimes \tilde{\gamma}^1\right]\right] = \ \left[(G^R_{\uparrow\uparrow} + G^A_{\uparrow\uparrow}) - (G^R_{\downarrow\downarrow} + G^A_{\downarrow\downarrow})\right] \tag{5.83}$$

and for the other four that contain $\tilde{\gamma}^2$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\mathbb{I}_2 \otimes \tilde{\gamma}^2\right]\right] = \ \left[G^K_{\uparrow\uparrow} + G^K_{\downarrow\downarrow}\right] \qquad \mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_x \otimes \tilde{\gamma}^2\right]\right] = \ \left[G^K_{\uparrow\downarrow} + G^K_{\downarrow\uparrow}\right]$$

$$\mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_y \otimes \tilde{\gamma}^2\right]\right] = i\left[G^K_{\uparrow\downarrow} - G^K_{\downarrow\uparrow}\right] \qquad \mathrm{tr}\left[\tilde{\hat{G}}^{SP}(x,x)\ \left[\sigma_z \otimes \tilde{\gamma}^2\right]\right] = \ \left[G^K_{\uparrow\uparrow} - G^K_{\downarrow\downarrow}\right]$$

There exists a notation which allows to write all this in a convenient form. Taking different linear combinations of the components we go over to the singlet/triplet representation.

$$G = G_0 \delta_{ss'} + \vec{G} \cdot \vec{\sigma}_{ss'} \tag{5.84}$$

where G_0 and $\vec{G}$ are defined as

$$G_0 := \frac{1}{2} \sum_s G_{ss} \qquad \vec{G} := \frac{1}{2} \sum_{ss'} \vec{\sigma}_{ss'} G_{s's}. \tag{5.85}$$

More explicitly

$$G_0 = \frac{1}{2}(G_{\uparrow\uparrow} + G_{\downarrow\downarrow}) \qquad \vec{G} = \frac{1}{2}\sum_{ss'} \begin{pmatrix} \sigma^x_{ss'} G_{s's} \\ \sigma^y_{ss'} G_{s's} \\ \sigma^z_{ss'} G_{s's} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} G_{\uparrow\downarrow} + G_{\downarrow\uparrow} \\ iG_{\uparrow\downarrow} - iG_{\downarrow\uparrow} \\ G_{\uparrow\uparrow} - G_{\downarrow\downarrow} \end{pmatrix} =: \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix}. \tag{5.86}$$

G_0 behaves like a scalar under rotations whereas $\vec{G}$ behaves like a coordinate vector. Comparing the eight equations above with our definitions in equation (5.86) we see that we exactly encounter these combinations. The left hand side of (5.76) and (5.77) is easily evaluated and we obtain

$$\tilde{\phi}_1^{SP}(x) = \frac{2iV_s}{e}\, G_0^{K,SP}(x,x) \tag{5.87}$$

and

$$\tilde{\phi}_2^{SP}(x) = \frac{2iV_s}{e}\, \left[G_0^{R,SP}(x,x) + G_0^{A,SP}(x,x)\right] \tag{5.88}$$

for the components of Φ and

$$\vec{\tilde{B}}_1^{SP}(x) = \frac{2iV_t}{\mu_B}\, \vec{G}^{K,SP}(x,x) \tag{5.89}$$

$$\vec{\tilde{B}}_2^{SP}(x) = \frac{2iV_t}{\mu_B}\, \left[\vec{G}^{R,SP}(x,x) + \vec{G}^{A,SP}(x,x)\right] \tag{5.90}$$

for the components of $\vec{B}$. Now we are at a stage where we can verify, that the solution of these saddle point equations is given by (5.78). For $\tilde{\Phi}^{SP} = 0, \vec{\tilde{B}}^{SP} = 0$ we know that our Green's function is nothing else but the Green's function of non–interacting electrons.

$$\tilde{\hat{G}}^{SP}[\tilde{\Phi}^{SP} = 0, \vec{\tilde{B}}^{SP} = 0](\epsilon, p) = \begin{pmatrix} \frac{1}{\epsilon - \epsilon_p + i\eta} & \frac{2i\eta F(\epsilon)}{(\epsilon-\epsilon_p)^2+\eta^2} 0 & 0 & \\ 0 & \frac{1}{\epsilon - \epsilon_p - i\eta} & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon - \epsilon_p + i\eta} & \frac{2i\eta F(\epsilon)}{(\epsilon-\epsilon_p)^2+\eta^2} \\ 0 & 0 & 0 & \frac{1}{\epsilon - \epsilon_p - i\eta} \end{pmatrix}. \tag{5.91}$$

where $\eta = 0^+$ and $\epsilon_p = p^2/2m$. The only difference compared to (4.45) is the replacement of $\frac{1}{2\tau}$ by η since we have no disorder potential in this section. There is no mechanism that could flip the spin. Hence all the components $G_{\uparrow\downarrow}$ and $G_{\downarrow\uparrow}$ are zero ($\rightarrow G_x = G_y = 0$)! Furthermore there is no mechanism that lifts the degeneracy of $G_{\uparrow\uparrow}$ and $G_{\downarrow\downarrow}$ one has $G_{\uparrow\uparrow} = G_{\downarrow\downarrow}$ and thus $G_z = 0$. Therefore the right hand sides of (5.89) and (5.90) are zero and we conclude that $\vec{B}^{SP} = 0$ is a self–consistent solution. Let us continue with equation (5.87). We have

$$G_0^{K,SP}(x,x) = \int \frac{d\epsilon}{2\pi} \int \frac{d^3p}{(2\pi)^3}\, G_0^{K,SP}(\epsilon, p). \tag{5.92}$$

For the p–integration we can take our result from equation (4.48). As $F(\epsilon)$ is an odd function in ϵ, the ϵ–integration gives 0.

For the expression $G_0^{R,SP}(x,x) + G_0^{A,SP}(x,x)$ we refer the reader to equation (4.47). The expression vanishes already after the p–integration as it is proportional to $1-1=0$. In summary we come to the conclusion that also the right hand sides of (5.87) and (5.88) vanish. Hence $\Phi^{SP} = 0$ is a solution of (5.87) and (5.88).

5.5 Fluctuations

In this section we discuss the Gaussian fluctuations or mathematically spoken the quadratic corrections to the saddle point. Remember that the linear order vanishes at the saddle point. We take the action (5.57) in the rotated representation and calculate the second variational derivatives. We define

$$\delta\tilde{\phi}_\alpha := (\tilde{\phi}_\alpha - \tilde{\phi}_\alpha^{SP}) \qquad \delta\tilde{B}_\alpha^n := \big(\tilde{B}_\alpha^n - \big(\tilde{B}_\alpha^{SP}\big)^n\big). \tag{5.93}$$

Let us begin with the quadratic term in $\tilde{\Phi}$ in equation (5.60)

$$S_{2,\phi} = \frac{1}{2}\int dx \int dx' \frac{\delta^2 S_\phi[\tilde{\Phi}]}{\delta\tilde{\phi}_\alpha(x)\delta\tilde{\phi}_{\alpha'}(x')}\bigg|_{\Phi^{SP}} \delta\tilde{\phi}_\alpha(x)\delta\tilde{\phi}_{\alpha'}(x'). \tag{5.94}$$

In linear order we found

$$\frac{\delta S_\phi[\tilde{\Phi}]}{\delta\tilde{\phi}_\alpha(x)} = \bigg[\big[\,\sigma_x\, V_{0,s}^{-1}(x)\big]\tilde{\Phi}^{SP}(x)\bigg]^\alpha \tag{5.95}$$

hence we get for the second derivative

$$\frac{\delta^2 S_\phi[\tilde{\Phi}]}{\delta\tilde{\phi}_\alpha(x)\;\delta\tilde{\phi}_{\alpha'}(x')} = \delta(x-x')\big[\,\sigma_x\, V_{0,s}^{-1}(x)\big]^{\alpha,\alpha'}. \tag{5.96}$$

For the quadratic terms in equation (5.60) containing the inverse triplet interaction $V_{0,t}^{-1}(x)$ we find analogously for the second functional derivative

$$\frac{\delta^2 S_B[\vec{\tilde{B}}]}{\delta\tilde{B}_\alpha^n(x)\delta\tilde{B}_{\alpha'}^m(x')} = \delta(x-x')\big[\,\sigma_x\, V_{0,t}^{-1}(x)\big]^{\alpha,\alpha'}\delta_{n,m} \tag{5.97}$$

Note the additional factor $\delta_{n,m}$. The other term we have to look at is

$$-i\mathrm{Tr}\ln\big[\tilde{G}^{-1}\big[\tilde{\Phi},\vec{\tilde{B}}\big]\big]. \tag{5.98}$$

Remember that we set $J=0$. Therefore the second term in the first line of equation (5.60) vanishes. We use the definitions given in (5.72) and begin the calculation with equation (5.73) adding the second order term

$$\begin{aligned}\ln\hat{\tilde{G}}^{-1}[\tilde{\Phi},\vec{\tilde{B}}] \approx \bigg[&\ln(\hat{\tilde{G}}^{SP})^{-1} + \int dx'\,\tilde{G}^{SP}(x,x')\big[\mathbb{I}_2\otimes\delta\tilde{\phi}_\alpha(x')\tilde{\gamma}^\alpha + \vec{\sigma}\otimes\delta\vec{\tilde{B}}_\alpha(x')\tilde{\gamma}^\alpha\big]\\
&-\frac{1}{2}\int dx'\int dx''\,\bigg(\hat{\tilde{G}}^{SP}(x,x')\big[\mathbb{I}_2\otimes\delta\tilde{\phi}_{\alpha'}(x')\tilde{\gamma}^{\alpha'} + \vec{\sigma}\otimes\delta\vec{\tilde{B}}_{\alpha'}(x')\tilde{\gamma}^{\alpha'}\big]\\
&\qquad\hat{\tilde{G}}^{SP}(x',x'')\big[\mathbb{I}_2\otimes\delta\tilde{\phi}_\alpha(x'')\tilde{\gamma}^\alpha + \vec{\sigma}\otimes\delta\vec{\tilde{B}}_\alpha(x'')\tilde{\gamma}^\alpha\big]\bigg)\bigg]\end{aligned} \tag{5.99}$$

where we used $\ln(AB)=\ln(A)+\ln(B)$ and $\ln(1+A)=A-A^2/2+\mathcal{O}\big(A^3\big)$. Note the arguments of the second Green's function in the quadratic term. We have $\tilde{G}^{SP}(x',x'')$ and not $\tilde{G}^{SP}(x,x'')$ as one might think writing down naively the A^2 term. This is because A has to be interpreted as an operator acting on something to its right. We define

$$M_0^\alpha = \mathbb{I}_2 \otimes \gamma^\alpha \qquad\qquad M_n^\alpha = \sigma_n \otimes \gamma^\alpha \tag{5.100}$$

for $n = x, y, z$. Including the trace Tr [11] we obtain for the quadratic term in the fields

$$\begin{aligned} &-\frac{1}{2}\int dx dx' \,\mathrm{tr}\big[\tilde{\hat{G}}^{SP}(x,x')\delta\tilde{\phi}_{\alpha'}(x')M_0^{\alpha'}\tilde{\hat{G}}^{SP}(x',x)\delta\tilde{\phi}_\alpha(x)M_0^\alpha\big] \\ &-\frac{1}{2}\sum_n\int dx dx' \,\mathrm{tr}\big[\tilde{\hat{G}}^{SP}(x,x')\delta\tilde{B}^n_{\alpha'}(x')M_n^{\alpha'}\tilde{\hat{G}}^{SP}(x',x)\delta\tilde{B}^n_{\alpha}(x)M_n^{\alpha}\big] + \text{mixed terms.} \end{aligned} \tag{5.101}$$

With mixed terms we mean all combinations proportional to $\delta\tilde{\phi}_\alpha\delta\tilde{B}^n_{\alpha'}$ or $\delta\tilde{B}^n_\alpha\delta\tilde{B}^m_{\alpha'}$ where $n \neq m$. We do not to write them down explicitly because they all vanish according to the following simple argument. Remember that the Green's function for our situation is given in equation (5.91).

$$\tilde{\hat{G}}^{SP}(x,x') = \begin{pmatrix} G^R & G^K & 0 & 0 \\ 0 & G^A & 0 & 0 \\ 0 & 0 & G^R & G^K \\ 0 & 0 & 0 & G^A \end{pmatrix} = \mathbb{I}_2 \otimes \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} =: \mathbb{I}_2 \otimes \tilde{G}_0 \tag{5.102}$$

Hence the general form of a mixed term is

$$\mathrm{tr}\,\big[M_i^\alpha\,\tilde{\hat{G}}^{SP}\,M_j^{\alpha'}\,\tilde{\hat{G}}^{SP}\big] \tag{5.103}$$

where $i, j = 0, x, y, z$ **but** $i \neq j$ and $\alpha, \alpha' = 1, 2$. Let us consider for example $i = 0$ and $j = x, y$ or z and evaluate the product

$$\begin{aligned} M_0^\alpha\,\tilde{\hat{G}}^{SP}\,M_n^{\alpha'}\,\tilde{\hat{G}}^{SP} &= (\mathbb{I}_2 \otimes \gamma^\alpha)(\mathbb{I}_2 \otimes \tilde{G}_0)(\sigma_n \otimes \gamma^{\alpha'})(\mathbb{I}_2 \otimes \tilde{G}_0) \\ &= (\mathbb{I}_2 \otimes \gamma^\alpha\tilde{G}_0)(\sigma_n \otimes \gamma^{\alpha'}\tilde{G}_0) = \sigma_n \otimes \gamma^\alpha\tilde{G}_0\gamma^{\alpha'}\tilde{G}_0. \end{aligned} \tag{5.104}$$

For direct products one has $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$. Therefore we find for arbitrary α and α'

$$\mathrm{tr}\,(\sigma_n \otimes \gamma^\alpha\tilde{G}_0\gamma^{\alpha'}\tilde{G}_0) = \mathrm{tr}\,(\sigma_n)\,\mathrm{tr}\,(\gamma^\alpha\tilde{G}_0\gamma^{\alpha'}\tilde{G}_0) = 0 \tag{5.105}$$

as the trace of the Pauli matrix σ_n vanishes. Obviously we obtain zero for any combination as long as $i \neq j$ because we will find always something proportional to a Pauli matrix in the trace over the spin space. We conclude that **all mixed terms vanish.**

So we can turn to the remaining expressions in equation (5.101). Remark that they are characterized by $i = j$.

$$\mathrm{tr}\,\big[M_j^\alpha\,\tilde{\hat{G}}^{SP}\,M_j^{\alpha'}\,\tilde{\hat{G}}^{SP}\big] = \mathrm{tr}\,(\mathbb{I}_2)\,\mathrm{tr}\,(\gamma^\alpha\tilde{G}_0\,\gamma^{\alpha'}\tilde{G}_0) = 2\,\mathrm{tr}\,\big[\gamma^\alpha\tilde{G}_0\,\gamma^{\alpha'}\tilde{G}_0\big] \tag{5.106}$$

We define the *Polarization function* [12]

$$P^{\alpha,\alpha'}(x,x') := -i\,\mathrm{tr}\big[\gamma^\alpha\tilde{G}_0(x,x')\gamma^{\alpha'}\tilde{G}_0(x',x)\big]. \tag{5.107}$$

[11] which means effectively setting $x = x''$, summing over x and summing over the 4×4 matric structure.

[12] It can be shown that $P^{\alpha,\alpha'}$ coincides with the polarization function of the RPA approximation (clean case).

Combining (5.96), (5.97) and (5.101) and using the definition of $P^{\alpha,\alpha'}(x,x')$ the quadratic correction to the saddle point reads

$$\begin{aligned} S_2 = \int dx \int dx' \; \tilde{\Phi}^T(x) \Big[\sigma_x V_{0,s}^{-1}(x)\delta(x-x') + P(x,x') \Big] \Phi(x') \\ + \sum_n (\tilde{B}^n)^T(x) \Big[\sigma_x V_{0,t}^{-1}(x)\delta(x-x') + P(x,x') \Big] \tilde{B}^n(x') \end{aligned} \tag{5.108}$$

where we used that $\delta\tilde{\Phi} = \tilde{\Phi}$ and $\delta\vec{\tilde{B}} = \vec{\tilde{B}}$ since $\tilde{\Phi}^{SP} = 0$ and $\vec{\tilde{B}}^{SP} = 0$. We will not determine $P^{\alpha,\alpha'}(x,x')$ in this work as we do not need it for our further calculations (Apart from interactions we want to include the effect of disorder.). For $P^{\alpha,\alpha'}(x,x')$ we refer the reader to [19] where the detailed calculation can be found. One obtains

$$P(q,\omega) = \begin{pmatrix} 0 & P^A(q,\omega) \\ P^R(q,\omega) & P^K(q,\omega) \end{pmatrix} \tag{5.109}$$

where $P^A(q,\omega) = (P^R)^*(q,\omega)$ and $P^K(q,\omega) = \coth(\beta\epsilon/2)(P^R(q,\omega) - P^A(q,\omega))$. Thus the matrix $P(q,\omega)$ is determined by $P^R(q,\omega)$ which is

$$P^R(q,\omega) = \nu_3 + \frac{i\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \; G^R(p,\epsilon) G^A(p-q,\epsilon-\omega). \tag{5.110}$$

Defining

$$V_s^{-1}(x,x') := \Big[\sigma_x V_{0,s}^{-1}(x)\delta(x-x') + P(x,x') \Big] \tag{5.111}$$

and

$$V_t^{-1}(x,x') := \Big[\sigma_x V_{0,t}^{-1}(x)\delta(x-x') + P(x,x') \Big] \tag{5.112}$$

we obtain for the effective quadratic action the compact form

$$\boxed{S[\tilde{\Phi}, \vec{\tilde{B}}] = \mathrm{Tr} \Big[\tilde{\Phi}^T(x) \; V_s^{-1}(x,x') \tilde{\Phi}(x') + \sum_n (\tilde{B}^n)^T(x) \; V_t^{-1}(x,x') \tilde{B}^n(x') \Big].} \tag{5.113}$$

It turns out that the dynamically screened interactions (inverse of (5.111) and (5.112)) in the singlet and triplet channel corresponds to the standard *random phase approximation* (RPA) (for the clean case). See [19] for more information.

In the following chapter where we will take into account disorder **and** interactions we will determine a corresponding expression for the quadratic corrections in the action. The dynamically screened interaction in the singlet and triplet channel will turn out to be the *diffusive random phase approximation* interactions of equation (2.45).

Chapter 6

Interaction and Disorder

After the three preceding chapters that were rather technical but provided the necessary mathematical tools, let us remind what we are heading for. Our goal was to calculate the DOS near the Fermi edge of a diffusive quasi one–dimensional wire non perturbatively. In particular we wanted to study the dependence of the DOS on an external magnetic field which makes it necessary to take into account the spin of the electrons.

The path integral formulation of the fermionic many–body problem will allow us to reach this goal. By calculating the necessary path integrals for the generating functional $Z[J,\hat{J}]$ we perform a summation over an infinite number of diagrams which will lead to a non divergent result for the DOS.

Before setting up the action for interacting electrons in an diffusive environment let us repeat the key quantities and how they are connected. According to equation (1.11) the DOS is connected with the imaginary part of the retarded Green's function. This can also be expressed in a different way:

$$\mathrm{Im}G^R = \frac{i}{2}\big(G^R - G^A\big) = \frac{i}{2}\big(G^> - G^<\big). \tag{6.1}$$

Taking into account the spin of the electrons explicitly we have to sum the up–spin and down–spin DOS:

$$\nu(\epsilon) = \nu_\uparrow(\epsilon) + \nu_\downarrow(\epsilon) = \frac{i}{2\pi\mathcal{V}} \sum_p \Big(G^>_{\uparrow\uparrow}(p,\epsilon) - G^<_{\uparrow\uparrow}(p,\epsilon) + G^>_{\downarrow\downarrow}(p,\epsilon) - G^<_{\downarrow\downarrow}(p,\epsilon) \Big). \tag{6.2}$$

Thus in order to calculate the DOS non perturbatively we need a non perturbative result for the Green's functions appearing on the right hand side of the equation above. An elegant method to determine Green's functions is the calculation of the generating functional Z of the system including source fields J and $\hat{J}$ that are coupled linearly to the electrons. Then the Green's function can be obtained as functional derivative of $Z[J,\hat{J}]$. We demonstrated this procedure in section (3.5).

Ok, so let us write down the action S which determines the generating functional:

$$Z[J,\hat{J}] = \frac{1}{\mathcal{N}_U} \int \mathcal{D}\hat{\Psi} \int \mathcal{D}\Psi \int \mathcal{D}\Phi \int \mathcal{D}\vec{B} \int \mathcal{D}U \; e^{iS[\Psi,\hat{\Psi},\Phi,U,\vec{B},J,\hat{J}]}. \tag{6.3}$$

In the unrotated representation the action that takes into account disorder and interactions reads

$$S[\Psi,\hat{\Psi},\Phi,U,\vec{B},J,\hat{J}] = \int dx\ \hat{\Psi}(x)\hat{G}^{-1}[\Phi,\vec{B},U](x)\Psi(x) + \int dx\ \left(\hat{J}(x)\Psi(x) + \hat{\Psi}J(x)\right)$$
$$+ \frac{e^2}{2}\int dx\ \Phi^T(x)\ \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{\mu_B^2}{2}\int dx \sum_n (B^n)^T(x)\ \sigma_z V_{0,t}^{-1}(x) B^n(x) \tag{6.4}$$

with

$$\hat{G}^{-1}[\Phi,\vec{B},U](x) = \left(i\frac{\partial}{\partial t} + \frac{\Delta}{2m}\right)\ \mathbb{I}_4 + \mathbb{I}_2 \otimes e\phi_\alpha(x)\gamma^\alpha + \vec{\sigma}\otimes\mu_B\vec{B}_\alpha(x)\gamma^\alpha + U(r)\mathbb{I}_4. \tag{6.5}$$

The first 2×2 matrix gives the structure in spin space whereas the second 2×2 matrix determines the structure in Keldysh space. For the following steps we set again $e = \mu_B = 1$ and reinsert them in our final result.

Remark that we already introduced the fields Φ and $\vec{B}$ that decouple the four Fermion interaction terms. As next step we perform the disorder averaging (path integral over $U(r)$) like it was explained in chapter 4. The decoupling of the appearing quartic fermion terms leads to a path integral over a 4×4 matrix field Q.[1] The generating functional is then

$$Z[J,\hat{J}] = \int \mathcal{D}\hat{\Psi}\int\mathcal{D}\Psi\int\mathcal{D}\Phi\int\mathcal{D}\vec{B}\int\mathcal{D}\hat{Q}\ e^{iS[\Psi,\hat{\Psi},\Phi,\hat{Q},\vec{B},J,\hat{J}]} \tag{6.6}$$

with the action

$$S[\Psi,\hat{\Psi},\Phi,\vec{B},\hat{Q}] = \int dx\int dx'\ \hat{\Psi}(x)\hat{G}^{-1}[\Phi,\vec{B},\hat{Q}](x,x')\Psi(x') - \frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\hat{Q}^2\}$$
$$+ \frac{1}{2}\int dx\ \Phi^T(x)\ \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{1}{2}\int dx\sum_n (B^n)^T(x)\ \sigma_z V_{0,t}^{-1}(x)B^n(x)$$
$$+ \int dx\ \left(\hat{J}(x)\Psi(x) + \hat{\Psi}J(x)\right) \tag{6.7}$$

where

$$\hat{G}^{-1}[\Phi,\vec{B},\hat{Q}](x) = \left[\left(i\frac{\partial}{\partial t'} + \frac{\Delta}{2m}\right)\ \mathbb{I}_4 + \mathbb{I}_2\otimes\phi_\alpha(x')\gamma^\alpha + \vec{\sigma}\otimes\vec{B}_\alpha(x')\gamma^\alpha\right]\delta(x-x')$$
$$+ \frac{i}{2\tau}\hat{Q}(r,t,t')\delta(r-r'). \tag{6.8}$$

[1] We explained the procedure in chapter 4 for the spinless case. We will discuss how the form of $\hat{Q}$ changes including spin when we determine the saddle point solution.

6.1 Introduction of a rotation and local gauge factors

The action is quadratic in Ψ and $\hat{\Psi}$, hence we could integrate out the electrons. However before doing this, we use an idea that was first formulated by Kamenev and Andreev in [17]. They introduced gauge factors of the form $e^{ik_\alpha\gamma^\alpha}$, which allow to find an approximate saddle point for the combined problem of disorder and interaction. In their case they determined the functions k_α's such that they compensated approximately the scalar field Φ and they could use the result of the non–interacting saddle Λ in order to construct a solution for the interacting problem. [2]

In our work we want to use this trick in order to treat also the magnetic field non perturbatively. However the approach has to be modified for the following reason: The field matrix $F_{\phi,\vec{B}}$ (see equation (5.39)) in our case is **non diagonal**! This is due to the inclusion of the triplet interaction represented by the field $\vec{B}$.

Analyzing the calculation of Kamenev and Andreev with gauge factors shows that it is important that the field matrix $e\phi_\alpha\gamma^\alpha$ (it is diagonal !) commutes with the gauge factors $e^{ik_\alpha\gamma^\alpha}$ (which are also chosen diagonal). The necessity of this property becomes clear when one has to determine the gauge factors k_α explicitly as functions of the fields ϕ_α. We will remind the reader, where we need the commutativity, when we arrive at this stage in our calculations.

Unfortunately our field matrix $F_{\phi,\vec{B}}$ has a quite complicated structure. Hence our idea is to diagonalize our field matrix by an orthogonal transformation and then proceed with the trick of the gauge factors. The gauge factors can then also be chosen diagonal.

So let us first diagonalize the field matrix $F_{\phi,\vec{B}}$. The form of the rotation, which we denote by $\mathcal{R}$, can be calculated by determining the normalized eigensystem of the matrix $F_{\phi,\vec{B}}$. We skip the calculation which is just linear algebra and present the result. $\mathcal{R}$ reads explicitly

$$\mathcal{R} = \begin{pmatrix} \frac{B_1^\perp\sqrt{1+B_1^z/B_1}}{\sqrt{2}(B_1^x+iB_1^y)} & 0 & -\frac{B_1^\perp\sqrt{1-B_1^z/B_1}}{\sqrt{2}(B_1^x+iB_1^y)} & 0 \\ 0 & \frac{B_2^\perp\sqrt{1+B_2^z/B_2}}{\sqrt{2}(B_2^x+iB_2^y)} & 0 & -\frac{B_2^\perp\sqrt{1-B_2^z/B_2}}{\sqrt{2}(B_2^x+iB_2^y)} \\ \frac{B_1^\perp}{\sqrt{2}B_1\sqrt{1+B_1^z/B_1}} & 0 & \frac{B_1^\perp}{\sqrt{2}B_1\sqrt{1-B_1^z/B_1}} & 0 \\ 0 & \frac{B_2^\perp}{\sqrt{2}B_2\sqrt{1+B_2^z/B_2}} & 0 & \frac{B_2^\perp}{\sqrt{2}B_2\sqrt{1-B_2^z/B_2}} \end{pmatrix}. \tag{6.9}$$

$\mathcal{R}$ is unitary which allows to calculate the inverse of $\mathcal{R}$ simply by complex conjugation and transposition: $\mathcal{R}^{-1}=(\mathcal{R}^*)^T$. In further calculations we will denote the matrix elements of $\mathcal{R}$ by $\mathcal{R}_{ij}$. Note that $\mathcal{R}$ is a function of the components of $\vec{B}$ only and independent of Φ.

Assumption: Throughout this derivation we will encounter many additional terms compared to the treatment of Kamenev and Andreev due to the additional rotation matrix $\mathcal{R}$. Our general philosophy will be to assume that the variations of $\vec{B}$ in space and time are small and terms proportional to the derivatives of the components $\partial_t B^n$ and ∇B^n can be neglected compared to the contributions. Applying the rotation one obtains a diagonal matrix

$$D_{\phi,B} = \mathcal{R}^{-1}F_{\phi,\vec{B}}\mathcal{R} \tag{6.10}$$

which reads

[2] For further information see [17] (or [19] which is more detailed).

$$D_{\phi,B} = \begin{pmatrix} \phi_1 + B_1 & 0 & 0 & 0 \\ 0 & \phi_2 + B_2 & 0 & 0 \\ 0 & 0 & \phi_1 - B_1 & 0 \\ 0 & 0 & & \phi_2 - B_2 \end{pmatrix} \tag{6.11}$$

with $B_i = \sqrt{(B_i^x)^2 + (B_i^y)^2 + (B_i^z)^2}$.[3] We chose the rotation such that the magnetic field has only a z–component and its magnitude is the absolute value of the fields. One can use equation (6.10) to rewrite equation (6.8) as

$$\hat{G}^{-1}[\Phi, \vec{B}, \hat{Q}](x) = \left[\left(i\frac{\partial}{\partial t'} + \frac{\Delta}{2m} \right) \mathbb{I}_4 + \mathcal{R} D_{\phi,B} \mathcal{R}^{-1} \right] \delta(x - x') + \frac{i}{2\tau} \hat{Q}(r, t, t') \delta(r - r'). \tag{6.12}$$

We introduce rotated fields $\Psi_\mathcal{R}$ and $\hat{\Psi}_\mathcal{R}$

$$\Psi_\mathcal{R}(x) = \mathcal{R}^{-1}(x)\Psi(x) \qquad\qquad \hat{\Psi}_\mathcal{R}(x) = \hat{\Psi}(x)\mathcal{R}(x). \tag{6.13}$$

The integration measure remains unchanged $\mathcal{D}\hat{\Psi}\mathcal{D}\Psi = \mathcal{D}\hat{\Psi}_\mathcal{R}\mathcal{D}\Psi_\mathcal{R}$. However the two parts of the action S given in equation (6.7) which contain the fields Ψ change. We obtain the action

$$\begin{aligned} S_\mathcal{R}[\Psi, \hat{\Psi}, \Phi, \vec{B}, \hat{Q}] = &\int dx \ \left(\hat{J}(x)\mathcal{R}(x)\Psi_\mathcal{R}(x) + \hat{\Psi}_\mathcal{R}\mathcal{R}^{-1}(x)J(x) \right) + S[\Phi^2, B^2, \hat{Q}^2] \\ &+ \int dx \int dx' \ \hat{\Psi}_\mathcal{R}(x) \left(\delta(x - x') \left[\mathcal{R}^{-1} G_0^{-1} \mathcal{R} + D_{\phi,B} \right] + \frac{i}{2\tau} \mathcal{R}^{-1} \hat{Q}(r, t, t') \mathcal{R} \delta(r - r') \right) \Psi_\mathcal{R}(x'). \end{aligned} \tag{6.14}$$

By $S[\Phi^2, B^2, \hat{Q}^2]$ we denoted all the unchanged quadratic field terms that appear in the action. After having diagonalized the field matrix we choose the gauge transformation as

$$U(x) = \exp\left[i \begin{pmatrix} k_1 + f_1 & & 0 & 0 \\ 0 & k_2 + f_2 & 0 & 0 \\ 0 & 0 & k_1 - f_1 & 0 \\ 0 & 0 & & k_2 - f_2 \end{pmatrix} \right] \tag{6.15}$$

or in tensor notation

$$U(x) = e^{i(\mathbb{I}_2 \otimes k_\alpha \gamma^\alpha + \sigma_z \otimes f_\alpha \gamma^\alpha)}. \tag{6.16}$$

The quantities k_1 and k_2 have the same purpose as in the paper by Kamenev and Andreev where U reads

$$U(x) = e^{ik_\alpha \gamma^\alpha} = \exp\left[i \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \right] \tag{6.17}$$

They will account for the field Φ. Additionally we introduced the function f_1 and f_2 which will compensate the magnetic fields B_1 and B_2. What we mean by compensate will become clear

[3] $i = 1, 2$ is the Keldysh index

when we determine below the k's and f's. In the action we introduce factors of UU^{-1} wherever a Ψ appears and define the rotated and gauged fields $\Psi_{\mathcal{R}U}$ and $\bar{\Psi}_{\mathcal{R}U}$

$$\Psi_{\mathcal{R}U}(x) = U(x)\mathcal{R}^{-1}(x)\Psi(x) \qquad \hat{\Psi}_{\mathcal{R}U}(x) = \hat{\Psi}(x)\mathcal{R}(x)U^{-1}(x) \tag{6.18}$$

The action then becomes

$$\begin{aligned} S_{\mathcal{R}U}[\Psi,\hat{\Psi},\Phi,\hat{Q},\vec{B}] = &\int dx \int dx'\ \hat{\Psi}_{\mathcal{R}U}(x)\hat{G}^{-1}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}](x,x')\Psi_{\mathcal{R}U}(x') + S[\Phi^2,B^2,\hat{Q}^2] \\ &+ \int dx \left(\hat{J}(x)\mathcal{R}(x)U^{-1}(x)\Psi_{\mathcal{R}U}(x) + \hat{\Psi}_{\mathcal{R}U}(x)U(x)\mathcal{R}^{-1}(x)J(x)\right) \end{aligned} \tag{6.19}$$

with

$$\begin{aligned} \hat{G}^{-1}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}](x) = &\ U(x)\ \hat{G}^{-1}[\Phi,\hat{Q}](x,x')\ U^{-1}(x') \\ &= \left[U(x)\mathcal{R}^{-1}(x)G_0^{-1}(x)\mathcal{R}(x)U^{-1}(x) + D_{\phi,B}(x)\right]\delta(x-x') \\ &+ \frac{i}{2\tau}U(x)\mathcal{R}^{-1}(x)\hat{Q}(r,t,t')\mathcal{R}(x')U^{-1}(x')\delta(r-r') \end{aligned} \tag{6.20}$$

where we used that

$$U(x)D_{\phi,B}(x)U^{-1}(x) = D_{\phi,B}(x). \tag{6.21}$$

Note that the integration measure $\mathcal{D}\bar{\Psi}_{\mathcal{R}}\mathcal{D}\Psi_{\mathcal{R}} = \mathcal{D}\bar{\Psi}_{\mathcal{R}\mathcal{U}}\ \mathcal{D}\Psi_{\mathcal{R}\mathcal{U}}$ remains unchanged under the transformation with U.

Integrating out the Fermions

We integrate out the fermionic fields $\Psi_{\mathcal{R}\mathcal{U}}$ and $\bar{\Psi}_{\mathcal{R}\mathcal{U}}$ and are left with a path integral over Φ, $\vec{B}$ and $\hat{Q}$ which cannot be done exactly. The generating functional reads

$$Z[J,\hat{J}] = \int \mathcal{D}\Phi \int \mathcal{D}\vec{B} \int \mathcal{D}\hat{Q}\ e^{iS[\Phi,\vec{B}\hat{Q},,J,\hat{J}]} \tag{6.22}$$

with the action

$$\begin{aligned} S[\Phi,\vec{B},\hat{Q}] = &\int dx \int dx'\ \hat{J}(x)\mathcal{R}(x)U^{-1}(x)\ \hat{G}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}](x,x')\ U(x')\mathcal{R}^{-1}(x')J(x') \\ &+ \frac{1}{2}\int dx\ \Phi^T(x)\ \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x)\ \sigma_z V_{0,t}^{-1}(x)B^n(x) \\ &- \frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\hat{Q}^2\} - i\mathrm{Tr}\ln\left[\hat{G}^{-1}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}](x,x')\right] \end{aligned} \tag{6.23}$$

where $\hat{G}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}](x,x')$ is the inverse operator of $\hat{G}^{-1}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}]$ given in equation (6.20).

Our strategy to proceed is the following: We determine an approximate saddle point for $\hat{Q}$ for the combined problem with disorder and interaction. The path integral over $\hat{Q}$ we perform then as previously by simply taking its value at the approximate saddle point $\hat{Q}^{SP}$.[4] For the fields Φ and $\vec{B}$ we will take into account the quadratic fluctuation corrections around the approximate saddle point.

6.2 Determination of a saddle point for $\hat{Q}$

We have to evaluate

$$\frac{\delta}{\delta Q} S[\Phi, B, \hat{Q}, J = 0] = 0. \tag{6.24}$$

We do this like already demonstrated in chapter 4. We write $\hat{Q} = \hat{Q}^{SP} + \delta\hat{Q}$ and develop $S[\hat{Q}, \Phi, B, J = 0]$ in orders of $\delta\hat{Q}$. For the quadratic term in $\hat{Q}$ there is no change

$$-\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\hat{Q}^2\} = -\frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\left(\hat{Q}^{SP}\right)^2\} - \frac{\pi\nu_3}{2\tau}\mathrm{Tr}\{\hat{Q}^{SP}\delta\hat{Q}\} + \mathcal{O}(\delta\hat{Q}^2) \tag{6.25}$$

except that $\hat{Q}$ now denotes a 4×4 matrix. The second term in which $\hat{Q}$ appears is

$$\begin{aligned} \hat{G}^{-1}_{\mathcal{R}U}[\Phi, \vec{B}, \hat{Q}] &= U\mathcal{R}^{-1}G_0^{-1}\mathcal{R}U^{-1} + D_{\phi,B} + \frac{i}{2\tau}U\mathcal{R}^{-1}(\hat{Q}^{SP} + \delta\hat{Q})\mathcal{R}U^{-1} \\ &= \hat{G}^{-1}_{\mathcal{R}U}[\Phi, \vec{B}, \hat{Q}^{SP}] + \frac{i}{2\tau}U\mathcal{R}^{-1}\delta\hat{Q}\,\mathcal{R}U^{-1} \end{aligned} \tag{6.26}$$

where the inverse of $\hat{G}^{-1}_{\mathcal{R}U}[\Phi, \vec{B}, \hat{Q}^{SP}]$ is given by

$$\boxed{\begin{aligned} \hat{G}_{\mathcal{R}U}[\Phi, \vec{B}, \hat{Q}^{SP}](x) = \Bigg[\Big(U(x)\mathcal{R}^{-1}(x)G_0^{-1}(x)\mathcal{R}(x)U^{-1}(x) + D_{\phi,B}(x)\Big)\delta(x - x') \\ + \frac{i}{2\tau}U(x)\mathcal{R}^{-1}(x)\hat{Q}^{SP}\mathcal{R}(x')U^{-1}(x')\delta(r - r')\Bigg]^{-1}. \end{aligned}} \tag{6.27}$$

We abbreviate $\hat{G}^{-1}_{\mathcal{R}U}[\Phi, \vec{B}, \hat{Q}^{SP}] = \hat{G}^{-1}_{\mathcal{R}U}[\hat{Q}^{SP}]$. Again we expand the logarithm

$$\begin{aligned} \mathrm{Tr}\ln\big[\hat{G}^{-1}_{\mathcal{R}U}[\hat{Q}]\big] &= \mathrm{Tr}\ln\left[\hat{G}^{-1}_{\mathcal{R}U}[\hat{Q}^{SP}] + \frac{i}{2\tau}U\mathcal{R}^{-1}\delta\hat{Q}\,\mathcal{R}U^{-1}\right] \\ &= \mathrm{Tr}\ln\left[\hat{G}^{-1}_{\mathcal{R}U}[\hat{Q}^{SP}]\Big(\,\mathbb{I}_4 + \hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]\;\frac{i}{2\tau}U\mathcal{R}^{-1}\delta\hat{Q}\,\mathcal{R}U^{-1}\Big)\right] \\ &= \mathrm{Tr}\ln\left[\hat{G}^{-1}_{\mathcal{R}U}[\hat{Q}^{SP}]\right] + \mathrm{Tr}\left[\hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]\;\frac{i}{2\tau}U\mathcal{R}^{-1}\delta\hat{Q}\,\mathcal{R}U^{-1}\right] + \hat{\mathcal{O}}(\delta\hat{Q}^2). \end{aligned} \tag{6.28}$$

[4] We already mentioned in chapter (4) that the fluctuations of $\hat{Q}$ are not relevant for our problem and explained what kind of physics they contain.

We focus on the second term which is proportional to $\delta\hat{Q}$ and transform it using the cyclic invariance of the trace

$$\mathrm{Tr}\left[\hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]\frac{i}{2\tau}U\mathcal{R}^{-1}\delta\hat{Q}\;\mathcal{R}U^{-1}\right]=\mathrm{Tr}\left[\frac{i}{2\tau}\mathcal{R}U^{-1}\hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]U\mathcal{R}^{-1}\;\delta\hat{Q}\right]. \tag{6.29}$$

Thus we have in linear order

$$\int dr\;dr'\;dt\;dt'\left(-\frac{\pi\nu_3}{2\tau}\hat{Q}^{SP}+\frac{i}{2\tau}\mathcal{R}U^{-1}\hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]U\mathcal{R}^{-1}\;\right)(r,t,r',t')\;\delta\hat{Q}(r',t',t)\delta(r-r'). \tag{6.30}$$

The term in the brackets has to be zero in order that the whole expression vanishes for arbitrary $\delta\hat{Q}$:

$$\left(-\frac{\pi\nu_3}{2\tau}\hat{Q}^{SP}+\frac{i}{2\tau}\mathcal{R}U^{-1}\hat{G}_{\mathcal{R}U}[\hat{Q}^{SP}]U\mathcal{R}^{-1}\;\right)(r,t,r,t')=0 \tag{6.31}$$

which gives the saddle point equation

$$\boxed{U\mathcal{R}^{-1}\;\hat{Q}^{SP}\mathcal{R}U^{-1}(r,t,t')=\frac{i}{\pi\nu_3}\hat{G}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}^{SP}](r,t,t')} \tag{6.32}$$

How do we solve this equation? Consider $\hat{G}_{\mathcal{R}U}$ and assume that we could choose the functions k_α and f_α such that they would gauge away the fields ϕ_α and B_α. Then U would chosen such that

$$U(x)\mathcal{R}^{-1}(x)G_0^{-1}(x)\mathcal{R}(x)U^{-1}(x)+D_{\phi,B}(x)=G_0^{-1}(x)\mathbb{I}_4. \tag{6.33}$$

Whether this can be fulfilled we will study below. Plugging (6.33) into the Green's function (6.27), we find

$$\hat{G}_{\mathcal{R}U}[\Phi,\vec{B},\hat{Q}^{SP}](x)=\left[G_0^{-1}(x)\delta(x-x')+\frac{i}{2\tau}U\mathcal{R}^{-1}\hat{Q}^{SP}\mathcal{R}U^{-1}\delta(r-r')\right]^{-1} \tag{6.34}$$

and the saddle point equation reads

$$U\mathcal{R}^{-1}\;\hat{Q}^{SP}\mathcal{R}U^{-1}(r,t,t')=\frac{i}{\pi\nu_3}\left[G_0^{-1}(x)\delta(x-x')+\frac{i}{2\tau}U\mathcal{R}^{-1}\hat{Q}^{SP}\mathcal{R}U^{-1}\delta(r-r')\right]^{-1}. \tag{6.35}$$

For this saddle point equation we can construct a solution using the saddle point solution Q^{SP} of the non–interacting spinless case derived in chapter 4, which we denoted by Λ. It was given in chapter 4 in the rotated representation. In the unrotated representation, in which we are currently working, one finds[5]

$$\Lambda(\epsilon)=\begin{pmatrix}F(\epsilon) & 1-F(\epsilon)\\ 1+F(\epsilon) & -F(\epsilon)\end{pmatrix}\qquad\qquad \Lambda(t)=\begin{pmatrix}\tilde{F}(t) & \delta(t)-\tilde{F}(t)\\ \delta(t)+\tilde{F}(t) & -\tilde{F}(t)\end{pmatrix}. \tag{6.36}$$

[5]Remember that the two representations are connected by $L\Lambda L=\tilde{\Lambda}$.

In the situation without interactions and external magnetic field the 4×4 saddle point solution reads

$$\hat{\Lambda} = \mathbb{I}_2 \otimes \Lambda = \mathbb{I}_2 \otimes \begin{pmatrix} \Lambda^T & -\Lambda^< \\ \Lambda^> & -\Lambda^{\bar{T}} \end{pmatrix}. \tag{6.37}$$

This can be immediately understood if one repeats the calculations of section (4.3) including the spin degree of freedom and replaces (4.46) by (5.91). With the help of $\hat{\Lambda}$ the solution for the interacting saddle point can be written as

$$\boxed{\hat{Q}^{SP} = \mathcal{R} U^{-1} \, \hat{\Lambda} \, U \mathcal{R}^{-1}.} \tag{6.38}$$

Equation (6.35) reduces to

$$\hat{\Lambda}(r,t,t') = \frac{i}{\pi \nu_3} \left[G_0^{-1}(x) \delta(x-x') + \frac{i}{2\tau} \hat{\Lambda} \right]^{-1}_{r=r'} \tag{6.39}$$

and $\hat{\Lambda}$ is obviously the self–consistent solution. Thus we found a saddle point solution for the $\hat{Q}$ field. What remains to be done is the determination of the gauge functions k_α and f_α in such a way that equation (6.33) is satisfied. The operator $G_0^{-1} = \left(i \frac{\partial}{\partial t} + \frac{\Delta}{2m} \right)$ acts on the product $\mathcal{R}(x) U^{-1}(x)$ and generates several terms. We separate these terms in a part where G_0^{-1} acts only on U^{-1} and a rest term N where derivatives of $\mathcal{R}$ and U^{-1} appear.

$$U \mathcal{R}^{-1} G_0^{-1} \mathcal{R} U^{-1} = U G_0^{-1} U^{-1} + U \mathcal{R}^{-1} N \tag{6.40}$$

We begin with the first terms on the right hand side

$$U G_0^{-1} U^{-1} = \left[i \partial_t \mathbb{I}_4 + \mathbb{I}_2 \otimes \dot{k}_\alpha \gamma^\alpha + \sigma_z \otimes \dot{f}_\alpha \gamma^\alpha - \frac{1}{2m} \left(-\nabla \, \mathbb{I}_4 - \mathbb{I}_2 \otimes k'_\alpha \gamma^\alpha - \sigma_z \otimes f'_\alpha \gamma^\alpha \right)^2 \right]. \tag{6.41}$$

We restrict ourselves to slowly varying fields, which means $|\nabla \phi_\alpha / \phi_\alpha| \ll k_F$ or $1/l$ and $|\nabla B_\alpha / B_\alpha| \ll k_F$. As the k's and f's depend on the ϕ's and B's, we can assume that $|\nabla k_\alpha / k_\alpha| \ll k_F$ and $|\nabla f_\alpha / f_\alpha| \ll k_F$. We neglect therefore the quadratic terms in the k's and f's. Hence we get approximately

$$U G_0^{-1} U^{-1} + D_{\phi, \vec{B}} \approx G_0^{-1}(x) \, \mathbb{I}_4 + C(x) \tag{6.42}$$

where

$$C(x) = \mathbb{I}_2 \otimes C_\alpha^\phi(x) \gamma^\alpha + \sigma_z \otimes C_\alpha^B(x) \gamma^\alpha \tag{6.43}$$

with C_α^ϕ containing the ϕ–dependent part

$$C_\alpha^\phi(x) = C_\alpha^\phi(x, p_F) = \partial_t k_\alpha + v_F(p_F) \cdot \nabla k_\alpha + \phi_\alpha \tag{6.44}$$

and C_α^B containing the B dependent part

$$C_\alpha^B(x) = C_\alpha^B(x, p_F) = \partial_t f_\alpha + v_F(p_F) \cdot \nabla f_\alpha + B_\alpha. \tag{6.45}$$

$v_F(p_F)$ is the Fermi velocity in the direction of p_F. The second term in equation (6.40) is negligible. In all terms contributing to N appears by definition at least one partial derivatives $\partial_t\mathcal{R}$ or $\nabla\mathcal{R}$. Note that $\mathcal{R}$ is an explicit function of the components of $\vec{B}$. Hence each partial derivative is proportional to either $\partial_t B$ or ∇B. Thus according to our general assumption the N term can be neglected. The validity of (6.33) is then connected with a choice of the gauge fields that leads to a vanishing $C(x)$ in (6.42).

Maybe here is the best point to explain our wish to construct gauge factors that commute with the field matrix. Imagine we would have instead of (6.21) an expression containing $UF_{\phi,\vec{B}}U^{-1}$. Then the equations (6.44) and (6.45) would look much more complicated as the $k's$ and $f's$ would appear in all powers. Yet in our equations (6.44) and (6.45) the gauge factors appear only linearly. However even these equations can only be solved exactly in some special cases. The complicated term is the one containing $v_F(p_F)$ which depends on the direction of p. In a disordered system the particles keep their momentum only on a time scale τ. For time scales that are bigger than τ there is no information about the initial direction left and the problem becomes isotropic. We will not determine the k_α and f_α from the equations (6.44) and (6.45) but develop in a few lines a weaker condition that contains an angle average. As our object of investigation is a diffusive system that posses isotropy in the sense explained above, even working with the weaker condition leads to reasonable results.

We approximate the Green's function at the saddle point given in (6.27) using the right hand side of equation (6.42) and the saddle point $\hat{Q}^{SP}$ of equation (7.21).

$$\hat{G}^{SP}(x,x') = \left[\hat{G}_0^{-1} + \frac{i}{2\tau}\,\hat{\Lambda} + C\right]^{-1} \tag{6.46}$$

We define

$$\hat{G}_\tau = \left[\hat{G}_0^{-1} + \frac{i}{2\tau}\,\hat{\Lambda}\right]^{-1} \tag{6.47}$$

which is nothing else but the matrix of the well known disorder damped Green's functions. It is connected to $\tilde{G}$ of equation (4.45) by the relation

$$\hat{G}_\tau = \mathbb{I}_2 \otimes L\tilde{G}L. \tag{6.48}$$

We develop the Green's function $\hat{G}^{SP}$ in equation (6.46) in powers of the small field C.

$$\begin{aligned}\hat{G}^{SP}(x,x') &= \left[\hat{G}_\tau^{-1} + C\right]^{-1} = \left[\mathbb{I}_4 + \hat{G}_\tau C\right]^{-1}\hat{G}_\tau \\ &= \hat{G}_\tau - \hat{G}_\tau C\hat{G}_\tau + \hat{G}_\tau C\hat{G}_\tau C\hat{G}_\tau - ...\end{aligned} \tag{6.49}$$

From this series arises a new condition. Instead of fulfilling $C = 0$ it is sufficient to fulfill the weaker condition

$$\hat{G}_\tau C\hat{G}_\tau|_{r=r'} = 0. \tag{6.50}$$

As we need the Green's function only at equal points in space (see the saddle point equation) one can demand $r = r'$. From this weaker condition we will determine in the next section the functions k_α and f_α.

6.3 Determination of the gauge factors k_α and f_α

We Fourier transform equation (6.50) to energy and momentum space

$$\frac{1}{\mathcal{V}}\sum_p \hat{G}_\tau(p,\epsilon)C(q,\omega)\hat{G}_\tau(p-q,\epsilon-\omega)=0. \tag{6.51}$$

Already in the previous section we threw away quadratic variations in k_α and f_α (see discussion below equation (6.41)) with the argument that we restrict ourselves to the regime $|\nabla k_\alpha/k_\alpha| \ll k_F$ and $|\nabla f_\alpha/f_\alpha| \ll k_F$. That restriction translates to the condition $q \ll p_F$ or $q \ll 1/l$ in momentum space. This can be seen comparing (6.45) to

$$C^B_\alpha(q,\omega) = i\omega f_\alpha - iv_F\cdot q f_\alpha + B_\alpha. \tag{6.52}$$

Note that the small q condition is in accordance with the diffusive phenomena which we want to treat. We look at the problem in spin space

$$\begin{aligned}\frac{1}{\mathcal{V}}\sum_p \begin{pmatrix} G_{\uparrow\uparrow}(p,\epsilon) & G_{\uparrow\downarrow}(p,\epsilon)\\ G_{\downarrow\uparrow}(p,\epsilon) & G_{\downarrow\downarrow}(p,\epsilon)\end{pmatrix}&\begin{pmatrix}(C^\phi_\alpha+C^B_\alpha)(q,\omega)\gamma^\alpha & 0\\ 0 & (C^\phi_\alpha-C^B_\alpha)(q,\omega)\gamma^\alpha\end{pmatrix}\\ &\times\begin{pmatrix} G_{\uparrow\uparrow}(p-q,\epsilon-\omega) & G_{\uparrow\downarrow}(p-q,\epsilon-\omega)\\ G_{\downarrow\uparrow}(p-q,\epsilon-\omega) & G_{\downarrow\downarrow}(p-q,\epsilon-\omega)\end{pmatrix}=0\end{aligned} \tag{6.53}$$

The Hamiltonian contains no spin–flip mechanism, so all the G_{ij} with $i\neq j$ are 0 and we are left with

$$\frac{1}{\mathcal{V}}\sum_p \begin{pmatrix} G_{\uparrow\uparrow}(C^\phi_\alpha+C^B_\alpha)\gamma^\alpha G_{\uparrow\uparrow} & 0\\ 0 & G_{\downarrow\downarrow}(C^\phi_\alpha-C^B_\alpha)\gamma^\alpha G_{\downarrow\downarrow}\end{pmatrix}=0. \tag{6.54}$$

It would suffice to demand that each component vanishes but we will determine the gauge factors k_α and f_α such that every single term in each component vanishes

$$\frac{1}{\mathcal{V}}\sum_p G_{ii}(p,\epsilon)C^\phi_\alpha(q,\omega)\gamma^\alpha G_{ii}(p-q,\epsilon-\omega)=0 \tag{6.55}$$

$$\frac{1}{\mathcal{V}}\sum_p G_{ii}(p,\epsilon)C^B_\alpha(q,\omega)\gamma^\alpha G_{ii}(p-q,\epsilon-\omega)=0 \tag{6.56}$$

where $i=\uparrow$ or $\downarrow$. Without external magnetic field $G_{\uparrow\uparrow}$ and $G_{\downarrow\downarrow}$ are equal and we are left with only 2 conditions. Let us discuss the second condition. The calculation for $C^\phi_\alpha(q,\omega)$ is completely equivalent. We examine

$$\frac{1}{\mathcal{V}}\sum_p \tilde{G}(p,\epsilon)\tilde{C}^B(q,\omega)\tilde{G}(p-q,\epsilon-\omega)=0 \tag{6.57}$$

in the **Keldysh rotated representation** and use the decomposition

$$\tilde{G}(p,\epsilon)=\begin{pmatrix}G^R(p,\epsilon) & F(\epsilon)(G^R(p,\epsilon)-G^A(p,\epsilon))\\ 0 & G^A(p,\epsilon)\end{pmatrix}=\frac{G^R(p,\epsilon)}{2}\big(\mathbb{I}_2+\tilde{\Lambda}(\epsilon)\big)+\frac{G^A(p,\epsilon)}{2}\big(\mathbb{I}_2-\tilde{\Lambda}(\epsilon)\big). \tag{6.58}$$

Then (6.57) reads

$$\begin{aligned}0 = \frac{1}{4}\sum_p & G^R(p,\epsilon)\tilde{C}^B_\alpha(q,\omega)G^R(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2+\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2+\tilde{\Lambda}(\epsilon-\omega))\Big] \\ &+ G^R(p,\epsilon)\tilde{C}^B_\alpha(q,\omega)G^A(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2+\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2-\tilde{\Lambda}(\epsilon-\omega))\Big] \\ &+ G^A(p,\epsilon)\tilde{C}^B_\alpha(q,\omega)G^R(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2-\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2+\tilde{\Lambda}(\epsilon-\omega))\Big] \\ &+ G^A(p,\epsilon)\tilde{C}^B_\alpha(q,\omega)G^A(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2-\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2-\tilde{\Lambda}(\epsilon-\omega))\Big].\end{aligned} \tag{6.59}$$

For the terms containing products of G^RG^R or G^AG^A the poles lie on equal sides of the real axis. Their contribution is negligible and we are left with

$$\begin{aligned}0 =& \sum_p \tilde{C}^B_\alpha(q,\omega)\Big(G^R(p,\epsilon)G^A(p-q,\epsilon-\omega)+G^A(p,\epsilon)G^R(p-q,\epsilon-\omega)\Big)\Big[\tilde{\gamma}^\alpha-\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\Big] \\ &+ \tilde{C}^B_\alpha(q,\omega)\Big(G^R(p,\epsilon)G^A(p-q,\epsilon-\omega)-G^A(p,\epsilon)G^R(p-q,\epsilon-\omega)\Big)\Big[\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha-\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\Big]\end{aligned} \tag{6.60}$$

It is useful to define

$$\Pi_1(q,\omega) := \frac{1}{\mathcal{V}}\sum_p G^R(p,\epsilon)G^A(p-q,\epsilon-\omega) \tag{6.61}$$

$$\Pi_2(q,\omega) := \frac{1}{\mathcal{V}}\sum_p G^R(p,\epsilon)(v_F\cdot q)G^A(p-q,\epsilon-\omega). \tag{6.62}$$

We then get

$$\begin{aligned}0 =& \Big(\big[\tilde{B}_\alpha(q,\omega)+i\omega\tilde{f}_\alpha(q,\omega)\big]\Pi_1(q,\omega)-i\tilde{f}_\alpha(q,\omega)\Pi_2(q,\omega) \\ &+ \big[\tilde{B}_\alpha(q,\omega)+i\omega\tilde{f}_\alpha(q,\omega)\big]\Pi_1^*(q,\omega)-i\tilde{f}_\alpha(q,\omega)\Pi_2^*(q,\omega)\Big)\Big[\tilde{\gamma}^\alpha-\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\Big] \\ &+ \Big(\big[\tilde{B}_\alpha(q,\omega)+i\omega\tilde{f}_\alpha(q,\omega)\big]\Pi_1(q,\omega)-i\tilde{f}_\alpha(q,\omega)\Pi_2(q,\omega) \\ &- \big[\tilde{B}_\alpha(q,\omega)+i\omega\tilde{f}_\alpha(q,\omega)\big]\Pi_1^*(q,\omega)-i\tilde{f}_\alpha(q,\omega)\Pi_2^*(q,\omega)\Big)\Big[\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha-\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\Big].\end{aligned} \tag{6.63}$$

We introduce the abbreviations

$$R_1(q,\omega) := \mathrm{Re}\ \Pi_1(q,\omega) \qquad R_2(q,\omega) := i\omega \mathrm{Re}\ \Pi_1(q,\omega) - i\mathrm{Re}\ \Pi_2(q,\omega) \tag{6.64}$$

$$I_1(q,\omega) := \mathrm{Im}\ \Pi_1(q,\omega) \qquad R_2(q,\omega) := i\omega \mathrm{Im}\ \Pi_1(q,\omega) - i\mathrm{Im}\ \Pi_2(q,\omega). \tag{6.65}$$

Multiplying (6.63) by $\tilde{\Lambda}(\epsilon)$ from the left side and using that $\tilde{\Lambda}(\epsilon)^2 = \mathbb{I}_2$, we obtain

$$\begin{aligned} 0 = {} & 2\big(R_1(q,\omega)\tilde{B}_\alpha + R_2(q,\omega)\tilde{f}_\alpha\big)\big[\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha - \tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\big] \\ & - 2i\big(I_1(q,\omega)\tilde{B}_\alpha + I_2(q,\omega)\tilde{f}_\alpha\big)\big[\tilde{\gamma}^\alpha - \tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega)\big]. \end{aligned} \tag{6.66}$$

[6] The matrix products are

$$\tilde{\Lambda}(\epsilon)\tilde{\gamma}^1 - \tilde{\gamma}^1\tilde{\Lambda}(\epsilon-\omega) = [F(\epsilon) - F(\epsilon-\omega)]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.67}$$

$$\tilde{\Lambda}(\epsilon)\tilde{\gamma}^2 - \tilde{\gamma}^2\tilde{\Lambda}(\epsilon-\omega) = \begin{pmatrix} F(\epsilon) & 1 \\ -1 & -F(\epsilon-\omega) \end{pmatrix} \tag{6.68}$$

$$\tilde{\Lambda}(\epsilon)\tilde{\gamma}^1\tilde{\Lambda}(\epsilon-\omega) - \tilde{\gamma}^1 = -[F(\epsilon) - F(\epsilon-\omega)]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.69}$$

$$\tilde{\Lambda}(\epsilon)\tilde{\gamma}^2\tilde{\Lambda}(\epsilon-\omega) - \tilde{\gamma}^2 = \begin{pmatrix} F(\epsilon) & 2F(\epsilon)F(\epsilon-\omega) - 1 \\ -1 & -F(\epsilon-\omega) \end{pmatrix}. \tag{6.70}$$

Defining

$$m_{11} := \big[(R_1\tilde{B}_2 + R_2\tilde{f}_2) - i(I_1\tilde{B}_2 + I_2\tilde{f}_2)\big]F(\epsilon) \tag{6.71}$$

$$\begin{aligned} m_{12} := {} & \big[(R_1\tilde{B}_1 + R_2\tilde{f}_1) + i(I_1\tilde{B}_1 + I_2\tilde{f}_1)\big]\big(F(\epsilon) - F(\epsilon-\omega)\big) \\ & + \big[(R_1\tilde{B}_2 + R_2\tilde{f}_2) + i(I_1\tilde{B}_2 + I_2\tilde{f}_2)\big] - i(I_1\tilde{B}_2 + I_2\tilde{f}_2)\big(2F(\epsilon)F(\epsilon-\omega) - 1\big) \end{aligned} \tag{6.72}$$

$$m_{21} := -\big[(R_1\tilde{B}_2 + R_2\tilde{f}_2) - i(I_1\tilde{B}_2 + I_2\tilde{f}_2)\big] \tag{6.73}$$

$$m_{22} := -\big[(R_1\tilde{B}_2 + R_2\tilde{f}_2) - i(I_1\tilde{B}_2 + I_2\tilde{f}_2)\big]F(\epsilon-\omega) \tag{6.74}$$

we have the matrix equation

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = 0. \tag{6.75}$$

From $m_{11} = m_{21} = m_{22} = 0$ we obtain

[6] The minus sign in front of the second row appears due to the changed order of the matrices in the second bracket.

$$(R_1 - iI_1)\tilde{B}_2 + (R_2 - iI_2)\tilde{f}_2 = 0 \tag{6.76}$$

and from the condition $m_{12} = 0$ we get

$$(R_1 + iI_1)\tilde{B}_1 + (R_2 + iI_2)\tilde{f}_1 = -2i\,(I_1\tilde{B}_2 + I_2\tilde{f}_2)\,\frac{1 - F(\epsilon)F(\epsilon-\omega)}{F(\epsilon) - F(\epsilon-\omega)}. \tag{6.77}$$

In thermal equilibrium $F(\epsilon) = \tanh(\beta\epsilon/2)$ and

$$\frac{1 - F(\epsilon)F(\epsilon-\omega)}{F(\epsilon) - F(\epsilon-\omega)} = \coth\left(\frac{\beta\omega}{2}\right) =: B(\omega). \tag{6.78}$$

We can combine equations (6.76) and (6.77) to the matrix equation

$$\begin{pmatrix} R_2 + iI_2 & 2i\ I_2\ B(\omega) \\ 0 & R_2 - iI_2 \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = -\begin{pmatrix} R_1 + iI_1 & 2i\ I_1\ B(\omega) \\ 0 & R_1 - iI_1 \end{pmatrix} \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}. \tag{6.79}$$

Inverting the matrix on the left hand side we obtain

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R_1+iI_1}{R_2+iI_2} & 2i\ B(\omega)\ \frac{R_1I_2 - R_2I_1}{(R_2+iI_2)(R_2-iI_2)} \\ 0 & -\frac{R_1-iI_1}{R_2-iI_2} \end{pmatrix} \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \tag{6.80}$$

We introduce

$$\mathcal{D}^R := -\frac{R_1 + iI_1}{R_2 + iI_2} \qquad \text{and} \qquad \mathcal{D}^A := \frac{R_1 - iI_1}{R_2 - iI_2}. \tag{6.81}$$

From the definition of R_2 and I_2 one can see that $\mathcal{D}^R = (\mathcal{D}^A)^*$. With equation (6.81) one can write equation (6.80) in the form[7]

$$\boxed{\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \mu_B \begin{pmatrix} \mathcal{D}^R & B(\omega)(\mathcal{D}^R + \mathcal{D}^A) \\ 0 & -\mathcal{D}^A \end{pmatrix} \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}} \tag{6.82}$$

Let us introduce the short hand notation

$$\tilde{f}[\tilde{B}](q,\omega) = \mu_B \tilde{A}(q,\omega)\tilde{B}(q,\omega) \tag{6.83}$$

with $\tilde{f}\ [B] = (\tilde{f}_1, \tilde{f}_2)^T$, $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)^T$ and

$$\tilde{A}(q,\omega) = \begin{pmatrix} \mathcal{D}^R & B(\omega)(\mathcal{D}^R + \mathcal{D}^A) \\ 0 & -\mathcal{D}^A \end{pmatrix}. \tag{6.84}$$

The matrix $\tilde{A}(q,\omega)$ is uniquely determined by $\mathcal{D}^R$, which can be expressed as a function of Π_1 and Π_2.

$$\mathcal{D}^R(q,\omega) = \frac{\Pi_1(q,\omega)}{-i\Pi_2(q,\omega) - i\omega\Pi_1(q,\omega)} \tag{6.85}$$

For the scalar field Φ one obtains in the same way

[7] where we reinserted μ_B.

$$\boxed{\begin{pmatrix}\tilde{k}_1\\ \tilde{k}_2\end{pmatrix} = e\begin{pmatrix}\mathcal{D}^R & B(\omega)(\mathcal{D}^R+\mathcal{D}^A)\\ 0 & -\mathcal{D}^A\end{pmatrix}\begin{pmatrix}\tilde{\phi}_1\\ \tilde{\phi}_2\end{pmatrix}} \tag{6.86}$$

or in short notation

$$\tilde{k}[\tilde{\Phi}](q,\omega) = e\tilde{A}(q,\omega)\tilde{\Phi}(q,\omega). \tag{6.87}$$

The gauge factors k_α are proportional the components of Φ whereas the gauge factors f_α are proportional to the absolute value of $\vec{B}$. We will see later that the appearance of the absolute value of $\vec{B}$ in the f_α's complicates the evaluation of the path integral over the components of $\vec{B}$.

$\mathcal{D}^R(q,\omega)$ in the diffusive case

We evaluate Π_1 and Π_2 in the continuum approximation. The calculation is analoguous to the one of Π_0 in section (2.2.1) and one obtains in the diffusive limit ($ql = v_F q\tau \ll 1$ and $\omega\tau \ll 1$)

$$\Pi_1(q,\omega) \approx 2\pi\nu_3\tau\left(1 - i\omega\tau - (v_F q\tau)^2\right). \tag{6.88}$$

and

$$\Pi_2(q,\omega) \approx -2\pi i\nu_3(v_F q\tau)^2. \tag{6.89}$$

Putting the results for Π_1 and Π_2 in equation (6.85) and keeping only the lowest order terms one finally finds

$$\boxed{\mathcal{D}^R(q,\omega) = \frac{1}{Dq^2 - i\omega}} \tag{6.90}$$

where we introduced the diffusion constant $D = v_F^2\tau/3 = v_F l/3$.

The last thing we need to do before we can calculate Green's functions, is to determine the Gaussian fluctuations of the fields Φ and B. We will see that these quadratic corrections together with quadratic terms containing $V_{0,s}^{-1}$ and $V_{0,t}^{-1}$ will correspond to the dynamically screened RPA interactions, that we already derived using perturbation theory.

6.4 Gaussian fluctuations of the fields Φ and B

We approximated in section (6.2)

$$-i\mathrm{Tr}\ln\left[G_{\mathcal{R}U}^{-1}\right] \qquad \text{by} \qquad -i\mathrm{Tr}\ln\left[\tilde{\hat{G}}_\tau^{-1} + C\right]. \tag{6.91}$$

Expanding

$$-i\mathrm{Tr}\ln\left[\hat{G}_\tau^{-1} + C\right] = -i\mathrm{Tr}\ln\left[\hat{G}_\tau^{-1}\right] - i\mathrm{Tr}\ln\left[\mathbb{I}_4 + C\hat{G}_\tau\right] \tag{6.92}$$

in orders of C yields

$$- i\mathrm{Tr}\ln\big[\hat{G}_\tau^{-1}\big] - i\frac{(-1)^{n-1}}{n}\ \mathrm{Tr}\Big(\big[\hat{G}_\tau C\big]^n\Big) \qquad\qquad n\geq 2. \tag{6.93}$$

We restrict ourselves to the **Gaussian approximation** which means that we only take into account the term $n=2$ and neglect the higher order terms. That corresponds to the assumption that the fluctuations around the saddle point are small. For our translationally invariant system the second order term reads

$$S_2 = \frac{i}{2}\ \mathrm{Tr}\big[\hat{G}(x-y)C(y-y')\hat{G}(y'-y'')C(y''-x')\big]. \tag{6.94}$$

We Fourier transform S_2 and find

$$S_2 = \frac{i}{2}\ \mathrm{tr}\int\frac{d\omega}{2\pi}\int\frac{d\epsilon}{2\pi}\frac{1}{\mathcal{V}^2}\sum_{p,q}\Big[G(p,\epsilon)C_{\hat{p}}(q,\omega)G(p-q,\epsilon-\omega)C_{\hat{p}}(-q,-\omega)\Big] \tag{6.95}$$

where $\hat{p}=p/|p|$ is the unit vector in the direction of p. In this equation we already replaced $C_{p-q/|p-q|}$ by $C_{\hat{p}}$. This should be a good approximation as $q\ll p$. In the **Keldysh rotated representation** we have

$$S_2 = \frac{i}{2\mathcal{V}^2}\ \mathrm{tr}\int\frac{d\omega}{2\pi}\int\frac{d\epsilon}{2\pi}\sum_q\Bigg[\sum_p\begin{pmatrix}\tilde{G}_{\uparrow\uparrow}(\tilde{C}^\phi_\alpha+\tilde{C}^B_\alpha)\tilde{\gamma}^\alpha\tilde{G}_{\uparrow\uparrow} & 0\\ 0 & \tilde{G}_{\downarrow\downarrow}(\tilde{C}^\phi_\alpha-\tilde{C}^B_\alpha)\tilde{\gamma}^\alpha\tilde{G}_{\downarrow\downarrow}\end{pmatrix}\Bigg]\tilde{\gamma}^\beta\begin{pmatrix}\tilde{C}^\phi_\beta+\tilde{C}^B_\beta\\ \tilde{C}^\phi_\beta-\tilde{C}^B_\beta\end{pmatrix}. \tag{6.96}$$

Without external magnetic field $G_{\uparrow\uparrow}=G_{\downarrow\downarrow}$, and there remains the combination

$$2\,\tilde{G}\ \tilde{\gamma}^\alpha\tilde{G}\ \tilde{\gamma}^\beta\big[\tilde{C}^\phi_\alpha\ \tilde{C}^\phi_\beta+\tilde{C}^B_\alpha\ \tilde{C}^B_\beta\big] \tag{6.97}$$

where $\tilde{G}$ is given in (4.45). The expression $\sum_p GCG|_{r=r'}$ within the brackets of equation (6.96) was treated already in the section 6.3 where we determined the gauge factors. In section 6.3 we argued that the contribution relevant for small q comes from the G^RG^A and G^AG^R terms and neglected the G^RG^R and G^AG^A contributions. Here, however, we have an additional integration over ϵ which means that we have to take into account the G^RG^R and G^AG^A terms. The gauge factors f_α where chosen such that equation (6.60) is fulfilled. So we are left with[8]

$$\begin{aligned}\sum_\beta\Bigg\{\frac{1}{4\mathcal{V}}\sum_{p,\alpha}&G^R(p,\epsilon)\tilde{C}_\alpha(q,\omega)G^R(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2+\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2+\tilde{\Lambda}(\epsilon-\omega))\Big]\\ &+G^A(p,\epsilon)\tilde{C}_\alpha(q,\omega)G^A(p-q,\epsilon-\omega)\Big[(\mathbb{I}_2-\tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2-\tilde{\Lambda}(\epsilon-\omega))\Big]\Bigg\}\tilde{\gamma}^\beta\tilde{C}_\beta\end{aligned} \tag{6.98}$$

where $\tilde{C}_\alpha$ is either $\tilde{C}^\phi_\alpha$ or $\tilde{C}^B_\alpha$. The calculation for both terms proceeds identically. We define

$$M^\alpha_1 := \tilde{\gamma}^\alpha+\tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega) \qquad\text{and}\qquad M^\alpha_2 := \tilde{\Lambda}(\epsilon)\tilde{\gamma}^\alpha+\tilde{\gamma}^\alpha\tilde{\Lambda}(\epsilon-\omega) \tag{6.99}$$

in order to rewrite the matrix products in (6.98). One has

[8] The factor 1/4 stems from the decomposition given in equation (6.58).

$$M_1^\alpha + M_2^\alpha = (\mathbb{I}_2 + \tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2 + \tilde{\Lambda}(\epsilon - \omega)) \tag{6.100}$$

and

$$M_1^\alpha - M_2^\alpha = (\mathbb{I}_2 - \tilde{\Lambda}(\epsilon))\tilde{\gamma}^\alpha(\mathbb{I}_2 - \tilde{\Lambda}(\epsilon - \omega)). \tag{6.101}$$

We then have

$$\begin{aligned}\frac{1}{4}\sum_{\alpha,\beta}\tilde{C}_\alpha(q,\omega)\Big\{\int\frac{d^3p}{(2\pi)^3}\,\big[G^R(p,\epsilon)G^R(p-q,\epsilon-\omega)\big]\big[M_1^\alpha + M_2^\alpha\big]\tilde{\gamma}^\beta \\ + \int\frac{d^3p}{(2\pi)^3}\,\big[G^A(p,\epsilon)G^A(p-q,\epsilon-\omega)\big]\big[M_1^\alpha - M_2^\alpha\big]\tilde{\gamma}^\beta\Big\}\tilde{C}_\beta(-q,-\omega).\end{aligned} \tag{6.102}$$

As a next step we form combinations of G^RG^R and G^AG^A and define

$$\pi_\pm := \nu_3\int_{-\epsilon_F}^{\infty} d\epsilon_p\,\big[G^R(p,\epsilon)G^R(p-q,\epsilon-\omega) \pm G^A(p,\epsilon)G^A(p-q,\epsilon-\omega)\big] \tag{6.103}$$

where we split up the p–integration into the ϵ_p–integration and the angular integration.

$$\frac{1}{\mathcal{V}}\sum_p ... \approx \nu_3\int_{-\epsilon_F}^{\infty} d\epsilon_p\int\frac{d\Omega_p}{4\pi}\,... \tag{6.104}$$

This allows us to write the quadratic contribution S_2 as

$$S_2 = i\,\mathrm{tr}\frac{1}{\mathcal{V}}\sum_q\int\frac{d\omega}{2\pi}\int\frac{d\Omega_p}{4\pi}\int\frac{d\epsilon}{2\pi}\,\frac{1}{4}\sum_{\alpha,\beta}\,\tilde{C}_\alpha(q,\omega)\Big\{\pi_+\big[M_1^\alpha\tilde{\gamma}^\beta\big]+\pi_-\big[M_2^\alpha\tilde{\gamma}^\beta\big]\Big\}\tilde{C}_\beta(-q,-\omega). \tag{6.105}$$

We will not present explicitly here the calculation of π_+ and π_-. We just remark that one is not allowed to shift the lower boundary of the ϵ_p–integration from $-\epsilon_F \to -\infty$ (like in equation (2.15)) because there is a contribution coming from $-\epsilon_F$. Doing subsequently the ϵ–integration we get

$$\int\frac{d\epsilon}{2\pi}\pi_+(q,\epsilon,\omega) = 0 \qquad \text{and} \qquad \int\frac{d\epsilon}{2\pi}\pi_-(q,\epsilon,\omega) = \int\frac{d\epsilon}{2\pi}\,2\pi i\nu_3\delta(\epsilon+\epsilon_F) = i\nu_3. \tag{6.106}$$

Thus the first term in (6.105) vanishes. The trace of the expression proportional to π_- is

$$\mathrm{tr}\big[M_2^\alpha\gamma^\beta\big] = 2\begin{pmatrix}0 & F(\epsilon)+F(\epsilon-\omega)\\ F(\epsilon)+F(\epsilon-\omega) & 0\end{pmatrix}. \tag{6.107}$$

As π_- only contributes at $\epsilon = -\epsilon_F$, we need the matrices above only in the limit $\epsilon \to -\infty$.

$$\mathrm{tr}\big[M_2^\alpha\gamma^\beta\big]\Big|_{\epsilon\to-\infty} = -4\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{6.108}$$

because $\tanh(\beta\epsilon/2)|_{\epsilon\to\infty} = -1$ and we are left with

$$S_2 = \frac{\nu_3}{\mathcal{V}}\sum_q\int\frac{d\omega}{2\pi}\sum_{\alpha,\beta}\int\frac{d\Omega}{\Omega_p}\,\tilde{C}_\alpha(q,\omega)\,\sigma_x^{\alpha\beta}\,\tilde{C}_\beta(-q,-\omega). \tag{6.109}$$

The angular average has to performed over products of the $\tilde{C}$'s. Let us take for example the $\tilde{C}^B$'s

$$\tilde{C}^B_\alpha(q,\omega)\,\tilde{C}^B_\beta(-q,-\omega) = \big[i\omega\tilde{f}_\alpha(q,\omega)+i(v_F\cdot q)\tilde{f}_\alpha(q,\omega)+\tilde{B}_\alpha(q,\omega)\big] \\ \big[-i\omega\tilde{f}_\beta(-q,-\omega)-i(v_F\cdot q)f_\beta(-q,-\omega)+\tilde{B}_\beta(-q,-\omega)\big] \tag{6.110}$$

We get 3 types of expressions. First we have expressions that do not depend on $\hat{p}$ at all. They remain unchanged as the angular average is normalized. In (6.110) these terms are

$$\big[i\omega\tilde{f}_\alpha(q,\omega)+\tilde{B}_\alpha(q,\omega)\big]\big[-i\omega\tilde{f}_\beta(-q,-\omega)+\tilde{B}_\beta(-q,-\omega)\big]. \tag{6.111}$$

Second we get terms proportional to $(v_{\hat{p}}\cdot q)$ which vanish. And third there is a term which is proportional $(v_{\hat{p}}\cdot q)^2$

$$\tilde{f}_\alpha(q,\omega)\tilde{f}_\beta(-q,-\omega)\int\frac{d\Omega}{\Omega_p}(v_{\hat{p}}\cdot q)^2 = \tilde{f}_\alpha(q,\omega)\tilde{f}_\beta(-q,-\omega)\frac{v_F^2}{3}q^2 \tag{6.112}$$

Dividing S_2 into two terms

$$S_2^B := S_{2,I}^B + S_{2,II}^B \tag{6.113}$$

we finally obtain

$$S_{2,I}^B = \frac{\nu_3}{\mathcal{V}}\sum_q\int\frac{d\omega}{2\pi}\sum_{\alpha,\beta}\big[i\omega\tilde{f}_\alpha(q,\omega)+\tilde{B}_\alpha(q,\omega)\big]\sigma_x^{\alpha,\beta}\big[-i\omega\tilde{f}_\beta(-q,-\omega)+\tilde{B}_\beta(-q,-\omega)\big] \tag{6.114}$$

and

$$S_{2,II}^B = \frac{\nu_3 v_F^2}{3}\frac{1}{\mathcal{V}}\sum_q\int\frac{d\omega}{2\pi}\sum_{\alpha,\beta}\tilde{f}_\alpha(q,\omega)\;\sigma_x^{\alpha\beta}\;q^2\;\tilde{f}_\beta(-q,-\omega). \tag{6.115}$$

Using $\tilde{f}[B](q,\omega)=\tilde{A}(q,\omega)\tilde{B}(q,\omega)$ we can write

$$S_{2,I}^B = \frac{\nu_3}{\mathcal{V}}\sum_q\int\frac{d\omega}{2\pi}\big[\tilde{\Phi}^T(-q,-\omega)\Big[(\mathbb{I}_2-i\omega A(-q,-\omega))^T\sigma_x(\mathbb{I}_2-i\omega A(q,\omega))\Big]\tilde{\Phi}(q,\omega)\big]. \tag{6.116}$$

We define the inner part as

$$\Pi_g = (\mathbb{I}_2-i\omega A(-q,-\omega))^T\sigma_x(\mathbb{I}_2-i\omega A(q,\omega)). \tag{6.117}$$

Evaluating Π_g using the explicit form of $\mathcal{D}(q,\omega)$ given in equation (6.90) one arrives at

$$\Pi_g(q,\omega) = \tilde{P}_0+\tilde{P}_+ = \begin{pmatrix}0 & P_0^A(q,\omega)\\ P_0^R(q,\omega) & P_0^K(q,\omega)\end{pmatrix}+\begin{pmatrix}0 & P_+^A(q,\omega)\\ P_+^R(q,\omega) & P_+^K(q,\omega)\end{pmatrix} \tag{6.118}$$

where

$$P_0^R(q,\omega) = \nu_3\frac{i\omega}{Dq^2-i\omega} \tag{6.119}$$

and

$$P_0^A = (P_0^R)^* \tag{6.120}$$

and

$$P_0^K(q,\omega) = B(\omega)(P_0^R(q,\omega) - P_0^A(q,\omega)). \tag{6.121}$$

The same relations are valid for the components of the second matrix with entries P_+ where $P_+^R(q,\omega) = \nu_3 \frac{Dq^2}{Dq^2 - i\omega}$. We will need only the first matrix in the sequel. The term containing $\tilde{P}_+$ will disappear as well as $S_{2,II}^B$ when we write down the normalized generating functional $Z[J,\hat{J}]$.

Taking the quadratic term of equation (6.23) in the Keldysh rotated version ($\sigma_z \to \sigma_x$ see equation (5.56)) and combining it with the part containing $\tilde{P}_0$, we have in (q,ω)–representation

$$S_2^B = \frac{1}{\mathcal{V}} \sum_q \int \frac{d\omega}{2\pi} \Big[\sum_n (\tilde{B}^n)^T(q) \big[\sigma_x V_{0,t}^{-1}\big] \tilde{B}^n(-q) + \tilde{B}^T(q) \tilde{P}_0(q,\omega) \tilde{B}(-q) \Big]. \tag{6.122}$$

In our case $V_{0,t}^{-1}$ is a constant and does not depend on q. In order to relate the absolute value of B with the field components we need two things. First we remember that the field $\vec{B}$ was real so that we have $B(q) = B(-q)$. Furthermore treating B as an effectively classical field coming from outside like the potential $V(x)$ in the Schrödinger equation in first quantization, one has that $B_1(q) = B_2(q)$. Then we have

$$S_2^B = \frac{1}{\mathcal{V}} \sum_q \int \frac{d\omega}{2\pi} \Big[\sum_n (\tilde{B}^n)^T(q) \big[\sigma_1 V_{0,t}^{-1} + \tilde{P}_0(q,\omega)\big] \tilde{B}^n(-q) \Big]. \tag{6.123}$$

Let us evaluate the inner part:

$$\begin{aligned}
\tilde{V}_t(q,\omega) &= [V_{0,t}^{-1}\ \sigma_1 + \tilde{P}_0(q,\omega)]^{-1} \\
&= \begin{pmatrix} 0 & P_0^A(q,\omega) + V_{0,t}^{-1} \\ P_0^R(q,\omega) + V_{0,t}^{-1} & P_0^K(q,\omega) \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{-V_t^2 P_0^K(q,\omega)}{(1+P_0^A(q,\omega)V_t)(1+P_0^R(q,\omega)V_t)} & \frac{V_t}{1+P_0^R(q,\omega)V_t} \\ \frac{V_t}{1+P_0^A(q,\omega)V_t} & 0 \end{pmatrix} \\
&= \begin{pmatrix} V_t^K(q,\omega) & V_t^R(q,\omega) \\ V_t^A(q,\omega) & 0 \end{pmatrix}
\end{aligned} \tag{6.124}$$

where $V^K(q,\omega) = B(\omega)[V^R(q,\omega) - V^A(q,\omega)]$. Introducing now instead of V_t the renormalized vertex Γ_t as explained in chapter 2 we obtain for the retarded component

$$V^R = \frac{\Gamma_t}{1 + P_0^R(q,\omega)\Gamma_t} = \frac{\Gamma_t}{1 + \nu_3 \frac{i\omega}{Dq^2 - i\omega}\Gamma_t} = \Gamma_t \frac{Dq^2 - i\omega}{Dq^2 - i\omega Z_t} \tag{6.125}$$

where $Z_t = 1 - 2\nu_3\Gamma_t = 1 - A_0^a$. This result is identical to the RPA summed result in equation (2.44). Going likewise through all steps for the combination $C_\alpha^\phi C_\beta^\phi$ we find a dynamically screened singlet interaction

$$\Gamma_s(q,\omega) = \Gamma_s \frac{Dq^2 - i\omega}{Dq^2 - i\omega Z_s} \tag{6.126}$$

where $Z_s = 1 - 2\nu_3\Gamma_s = 1 - A_0^s$ which coincides with the result of equation (2.43).

We go back to the unrotated representation in order to compare our result with the action given in (6.23). The quadratic terms in the field now read

$$\boxed{S[\Phi^2, B^2] = \frac{1}{2}\int dx \int dx' \left[\Phi^T(x)\; V_s^{-1}(x,x')\Phi(x) + \sum_n (B^n)^T(x)\; V_t^{-1}(x,x')B^n(x')\right]} \tag{6.127}$$

where

$$V_s^{-1}(x) = [\sigma_z V_{0,s}^{-1}\; \delta(x-x') + P_0(x-x')] \tag{6.128}$$

and

$$V_t^{-1}(x) = [\sigma_z V_{0,t}^{-1}\; \delta(x-x') + P_0(x-x')]. \tag{6.129}$$

6.5 Calculation of the single particle Green's function

For the calculation of the single particle Green's function we take the normalized generating functional $Z[J,\hat{J}]$. [9]

$$Z[J,\hat{J}] = \frac{1}{\mathcal{N}_\phi \mathcal{N}_B}\int \mathcal{D}\Phi \int \mathcal{D}\vec{B}\; e^{iS[\Phi,\hat{Q}^{SP},\vec{B},J,\hat{J}]} \tag{6.130}$$

with the action

$$\begin{aligned} S[\Phi,\hat{Q}^{SP},\vec{B}] = &\int dx \int dx'\; \hat{J}(x)\mathcal{R}(x)U^{-1}(x)\; \hat{G}_\tau(x,x')\; U(x')\mathcal{R}^{-1}(x')J(x') \\ &+ \frac{1}{2}\int dx\; \Phi^T(x)\; V_s^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x)\; V_t^{-1}(x)B^n(x) \end{aligned} \tag{6.131}$$

where $\hat{G}_\tau(x,x')$ is given by equation (6.48). Explicitely

$$\hat{G}_\tau(t-t') = \mathbb{I}_2 \otimes \begin{pmatrix} G_\tau^T(t-t') & -G_\tau^<(t-t') \\ G_\tau^>(t-t') & -G_\tau^{\bar{T}}(t-t') \end{pmatrix}. \tag{6.132}$$

Differentiating the normalized generating functional with respect to the source fields J and $\hat{J}$, we obtain the Green's function in the presence of interaction and disorder

$$\boxed{G(x,x') = i\frac{\delta}{\delta J(x)}\frac{\delta}{\delta \hat{J}(x')}Z[J(x),\hat{J}(x')] = \left\langle \mathcal{R}(x)\; U^{-1}(x)\hat{G}_\tau(x,x')\; U(x')\; \mathcal{R}^{-1}(x')\right\rangle_{\phi,B}.} \tag{6.133}$$

[9] where normalized means that $Z[0,0] = 1$.

where[10]

$$\langle\, ...\rangle_\phi = \frac{1}{\mathcal{N}_\phi}\int \mathcal{D}\Phi\; ...e^{i/2\mathrm{Tr}[\Phi^T V_s^{-1}\Phi]} \qquad \text{and} \qquad \langle\, ...\rangle_B = \frac{1}{\mathcal{N}_B}\int \mathcal{D}\vec{B}\; ...e^{i/2\sum_n \mathrm{Tr}[(B^n)^T V_t^{-1} B^n]} \tag{6.134}$$

The considerable task that remains to be done is to average over the fluctuating fields Φ and $\vec{B}$. But before evaluating the path integrals one has to multiply the five 4×4 matrices. We begin with the inner matrices and denote the result by $G_F(x,x')$ indicating that these Green's already include the exponential gauge factors.

$$G_F(x,x') := U^{-1}(x)\hat{G}_\tau(x,x')U(x')$$

$$= \begin{pmatrix} e^{-i(k_1+f_1)}G_\tau^T e^{i(k_1+f_1)} & -e^{-i(k_1+f_1)}G_\tau^< e^{i(k_2+f_2)} & 0 & 0 \\ e^{-i(k_2+f_2)}G_\tau^> e^{i(k_1+f_1)} & -e^{-i(k_2+f_2)}G_\tau^{\bar{T}} e^{i(k_2+f_2)} & 0 & 0 \\ 0 & 0 & e^{-i(k_1-f_1)}G_\tau^T e^{i(k_1-f_1)} & -e^{-i(k_1-f_1)}G_\tau^< e^{i(k_2-f_2)} \\ 0 & 0 & e^{-i(k_2-f_2)}G_\tau^> e^{i(k_1-f_1)} & -e^{-i(k_2-f_2)}G_\tau^{\bar{T}} e^{i(k_2-f_2)} \end{pmatrix} \tag{6.135}$$

where the left exponential factor always depends on x and the right on x'. Next we multiply the rotation matrices on both sides. $\mathcal{R}$ depends on x and $\mathcal{R}^{-1}$ depends on x'. This gives

$$G(x,x') = \mathcal{R}(x)\; G_F(x,x')\; \mathcal{R}^{-1}(x') =$$

$$\begin{pmatrix} \mathcal{R}_{11} & 0 & \mathcal{R}_{13} & 0 \\ 0 & \mathcal{R}_{22} & 0 & \mathcal{R}_{24} \\ \mathcal{R}_{31} & 0 & \mathcal{R}_{33} & 0 \\ 0 & \mathcal{R}_{42} & 0 & \mathcal{R}_{44} \end{pmatrix} \begin{pmatrix} G_{F\uparrow\uparrow}^T & -G_{F\uparrow\uparrow}^< & 0 & 0 \\ G_{F\uparrow\uparrow}^> & -G_{F\uparrow\uparrow}^{\bar{T}} & 0 & 0 \\ 0 & 0 & G_{F\downarrow\downarrow}^T & -G_{F\downarrow\downarrow}^< \\ 0 & 0 & G_{F\downarrow\downarrow}^> & -G_{F\downarrow\downarrow}^{\bar{T}} \end{pmatrix} \begin{pmatrix} \mathcal{R}_{11}^* & 0 & \mathcal{R}_{31} & 0 \\ 0 & \mathcal{R}_{22}^* & 0 & \mathcal{R}_{42} \\ \mathcal{R}_{13}^* & 0 & \mathcal{R}_{33} & 0 \\ 0 & \mathcal{R}_{24}^* & 0 & \mathcal{R}_{44} \end{pmatrix}. \tag{6.136}$$

We denote the matrix elements as follows:

$$G(x,x') = \begin{pmatrix} G_{\uparrow\uparrow}^T & -G_{\uparrow\downarrow}^< & G_{\uparrow\downarrow}^> & -G_{\uparrow\downarrow}^{\bar{T}} \\ G_{\uparrow\uparrow}^> & -G_{\uparrow\uparrow}^T & G_{\uparrow\downarrow}^> & -G_{\uparrow\downarrow}^T \\ G_{\downarrow\uparrow}^T & -G_{\downarrow\uparrow}^< & G_{\downarrow\downarrow}^T & -G_{\downarrow\downarrow}^< \\ G_{\downarrow\uparrow}^> & -G_{\downarrow\uparrow}^{\bar{T}} & G_{\downarrow\downarrow}^> & -G_{\downarrow\downarrow}^{\bar{T}} \end{pmatrix}(x,x'). \tag{6.137}$$

In particular one finds

$$G_{\uparrow\uparrow}^> = \mathcal{R}_{22}\mathcal{R}_{11}^* G_{F\uparrow\uparrow}^> + \mathcal{R}_{24}\mathcal{R}_{13}^* G_{F\downarrow\downarrow}^> \qquad G_{\uparrow\uparrow}^< = \mathcal{R}_{11}\mathcal{R}_{22}^* G_{F\uparrow\uparrow}^< + \mathcal{R}_{13}\mathcal{R}_{24}^* G_{F\downarrow\downarrow}^< \tag{6.138}$$

$$G_{\downarrow\downarrow}^> = \mathcal{R}_{42}\mathcal{R}_{31} G_{F\uparrow\uparrow}^> + \mathcal{R}_{44}\mathcal{R}_{33} G_{F\downarrow\downarrow}^> \qquad G_{\downarrow\downarrow}^< = \mathcal{R}_{31}\mathcal{R}_{42} G_{F\uparrow\uparrow}^< + \mathcal{R}_{33}\mathcal{R}_{44} G_{F\downarrow\downarrow}^< \tag{6.139}$$

These are the four Green's functions we need in order to determine the DOS from equation (6.2). Let us pick $G_{\uparrow\uparrow}^>(x,x')$ and calculate the averages over ϕ and $\vec{B}$. Afterwards, we will

[10] with a normalization constants $\mathcal{N}_\phi = \mathrm{Det}[i\sigma_z V_s^{-1}]^{1/2}$ and $\mathcal{N}_B = \mathrm{Det}[i\sigma_z V_t^{-1}]^{3/2}$

shortly comment on what is different for the other three Green's functions. Writing down explicitly the dependence on the fields, the Green's function $G^{>}_{\uparrow\uparrow}$ reads

$$G^{>}_{\uparrow\uparrow} = \Big\langle \mathcal{R}_{22}[\vec{B}_2(x)]\mathcal{R}^*_{11}[\vec{B}_1(x')]e^{-if_2(B(x))}e^{if_1(B(x'))} + \mathcal{R}_{24}[\vec{B}_2(x)]\mathcal{R}^*_{13}[\vec{B}_1(x')]e^{if_2(B(x))}e^{-if_1(B(x'))}\Big\rangle_B \times \left\langle e^{-ik_2(\Phi(x))}e^{ik_1(\Phi(x'))}\right\rangle_\Phi G^{>}_\tau(x-x'). \tag{6.140}$$

So $\mathcal{R}^*_{11}$ depends on the components of the magnetic field $\vec{B}_1(x')$ on the upper branch while $\mathcal{R}_{22}$ depends on the components of the magnetic field $\vec{B}_2(x)$ on the lower branch. Remember that the lower index stands for the Keldysh branch. The gauge factors f_α, however, only depend on the absolute values of $\vec{B}_\alpha$ but of both branches (see equation (6.82)).[11] Finally the gauge factors k_α depend on $\Phi = (\phi_1, \phi_2)^T$. In the following we shorten the notation by omitting the $\vec{B}$'s and ϕ's, so that

$$G^{>}_{\uparrow\uparrow} = \left\langle \mathcal{R}_{22}(x)\mathcal{R}^*_{11}(x')\, e^{-if_2(x)}e^{if_1(x')} + \mathcal{R}_{24}(x)\mathcal{R}^*_{13}(x')\, e^{if_2(x)}e^{-if_1(x')}\right\rangle_B \times \left\langle e^{-ik_2(x)}e^{ik_1(x')}\right\rangle_\phi G^{>}_\tau(x-x') \tag{6.141}$$

The averaging over the Φ fields can be done exactly as we will see in the next part. However the path integral over the components of the magnetic field is not Gaussian and we will have to approximate it.

6.5.1 Path integral over scalar field Φ

We have to determine:

$$P_\phi := \left\langle e^{-ik_2(x)}e^{ik_1(x')}\right\rangle_\phi = \frac{1}{\mathcal{N}_\phi}\int \mathcal{D}\Phi\; e^{\frac{i}{2}\int dy \int dy' \Phi^T(y)V_s^{-1}(y,y')\Phi(y')}e^{-ik_2(x)}e^{ik_1(x')} \tag{6.142}$$

where

$$V_s^{-1}(y,y') = \left[\sigma_3 V_{0,s}^{-1}\delta(y-y') + P_0(y-y')\right] \tag{6.143}$$

and $k_\alpha(x)$ is given in the unrotated representation by[12]

$$k_\alpha(x) = e\sum_{\beta=1}^{2}\int dy\; A_{\alpha\beta}(x,y)\phi_\beta(y). \tag{6.144}$$

The Gaussian path integral will lead to an exponential. Let us denote the exponent by $J^\phi_{\alpha\alpha'}(x,x')$ defined by

[11] Our notation is $B = (B_1, B_2)^T$.
[12] $A = L\tilde{A}L^{-1}$.

$$e^{J^{\phi}_{\alpha\alpha'}(x,x')} := \left\langle e^{ik_\alpha(x)} e^{-ik_{\alpha'}(x')} \right\rangle_\phi = \frac{1}{\mathcal{N}_\phi} \int \mathcal{D}\Phi \; e^{\frac{i}{2}\int dy \int dy' \Phi^T(y) V_s^{-1}(y,y') \Phi(y') - i[k_\alpha(x) - k_{\alpha'}(x')]}. \tag{6.145}$$

We will skip the index of the interaction for the rest of this chapter writing simply V^{-1} instead of V_s^{-1}. Employing the definition of the $k's$ one has

$$e^{J^{\phi}_{\alpha\alpha'}(x,x')} = \frac{1}{\mathcal{N}_\phi} \int \mathcal{D}\Phi \; \exp\Big(\frac{i}{2} \sum_{\beta\beta'} \int dy \int dy' \phi_\beta(y) V^{-1}_{\beta\beta'}(y,y') \phi_{\beta'}(y') + i \sum_\beta \int dy \; \big[A_{\alpha\beta}(x,y) - A_{\alpha'\beta}(x',y) \big] \phi_\beta(y) \Big) \tag{6.146}$$

where the right hand side combines to give

$$= \exp\Big(-\frac{i}{2} \sum_{\beta\beta'} \int dy \; dy' \big[A_{\alpha\beta}(x,y) - A_{\alpha'\beta}(x',y) \big] \breve{V}_{\beta\beta'}(y,y') \big[A_{\alpha\beta'}(x,y') - A_{\alpha'\beta'}(x',y') \big] \Big). \tag{6.147}$$

Performing the path integral effectively inverts $V^{-1}_{\beta\beta'}(y,y')$. In the unrotated representation one has [13]

$$\breve{V}(x,x') = \begin{pmatrix} V^T(x,x') & V^<(x,x') \\ V^>(x,x') & V^{\bar{T}}(x,x') \end{pmatrix}. \tag{6.148}$$

The exponent in (6.147) contains terms of the form:

$$\mathcal{V}(x,x') = \int dy \int dy' A(x,y) \breve{V}(y,y') A^T(x',y') \tag{6.149}$$

It is straightforward to show that $\mathcal{V}(x,x')$ has the same time structure as $\breve{V}(x,x')$

$$\mathcal{V}(x,x') = \begin{pmatrix} \mathcal{V}^T(x,x') & \mathcal{V}^<(x,x') \\ \mathcal{V}^>(x,x') & \mathcal{V}^{\bar{T}}(x,x') \end{pmatrix}. \tag{6.150}$$

We write $J^{\phi}_{\alpha\alpha'}(x,x')$ in terms of $\mathcal{V}$ as

$$J^{\phi}_{\alpha\alpha'}(x,x') = \frac{1}{2i} \big[\mathcal{V}_{\alpha\alpha}(x,x) - \mathcal{V}_{\alpha\alpha'}(x,x') - \mathcal{V}_{\alpha'\alpha}(x',x) + \mathcal{V}_{\alpha'\alpha'}(x',x') \big] \tag{6.151}$$

or

$$J^{\phi}_{\alpha\alpha'}(x,x') = -\frac{1}{2i} \big[\mathcal{V}_{\alpha\alpha'}(x,x') + \mathcal{V}_{\alpha'\alpha}(x',x) - \mathcal{V}_{\alpha\alpha}(x,x) - \mathcal{V}_{\alpha'\alpha'}(x',x') \big]. \tag{6.152}$$

This is the general result for $J^{\phi}_{\alpha\alpha'}(x,x')$. Let us determine the explicit expression for $J^{\phi}_{21}(x,x')$ as we need only this component for our Green's function $G^{>}_{\uparrow\uparrow}(x,x')$.

$$J^{\phi}_{21}(x,x') = \frac{i}{2} \big[\mathcal{V}_{21}(x,x') + \mathcal{V}_{12}(x',x) - \mathcal{V}_{11}(x,x) - \mathcal{V}_{22}(x',x') \big] \tag{6.153}$$

[13] See section (5.1).

Using (6.150) we have

$$J_{21}^{\phi}(x,x') = \frac{i}{2}\big[\mathcal{V}^{>}(x,x') + \mathcal{V}^{<}(x',x) - \mathcal{V}^{T}(x,x) - \mathcal{V}^{\bar{T}}(x',x')\big]. \tag{6.154}$$

This can be rewritten using the relations (5.53), (5.54) and (5.55) as

$$J_{21}^{\phi}(x,x') = \frac{i}{2}\left[\frac{1}{2}\left(\mathcal{V}^{K} + \mathcal{V}^{R} - \mathcal{V}^{A}\right)(x,x') + \frac{1}{2}\left(\mathcal{V}^{K} - \mathcal{V}^{R} + \mathcal{V}^{A}\right)(x',x) - \mathcal{V}^{K}(x,x)\right] \tag{6.155}$$

where we also used the thermal equilibrium property $\mathcal{V}(x,x) = \mathcal{V}(x',x')$. In order to simplify this further we assume in addition a translationally invariant system which means that $J^{\phi}(x,x')$ depends only on $x - x' =: (r,t)$.

$$J_{21}^{\phi}(r,t) = \frac{i}{4\mathcal{V}}\int\frac{d\omega}{2\pi}\sum_{q}\Big[(\mathcal{V}^{K}+\mathcal{V}^{R}-\mathcal{V}^{A})(q,\omega)\; e^{iqr-i\omega t} + (\mathcal{V}^{K}-\mathcal{V}^{R}+\mathcal{V}^{A})(q,\omega)\; e^{-iqr+i\omega t} - 2\mathcal{V}^{K}(q,\omega)\Big] \tag{6.156}$$

We rearrange the terms in the form

$$\begin{aligned} J_{21}^{\phi}(r,t) = \frac{i}{4}\int\frac{d\omega}{2\pi}\frac{1}{\mathcal{V}}\sum_{q}\Big[&\mathcal{V}^{K}(q,\omega)\;(e^{iq\cdot r-i\omega t} + e^{-iq\cdot r+i\omega t}) \\ &+ (\mathcal{V}^{R} - \mathcal{V}^{A})(q,\omega)\;(e^{iq\cdot r-i\omega t} - e^{-iq\cdot r+i\omega t}) - 2\mathcal{V}^{K}(q,\omega)\Big] \end{aligned} \tag{6.157}$$

and get

$$J_{21}^{\phi}(r,t) = \frac{i}{2}\int\frac{d\omega}{2\pi}\frac{1}{\mathcal{V}}\sum_{q}\Big[\mathcal{V}^{K}(q,\omega)\cos(q\cdot r-\omega t) + i(\mathcal{V}^{R}-\mathcal{V}^{A})(q,\omega)\sin(q\cdot r-\omega t) - 2\mathcal{V}^{K}(q,\omega)\Big]. \tag{6.158}$$

Now using the result of the following section (6.5.2)[14]

$$\mathcal{V}^{K}(q,\omega) = B(\omega)\big[\mathcal{V}^{R} - \mathcal{V}^{A}\big](q,\omega) = 2iB(\omega)\mathrm{Im}\mathcal{V}^{R} \tag{6.159}$$

where used the relation $(\mathcal{V}^{A})^{*} = \mathcal{V}^{R}$ and $B(\omega) = \coth(\beta\omega/2)$. One obtains

$$J_{21}^{\phi}(r,t) = -\int\frac{d\omega}{2\pi}\underbrace{\frac{1}{\mathcal{V}}\sum_{q}\mathrm{Im}\mathcal{V}_{s}^{R}(q,\omega)}_{=:\mathrm{Im}Y_{s}(\omega)}\Big[B(\omega)\left(\cos(q\cdot r-\omega t)-1\right) + i\sin(q\cdot r-\omega t)\Big]. \tag{6.160}$$

where we defined the function

$$Y_{s}(\omega) = \frac{1}{\mathcal{V}}\sum_{q}\mathcal{V}_{s}^{R}(q,\omega). \tag{6.161}$$

[14] See equation (6.170).

We put the index s for singlet because $\mathcal{V}_s$ contains the singlet interaction. If one looks especially at equal positions in space (which means $r = 0$) and also notes that the imaginary part of $\mathcal{V}_s^R$ is an odd function, which can be seen from the explicit form of $\mathcal{V}^R$ given in equation (6.171), one gets

$$\boxed{J_{21}^{\phi}(0,t) = -2\int_0^{\infty} \frac{d\omega}{2\pi}\, \mathrm{Im}Y_s(\omega)\Big[B(\omega)\left(\cos(\omega t) - 1\right) - i\sin(\omega t)\Big].} \tag{6.162}$$

We abbreviate J_{21}^{ϕ} for future calculations as J_{ϕ}. Our final result then reads

$$\boxed{P_{\phi}(t) = e^{J_{\phi}(t)}.} \tag{6.163}$$

6.5.2 Determination of $\mathcal{V}^R$

We Fourier transform equation (6.149) in (q,ω)–space and go over to the Keldysh rotated representation

$$\tilde{\mathcal{V}}(q,\omega) = \tilde{A}(q,\omega)\tilde{V}(q,\omega)\tilde{A}^T(-q,-\omega) \tag{6.164}$$

In components we have

$$\tilde{\mathcal{V}}(q,\omega) = \begin{pmatrix} \mathcal{D}^R(q,\omega) & \mathcal{D}^K(q,\omega) \\ 0 & -\mathcal{D}^A(q,\omega) \end{pmatrix} \begin{pmatrix} V^K(q,\omega) & V^R(q,\omega) \\ V^A(q,\omega) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}^R(-q,-\omega) & 0 \\ \mathcal{D}^K(-q,-\omega) & -\mathcal{D}^A(-q,-\omega) \end{pmatrix}. \tag{6.165}$$

Looking at the explicit form of $\mathcal{D}^R(q,\omega)$ given in equation (6.90) one finds

$$\mathcal{D}^R(-q,-\omega) = \mathcal{D}^A(q,\omega), \qquad \mathcal{D}^A(-q,-\omega) = \mathcal{D}^R(q,\omega) \tag{6.166}$$

and

$$\mathcal{D}^K(-q,-\omega) = -\mathcal{D}^K(q,\omega), \qquad \mathcal{D}^K(q,\omega) = B(\omega)\big[\mathcal{D}^R(q,\omega) + \mathcal{D}^A(q,\omega)\big]. \tag{6.167}$$

Using these relations to express $\tilde{\mathcal{V}}$ in terms of functions with the arguments (q,ω), we find

$$\begin{aligned} \tilde{\mathcal{V}}(q,\omega) &= \begin{pmatrix} \mathcal{D}^R(q,\omega) & \mathcal{D}^K(q,\omega) \\ 0 & -\mathcal{D}^A(q,\omega) \end{pmatrix} \begin{pmatrix} V^K(q,\omega) & V^R(q,\omega) \\ V^A(q,\omega) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{D}^A(q,\omega) & 0 \\ -\mathcal{D}^K(q,\omega) & -\mathcal{D}^R(q,\omega) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{D}^R V^K \mathcal{D}^A - (\mathcal{D}^R V^R - \mathcal{D}^A V^A)B(\omega)\big[\mathcal{D}^R + \mathcal{D}^A\big] & -\mathcal{D}^R V^R \mathcal{D}^R \\ -\mathcal{D}^A V^A \mathcal{D}^A & 0 \end{pmatrix}(q,\omega). \end{aligned} \tag{6.168}$$

This can be simplified to read

$$\tilde{\mathcal{V}}(q,\omega) = \begin{pmatrix} B(\omega)\big[-\mathcal{D}^R V^R \mathcal{D}^R + \mathcal{D}^A V^A \mathcal{D}^A\big] & -\mathcal{D}^R V^R \mathcal{D}^R \\ -\mathcal{D}^A V^A \mathcal{D}^A & 0 \end{pmatrix}(q,\omega) = \begin{pmatrix} \mathcal{V}^K & \mathcal{V}^R \\ \mathcal{V}^A & 0 \end{pmatrix}(q,\omega) \tag{6.169}$$

where

$$\mathcal{V}^K(q,\omega) = B(\omega)(\mathcal{V}^R(q,\omega) - \mathcal{V}^A(q,\omega)). \tag{6.170}$$

The explicit expression of $\mathcal{V}^R(q,\omega)$ which we need for the calculation of the J's is then

$$\boxed{\mathcal{V}^R(q,\omega) = -\frac{V^R(q,\omega)}{(Dq^2 - i\omega)^2}} \tag{6.171}$$

Let us now turn to the path integral over the magnetic fluctuations.

6.5.3 Path integral over B

We denote the factor coming from the path integral over B by

$$P_B(x,x') := \left\langle \mathcal{R}_{22}(x)\mathcal{R}^*_{11}(x')\, e^{-if_2(x)}e^{if_1(x')} + \mathcal{R}_{24}(x)\mathcal{R}^*_{13}(x')\, e^{if_2(x)}e^{-if_1(x')} \right\rangle_B . \tag{6.172}$$

The path integral over the components of the magnetic field cannot be solved exactly. In particular since the matrix elements $\mathcal{R}_{ij}$ are complicated functions of the components of the magnetic field. Hence we make use of our assumption formulated in the beginning of section (6.1) (The variations of $\vec{B}$ are small.)

Furthermore we treat $\vec{B}_1(x')$ and $\vec{B}_2(x)$ semiclassically[15] which means that we can neglect the Keldysh index as the field components are assumed to take the same values on the upper and lower branch. We write down the first three terms of a Taylor expansion of $\mathcal{R}^*_{11}(B(x'))$ around $B(x)$ [16]

$$\mathcal{R}^*_{11}(B(x')) = \mathcal{R}^*_{11}(B(x)) + C_1(B(x))\partial_t B(x)\cdot(t-t') + C_2(B(x))\nabla B(x)\cdot(r-r') + ... \tag{6.173}$$

where C_1 and C_2 are coefficients at $B(x') = B(x)$. As for the calculation of the DOS we need the Green's function only at equal points ($r = r'$) in space all terms containing the difference $(r - r')$ vanish. According to our assumption $\partial_t B(x)$ is small. However $\partial_t B(x)$ is multiplied by $t - t'$ which could become large. Fortunately we have a factor e^{J_ϕ} from the scalar field Φ which falls off exponentially. At $T = 0$ it roughly goes like $e^{-\sqrt{E_s|t-t'|}(1+i)}$ as we will show in the next section. $\sqrt{E_s}$ is the energy scale of the singlet which is much bigger than $\sqrt{E_t}$ the corresponding triplet energy. Hence we can restrict ourselves in good approximation to the first term of equation (6.173). The products of matrix elements then read approximately

$$\mathcal{R}_{22}(x)\mathcal{R}^*_{11}(x') \approx \frac{1}{2}\left(1 + \frac{B^z(x)}{B(x)}\right) + \mathcal{O}\big(\dot{B}(x), \nabla B(x)\big) \tag{6.174}$$

$$\mathcal{R}_{24}(x)\mathcal{R}^*_{13}(x') \approx \frac{1}{2}\left(1 - \frac{B^z(x)}{B(x)}\right) + \mathcal{O}\big(\dot{B}(x), \nabla B(x)\big) \tag{6.175}$$

where $B = \sqrt{(B^x)^2 + (B^y)^2 + (B^z)^2}$ is the absolute value of the magnetic field. This simplifies the expression that we have to calculate considerably

[15] like the potential $eV(x)$ in the one particle Schrödinger equation in first quantization.
[16] $x = (r,t)$ and $x' = (r',t')$

$$P_B \approx \frac{1}{2}\left\langle e^{-if_2(x)}e^{if_1(x')} + e^{if_2(x)}e^{-if_1(x')}\right\rangle_B + \frac{1}{2}\left\langle \frac{B^z(x)}{B(x)}\left(e^{-if_2(x)}e^{if_1(x')} - e^{if_2(x)}e^{-if_1(x')}\right)\right\rangle_B . \tag{6.176}$$

It can be shown that the second term vanishes exactly. However there is another argument which is shorter and thus easier to understand: Calculating the DOS one has to sum $G^{>}_{\uparrow\uparrow}$ and $G^{>}_{\downarrow\downarrow}$ and the second term in the equation above proportional to $\frac{B^z(x)}{B(x)}$ cancels exactly with a contribution of $G^{>}_{\downarrow\downarrow}$. This can be seen from equations (7.37) and (7.39) below in which one has to set $B_{ex} = 0$ in order to have the correct expressions for our current problem. Hence we only have to evaluate the first term on the right hand side of equation (6.176):

$$P_B \approx \frac{1}{2}\left\langle e^{if_2(x)}e^{-if_1(x')} + e^{-if_2(x)}e^{-if_1(x')} = \right\rangle_B = \left\langle \cos\left(f_2(x) - f_1(x')\right)\right\rangle_B \tag{6.177}$$

or in a power series

$$P_B(x,x') = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left\langle \left(f_2(x) - f_1(x')\right)^{2n}\right\rangle_B \tag{6.178}$$

Unfortunately the f's are not linear in the field components but linear in the absolute value. $f_\alpha(x)$ reads in the unrotated representation

$$f_\alpha(x) = \mu_B \sum_{\beta=1}^{2}\int dy\; A_{\alpha\beta}(x,y)\sqrt{(B^x_\beta)^2 + (B^y_\beta)^2 + (B^z_\beta)^2}\;(y). \tag{6.179}$$

Thus we have for P_B:

$$P_B(x,x') = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\sum_{\beta_1\ldots\beta_{2n}}\int dy_1\ldots dy_{2n}\left[A_{2\beta_1}(x,y_1) - A_{1\beta_1}(x',y_1)\right]\ldots$$
$$\left[A_{2\beta_{2n}}(x,y_{2n}) - A_{1\beta_{2n}}(x',y_{2n})\right]\mu_B^{2n}\left\langle B_{\beta_1}(y_1)\ldots B_{\beta_{2n}}(y_{2n})\right\rangle_B . \tag{6.180}$$

The path integral $\left\langle B_{\beta_1}(y_1)\ldots B_{\beta_{2n}}(y_{2n})\right\rangle_B$ is not exactly solvable for an arbitrary number of B's at different points. In Appendix (9.2) we discuss this in some detail. Here we only summarize the results. For example one can obtain the exact result for the correlator of two absolute values of B at different points

$$\langle B(x)B(x')\rangle_B = \frac{2}{\pi}\left(3\sqrt{V_t(x,x)^2 - V_t(x,x')^2} + \frac{(V_t(x,x)^2 + 2V_t(x,x')^2)\arcsin(V_t(x,x')/V_t(x,x))}{V_t(x,x')}\right). \tag{6.181}$$

This is a rather inconvenient expression for further calculations so we look at the limiting case of a long range interaction (constant in space and time). We remind that the interaction in the triplet channel is connected to the fluctuations of the components of the magnetic field

$$V_t^R(x,x')\delta_{ij} = -i\mu_B^2 \big\langle B_1^i(x) B_2^j(x') \big\rangle_B \qquad i,j = x,y,z. \tag{6.182}$$

We already assumed weakly varying magnetic field components when we approximated the elements of the rotation matrix. Here one can see that weakly varying magnetic fields translate also into a slowly varying interaction $V_t(x,x')$. Hence we are using the same approximation as before. In the limit of a constant interaction $V(x,x') = V_{\text{long}}$ the arcsin in equation (6.181) becomes $\pi/2$ and one obtains

$$\big\langle B(x)B(x') \big\rangle_B = 3V_{\text{long}}. \tag{6.183}$$

Proceeding along the same line of reasoning, the four point correlator reads

$$\boxed{\big\langle B(x)B(x_1)B(x_2)B(x_3) \big\rangle_B = \big\langle B(x)^4 \big\rangle_B = 15V_{\text{long}}.} \tag{6.184}$$

This is the decisive idea that allows us to calculate arbitrary powers of B at different points. Thus we will determine the remaining average in the limit of a long range interaction in which we can use the results of the higher moments at one point in space/time. The general result for $\big\langle B(x)^{2n} \big\rangle_B$ for a constant interaction V_{long} we get from formula (9.45) in the appendix for $l = 2n$ adding the factor μ_B^{2n} which was neglected in the calculation in the Appendix.

$$\mu_B^{2n} \Big\langle \underbrace{B(x)...B(x)}_{2n} \Big\rangle_B = \frac{2^{n+1}}{\sqrt{\pi}}\, \Gamma\left(\frac{2n+3}{2}\right) (iV_{\text{long}})^n \tag{6.185}$$

Putting this result into (6.180), we have

$$P_B(x,x') = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{2^{n+1}}{\sqrt{\pi}}\, \Gamma\left(\frac{2n+3}{2}\right) 2^n \Bigg(\frac{1}{2} \sum_{\beta\beta'} \int dy\, dy' \Big[A_{2\beta}(x,y) - A_{1\beta}(x',y) \Big] \\ iV_{\text{long}} \Big[A_{2\beta'}(x,y') - A_{1\beta'}(x',y') \Big] \Bigg)^n. \tag{6.186}$$

We define analogously to the singlet case (see equation (6.149)) a propagator containing a triplet interaction plus two factors of A

$$\mathcal{V}_t(x,x') = \int dy \int dy' A(x,y) \breve{V}_{\text{long}} A^T(x',y'). \tag{6.187}$$

The 2×2 matrix $\breve{V}_{\text{long}}$ has as all matrix elements the constant V_{long}. $\mathcal{V}_t(x,x')$ has the same time structure as in the singlet case:

$$\mathcal{V}_t(x,x') = \begin{pmatrix} \mathcal{V}_t^T(x,x') & \mathcal{V}_t^<(x,x') \\ \mathcal{V}_t^>(x,x') & \mathcal{V}_t^{\overline{T}}(x,x') \end{pmatrix}. \tag{6.188}$$

Also we introduce a function

$$J^B_{\alpha\alpha'}(x,x') = -\frac{1}{2i} \big[\mathcal{V}^t_{\alpha\alpha'}(x,x') + \mathcal{V}^t_{\alpha'\alpha}(x',x) - \mathcal{V}^t_{\alpha\alpha}(x,x) - \mathcal{V}^t_{\alpha'\alpha'}(x',x') \big]. \tag{6.189}$$

Restricting ourselves, as in the singlet case, to a translationally invariant system we obtain at equal positions in space

$$\boxed{J_{21}^B(0,t) = 2\int_0^\infty \frac{d\omega}{2\pi}\frac{1}{\mathcal{V}}\sum_q \mathrm{Im}\mathcal{V}_t^R(q,\omega)\Big[B(\omega)\left(\cos(\omega t)-1\right) - i\sin(\omega t)\Big].} \tag{6.190}$$

We abbreviate J_{21}^B by J_B. The only difference with J_ϕ is that the singlet interaction is replaced by the triplet interaction. We define

$$Y_t(\omega) = \frac{1}{\mathcal{V}}\sum_q \mathcal{V}_t^R(q,\omega). \tag{6.191}$$

Expressing the series of equation (6.186) in J_B we find

$$P_B(t) = \sum_{n=0}^\infty \frac{2^{2n+1}}{(2n)!\sqrt{\pi}}\,\Gamma\left(\frac{2n+3}{2}\right) J_B(t)^n. \tag{6.192}$$

Rewriting the Γ–function

$$\Gamma\left(n+1+\frac{1}{2}\right) = \frac{\Gamma\left(2(n+1)\right)}{\Gamma\left(n+1\right)}\,\sqrt{\pi}\,2^{-2(n+1)+1} \tag{6.193}$$

brings us to

$$P_B(t) = \sum_{n=0}^\infty \frac{\Gamma\left(2(n+1)\right)}{\Gamma\left(n+1\right)}\frac{1}{(2n)!}\,J_B(t)^n = \sum_{n=0}^\infty (2n+1)\,\frac{J_B(t)^n}{n!}. \tag{6.194}$$

This can be written in compact form since

$$\sum_0^\infty (2n+1)\frac{x^n}{n!} = e^x + 2x\sum_0^\infty \frac{x^{n-1}}{(n-1)!} = (1+2x)\,e^x \tag{6.195}$$

Thus we obtain a quite simple result

$$\boxed{P_B(t) = \left[1+2J_B(t)\right]\,e^{J_B(t)}.} \tag{6.196}$$

Equation (6.196) is a central result of this thesis. It is exact for the case of a constant magnetic field ($\rightarrow$ constant interaction). As there is no interaction which is absolutely constant in space and time let us go back to equation (6.190). $\mathcal{V}_t^R(q,\omega)$ contains a factor $1/(Dq^2 - i\omega)^2$ (see equation (6.171)) which is highly peaked in (q,ω)–space. Thus the factor $1/(Dq^2 - i\omega)^2$ can serve as a cut off.

An interaction that is constant in real space is a δ–function in (q,ω)–space. If however the interaction falls off over a finite length in real space it has also a finite width in (q,ω)–space. If the width of the factors $A(q,\omega)$ is smaller than the one of $V(q,\omega)$ they cut off the q and ω

integral in (6.190). So we can allow a certain space and time dependence of the interaction and replace $V_{long} \to V_t(x,x')$. We generalize (6.187) to

$$\mathcal{V}_t(x,x') = \int dy \int dy' A(x,y)\breve{V}_t(y,y')A^T(x',y'). \tag{6.197}$$

6.5.4 Result for the Green's functions

Let us summarize what we have found for the Green's function $G^{>}_{\uparrow\uparrow}(r,r',t-t')|_{r=r'}$ in the limit of weakly varying magnetic fields. Putting equations (6.163) and (6.196) in equation (6.141) we obtain

$$G^{>}_{\uparrow\uparrow}(t,p) = P_\phi(t)P_B(t)G^{>}_\tau(t,p) = e^{J_\phi(t)+J_B(t)}\big[1+2J_B(t)\big]G^{>}_\tau. \tag{6.198}$$

Going through the corresponding steps of the calculation of $G^{<}_\tau$, one finds the complex conjugated $J's$ ($J^\phi_{12} = (J^\phi_{21})^*$ and $J^B_{12} = (J^B_{21})^*$) and the result

$$G^{<}_{\uparrow\uparrow}(t,p) = P^*_\phi(t)P^*_B(t)G^{>}_\tau(t,p) = e^{J^*_\phi(t)+J^*_B(t)}\big[1+2J^*_B(t)\big]G^{<}_\tau(t,p). \tag{6.199}$$

Without external magnetic field it follows directly that

$$G^{>}_{\downarrow\downarrow}(t,p) = G^{>}_{\uparrow\uparrow}(t,p) \qquad \text{and} \qquad G^{<}_{\downarrow\downarrow}(t,p) = G^{<}_{\uparrow\uparrow}(t,p). \tag{6.200}$$

6.6 Density of states

In this section we calculate the DOS of an infinite quasi one–dimensional wire at zero temperature. Taking the expressions for the Green's functions of the previous section we are now able to calculate the DOS non perturbatively. Without external magnetic field equation (6.2) simplifies to

$$\nu(\epsilon) = \nu_\uparrow(\epsilon) + \nu_\downarrow(\epsilon) = \frac{i}{\pi\mathcal{V}}\sum_p \left(G^{>}_{\uparrow\uparrow}(p,\epsilon) - G^{<}_{\uparrow\uparrow}(p,\epsilon)\right) \tag{6.201}$$

We have[17]

$$\Lambda(t) = \frac{i}{\pi\nu_3\mathcal{V}}\sum_p G_\tau(t,p) = \begin{pmatrix} \tilde{F}(t) & \delta(t)-\tilde{F}(t) \\ \delta(t)+\tilde{F}(t) & -\tilde{F}(t)\end{pmatrix} \tag{6.202}$$

and thus[18]

$$\nu_1(t) = \nu_1\Big[\Big(\delta(t)+F(t)\Big)e^{J_\phi+J_B}(1+2J_B) + \Big(\delta(t)-F(t)\Big)e^{J^*_\phi+J^*_B}(1+2J^*_B)\Big] \tag{6.203}$$

which gives

$$\nu_1(t) = 2\nu_1\left[\delta(t) + \frac{1}{2}\left(e^{J_\phi+J_B}(1+2J_B) - e^{J^*_\phi+J^*_B}(1+2J^*_B)\right)F(t)\right]. \tag{6.204}$$

As we are interested in the DOS as a function of energy we Fourier transform $\nu_1(t)$.

[17] (4.43) is the corresponding equation in Keldysh rotated representation.

[18] We multiplied both sides of the equation by a^2 in order to obtain the quasi one–dimensional DOS $\nu_1 = a^2\nu_3$.

$$\nu_1(\epsilon) = 2\nu_1 \left[1 + \frac{1}{2}\int_{-\infty}^{\infty} dt\; 2\pi F(t)\big[P(t) - P^*(t)\big]e^{i\epsilon t}\right] \tag{6.205}$$

where we introduced the function

$$P(t) := \frac{e^{J_\phi(t)+J_B(t)}(1+2J_B)(t)}{2\pi}. \tag{6.206}$$

This may be written as

$$\nu_1(\epsilon) = 2\nu_1 \left[1 + \frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty}\frac{d\epsilon'}{2\pi}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\; 2\pi F(\epsilon')\big[P(E) - P^*(E)\big]e^{it(\epsilon-\epsilon'-E)}\right]. \tag{6.207}$$

Performing the t integration one gets a factor $\delta(\epsilon-\epsilon'-E)$ which we use for the integration over $d\epsilon'$ to obtain

$$\nu_1(\epsilon) = 2\nu_1 \left[1 + \frac{1}{2}\int_{-\infty}^{\infty} dE\; F(\epsilon - E)\big[P(E) - P^*(E)\big]\right]. \tag{6.208}$$

In order to simplify this expression further we have to determine the function $P(E)$ and relate $P^*(E)$ to $P(E)$.

6.6.1 Determination of $P(E)$

In order to calculate $P(E)$ let us begin with the calculation of $Y(\omega)$. We will treat singlet and triplet contributions in parallel. The function $Y_n(\omega)$ was defined as $Y_n(\omega) = \frac{1}{\mathcal{V}}\sum_q \mathcal{V}_n^R(q,\omega)$. [19] In the quasi one–dimensional system of infinite length in z–direction the sum over q is approximated as explained in section (2.2.5).

$$\frac{1}{\mathcal{V}}\sum_{q_x,q_y,q_z} \approx \frac{1}{a^2}\int \frac{dq_z}{2\pi} \tag{6.209}$$

Plugging the interactions

$$\Gamma_s(q,\omega) = \frac{\Gamma_s}{Z_s}\,\frac{Dq^2 - i\omega}{D_s^* q^2 - i\omega}, \qquad\qquad \Gamma_t(q,\omega) = \frac{\Gamma_t}{Z_t}\,\frac{Dq^2 - i\omega}{D_t^* q^2 - i\omega} \tag{6.210}$$

in equation (6.171), we obtain for $Y_n(\omega)$ the following integral

$$\mathrm{Im}Y_n(\omega) = c_n\,\mathrm{Im}\int \frac{dq}{2\pi}\frac{1}{(D_n^* q^2 - i\omega)(Dq^2 - i\omega)} \tag{6.211}$$

with $n = s$ or $n = t$. The constants are

$$D_s^* = \frac{D}{1 - A_0^s} \qquad\qquad D_t^* = \frac{D}{1 - A_0^a} \tag{6.212}$$

and

$$c_s = \frac{A_0^s}{2\nu_1(1 - A_0^s)} \qquad\qquad c_t = \frac{A_0^a}{2\nu_1(1 - A_0^a)} \tag{6.213}$$

[19] where $n = s$ or $n = t$.

where we combined $a^2\nu_3$ to ν_1. Performing the q–integral according to equation (2.55) we have

$$\mathrm{Im}Y_n(\omega) = \frac{c_n\,\omega^{-3/2}}{2\sqrt{2}(\sqrt{D}+\sqrt{D_n^*})}. \tag{6.214}$$

With the definitions of equation (2.102) it follows

$$\mathrm{Im}Y_s(\omega) = \frac{2\pi}{\omega}\sqrt{\frac{E_s}{4\pi\omega}} \qquad\qquad \mathrm{Im}Y_t(\omega) = -\frac{2\pi}{\omega}\sqrt{\frac{E_t}{4\pi\omega}}. \tag{6.215}$$

Having the $\mathrm{Im}Y_n(\omega)$ we can go on with the calculation of the J's from equations (6.162) and (6.190). At $T=0$ the function $J(t)$ can be evaluated exactly. In the singlet case one has

$$J_\phi(t) = 2\int_0^\infty \frac{d\omega}{\omega}\sqrt{\frac{E_s}{4\pi\omega}}(e^{-i\omega t}-1). \tag{6.216}$$

Using the integral

$$\int_0^\infty d\omega\;\omega^{-3/2}(e^{-i\omega t}-1) = -\sqrt{2\pi|t|}(1+i\mathrm{sign}(t)) \tag{6.217}$$

one obtains

$$J_\phi(t) = -\sqrt{2E_s|t|}(1+i\mathrm{sign}(t)). \tag{6.218}$$

In the triplet case we just have to add a minus sign.

$$J_B(t) = \sqrt{2E_t|t|}(1+i\mathrm{sign}(t)) \tag{6.219}$$

So we can address the computation of $P(E)$. We divide $P(E)$ into 2 parts:

$$P(E) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{J_\phi(t)+J_B(t)+iEt}}_{P_1(E)} + \underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} dt\; J_B(t)\; e^{J_\phi(t)+J_B(t)+iEt}}_{P_2(E)}. \tag{6.220}$$

Also we set

$$\sqrt{E_g} := \sqrt{E_s} - \sqrt{E_t} \tag{6.221}$$

which allows us to write the sum of the J's simply as

$$J := J_\phi(t) + J_B(t) = -\sqrt{2E_g|t|}(1+i\mathrm{sign}(t)) \tag{6.222}$$

So let us begin with the calculation of $P_1(E)$. As it can be seen directly from the definition $J(-t) = (J(t))^*$. We use this property in order to restrict the time integral to the positive real axis

$$P_1(E) = \frac{1}{2\pi}\int_0^\infty dt\,\left(e^{\mathrm{Re}J(t)-i\mathrm{Im}J(t)-iEt} + e^{\mathrm{Re}J(t)+i\mathrm{Im}J(t)+iEt}\right) = \frac{1}{\pi}\int_0^\infty dt\; e^{\mathrm{Re}J(t)}\cos(\mathrm{Im}J(t)+Et). \tag{6.223}$$

Now we plug in the explicit form of $J(t)$ and substitute $u = \sqrt{2E_g t}$. [20] This gives

$$\begin{aligned} P_1(E) &= \frac{1}{\pi E_g}\int_0^\infty du\; u\; e^{-u}\cos\left(\frac{E}{2E_g}u^2 - u\right) \\ &= \frac{1}{\pi E_g}\int_0^\infty du\; u\; e^{-u}\left[\cos\left(\frac{E}{2E_g}u^2\right)\cos(u) + \sin\left(\frac{E}{2E_g}u^2\right)\sin(u)\right] \end{aligned} \tag{6.225}$$

With[21]

$$I_1 = \int_0^\infty du\; u\; e^{-\beta u}\;\cos\left(\alpha u^2\right)\cos(\beta u) = \frac{\beta}{4}\sqrt{\frac{\pi}{2\alpha^3}}e^{-\beta^2/(2\alpha)} \tag{6.226}$$

and

$$I_2 = \int_0^\infty du\; u\; e^{-\beta u}\;\sin\left(\alpha u^2\right)\sin(\beta u) = \frac{\beta}{4}\sqrt{\frac{\pi}{2\alpha^3}}e^{-\beta^2/(2\alpha)} \tag{6.227}$$

the solution is

$$\boxed{P_1(E) = \frac{1}{\sqrt{\pi}E_g}\left(\frac{E_g}{E}\right)^{3/2}\; e^{-E_g/E}.} \tag{6.228}$$

Note that at $T = 0:\ P_1(E) = 0$ for $E < 0$. This can be seen by substituting $E' = -E$ into equation (6.225) and using (6.224). We will use this property for further manipulations of equation (6.208). We turn to $P_2(E)$ and use the definition (6.222). Again we substitute $u = \sqrt{2E_g t}$ and arrive at

$$P_2(E) = \frac{2\sqrt{E_t}}{\pi(E_g)^{3/2}}\int_0^\infty du\; u^2\; e^{-u}\left[\cos\left(\frac{E}{2E_g}u^2 - u\right) - \sin\left(\frac{E}{2E_g}u^2 - u\right)\right]. \tag{6.229}$$

Setting $\alpha = \frac{E}{2E_g}$ and $b_0 = \frac{2\sqrt{E_B}}{\pi(E_g)^{3/2}}$ we have

$$P_2(E) = b_0\int_0^\infty du\; u^2\; e^{-u}\left[\cos\left(\alpha u^2\right)\cos(u) + \sin\left(\alpha u^2\right)\sin(u) - \sin(\alpha u^2)\cos(u) + \cos(\alpha u^2)\sin(u)\right] \tag{6.230}$$

We can relate this integral to the integrals (6.226) and (6.227) given above. One has

$$-\frac{\partial}{\partial\beta}I_1\Big|_{\beta=1} = \int_0^\infty du\; u^2\; e^{-\beta u}\left[\cos(\alpha u)\cos(\beta u) + \cos(\alpha u)\sin(\beta u)\right]\Big|_{\beta=1} \tag{6.231}$$

and

[20] One has

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\beta)\sin(\alpha) \qquad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta). \tag{6.224}$$

[21] See [51].

$$-\frac{\partial}{\partial\beta}I_2\bigg|_{\beta=1} = \int_0^\infty du\, u^2\, e^{-\beta u}\left[\sin(\alpha u)\sin(\beta u) - \sin(\alpha u)\cos(\beta u)\right]\bigg|_{\beta=1} \tag{6.232}$$

which brings us to the final result

$$\boxed{P_2(E) = \frac{2\sqrt{E_t}}{\sqrt{\pi}(E_g)^{3/2}}\left(\frac{E_g}{E}\right)^{3/2} e^{-E_g/E}\left(\frac{2E_g}{E} - 1\right).} \tag{6.233}$$

Note that at $T = 0$ also $P_2(E) = 0$ for $E < 0$. This can be seen by substituting $\alpha' = -\alpha$ which is equivalent to $E' = -E$ into equation (6.230) and using (6.224). As P_1 and P_2 vanish for negative energies, $P(E)$ as a whole vanishes for negative energies. It is also easy to see that $P^*(E) = P(-E)$. A third property of $P(E)$ is its normalization: $\int dE\ P(E) = 1$.[22] We leave the proof to the reader. Using these properties we can further transform equation (6.208).

6.6.2 The DOS at $T = 0$

For $T = 0$ the function $F(\epsilon - E) = \tanh(\frac{\epsilon - E}{2T})$ becomes $\mathrm{sign}(\epsilon - E)$. Together with $P^*(E) = P(-E)$ equation (6.208) turns into

$$\nu_1(\epsilon) = 2\nu_1\left(1 - \frac{1}{2}\int_{-\infty}^{\infty} dE\ \mathrm{sign}(E - \epsilon)\left[P(E) - P(-E)\right]\right) \tag{6.234}$$

Substituting $E' = -E$ in the second integral we get

$$\nu_1(\epsilon) = 2\nu_1\left(1 - \frac{1}{2}\int_0^{\infty} dE\ P(E)\left[\mathrm{sign}(E - \epsilon) + \mathrm{sign}(E + \epsilon)\right]\right). \tag{6.235}$$

We assume that ϵ is positive. For $0 < E < \epsilon$ the first sign is negative. Furthermore we use that the integral of $P(E)$ over all energies is 1. In the case of $T = 0$ the integral only goes from 0 to ∞ and we have

$$\nu_1(\epsilon) = 2\nu_1\left(\int_0^{\infty} dE\ P(E) - \frac{1}{2}\left[-\int_0^{\epsilon} dE\ P(E) + \int_{\epsilon}^{\infty} dE\ P(E) + \int_0^{\infty} dE\ P(E)\right]\right). \tag{6.236}$$

Thus we finally obtain

$$\nu_1(\epsilon) = 2\nu_1\int_0^{\epsilon} dE\ P(E). \tag{6.237}$$

So we get the DOS simply by integrating $P(E)$ over dE from 0 to ϵ. This final integration is rather simple and yields[23]

[22] Formally our $P(E)$ is similar to the $P(E)$ that appears in the theory of Coulomb Blockade ([52], [53], [54], [55]). The connection of the microscopic approach to Coulomb Blockade Theory is discussed in [19].

[23] One uses the definition of the Error function

$$1 - \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} dv\, e^{-v^2} \tag{6.238}$$

See [56].

$$\boxed{\nu_1(\epsilon) = 2\nu_1 \left[1 - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}}\right) + \frac{4}{\sqrt{\pi}}\sqrt{\frac{E_t}{\epsilon}}\, e^{-E_g/\epsilon}\right]} \tag{6.239}$$

Expanding this expression for high energies ϵ and using the definition of $\sqrt{E_g}$ one obtains

$$\nu(\epsilon) = 2\nu_1 \left[1 - \frac{2}{\sqrt{\pi}}\left(\sqrt{\frac{E_s}{\epsilon}} - 3\sqrt{\frac{E_t}{\epsilon}}\right)\right]. \tag{6.240}$$

We see that $\delta\nu_1(\epsilon)/\nu_1$ exactly coincides with the result of perturbation theory in equation (2.103).

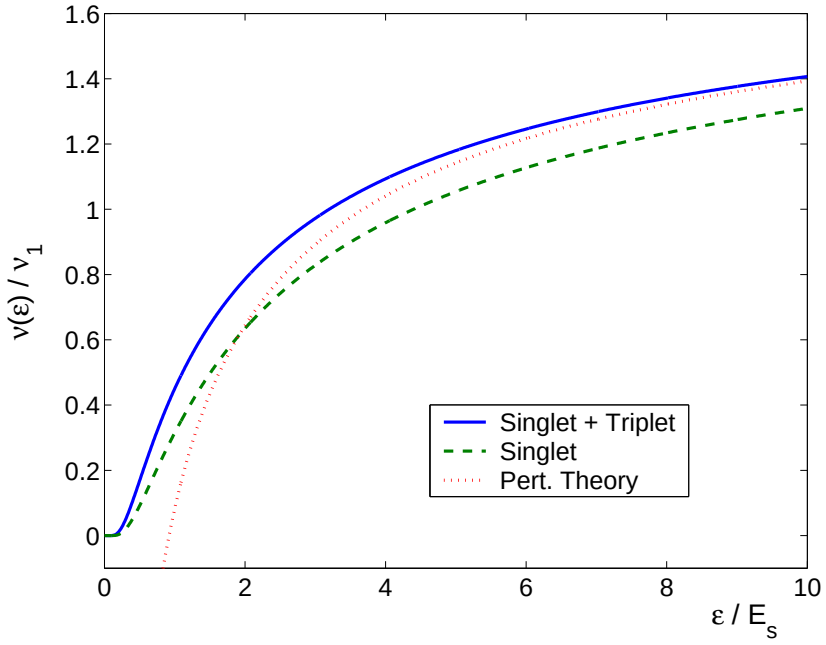

Figure 6.1: The DOS close to the Fermi energy. The drawn through curve is the full result containing the Triplets whereas the dashed curve is the non perturbative result for the Singlets only. The dotted curve is the perturbative result which diverges for small energies.

In figure (6.1) we show the full solution (6.239) and the perturbative result (6.240). Additionally we show only the singlet part of the non–perturbative result in order to make the contribution of the triplet term visible. For the plot we have chosen $R = \sqrt{E_t}/\sqrt{E_s} = 0.05$ which is a conservative value. The larger R the bigger is the effect of the triplet. In appendix (9.4) we estimated a value of $R = 0.12$ for Palladium.

Chapter 7

Interaction, Disorder and External Magnetic Field

In this chapter we add a constant external magnetic field B_{ex} in z–direction to our interacting and disordered system, that we discussed in the previous chapter. We will investigate how the DOS changes in the presence of the external field. The strategy to solve the problem remains the same. We introduce a rotation matrix and gauge factors in order to find an approximate saddle point solution. Of course, all the matrices change due to the presence of the external magnetic field.

In the case of a strong external magnetic field ($B_{ex} \to \infty$) we will be able to extract an analytic solution for the DOS at $T = 0$ that coincides with the predictions of perturbation theory for high energies. For an external magnetic field that is of order of the characteristic energies E_s and E_t we will have to resort to numerical treatment even at zero temperature. But also there the numerical curve converges for high energies towards the perturbative result.

Finally we will evaluate our formula for the DOS for finite temperatures. That will allow to estimate up to which temperatures triplets effects survive.

Let us begin again with the generating functional which now depends on the additional parameter B_{ex}:

$$Z[J, \hat{J}, B_{ex}] = \frac{1}{\mathcal{N}_U} \int \mathcal{D}\hat{\Psi} \int \mathcal{D}\Psi \int \mathcal{D}\Phi \int \mathcal{D}\vec{B} \int \mathcal{D}U \; e^{iS[\Psi,\hat{\Psi},\Phi,U,\vec{B},J,\hat{J},B_{ex}]}. \tag{7.1}$$

The external magnetic field couples linearly to the electron density. Thus it affects the action only through the first term in

$$S[\Psi, \hat{\Psi}, \Phi, U, \vec{B}, J, \hat{J}, B_{ex}] = \int dx \; \hat{\Psi}(x)\hat{G}^{-1}[\Phi, \vec{B}, U, B_{ex}](x)\Psi(x) + \int dx \; \left(\hat{J}(x)\Psi(x) + \bar{\Psi}J(x)\right)$$

$$+ \frac{1}{2}\int dx \; \Phi^T(x) \; \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x) \; \sigma_z V_{0,t}^{-1}(x) B^n(x) \tag{7.2}$$

with

$$\hat{G}^{-1}[\Phi, \vec{B}, U, B_{ex}](x) = G_0^{-1} \; \mathbb{I}_4 + \mathbb{I}_2 \otimes \phi_\alpha(x)\gamma^\alpha + \vec{\sigma} \otimes \vec{B}_\alpha(x)\gamma^\alpha + U(r)\mathbb{I}_4 + \sigma_z \otimes B_{ex}\mathbb{I}_2. \tag{7.3}$$

As B_{ex} is constant and points in z–direction, it couples to a σ_z matrix in spin space and a unity matrix in Keldysh space. The field matrix now reads

$$F_{\phi,\vec{B},B_{ex}} = \begin{pmatrix} \phi_1 + B_1^z + B_{ex} & 0 & B_1^x - iB_1^y & 0 \\ 0 & \phi_2 + B_2^z + B_{ex} & 0 & B_2^x - iB_2^y \\ B_1^x + iB_1^y & 0 & \phi_1 - B_1^z - B_{ex} & 0 \\ 0 & B_2^x + iB_2^y & & \phi_2 - B_2^z - B_{ex} \end{pmatrix}. \tag{7.4}$$

Again we do the disorder averaging first, like it was explained in chapter (4). The decoupling of the resulting quartic fermionic terms leads to a path integral over a 4×4 matrix field $\hat{Q}$. The generating functional then reads

$$Z[J, \hat{J}, B_{ex}] = \int \mathcal{D}\hat{\Psi} \int \mathcal{D}\Psi \int \mathcal{D}\Phi \int \mathcal{D}\vec{B} \int \mathcal{D}\hat{Q}\; e^{iS[\Psi,\hat{\Psi},\Phi,\hat{Q},\vec{B},J,\hat{J},B_{ex}]} \tag{7.5}$$

with the action

$$\begin{aligned} S[\Psi, \hat{\Psi}, \Phi, \hat{Q}, \vec{B}, B_{ex}] = &\int dx \int dx'\; \hat{\Psi}(x)\hat{G}^{-1}[\Phi, \vec{B}, \hat{Q}, B_{ex}](x, x')\Psi(x') - \frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\hat{Q}^2\} \\ &+ \frac{1}{2}\int dx\; \Phi^T(x)\; \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x)\; \sigma_z V_{0,t}^{-1}(x) B^n(x) \\ &+ \int dx \left(\bar{J}(x)\Psi(x) + \bar{\Psi}J(x)\right) \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} \hat{G}^{-1}[\Phi, \vec{B}, \hat{Q}, B_{ex}](x) = &\Big[G_0^{-1}\; \mathbb{I}_4 + \mathbb{I}_2 \otimes \phi_\alpha(x')\gamma^\alpha + \vec{\sigma} \otimes \vec{B}_\alpha(x')\gamma^\alpha + \sigma_z \otimes B_{ex}\mathbb{I}_2\Big]\delta(x - x') \\ &+ \frac{i}{2\tau}\hat{Q}(r, t, t')\delta(r - r'). \end{aligned} \tag{7.7}$$

7.1 Introduction of a rotation and local gauge factors

We proceed with the introduction of a rotation matrix $\mathcal{R}_{ex}$ in order to diagonalize the field matrix including the external field B_{ex}:

$$D_{\phi,B,B_{ex}} = \mathcal{R}_{ex}^{-1} F_{\phi,\vec{B},B_{ex}} \mathcal{R}_{ex}. \tag{7.8}$$

The modified rotation matrix reads

$$\mathcal{R}_{ex} = \begin{pmatrix} \frac{B_1^\perp B_1^{s+}}{(B_1^x+iB_1^y)\sqrt{(B_1^\perp)^2+(B_1^{s+})^2}} & 0 & \frac{-B_1^\perp B_1^{s-}}{(B_1^x+iB_1^y)\sqrt{(B_1^\perp)^2+(B_1^{s-})^2}} & 0 \\ 0 & \frac{B_2^\perp B_2^{s+}}{(B_2^x+iB_2^y)\sqrt{(B_2^\perp)^2+(B_2^{s+})^2}} & 0 & \frac{-B_2^\perp B_2^{s-}}{(B_2^x+iB_2^y)\sqrt{(B_2^\perp)^2+(B_2^{s-})^2}} \\ \frac{B_1^\perp}{\sqrt{(B_1^\perp)^2+(B_1^{s+})^2}} & 0 & \frac{B_1^\perp}{\sqrt{(B_1^\perp)^2+(B_1^{s-})^2}} & 0 \\ 0 & \frac{B_2^\perp}{\sqrt{(B_2^\perp)^2+(B_2^{s+})^2}} & 0 & \frac{B_2^\perp}{\sqrt{(B_2^\perp)^2+(B_2^{s-})^2}} \end{pmatrix} \tag{7.9}$$

where we introduced the abbreviations

$$B_i^\perp := \sqrt{(B_i^x)^2 + (B_i^y)^2} \qquad \widehat{B}_i := \sqrt{(B_{ex} + B_i^z)^2 + (B_i^\perp)^2} \tag{7.10}$$

and

$$B_i^{s+} := B_{ex} + B_i^z + \widehat{B}_i \qquad B_i^{s-} := B_{ex} + B_i^z - \widehat{B}_i. \tag{7.11}$$

where $i = 1, 2$. We see that we recover the matrix (6.9) in the limit of vanishing external magnetic field. Our new diagonal matrix is

$$D_{\phi,B,B_{ex}} = \begin{pmatrix} \phi_1 + \widehat{B}_1 & 0 & 0 & 0 \\ 0 & \phi_2 + \widehat{B}_2 & 0 & 0 \\ 0 & 0 & \phi_1 - \widehat{B}_1 & 0 \\ 0 & 0 & & \phi_2 - \widehat{B}_2 \end{pmatrix}. \tag{7.12}$$

The advantage of including B_{ex} in the diagonalization is that we can expand all quantities in the limit $B_{ex} \to \infty$ which will simplify the calculation. For finite external magnetic field however we can already anticipate at this early point of the calculation that the evaluation of the DOS is far more complicated although B_{ex} is assumed to be constant in space and time. For the gauge factors we use the same ansatz as before

$$U_{ex}(x) = \exp\left[i \begin{pmatrix} k_1 + f_1 & & 0 & 0 \\ 0 & k_2 + f_2 & 0 & 0 \\ 0 & 0 & k_1 - f_1 & 0 \\ 0 & 0 & & k_2 - f_2 \end{pmatrix} \right]. \tag{7.13}$$

The difference however is that the functions f_1 and f_2 should now compensate the magnetic fields $\widehat{B}_1$ and $\widehat{B}_2$ instead of the absolute values B_1 and B_2 before. We go exactly through the steps of section (6.1). All one has to do is the replacement $\mathcal{R} \to \mathcal{R}_{ex}$, $D_{\phi,B} \to D_{\phi,B,B_{ex}}$ and $U(x) \to U_{ex}(x)$. One integrates out the Fermions and obtains the modified generating functional

$$Z[J, \hat{J}, B_{ex}] = \int \mathcal{D}\Phi \int \mathcal{D}\vec{B} \int \mathcal{D}\hat{Q}\; e^{iS[\Phi,\hat{Q},\vec{B},J,\hat{J},B_{ex}]} \tag{7.14}$$

with the action

$$
\begin{aligned}
S[\Phi,\hat{Q},\vec{B},B_{ex}] =& \int dx \int dx' \hat{J}(x)\mathcal{R}_{ex}(x)U_{ex}^{-1}(x)\hat{G}_{ex}[\Phi,\vec{B},\hat{Q},B_{ex}](x,x')\; U_{ex}(x')\mathcal{R}_{ex}^{-1}(x')J(x') \\
&+ \frac{1}{2}\int dx\; \Phi^T(x)\; \sigma_z V_{0,s}^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x)\; \sigma_z V_{0,t}^{-1}(x) B^n(x) \\
&- \frac{\pi\nu_3}{4\tau}\mathrm{Tr}\{\hat{Q}^2\} - i\mathrm{Tr}\ln\left[\hat{G}_{ex}^{-1}[\Phi,\vec{B},\hat{Q}](x,x')\right]
\end{aligned}
\tag{7.15}
$$

where $\hat{G}_{ex}[\Phi,\vec{B},\hat{Q}](x,x')$ is the inverse of

$$
\begin{aligned}
\hat{G}_{ex}^{-1}[\Phi,\vec{B},\hat{Q},B_{ex}](x) =& \left[U_{ex}(x)\mathcal{R}_{ex}^{-1}(x)G_0^{-1}(x)\mathcal{R}_{ex}(x)U_{ex}^{-1}(x) + D_{\phi,B,B_{ex}}(x)\right]\delta(x-x') \\
&+ \frac{i}{2\tau}U_{ex}(x)\mathcal{R}_{ex}^{-1}(x)\hat{Q}(r,t,t')\mathcal{R}_{ex}(x')U_{ex}^{-1}(x')\delta(r-r').
\end{aligned}
\tag{7.16}
$$

7.2 Determination of a saddle point for $\hat{Q}$

The determination of the saddle point equation is completely analoguous to the case without external magnetic field and one arrives at

$$
\boxed{U_{ex}\mathcal{R}_{ex}^{-1}\;\hat{Q}_{ex}^{SP}\mathcal{R}_{ex}U_{ex}^{-1}(r,t,t') = \frac{i}{\pi\nu_3}\hat{G}_{ex}[\Phi,\vec{B},\hat{Q}_{ex}^{SP}](r,t,t')}
\tag{7.17}
$$

with

$$
\boxed{
\begin{aligned}
\hat{G}_{ex}[\Phi,\vec{B},\hat{Q}_{ex}^{SP}](x) =& \left[\left(U_{ex}(x)\mathcal{R}_{ex}^{-1}(x)G_0^{-1}(x)\mathcal{R}_{ex}(x)U_{ex}^{-1}(x) + D_{\phi,B,B_{ex}}(x)\right)\delta(x-x')\right. \\
&\left. + \frac{i}{2\tau}U_{ex}(x)\mathcal{R}_{ex}^{-1}(x)\hat{Q}_{ex}^{SP}\mathcal{R}_{ex}(x')U_{ex}^{-1}(x')\delta(r-r')\right]^{-1}.
\end{aligned}
}
\tag{7.18}
$$

Formally the two equations above are identical with the $B_{ex} = 0$ case except for the additional index $_{ex}$ at all quantities. We follow the same solution strategy for the saddle point equation. We adjust the gauge factors such that they approximately annihilate the diagonal field matrix $D_{\phi,B,B_{ex}}$. Then we are able to construct a solution of equation (7.17) starting from the saddle point solution of the non interacting problem.

In the case of a finite external magnetic field B_{ex} we have no longer a unity matrix in spin space (see equation (6.37)) but two different solutions for the spin–up and the spin–down part

due to the different occupation of the spin–up and the spin–down levels. In energy space the non interacting saddle point solution for $\hat{Q}$ reads

$$\hat{\Lambda}_{ex}(\epsilon) = \begin{pmatrix} \Lambda_\uparrow(\epsilon) & 0 \\ 0 & \Lambda_\downarrow(\epsilon) \end{pmatrix} = \begin{pmatrix} F_\uparrow(\epsilon) & 1-F_\uparrow(\epsilon) & 0 & 0 \\ 1+F_\uparrow(\epsilon) & -F_\uparrow(\epsilon) & 0 & 0 \\ 0 & 0 & F_\downarrow(\epsilon) & 1-F_\downarrow(\epsilon) \\ 0 & 0 & 1+F_\downarrow(\epsilon) & -F_\downarrow(\epsilon) \end{pmatrix} \tag{7.19}$$

where

$$F_\uparrow(\epsilon) = F(\epsilon - \mu_B B_{ex}) \qquad \text{and} \qquad F_\downarrow(\epsilon) = F(\epsilon + \mu_B B_{ex}). \tag{7.20}$$

An approximate saddle point for the interacting problems is then given by

$$\hat{Q}^{SP}_{ex} = \mathcal{R}_{ex} U_{ex}^{-1}\, \hat{\Lambda}_{ex}\, U_{ex} \mathcal{R}_{ex}^{-1} \tag{7.21}$$

which leads to the approximate Green's function

$$\hat{G}^{SP}_{ex}(x,x') = \left[\hat{G}_0^{-1} + \frac{i}{2\tau}\, \hat{\Lambda}_{ex} + C_{ex} \right]^{-1}. \tag{7.22}$$

Following the lines of argumentation in section (6.2), one arrives at

$$C_{ex}(x) = \mathbb{I}_2 \otimes C^\phi_\alpha(x)\gamma^\alpha + \sigma_z \otimes C^{B,ex}_\alpha(x)\gamma^\alpha \tag{7.23}$$

with C^ϕ_α given in equation (6.44) and

$$C^{B,ex}_\alpha(x) = \partial_t f_\alpha + v_F(p_F)\cdot\nabla f_\alpha + \widehat{B}_\alpha. \tag{7.24}$$

Compared to the equations (6.45) B_α is replaced by $\widehat{B}_\alpha$. In analogy to (6.47) we define

$$\hat{G}_{\tau,ex} = \left[\hat{G}_0^{-1} + \frac{i}{2\tau}\, \hat{\Lambda}_{ex} \right]^{-1} = \begin{pmatrix} G_0^{-1} + \frac{i}{2\tau}\, \Lambda_\uparrow & 0 \\ 0 & G_0^{-1} + \frac{i}{2\tau}\, \Lambda_\downarrow \end{pmatrix}^{-1} \tag{7.25}$$

and expand the Green's function $\hat{G}^{SP}_{ex}$ in equation (7.22) in powers of the small field C_{ex} which leads to the condition[1]

$$\hat{G}_{\tau,ex} C_{ex} \hat{G}_{\tau,ex}\big|_{r=r'} = 0 \tag{7.26}$$

The **determination of the gauge factors** from the condition proceeds as above. Looking at equation (6.54) one really has to distinguish now between $G_{\uparrow\uparrow}$ and $G_{\downarrow\downarrow}$. However as there is no mixing of the G's, the results for the respective Π_1's and Π_2's remain unchanged. The energy shift due to the external magnetic field simply disappears after a rescaling of the energies. For the gauge factors k_α we recover exactly equation (6.86) whereas for the gauge factors f_α one finds $\widehat{\tilde{B}}$ on the right side instead of $\tilde{B}$.

$$\boxed{ \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \mu_B \begin{pmatrix} \mathcal{D}^R & B(\omega)(\mathcal{D}^R + \mathcal{D}^A) \\ 0 & -\mathcal{D}^A \end{pmatrix} \begin{pmatrix} \widehat{\tilde{B}}_1 \\ \widehat{\tilde{B}}_2 \end{pmatrix} } \tag{7.27}$$

[1] Compare with (6.50).

Taking into account Gaussian fluctuations of the field B leads unfortunately to untractable expressions when the dependence on the external magnetic field of the screened interaction is retained. In the sequel we neglect the effect of the external field on the dynamically screened interaction.[2] Another option would be to neglect the quadratic corrections in $\vec{B}$ and work with the static amplitude Γ_t. We have shown in chapter 2 on perturbation theory that the modification of the interaction due to the external field is not decisive. Even with the static amplitudes one obtains the shifted poles at $\pm 2\mu_B B_{ex}$. We will perform the calculations with the effective interactions given in equation (6.131).

7.3 Calculation of the single particle Green's function

The normalized generating functional $Z[J, \hat{J}, B_{ex}]$ in the presence of an external magnetic field reads[3]

$$Z[J, \hat{J}, B_{ex}] = \frac{1}{\mathcal{N}_\phi \mathcal{N}_B} \int \mathcal{D}\Phi \int \mathcal{D}\vec{B}\; e^{iS[\Phi, \hat{Q}^{SP}, \vec{B}, J, \hat{J}, B_{ex}]} \tag{7.28}$$

where the action

$$\begin{aligned} S[\Phi, \hat{Q}^{SP}, \vec{B}, B_{ex}] = &\int dx \int dx'\; \hat{J}(x)\mathcal{R}_{ex}(x)U_{ex}^{-1}(x)\; \hat{G}_{\tau,ex}(x,x')\; U_{ex}(x')\mathcal{R}_{ex}^{-1}(x')J(x') \\ &+ \frac{1}{2}\int dx\; \Phi^T(x)\; V_s^{-1}(x)\Phi(x) + \frac{1}{2}\int dx \sum_n (B^n)^T(x)\; V_t^{-1}(x)B^n(x) \end{aligned} \tag{7.29}$$

and

$$\hat{G}_{\tau,ex}(t-t') = \begin{pmatrix} G_\uparrow^T(t-t') & -G_\downarrow^<(t-t') & 0 & 0 \\ G_\uparrow^>(t-t') & -G_\uparrow^T(t-t') & 0 & 0 \\ 0 & 0 & G_\downarrow^T(t-t') & -G_\downarrow^<(t-t') \\ 0 & 0 & G_\downarrow^>(t-t') & -G_\downarrow^T(t-t') \end{pmatrix}. \tag{7.30}$$

The explicit form of $G_\uparrow^R(\epsilon, p)$ and $G_\downarrow^R(\epsilon, p)$ is for example given in equation (2.63). We do not need the explicit form of the elements above as we go over to the Keldysh rotated representation for the calculation of the DOS. The Green's function in the presence of interaction, disorder and external magnetic field follows from[4]

$$G_{ex}(x,x') = i\frac{\delta}{\delta J(x)}\frac{\delta}{\delta \hat{J}(x')} Z[J(x), \hat{J}(x'), B_{ex}] = \left\langle \mathcal{R}_{ex}(x)U_{ex}^{-1}(x)\hat{G}_{\tau,ex}(x,x')U_{ex}(x')\mathcal{R}_{ex}^{-1}(x') \right\rangle_{\phi,B}. \tag{7.32}$$

The four components that are relevant for the calculation of the DOS are

$$G_{\uparrow\uparrow}^{ex,>} = \mathcal{R}_{22}(\mathcal{R}_{11}^{ex})^* G_{F\uparrow\uparrow}^{ex,>} + \mathcal{R}_{24}^{ex}(\mathcal{R}_{13}^{ex})^* G_{F\downarrow\downarrow}^{ex,>} \tag{7.33}$$

[2] We will take the result of section (6.2).

[3] with a normalization constants $\mathcal{N}_\phi = \mathrm{Det}[i\sigma_z V_s^{-1}]^{1/2}$ and $\mathcal{N}_B = \mathrm{Det}[i\sigma_z V_t^{-1}]^{3/2}$

[4]

$$\langle\, ...\rangle_\phi = \frac{1}{\mathcal{N}_\phi}\int \mathcal{D}\Phi\; ...e^{i/2\mathrm{Tr}[\Phi^T V_s^{-1}\Phi]} \qquad \text{and} \qquad \langle\, ...\rangle_B = \frac{1}{\mathcal{N}_B}\int \mathcal{D}\vec{B}\; ...e^{i/2 \sum_n \mathrm{Tr}[(B^n)^T V_t^{-1} B^n]} \tag{7.31}$$

$$G^{ex,<}_{\uparrow\uparrow} = \mathcal{R}^{ex}_{11}(\mathcal{R}^{ex}_{22})^* G^{ex,<}_{F\uparrow\uparrow} + \mathcal{R}^{ex}_{13}(\mathcal{R}^{ex}_{24})^* G^{ex,<}_{F\downarrow\downarrow} \tag{7.34}$$

$$G^{ex,>}_{\downarrow\downarrow} = \mathcal{R}^{ex}_{42}\mathcal{R}^{ex}_{31} G^{ex,>}_{F\uparrow\uparrow} + \mathcal{R}_{44}\mathcal{R}^{ex}_{33} G^{ex,>}_{F\downarrow\downarrow} \tag{7.35}$$

$$G^{ex,<}_{\downarrow\downarrow} = \mathcal{R}^{ex}_{31}\mathcal{R}^{ex}_{42} G^{ex,<}_{F\uparrow\uparrow} + \mathcal{R}^{ex}_{33}\mathcal{R}^{ex}_{44} G^{ex,<}_{F\downarrow\downarrow}. \tag{7.36}$$

The definitions are made in complete analogy to section (6.5). The index ^{ex} just indicates that we have now modified rotation matrix elements and the Green's functions $G^{ex,}_{Fs_is_i}$ contain modified exponentials e^{if_α}.[5]The Φ dependent part is of course unchanged and thus the path integral over Φ is performed like in the previous chapter.

For the path integral over $\vec{B}$ we use again the assumption of slow variations. As a first step we approximate the products of rotation matrix elements in that spirit. We obtain suppressing the arguments

$$G^{ex,>}_{\uparrow\uparrow} = \left[\frac{1}{2}\left(1 + \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{>}_{\uparrow}\, e^{-i(f_2 - f_1)} + \frac{1}{2}\left(1 - \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{>}_{\downarrow}\, e^{i(f_2 - f_1)}\right] e^{J_\phi} \tag{7.37}$$

$$G^{ex,<}_{\uparrow\uparrow} = \left[\frac{1}{2}\left(1 + \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{<}_{\uparrow}\, e^{-i(f_1 - f_2)} + \frac{1}{2}\left(1 - \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{<}_{\downarrow}\, e^{i(f_1 - f_2)}\right] e^{J^*_\phi} \tag{7.38}$$

$$G^{ex,>}_{\downarrow\downarrow} = \left[\frac{1}{2}\left(1 - \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{>}_{\uparrow}\, e^{-i(f_2 - f_1)} + \frac{1}{2}\left(1 + \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{>}_{\downarrow}\, e^{i(f_2 - f_1)}\right] e^{J_\phi} \tag{7.39}$$

$$G^{ex,<}_{\downarrow\downarrow} = \left[\frac{1}{2}\left(1 - \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{<}_{\uparrow}\, e^{-i(f_1 - f_2)} + \frac{1}{2}\left(1 + \frac{B^z + B_{ex}}{\widehat{B}}\right) G^{<}_{\downarrow}\, e^{i(f_1 - f_2)}\right] e^{J^*_\phi}. \tag{7.40}$$

Note that we recover the expressions in equtions (6.174) and (6.175) if we send $B_{ex} \to 0$ in the prefactors above. Let us repeat ones more the formula for the DOS

$$\nu(\epsilon) = \frac{i}{2\pi\mathcal{V}} \sum_p \left(G^{ex,>}_{\uparrow\uparrow}(p,\epsilon) - G^{ex,<}_{\uparrow\uparrow}(p,\epsilon) + G^{ex,>}_{\downarrow\downarrow}(p,\epsilon) - G^{ex,<}_{\downarrow\downarrow}(p,\epsilon) \right). \tag{7.41}$$

Now we cannot simply take the spin–up contribution and multiply it by two. However it is advantageous not to treat the spin–up and spin–down contribution completely separately. Adding equation (7.37) and equation (7.39) we see that the terms proportional to $\frac{B^z + B_{ex}}{\widehat{B}}$ cancel exactly and for the DOS remains

$$\nu(t) = \frac{i}{2\pi\mathcal{V}} \sum_p \left\langle \left(e^{-i(f_2 - f_1)} G^{>}_{\uparrow} e^{J_\phi} - e^{-i(f_1 - f_2)} G^{<}_{\uparrow} e^{J^*_\phi} \right) + \left(e^{i(f_2 - f_1)} G^{>}_{\downarrow} e^{J_\phi} - e^{i(f_1 - f_2)} G^{<}_{\downarrow} e^{J^*_\phi} \right) \right\rangle_B . \tag{7.42}$$

Performing the p–sum in the continuum limit like in (6.202) and multiplying both sides by a^2, we obtain

[5]The f_α's are given in equation (7.27).

$$\nu_1(t) = \frac{\nu_1}{2}\left\langle \left(e^{-i(f_2-f_1)}\Lambda_\uparrow^> e^{J_\phi} - e^{-i(f_1-f_2)}\Lambda_\uparrow^< e^{J_\phi^*}\right) + \left(e^{i(f_2-f_1)}\Lambda_\downarrow^> e^{J_\phi} - e^{i(f_1-f_2)}\Lambda_\downarrow^< e^{J_\phi^*}\right)\right\rangle_B . \tag{7.43}$$

The relevant matrix elements of (7.19) read in t–space:

$$\Lambda_\uparrow^> = \delta(t) + F_\uparrow(t) \qquad \Lambda_\uparrow^< = -(\delta(t) - F_\uparrow(t)) \tag{7.44}$$

$$\Lambda_\downarrow^> = \delta(t) + F_\downarrow(t) \qquad \Lambda_\downarrow^< = -(\delta(t) - F_\downarrow(t)). \tag{7.45}$$

Using furthermore the definitions

$$P_{B,\uparrow} := \left\langle e^{-i(f_2-f_1)}\right\rangle_B \qquad P_{B,\downarrow} := \left\langle e^{i(f_2-f_1)}\right\rangle_B \tag{7.46}$$

one has

$$\nu_1(t) = \frac{\nu_1}{2}\left[2\delta(t) + F_\uparrow(t)\left(P_{B,\uparrow}\, e^{J_\phi} - P_{B,\uparrow}^c\, e^{J_\phi^*}\right)\right] + \frac{\nu_1}{2}\left[2\delta(t) + F_\downarrow(t)\left(P_{B,\downarrow}\, e^{J_\phi} - P_{B,\downarrow}^c\, e^{J_\phi^*}\right)\right] \tag{7.47}$$

where $P_{B,\uparrow}^c$ and $P_{B,\downarrow}^c$ denote the expectation values $\left\langle e^{-i(f_1-f_2)}\right\rangle_B$ and $\left\langle e^{i(f_1-f_2)}\right\rangle_B$ respectively. They are not independent of the two expectation values defined in (7.46). One obtains $P_{B,\uparrow}^c$ simply by replacing J_B by J_B^* in $P_{B,\uparrow}$ and $P_{B,\downarrow}^c$ by replacing J_B by J_B^* in $P_{B,\downarrow}$. These replacements do not correspond to complex conjugation as $P_{B,\uparrow}$ depends also on iB_{ex}. It can be shown that $P_{B,\downarrow}$ can be obtained from $P_{B,\uparrow}$ by the replacement of B_{ex} by $-B_{ex}$. Hence we only need to determine one of the averages. Let us first discuss the limit of a strong external magnetic field and calculate the average over the fluctuations of $\vec{B}$ in that limit. Afterwards we will determine the path integrals for arbitrary external magnetic field for slowly varying magnetic fluctuations.

7.4 The DOS at $T = 0$ for strong external magnetic field

In this section we assume that $\mu_B B_{ex}$ is much bigger than all the other energies involved in our problem. Thus we expand our expressions in $1/B_{ex}$ and keep only the leading order term which corresponds to the limit $B_{ex} \to \infty$. We have

$$\widehat{B} = B_{ex} + B^z + \frac{B_\perp^2}{2B_{ex}} + \mathcal{O}(1/B_{ex}^3) \tag{7.48}$$

We approximate $\widehat{B}$ in the gauge factors f_α by $B_{ex} + B_z$. The gauge factors then simplify to read

$$f_\alpha(x) = \mu_B \sum_{\beta=1}^{2} \int dy\; A_{\alpha\beta}(x,y) B_\beta^z(y) - i\mu_B B_{ex} t. \tag{7.49}$$

This has the consequence that the path integrals over $\vec{B}$ become Gaussian because B^z appears now linear in the exponentials.[6] Thus the integral is completely analoguous to the path integral for the scalar field Φ.[7] We can calculate it exactly and do not have assume slow spatial variations

[6] B_{ex} is not involved in the averaging process.

[7] We set $e = \mu_B = 1$ in the following.

of the magnetic fluctuations like in section (6.5.3), where we calculated the average over $\vec{B}$ for zero external field. For $r = r'$ we obtain

$$P_{B,\uparrow}(t-t') = \left\langle e^{-i(f_2(r,t)-f_1(r,t'))} \right\rangle_B = e^{J_{21}^B(0,t-t')} e^{iB_{ex}(t-t')}. \tag{7.50}$$

Setting $t = t - t'$ and using our convention $J_{21}^B = J_B$, the four expectation values read

$$P_{B,\uparrow} = e^{J_B(t)} e^{iB_{ex}t} \qquad P_{B,\downarrow} = e^{J_B(t)} e^{-iB_{ex}t} \tag{7.51}$$

and

$$P^c_{B,\uparrow} = e^{J_B^*(t)} e^{iB_{ex}t} \qquad P^c_{B,\downarrow} = e^{J_B^*(t)} e^{-iB_{ex}t}. \tag{7.52}$$

Combining these averages with the factors e^{J_ϕ} and $e^{J_\phi^*}$ respectively we define

$$P_\uparrow := \frac{1}{2\pi}\, e^{J(t)-iB_{ex}t} \qquad P_\downarrow := \frac{1}{2\pi}\, e^{J(t)+iB_{ex}t} \tag{7.53}$$

and

$$P^c_\uparrow := \frac{1}{2\pi}\, e^{J^*(t)-iB_{ex}t} \qquad P^c_\downarrow := \frac{1}{2\pi}\, e^{J^*(t)+iB_{ex}t} \tag{7.54}$$

where we used again the abbreviation intoduced in equation (6.222): $J(t) = J_\phi(t) + J_B(t)$. Then the DOS can be written in the compact form

$$\nu_1(\epsilon) = \nu_1 \left(2 + \frac{1}{2} \int_{-\infty}^{\infty} dE\; F(\epsilon - E - B)\; \left[P_\uparrow(E) - P^c_\uparrow(E) \right] + F(\epsilon - E + B)\; \left[P_\downarrow(E) - P^c_\downarrow(E) \right] \right). \tag{7.55}$$

The definition of the P's in (7.53) and (7.54) is similar to the definition of P_1 in equation (6.220). One can relate the different P's as function of the energy E to $P_1(E)$ and $P_1^*(E)$ through a shift by $\pm B_{ex}$. Using then for $T = 0$ the properties $P_1(E) = 0$ for $E < 0$ and $P_1^*(E) = P_1(-E)$[8] we arrive at

$$\nu_1(\epsilon) = 2\nu_1 \int_0^\epsilon dE\; P_1(E) \tag{7.56}$$

which leads to

$$\boxed{\nu_1 = 2\nu_1 \left[1 - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}} \right) \right].} \tag{7.57}$$

Expanding this expression for high energies ϵ and using the definition of $\sqrt{E_g}$ one obtains

$$\nu(\epsilon) = 2\nu_1 \left[1 - \frac{2}{\sqrt{\pi}} \left(\sqrt{\frac{E_s}{\epsilon}} - \sqrt{\frac{E_t}{\epsilon}} \right) \right]. \tag{7.58}$$

We see that $\delta\nu_1(\epsilon)/\nu_1$ coincides with the result of perturbation theory in equation (2.88) because the terms containing the arctanh and arctan vanish for $B_{ex} \to \infty$. One could say: High magnetic fields simply suppress two of the three triplet terms.

[8] See section (6.6.1).

In figure (7.1) we plotted our result (7.57) for infinite field and the perturbative solution (7.58). Additionally we plotted the non–perturbative result (6.239) for $B_{ex} = 0$. Again we have chosen $R = \sqrt{E_t}/\sqrt{E_s} = 0.05$.

In figure (7.2) we plotted our non perturbative solutions for zero and infinite field for high energies for two different ratios of E_s and E_t. $1000E_s$ correspond to approximately 0.003mV which is still two orders of magnitude lower than the voltages in figure (1.2). Turning on and off a sufficiently high external magnetic field would allow to jump in between the two corresponding curves. We see that with increasing R the step height increases but even for a value of $R = 0.1$ the effect corresponds only to a change of about one percent.

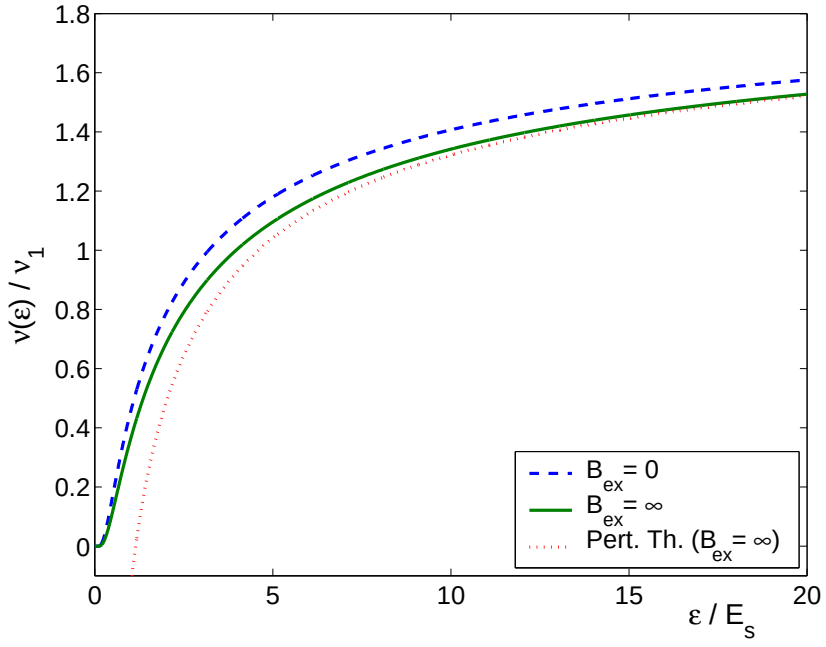

Figure 7.1: The DOS close to the Fermi energy. The solid line is the non–perturbative asymptotic result for $B_{ex} \to \infty$. The dashed curve is the non–perturbative result including the singlet and the triplet terms for $B_{ex} = 0$ and the dotted curve is the perturbative result in the limit of infinite external magnetic field which diverges for small energies.

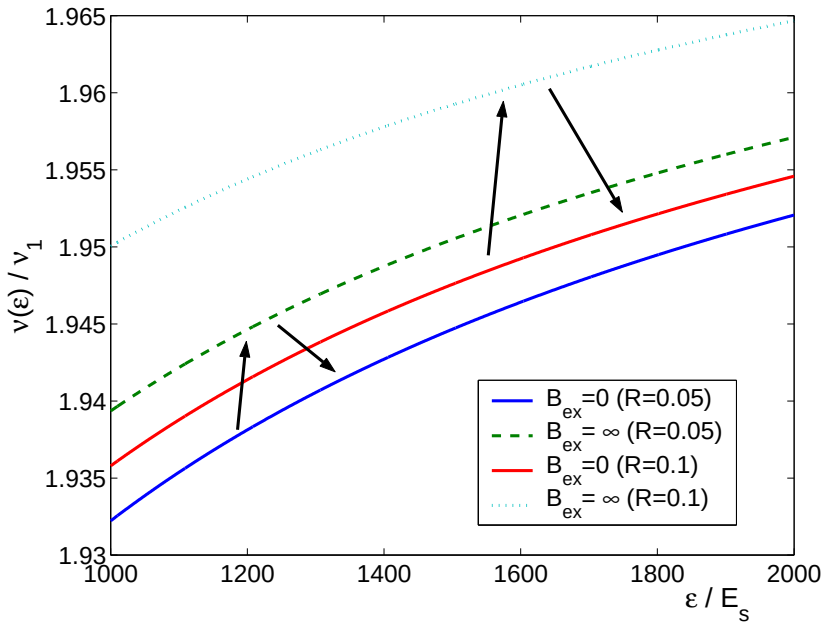

Figure 7.2: The DOS further away from the Fermi energy ($1000E_s$ correspond to approximately 0.003mV). The solid lines are the non–perturbative asymptotic results for $B_{ex} \to \infty$ and the dashed lines are the non–perturbative results including the singlet and the triplet terms for $B_{ex} = 0$.

7.5 The DOS at $T = 0$ for arbitrary external magnetic field

For an arbitrary magnetic field the calculation is much more complicated as we cannot develop the f_α's in B_{ex} and thus do not encounter a Gaussian integral for the fluctuations of $\vec{B}$. We have to make the same assumptions as in section (6.5.3) and evaluate $P_{B,\uparrow} := \left\langle e^{-i(f_2 - f_1)} \right\rangle_B$ in the limit of weakly varying magnetic fluctuations which simplified the path integral to an ordinary three dimensional integral. The explicit calculation can be found in appendix (9.2.3) . We just cite here the result

$$\begin{aligned} P_{B,\uparrow} = \frac{e^{J_B}}{2iB_{ex}t}\Bigg(& e^{iB_{ex}t}\,\left(iB_{ex}t + 2J_B\right)\left[1 + \mathrm{Erf}\left(\frac{iB_{ex}t + 2J_B}{2i\sqrt{J_B}}\right)\right] \\ & + e^{-iB_{ex}t}\,\left(iB_{ex}t - 2J_B\right)\left[1 - \mathrm{Erf}\left(\frac{iB_{ex}t - 2J_B}{2i\sqrt{J_B}}\right)\right]. \end{aligned} \tag{7.59}$$

$P_{B,\downarrow}$ is obtained by replacing $B_{ex} \to -B_{ex}$. In the case without external field $G^{>}_{\tau,\uparrow}$ and $G^{>}_{\tau,\downarrow}$ would be equal and the terms containing the Erf would cancel in equation (7.37) and we would recover $e^{J_B}\,(1 + 2J_B)$. Sending B_{ex} and $V_t \to 0$ $(= J_B \to 0)$ we obtain $P_{B,\uparrow} = 1$ which is also correct. $P^*_{B,\uparrow}$ and $P^*_{B,\downarrow}$ are obtained from $P_{B,\uparrow}$ and $P_{B,\downarrow}$ simply by replacing J_B by J^*_B. We decompose these expressions in two parts. For convenience, we add a factor e^{J_ϕ} emerging when

we average over the scalar field and a factor of $\frac{1}{2\pi}$. Putting

$$\frac{e^{J_\phi} P_{B,\uparrow}}{2\pi} = P_{g_1} + P_{g_2}, \qquad \frac{e^{J_\phi} P_{B,\downarrow}}{2\pi} = P_{g_1} - P_{g_2} \tag{7.60}$$

we have

$$P_{g_1} := \frac{1}{2\pi}\,\frac{e^{J_\phi+J_B}}{2iB_{ex}t}\Big(e^{iB_{ex}t}\big(iB_{ex}t+2J_B\big)+e^{-iB_{ex}t}\big(iB_{ex}t-2J_B\big)\Big) \tag{7.61}$$

$$\begin{aligned} P_{g_2} := \frac{1}{2\pi}\,\frac{e^{J_\phi+J_B}}{2iB_{ex}t}\Big[e^{iB_{ex}t}\big(iB_{ex}t+2J_B\big)\mathrm{Erf}\Big(\frac{iB_{ex}t+2J_B}{2i\sqrt{J_B}}\Big) \\ - e^{-iB_{ex}t}\big(iB_{ex}t-2J_B\big)\mathrm{Erf}\Big(\frac{iB_{ex}t-2J_B}{2i\sqrt{J_B}}\Big)\Big]. \end{aligned} \tag{7.62}$$

In the following we abbreviate again $J = J_\phi + J_B$. In the limit $T = 0$ we will be able to evaluate analytically the contribution of P_{g_1} to the DOS. The part coming from P_{g_2} is determined numerically.[9] With these definitions the formula for the DOS of (7.47) becomes

$$\nu_1(t) = \frac{\nu_1}{2}\Big[4\delta(t) + 2\pi F_\uparrow(t)\Big(\big[P_{g_1}+P_{g_2}\big]-\big[P^c_{g_1}+P^c_{g_2}\big]\Big)+2\pi F_\downarrow(t)\Big(\big[P_{g_1}-P_{g_2}\big]-\big[P^c_{g_1}-P^c_{g_2}\big]\Big)\Big]. \tag{7.63}$$

We separate the parts containing P_{g_1} and P_{g_2} defining

$$\nu_1 := \nu_{g_1} + \nu_{g_2} \tag{7.64}$$

where ν_{g_1} and ν_{g_2} read in energy representation

$$\nu_{g_1}(\epsilon) := \frac{\nu_1}{2}\Big[4 + \int_{-\infty}^{\infty} dE\,\big[F_\uparrow(\epsilon-E)+F_\downarrow(\epsilon-E)\big]\Big(P_{g_1}(E)-P^c_{g_1}(E)\Big)\Big] \tag{7.65}$$

and

$$\nu_{g_2}(\epsilon) := \frac{\nu_1}{2}\int_{-\infty}^{\infty} dE\,\big[F_\uparrow(\epsilon-E)-F_\downarrow(\epsilon-E)\big]\Big(P_{g_2}(E)-P^c_{g_2}(E)\Big). \tag{7.66}$$

Let us first discuss the evaluation of ν_{g_1}. In order to write down the Fourier transform of $P_{g_1}(t)$ we need to determine two basic Fourier transforms. The first one is already known

$$P_1(t) = \frac{e^{J(t)}}{2\pi} \qquad \rightarrow \qquad P_1(E) = \frac{1}{\sqrt{\pi}E_g}\Big(\frac{E_g}{E}\Big)^{3/2} e^{-E_g/E} \tag{7.67}$$

The second basic Fourier transform we denote by P_3:

$$P_3(t) := \frac{2J_B(t)e^{J(t)}}{2\pi i B_{ex}t} \qquad \rightarrow \qquad P_3(E) = \frac{4}{\sqrt{\pi}B_{ex}}\Big(\frac{E_B}{E}\Big)^{1/2} e^{-E_g/E} \tag{7.68}$$

[9] The definitions of $P^c_{g_1}$ and $P^c_{g_2}$ are completely analogous. The only difference is that they contain J^*_ϕ and J^*_B instead J_ϕ and J_B.

The derivation of $P_3(E)$ can be found in subsection (9.3). Note that $P_1^*(E) = P_1(-E)$. But $P_3^*(E) = -P_3(-E)$. This is due to the additional $1/t$ appearing in P_3. Using furthermore that at $T = 0$ the function $F(\epsilon - E) = \tanh(\frac{\epsilon - E}{2T})$ becomes $\mathrm{sign}(\epsilon - E)$ we can derive

$$\begin{aligned}\nu_{g_1}(\epsilon) = \frac{\nu_1}{2}\Bigg[2\int_0^{\epsilon} dE\ P_1(E) + \int_0^{\epsilon+2B_{ex}} dE\ P_1(E) + \int_0^{|\epsilon-2B_{ex}|} dE\ P_1(E) \\ - \mathrm{sign}(\epsilon - 2B_{ex})\int_0^{|\epsilon-2B_{ex}|} dE\ P_3(E) + \int_0^{\epsilon+2B_{ex}} dE\ P_3(E)\Bigg]\end{aligned} \tag{7.69}$$

The E–integrations can be performed using

$$\int_0^{\epsilon} dE\ P_1(E) = 1 - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}}\right) \tag{7.70}$$

and

$$\int_0^{\epsilon} dE\ P_3(E) = \frac{8\sqrt{E_g}\sqrt{E_B}}{B_{ex}}\left(\sqrt{\frac{\epsilon}{E_g}}\ \frac{e^{-E_g/\epsilon}}{\sqrt{\pi}} + \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}}\right) - 1\right). \tag{7.71}$$

Thus we obtain for $\nu_{g_1}(\epsilon)$:

$$\begin{aligned}\nu_{g_1}(\epsilon) = \frac{\nu_1}{2}\Bigg[4 - 2\mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}}\right) - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon + 2B_{ex}}}\right) - \mathrm{Erf}\left(\sqrt{\frac{E_g}{|\epsilon - 2B_{ex}|}}\right)) \\ - 8\frac{\sqrt{E_B}\sqrt{E_g}}{B_{ex}}\Bigg(\Bigg\{1 - \sqrt{\frac{\epsilon + 2B_{ex}}{E_g}}\ \frac{e^{-E_g/(\epsilon+2B_{ex})}}{\sqrt{\pi}} - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon + 2B_{ex}}}\right)\Bigg\} \\ - \mathrm{sign}(\epsilon - 2B_{ex})\Bigg\{1 - \sqrt{\frac{|\epsilon - 2B_{ex}|}{E_g}}\ \frac{e^{-E_g/|\epsilon-2B_{ex}|}}{\sqrt{\pi}} - \mathrm{Erf}\left(\sqrt{\frac{E_g}{|\epsilon - 2B_{ex}|}}\right)\Bigg\}\Bigg)\Bigg]\end{aligned} \tag{7.72}$$

In the limit $B_{ex} \to 0$ one gets

$$\nu_{g_1}(\epsilon, B_{ex} = 0) = 2\nu_1\left[1 - \mathrm{Erf}\left(\sqrt{\frac{E_g}{\epsilon}}\right) - \frac{4}{\sqrt{\pi}}\sqrt{\frac{E_B}{\epsilon}}\ e^{-E_g/\epsilon}\right] \tag{7.73}$$

which is exactly the result of equation (6.239) that we found for zero external magnetic field. Thus the contribution of ν_{g_2} has to vanish for $B_{ex} = 0$. Let us verify this property analytically. Without loss of generality we can assume $\epsilon > 0$ and $B_{ex} > 0$ as the problem is symmetric about $\epsilon = 0$. For $T = 0$ we have

$$\begin{aligned}F_\uparrow(\epsilon - E) - F_\downarrow(\epsilon - E) = F(\epsilon - E - B_{ex}) - F(\epsilon - E + B_{ex}) \\ = -\big[\mathrm{sign}(E - (\epsilon - B_{ex})) - \mathrm{sign}(E - (\epsilon + B_{ex}))\big] = \begin{cases} -2 & \epsilon - B_{ex} < E < \epsilon + B_{ex} \\ 0 & \text{else}\end{cases}\end{aligned} \tag{7.74}$$

and thus

$$\nu_{g_2}(\epsilon) = -\nu_1 \int_{\epsilon - B_{ex}}^{\epsilon + B_{ex}} dE \, \big[P_{g_2}(E) - P^c_{g_2}(E) \big]. \tag{7.75}$$

We see that the integration interval of length $2B_{ex}$ goes to zero for $B_{ex} \to 0$ and hence $\nu_{g_2} \to 0$. For the numerical evaluation of ν_{g_2} it is not useful to work with equation (7.75) because one would have to determine first P_{g_2} and $P^c_{g_2}$ as function of energy and then do the dE integration. What remains in t–representation is the Fourier transformation to ϵ:

$$\nu_{g_2}(\epsilon) = \nu_1 \int_{-\infty}^{\infty} dt \; 2\pi \big[F_\uparrow(t) - F_\downarrow(t) \big] \big(P_{g_2}(t) - P^c_{g_2}(t) \big) e^{i\epsilon t}. \tag{7.76}$$

In t–representation $F_\uparrow$ and $F_\downarrow$ are given by

$$F_\uparrow(t) = -\frac{i}{\pi t} \, e^{-iB_{ex}t} \qquad \text{and} \qquad F_\downarrow(t) = -\frac{i}{\pi t} \, e^{iB_{ex}t}. \tag{7.77}$$

One might wonder if the infinite integration interval poses a problem. This is not the case because $P_{g_2}(t)$ $(P^c_{g_2}(t))$ contains a factor $e^{J(t)}$ $(e^{J^*(t)})$ which falls off exponentially. So let us analyze the results of the numerical calculation. We assumed as before $R = 0.05$. We begin with the limits of strong and weak external magnetic field. We have chosen $B_{ex} = 10$ (in units of E_s) for the plot in figure (7.3) and $B_{ex} = 0.02$ for the plot in figure (7.4). As reference curves we plotted the two limits of zero and infinite external magnetic field, are given in equations (6.239) and (7.57). In figure (7.3) we see that the DOS for an external field of $B_{ex} = 10E_s$ is already close to the $B_{ex} = \infty$ limit. According to our estimate of E_s given in equation (9.82) the energy of $\mu_B B_{ex} = 10E_s$ corresponds to approximately 10^{-3}Tesla. Such a small field could already serve as switching field (see figure (7.2)).

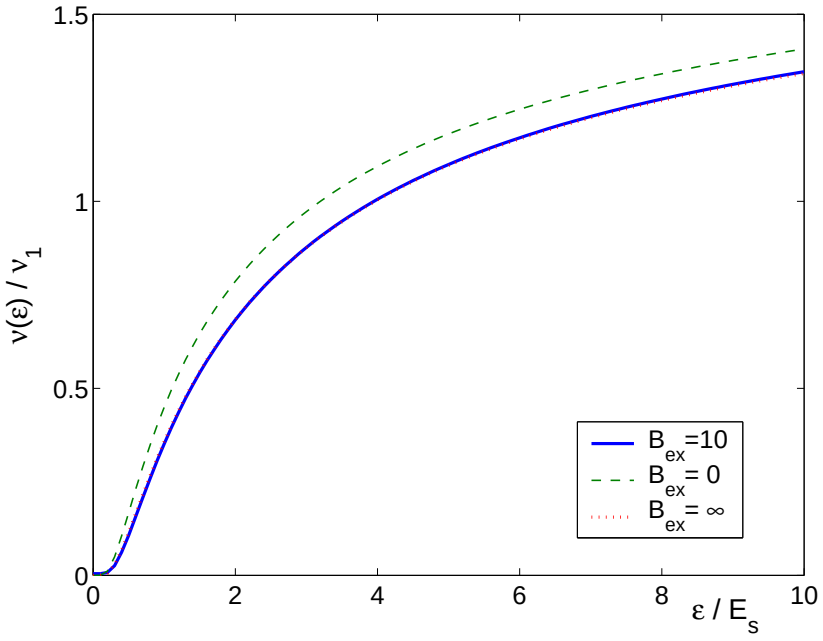

Figure 7.3: The DOS close to the Fermi energy. The solid line is the non–perturbative numerical solution for $B_{ex} = 10$. The dashed curve is the non–perturbative result for $B_{ex} = 0$ and the dotted curve is the non–perturbative result in the limit of infinite external magnetic field. (The dotted line is almost completely masked by the solid line.)

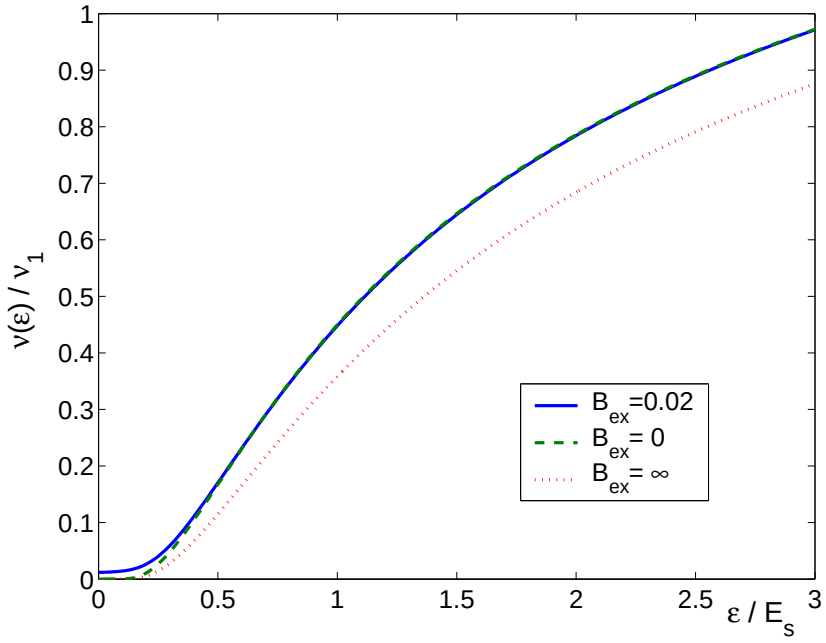

Figure 7.4: The DOS close to the Fermi energy. The solid line is the non–perturbative numerical solution for $B_{ex} = 0.02$. The dashed curve is the non–perturbative result for $B_{ex} = 0$ and the dotted curve is the non–perturbative result in the limit of infinite external magnetic field. One remarks a small offset at $\epsilon = 0$ for the finite field of $B_{ex} = 0.02$.

Figure (7.4) shows results for a weak external magnetic field. One notes that the DOS for $B_{ex} = 0.02$ almost lies on top of the $B_{ex} = 0$ curve. However for small energies there is a visible difference. The DOS for $B_{ex} = 0.02$ does not go to zero for $\epsilon = 0$. In figure (7.5) we analyzed this offset in greater detail. For zero and infinite external magnetic field the offset of the DOS is zero. For finite field, however, the DOS does not decrease to zero. This might be due to the appearance of an additional length scale in the system, the cyclotron radius $r = mv/eB_{ex}$, introduced by the external magnetic field.

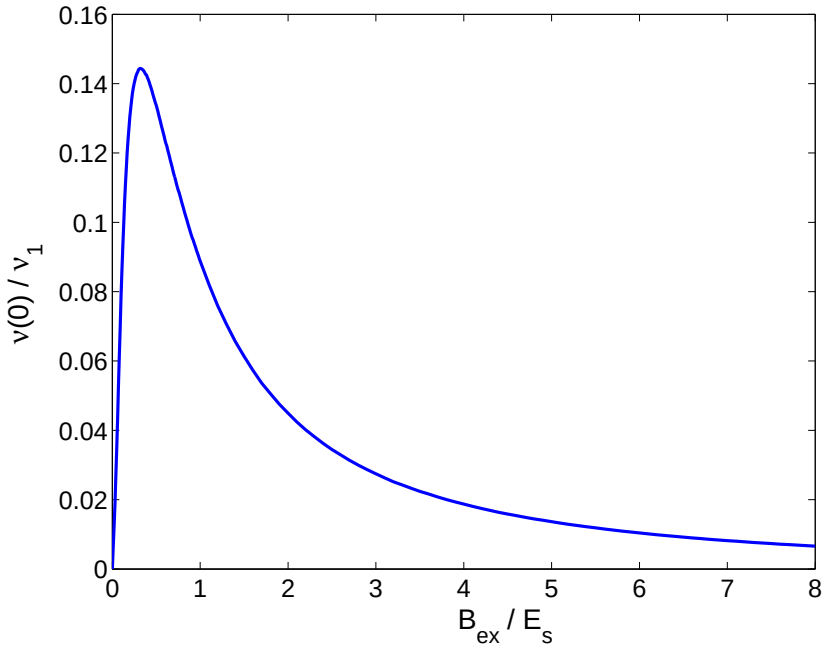

Figure 7.5: The DOS at the Fermi energy ($\epsilon = 0$) plotted as a function of the applied external field.

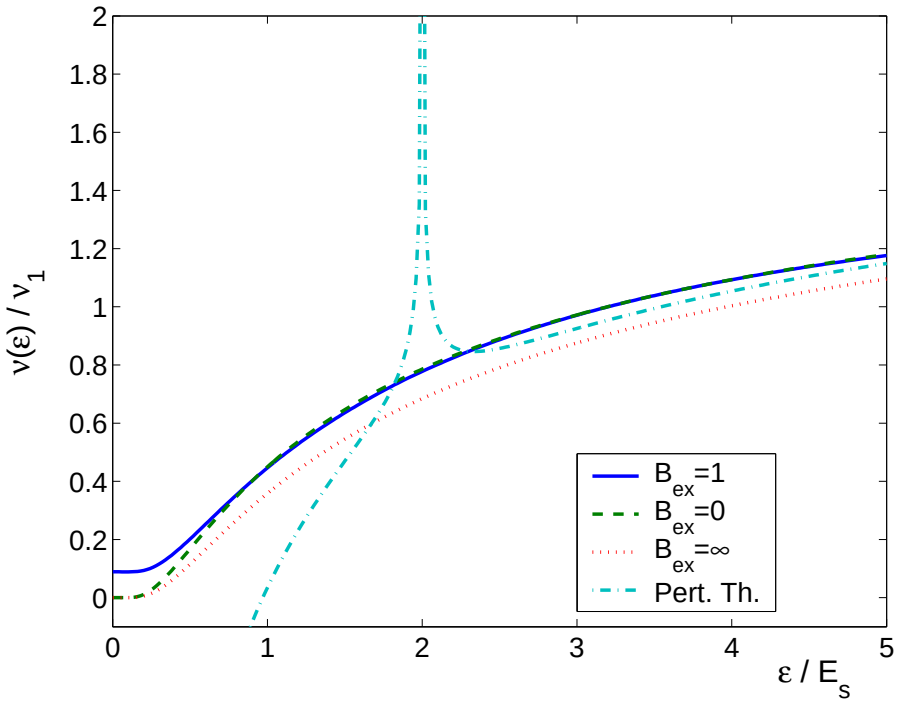

Figure 7.6: The DOS close to the Fermi energy. The solid line is the non–perturbative numerical solution for the intermediate field $B_{ex} = 1$. The dashed curve is the non–perturbative result for $B_{ex} = 0$ and the dotted curve is the non–perturbative result in the limit of infinite external magnetic field. The dashed–dotted line is the perturbative prediction.

In figure (7.6) we have chosen $B_{ex} = 1$ and plotted additionally to the zero and infinity field solution the perturbative prediction. We note that our non–perturbative curve does not have any feature at $\epsilon = 2\mu_B B_{ex}$. It is completely smooth and does not show even a dip like it was predicted by Raimondi, Castellani and Di Castro [25].

7.6 The DOS at $T > 0$ for arbitrary external magnetic field

Let us finally consider the case of finite temperatures. The question is how low the temperature has to be in order to see the effects we described above. For finite temperature we have to evaluate both ν_{g_1} and ν_{g_2} numerically. The temperature enters in our theory through the function F. We noted already that in energy representation $F(\epsilon) = \tanh(\epsilon/2T)$. In t space we get for the shifted functions $F_\uparrow$ and $F_\downarrow$:

$$F_\uparrow(t) = -i\,T\,\frac{e^{-iB_{ex}t}}{\sinh(\pi\,t\,T)} \qquad \text{and} \qquad F_\downarrow(t) = -i\,T\,\frac{e^{iB_{ex}t}}{\sinh(\pi\,t\,T)}. \tag{7.78}$$

In figure (7.7) we plotted the DOS obtained for various temperatures.

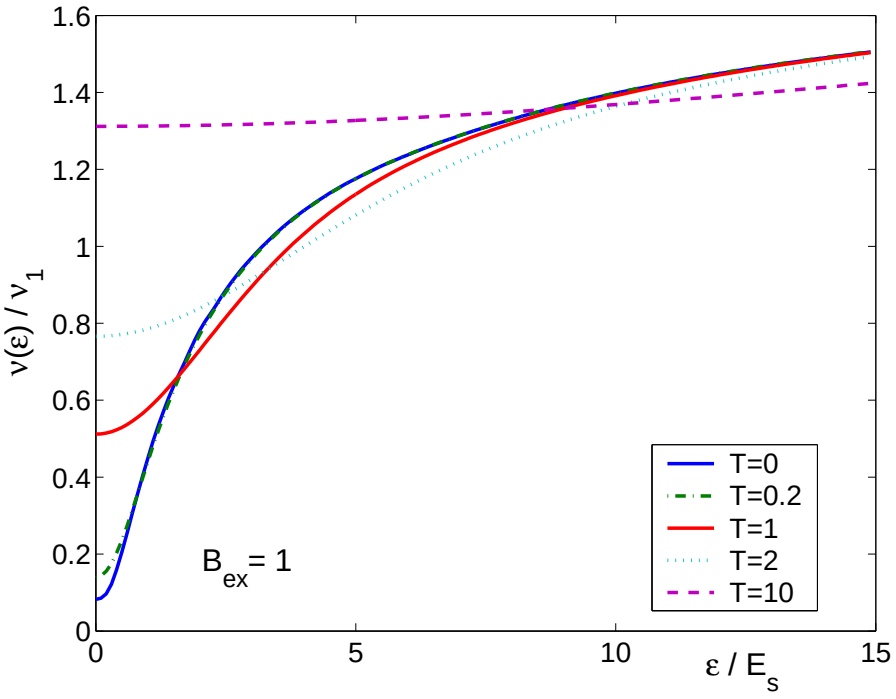

Figure 7.7: The DOS near the Fermi edge for $B_{ex} = 1$ shown for different temperatures. The zero bias anomaly is already smeared out for very low temperatures. $k_B T = E_s$ corresponds to approximately a temperature of $T = 0.03$mK.

Chapter 8

Conclusions

In this thesis we investigated the effect of the spin on the density of states (DOS) near the Fermi edge in quasi one–dimensional diffusive wires. In order to approach this task we had to set up a formalism which allows to treat interacting spinfull electrons in a disordered environment.

First we used the methods of classical diagrammatic perturbation theory. We discussed how the one particle Green's function, that is connected to the DOS, is modified in the presence of weak disorder. Afterwards, we added the Coulomb interaction between the electrons and constructed the two basic diagrams, namely the exchange and the Hartree diagrams, that lead to the famous *zero bias anomaly* which was first discovered by Altshuler and Aronov in 1979 ([7], [8]).

As next step we showed how to generalize this result to higher orders in the interaction by taking effective amplitudes with values determined by Landau parameters of the Fermi liquid theory. We rewrote the exchange and the Hartree terms into a singlet and a triplet contribution that allowed us to isolate the effect of the spin triplet on the DOS.

As the Landau parameters are connected to response functions like the specific heat and spin susceptibility, one is able to estimate the magnitude of the suppression of the DOS due to the singlet and triplet terms on the basis of measured properties of the metal under consideration. In general, one can say that the impact of the triplet terms in metals on the DOS is very small. Palladium however is a promising material as it has an extraordinarily large magnetic susceptibility for low temperatures. Approaching the magnetic transition the magnetic susceptibility increases which corresponds to an increase of the triplet amplitude whereas the singlet amplitude changes only very little. From the values of the response functions taken from different experimental works we estimated that the triplet contribution could be of the order of some percent of the singlet contribution. However the absolute energies E_s and E_t, that are connected to the interaction in the singlet and triplet channels, are very small and one would have to go to very low temperatures (mK or less), because otherwise thermal fluctuations simply smear out the effect. Doping Palladium with impurities in order to increase the magnetic susceptibility is not a real option because scattering from magnetic impurities suppresses the triplet contribution as we have also discussed. Additional impurities could also increase the spin–orbit scattering which cuts off the triplet contribution.

Furthermore we investigated in the framework of perturbation theory also the effect of an external magnetic field. We showed that the singlet term remains unchanged whereas the triplet term splits up into three different terms which reminds of the Zeeman splitting of a spin triplet in an atom in the presence of an external field. By splitting it is meant here that within perturbation theory one obtains three different poles in the quasi one–dimensional DOS at

$1/\sqrt{\epsilon}$, $1/\sqrt{|\epsilon - 2\mu_B B_{ex}|}$ and $1/\sqrt{\epsilon + 2\mu_B B_{ex}}$. Also this result was first derived by Altshuler and Aronov ([20], [21], [22], [23] and [24]). But the theory is not satisfactory because one can trust the expression for the DOS only for sufficiently high energies ϵ that means far from the divergencies. The question is: What happens if one takes into account more diagrams in the Coulomb interaction? Do the divergencies at $\pm 2\mu_B B_{ex}$ disappear completely or do they transform into peaks of finite height? There is an approach using perturbation theory combined with a renormalization group analysis by Raimondi, Castellani and Di Castro [25] that predicts that the divergencies disappear. In their work they obtained finite peaks at the energies $\pm 2\mu_B B_{ex}$.

We tried to approach the task of including higher order diagrams in the Coulomb interaction and going beyond the perturbative result by different means. We used a path integral formalism that allows to describe interacting Fermions in the presence of disorder. For this purpose we presented in chapter 3 the construction of a coherent state path integral for Fermions on the Keldysh contour. In chapters 4 and 5 we showed how to incorporate disorder and interactions (including spin effects).

In chapter 6 we calculated for a wire of infinite length at zero temperature the DOS without an external magnetic field. We generalized the idea of a gauge ansatz presented by Kamenev and Andreev [17] by introducing an additional rotation matrix. The additional complications due to the non–commutativity of the spin matrices allowed us to derive the DOS in the limit of quasi static and spatially weakly varying magnetic fluctuations only. Our result reproduces the perturbative result of the DOS for higher energies. For low energies there appears instead of the unphysical divergency in the perturbative approach an exponential suppression of the DOS. Including only the singlet part of the interaction[1] we recover results first derived by Rollbühler and Grabert [18].[2] Our discussion of the triplet interactions, which enormously complicate the problem, remains incomplete since non–perturbative results for arbitrary magnetic fluctuations could not be derived. This should be subject of future work. Maybe a different gauge approach could circumvent the technical problems we encountered.

In chapter 7 we introduced a constant external magnetic field in the path integral formalism. We could determine analytically the DOS in the case of a large external magnetic field at zero temperature. Also here we could prove that we recover the perturbative expression for high enough energies. The calculation was even easier than in the zero field case. We only had to assume that the magnetic variations are slow in time but not in space because the path integral over the magnetic fluctuations became Gaussian and could be done exactly.

In the case of a finite external magnetic field one has to determine the DOS partially numerically even at zero temperature. The numerical data that we obtained show the correct asymptotic behavior for high energies meaning that they approached the perturbative result. Furthermore our results have two interesting and remarkable properties: First, there are no features at the energies $\pm 2\mu_B B_{ex}$. The curve is completely smooth. In contrast to the findings in [25]. This should not be over–interpreted as every approach is only able to take into account a certain subset of diagrams and we are not able to prove that we included all the decisive ones. However, we recover the correct low energy behaviour of the singlet channel which indicates that our approach is powerful. Second, at zero temperature the value of the DOS is different from zero at $\epsilon = 0$. This could be due to the appearance of a new length scale in the system which is the magnetic length associated with the cyclotron radius ($r = mv/eB_{ex}$). Finally we investigated the DOS for finite temperatures. It turned out that already temperatures of a few

[1] but taking into account additional interactions between the electrodes

[2] See figure (6.1).

mK mask the small offset at $\epsilon = 0$ of the DOS.

In summary one can say that the spin triplet effects in the DOS of a quasi one–dimensional wire are small. In order to make them visible, for example by turning on and off an external magnetic field, one would need to examine a metal with large spin density fluctuations.

Regarding the theoretical approach employed we conclude that the path integral technique allows to go substantially beyond previous work by including a non–perturbative summation of diagrams. However, a complete solution of the problem was not yet achieved since our analysis could only be justified for slow and weakly varying magnetic fluctuations. Although the results might be valid more generally, alternative approaches should be pursued in the future to establish the solution for arbitrary fluctuations.

Chapter 9

Appendices

9.1 Connection between tunnelling conductance and DOS

The modification of the DOS near the Fermi energy due to the interaction between the electrons can be observed by measuring the tunnelling conductance. In the experiment one places the conductor one wishes to study close to another metal whose DOS ν_A is well known. Measuring the tunnelling current which is proportional to the DOS of the two metals allows to observe the anomaly in the DOS. The suppression of the tunnelling conductance is not restricted to the case of weak disorder that we consider here. In order to understand the connection between conductance and DOS remember that the current $I(V)$ for a tension of $V > 0$ applied between two metals A and B depends on the tunnelling probability of electrons between the two metals. The tunnelling rate between an initial state i of metal A and a final state f of metal B is given by Fermi's Golden Rule

$$\Gamma_{i\to f}(V) = \frac{2\pi}{\hbar}|t_{if}|^2\ \delta(\epsilon_i - \epsilon_f + eV) \tag{9.1}$$

where t_{if} is the matrix element describing the coupling of the two metals. The tunnelling rate between the two metals A and B depends on the occupation numbers of the initial and the final state. It is given by

$$\Gamma_{AB}(V) = \frac{2\pi}{\hbar}\sum_{i,f}|t_{if}|^2\ n_F(\epsilon_i)\ \left[1 - n_F(\epsilon_f)\right]\ \delta(\epsilon_i - \epsilon_f + eV) \tag{9.2}$$

where $n_F(\epsilon)$ is the Fermi distribution. At finite temperature there is also a finite probability for electrons to tunnel from metal B to metal A. Summing these two contributions and multiplying by the elementary charge we get for the current at finite temperature T

$$I(V) = e(\Gamma_{AB} - \Gamma_{BA}) = \frac{2\pi e}{\hbar}\sum_{i,f}|t_{if}|^2\ \left[n_F(\epsilon_i) - n_F(\epsilon_f)\right]\ \delta(\epsilon_i - \epsilon_f + eV). \tag{9.3}$$

Assuming that the matrix element depends only very weakly on the energy and that the tension V and the temperature T are both small compared to the Fermi energy and the barrier height we can replace the sums by integrals introducing densities of states $\nu_A(\epsilon)$ and $\nu_B(\epsilon)$. On has

$$I(V) = \frac{2\pi e}{\hbar}|t|^2\int_{-\infty}^{\infty} d\epsilon\ \nu_A(\epsilon)\ \nu_B(\epsilon + eV)\ \left[n_F(\epsilon) - n_F(\epsilon + eV)\right]. \tag{9.4}$$

If the two density of states vary weakly near the Fermi energy we can replace them by their value at the Fermi energy $\nu_A = \nu_A(\epsilon_\mathrm{F})$ and $\nu_B = \nu_B(\epsilon_\mathrm{F})$ and obtain

$$I(V) = \frac{2\pi e^2}{\hbar}|t|^2\ \nu_A\ \nu_B\ V \qquad \rightarrow \qquad G_0 = \frac{2\pi e^2}{\hbar}|t|^2\ \nu_A\ \nu_B \tag{9.5}$$

Assuming that the DOS of the reference metal A is constant as a function of energy the variation $\delta\nu_A$ of the metal to study leads to a variation $\delta I(V)$ of the current and thus to a variation $\delta G(V)$ in the conductance.

$$\delta G = \frac{d\delta I}{dV} = -\frac{2\pi e^2}{\hbar}\ |t|^2\ \nu_A \int_{-\infty}^{\infty} d\epsilon\ \delta\nu(\epsilon)\ n'_F(\epsilon - eV) \tag{9.6}$$

For small temperatures the derivative n'_F is in good approximation a δ–function. Dividing by the conductance given in equation (9.5) we get

$$\frac{\delta G(V)}{G_0} = \frac{\delta\nu(V)}{\nu_B}. \tag{9.7}$$

The reduction of the conductance is a direct measure of the change of the density of states due to the Coulomb interactions.

9.2 Path integral over the magnetic fluctuations

The problem in section (6.5.3) is to calculate averages of the form $\left\langle B(x_1)...B(x_n)\right\rangle_B$ where $B(x_i)$ is the absolute value of $\vec{B}(x_i)$. Unfortunately we are not able to solve this problem for an arbitrary number of B's at different points in space/time. One can also reformulate the problem in the language of generating functionals. It reads written as discrete version

$$Z = N_0 \prod_{i=1}^{N} \left\{ \int_{-\infty}^{\infty} \frac{dB_i^x}{\sqrt{2\pi}} \frac{dB_i^y}{\sqrt{2\pi}} \frac{dB_i^z}{\sqrt{2\pi}} \right\} \exp\left(-\frac{1}{2} \sum_{k=x,y,z} B_i^k V_{ij}^{-1} B_j^k + \lambda_i \sqrt{\sum_{k=x,y,z} (B_i^k)^2} \right) \tag{9.8}$$

where $N_0 = (\det V^{-1})^{3/2}$. Averages of components of $\vec{B}$ could be calculated using the idea of diagonalization which we applied in section (3.1). This method however fails for the absolute values of $\vec{B}$ as one can see: [1]

$$Z = N_0 \prod_{i=1}^{N} \left\{ \int_{-\infty}^{\infty} \frac{d\chi_i^x}{\sqrt{2\pi}} \frac{d\chi_i^y}{\sqrt{2\pi}} \frac{d\chi_i^z}{\sqrt{2\pi}} \right\} \exp\left(-\frac{1}{2} \sum_{k=x,y,z} \chi_i^k D_{ii} \chi_j^k + (\beta_j M_{ji}) \sqrt{\sum_{k=x,y,z} (M_{ip}\chi_p^k)^2} \right) \tag{9.9}$$

The rotation M diagonalizes $V \to D$ but in the second part in the exponential all components are entangled.

So let us begin with a simpler problem of just two absolute values of $\vec{B}$ at two different points: $\left\langle B_i B_j \right\rangle_B$.[2] As there appear only two points in the problem one might ask if one has to use (or to know) the whole matrix V (or V^{-1}) for calculation. The answer is no. Actually one can reduce

$$V = \begin{pmatrix} V_{11} & ... & ... & ... & ... & ... & V_{1N} \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & V_{ii} & ... & V_{ij} & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ ... & ... & V_{ji} & ... & V_{jj} & ... & ... \\ ... & ... & ... & ... & ... & ... & ... \\ V_{N1} & ... & ... & ... & ... & ... & V_{NN} \end{pmatrix} \quad \to \quad \begin{pmatrix} V_{ii} & V_{ij} \\ V_{ji} & V_{jj} \end{pmatrix}. \tag{9.10}$$

The proof is given in the following subsection.

9.2.1 Theorem

In this subsection we present a mathematical theorem which allows to reduce considerably the dimension of a multidimensional integral. Let $\mathbf{x} = (x_1, ...x_N)$ be random variables with values in the N–dimensional real space and $\Delta(x_1...x_N)$ the probability density. Then the probability that $x_1 < t_1$, $x_2 < t_2$,...,$x_N < t_N$ is given by

$$P(x_1 < t_1, x_2 < t_2, ..., x_N < t_N) = \int_{-\infty}^{t_1} dx_1 \int_{-\infty}^{t_2} dx_2 ... \int_{-\infty}^{t_N} dx_N \; \Delta(x_1, x_2, ..., x_N). \tag{9.11}$$

[1] Let M be the rotation matrix. Then $V^{-1} = M^{-1} D M$, $\chi^k = M B^k$ and $\lambda^t = \beta^t M$.

[2] without loss of generality $i < j$

Furthermore the expectation value of a function $f(x_1, x_2, ..., x_N)$ of the variables $x_1, ..., x_N$ is

$$\big\langle f(x_1, x_2, ..., x_N)\big\rangle = \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_N \; f(x_1, x_2, ..., x_N)\Delta(x_1, x_2, ..., x_N). \tag{9.12}$$

Definition: A N-dimensional vector of random variables $x = (x_1, ..., x_N)^T$ is called gaussian (or normally) distributed with expectation value 0 if its probability density is given by

$$\Delta(x) = N_0 \; e^{-\frac{1}{2}x^T V^{-1} x} \tag{9.13}$$

where V is the symmetric positive definite covariance matrix ($V_{ij} = \langle x_i x_j\rangle$) and the normalization constant $N_0 = (2\pi)^{-N/2}/\sqrt{\det V}$.

Lemma:

$$\big\langle \exp\left(it^T x\right)\big\rangle = \exp\left(-\frac{1}{2}t^T V t\right). \tag{9.14}$$

This is just an application of the Gaussian integrals (see (3.18)). The expectation value of $\exp(it^T x)$ is called *Characteristic function* of the random vector $\mathbf{x}$. Now to the important theorem which reads:

Theorem: If $\mathbf{x} = (x_1, ..., x_N)^T$ is gaussian distributed then also $\mathbf{y} = (x_1, ..., x_p)^T$ with $p < N$ is gaussian distributed with the probability density $\tilde{\Delta}(\vec{y}) = \tilde{N}_0 \exp[-\frac{1}{2}(x_1, ..., x_p)\tilde{V}^{-1}(x_1 ... x_p)^T]$ with the reduced $p \times p$ covariance matrix $\tilde{V}_{ij} = V_{ij}$.

Proof: Let $\tilde{\Delta}$ be the probability density of the vector $y = (x_1, ..., x_p)$ then we can write

$$P(x_1 < t_1, x_2 < t_2, ..., x_p < t_p, x_{p+1} < \infty, ..., x_N < \infty) = \int_{-\infty}^{t_1} dx_1 ... \int_{-\infty}^{t_p} dx_p \; \tilde{\Delta}(x_1, ..., x_p). \tag{9.15}$$

This probability can also be expressed with the original total probability density.

$$P(x_1 < t_1, ..., x_p < t_p, x_{p+1} < \infty, ..., x_N < \infty) = \int_{-\infty}^{t_1} dx_1 ... \int_{-\infty}^{t_p} dx_p \int_{-\infty}^{\infty} dx_{p+1} ... \int_{-\infty}^{\infty} dx_N \; \Delta(x_1, ..., x_N) \tag{9.16}$$

From equation (9.15) we get

$$\frac{\partial}{\partial t_1} ... \frac{\partial}{\partial t_p} P(x_1 < t_1, ..., x_p < t_p, x_{p+1} < \infty, ..., x_N < \infty) = \tilde{\Delta}(t_1, ..., t_p) \tag{9.17}$$

and from equation (9.16) one obtains for the same derivative

$$\begin{aligned} &\frac{\partial}{\partial t_1} ... \frac{\partial}{\partial t_p} P(x_1 < t_1, ..., x_p < t_p, x_{p+1} < \infty, ..., x_N < \infty) \\ &\qquad = \int_{-\infty}^{\infty} dx_{p+1} ... \int_{-\infty}^{\infty} dx_N \; \Delta(t_1, ...t_p, x_{p+1}, ..., x_N). \end{aligned} \tag{9.18}$$

The last two equations allow to connect the densities $\Delta(x_1, ..., x_N)$ and $\tilde{\Delta}(x_1, ..., x_p)$

$$\tilde{\Delta}(t_1,...,t_p)=\int_{-\infty}^{\infty}dx_{p+1}...\int_{-\infty}^{\infty}dx_N\ \Delta(t_1,...t_p,x_{p+1},...,x_N) \tag{9.19}$$

So if we want to calculate an expectation value of the function $\exp\left(i\sum_{i=1}^{p}t_ix_i\right)$ of the first p variables we can write this using (9.19)

$$\left\langle \exp\left(i\sum_{i=1}^{p}t_ix_i\right)\right\rangle=\int_{-\infty}^{\infty}dx_1...\int_{-\infty}^{\infty}dx_p\exp\left(i\sum_{i=1}^{p}t_ix_i\right)\int_{-\infty}^{\infty}dx_{p+1}...\int_{-\infty}^{\infty}dx_N\ \Delta(x_1,...x_p,x_{p+1},...,x_N) \tag{9.20}$$

which is

$$\left\langle \exp\left(i\sum_{i=1}^{p}t_ix_i\right)\right\rangle=\int_{-\infty}^{\infty}dx_1...\int_{-\infty}^{\infty}dx_N\ \exp\left(i\sum_{i=1}^{N}t_ix_i\right)\Delta(x_1,...,x_N)\Big|_{t_{p+1}\to 0,...t_N\to 0}. \tag{9.21}$$

The result we can take from the Lemma above

$$\left\langle \exp\left(i\sum_{i=1}^{p}t_ix_i\right)\right\rangle=\exp\left(-\frac{1}{2}\sum_{i,j=1}^{N}t_iV_{ij}t_j\right)\Big|_{t_{p+1}\to 0,...t_N\to 0} \tag{9.22}$$

which is finally

$$\left\langle \exp\left(i\sum_{i=1}^{p}t_ix_i\right)\right\rangle=\exp\left(-\frac{1}{2}\sum_{i,j=1}^{p}t_iV_{ij}t_j\right). \tag{9.23}$$

That's it. We determined so to speak the Fourier transform of the probability density. But as the Fourier transform determines uniquely the probability density we can stop here. We remark that the entries of the reduced covariance matrix $\tilde{V}_{ij}$ are in fact the V_{ij} of the original covariance matrix. The theorem also holds if the expectation value is different from 0.

9.2.2 Two point correlators

In this subsection we show how to calculate with the help of the theorem expectation values of the form

$$\left\langle B_n^lB_m^k\right\rangle=\left\langle \sqrt{(B_n^x)^2+(B_n^y)^2+(B_n^z)^2}^{\,l}\sqrt{(B_n^x)^2+(B_n^y)^2+(B_n^z)^2}^{\,k}\right\rangle \tag{9.24}$$

where l and k are ≥ 0. Unfortunately this is not enough to solve the non Gaussian case given above as there appear products of absolute values of B's at more than two different points in space/time. However from this calculation arises the idea in which limiting case the non Gaussian path integral can be solved approximately. We define

$$\begin{aligned}E_{l,k}:=\left\langle B_n^lB_m^k\right\rangle=N_0\prod_{i=1}^{N}\left\{\int_{-\infty}^{\infty}\frac{dB_i^x}{\sqrt{2\pi}}\frac{dB_i^y}{\sqrt{2\pi}}\frac{dB_i^z}{\sqrt{2\pi}}\right\}\left(\sqrt{\sum_{k=x,y,z}(B_n^k)^2}\right)^l\\ \times\left(\sqrt{\sum_{k=x,y,z}(B_m^k)^2}\right)^k\exp\left(-\frac{1}{2}\sum_{k=x,y,z}B_i^kV_{ij}^{-1}B_j^k\right)\end{aligned} \tag{9.25}$$

and introduce a bunch of abbreviations in order to shorten the notation.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} := \begin{pmatrix} B_n^x \\ B_n^y \\ B_n^z \end{pmatrix} \qquad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} := \begin{pmatrix} B_m^x \\ B_m^y \\ B_m^z \end{pmatrix} \tag{9.26}$$

and

$$R = \sqrt{x_1^2 + y_1^2 + z_1^2} \qquad S = \sqrt{x_2^2 + y_2^2 + z_2^2}. \tag{9.27}$$

Finally we set

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} := \begin{pmatrix} V_{nn} & V_{nm} \\ V_{nm} & V_{mm} \end{pmatrix}^{-1} = \frac{1}{V_{nn}V_{mm} - V_{nm}^2} \begin{pmatrix} V_{mm} & -V_{nm} \\ -V_{nm} & V_{nn} \end{pmatrix}. \tag{9.28}$$

As we have only a two point function we do not need the whole $3N \times 3N$ covariance matrix. It can be shown (see theorem in section (9.2.1)) that only the following 6×6 matrix enters the problem:

$$V_{red} = \begin{pmatrix} V_{nn} & V_{nm} & 0 & 0 & 0 & 0 \\ V_{nm} & V_{mm} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{nn} & V_{nm} & 0 & 0 \\ 0 & 0 & V_{nm} & V_{mm} & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{nn} & V_{nm} \\ 0 & 0 & 0 & 0 & V_{nm} & V_{mm} \end{pmatrix} \tag{9.29}$$

which has the inverse

$$V_{red}^{-1} = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} & 0 & 0 \\ 0 & 0 & A_{12} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{11} & A_{12} \\ 0 & 0 & 0 & 0 & A_{12} & A_{22} \end{pmatrix}. \tag{9.30}$$

Instead of equation (9.25) we get

$$E_{l,k} = \frac{\tilde{N}_0}{(2\pi)^3} \int dx_1\, dy_1\, dz_1 dx_2\, dy_2\, dz_2\; R^l S^k\; e^{-\frac{1}{2}(A_{11}(x_1^2+y_1^2+z_1^2)+2A_{12}(x_1x_2+y_1y+z_1z_2)+A_{22}(x_2^2+y_2^2+z_2^2))} \tag{9.31}$$

with $\tilde{N}_0$ such that for $l = 0$ and $k = 0$ the integral is normalized to 1. We introduce two sets of spherical coordinates and choose θ_1 to be the angle between the vector in between $(x_1, y_1, z_1)^t$ and $(x_2, y_2, z_2)^t$.

$$E_{l,k} = \frac{\tilde{N}_0}{\pi} \int_0^\infty dR \int_0^\infty dS \int_0^\pi d\theta_1\; R^{l+2}\; S^{k+2}\; \sin\theta_1 e^{-\frac{1}{2}A_{11}R^2} e^{-\frac{1}{2}A_{22}S^2} e^{-A_{12}RS\cos\theta_1}. \tag{9.32}$$

where we performed already the integrations over ϕ_1, ϕ_2 and θ_2 which are easy. Next we integrate over θ_1.

$$E_{l,k} = \frac{\tilde{N}_0}{\pi} \int_0^\infty dR \int_0^\infty dS\ R^{l+2}\ S^{k+2}\ e^{-\frac{1}{2}A_{11}R^2} e^{-\frac{1}{2}A_{22}S^2} \left[\frac{-1}{A_{12}RS} e^{-A_{12}RS\cos\theta_1} \right]_0^\pi \tag{9.33}$$

which is

$$E_{l,k} = \frac{\tilde{N}_0}{A_{12}\pi} \int_0^\infty dR \int_0^\infty dS\ R^{l+1}\ S^{k+1}\ e^{-\frac{1}{2}A_{11}R^2} e^{-\frac{1}{2}A_{22}S^2} [e^{A_{12}RS} - e^{-A_{12}RS}]. \tag{9.34}$$

We replace the last term by its power series. It is

$$e^{A_{12}RS} - e^{-A_{12}RS} = 2\sum_{\nu=0}^{\infty} \frac{(A_{12}RS)^{(2\nu+1)}}{(2\nu+1)!}. \tag{9.35}$$

Now we integrate term by term and exchange the sum and the 2 integrals. This can be done as the expressions converge very well due to the exponentials.

$$E_{l,k} = \frac{2\tilde{N}_0}{A_{12}\pi} \sum_{\nu=0}^{\infty} \int_0^\infty dR \int_0^\infty dS\ \frac{A_{12}^{2\nu+1}}{(2\nu+1)!} R^{l+2\nu+2}\ S^{k+2\nu+2}\ e^{-\frac{1}{2}A_{11}R^2} e^{-\frac{1}{2}A_{22}S^2} \tag{9.36}$$

It is[3]

$$\int_0^\infty dx\ x^\alpha\ e^{-\frac{1}{2}\beta x^2} = 2^{\frac{\alpha-1}{2}}\ \Gamma\left(\frac{\alpha+1}{2}\right) \beta^{-\frac{\alpha+1}{2}} \tag{9.37}$$

Using this general result we find

$$E_{l,k} = \frac{2\tilde{N}_0}{\pi} \sum_{\nu=0}^{\infty} \frac{A_{12}^{2\nu}}{(2\nu+1)!} \left(2^{\frac{l+2\nu+1}{2}} \Gamma\left(\frac{l+2\nu+3}{2}\right) A_{11}^{-\frac{l+2\nu+3}{2}}\right) \left(2^{\frac{k+2\nu+1}{2}} \Gamma\left(\frac{k+2\nu+3}{2}\right) A_{22}^{-\frac{k+2\nu+3}{2}}\right). \tag{9.38}$$

We arrange the terms in order to identify the following Hypergeometric function

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\ \frac{z^n}{n!}. \tag{9.39}$$

Applying several identities for the Γ–function[4] and resubstituting to the original quantities one arrives at

$$\boxed{E_{l,k} = \frac{4}{\pi} \frac{(V_{nn}^2 - V_{nm}^2)^{(3+l+k)/2}}{V_{nn}^{3+(l+k)/2}}\ 2^{\frac{l+k}{2}}\ \Gamma\left(\frac{l+3}{2}\right) \Gamma\left(\frac{k+3}{2}\right)\ F\left(\frac{l+3}{2}, \frac{k+3}{2}, \frac{3}{2}, \left(\frac{V_{nm}}{V_{nn}}\right)^2\right).} \tag{9.41}$$

[3] See [51].

[4]
$$\Gamma(n+1) = n! \qquad \Gamma(n+1) = n\Gamma(n) \qquad \Gamma(2n) = \frac{1}{\sqrt{2\pi}}\ 2^{2n-1/2}\ \Gamma(n)\ \Gamma(n+1/2) \tag{9.40}$$

This is the general result for two point correlator. Let us evaluate the formula for two special cases.

$$E_{2,0} = \left\langle B_n B_n \right\rangle_B = 3V_{nn} \tag{9.42}$$

and

$$E_{1,1} = \left\langle B_n B_m \right\rangle_B = \frac{2}{\pi}\left(3(V_{nn}^2 - V_{nm}^2)^{1/2} + \frac{(V_{nn}^2 + 2V_{nm}^2)\arcsin(V_{nm}/V_{nn})}{V_{nm}}\right). \tag{9.43}$$

$E_{1,1}$ is a rather inconvenient expression for further calculations so we look at the limiting case that $V_{nn} \approx V_{nm}$ which corresponds to picture of a long range interaction. The arcsin becomes approximately $\pi/2$ and we get

$$E_{1,1}^{\text{long}} \approx 3V_{nm} \approx 3V_{nn} \tag{9.44}$$

This is the crucial point of the calculation. In the case of long range interaction the expectation value $E_{1,1}$ approaches the value of $E_{2,0}$. Of course this also has to be true for any higher moment. All correlators with B's at different points $\langle B_{n_1} B_{n_2} .... B_{n_l}\rangle$ in space/time can be approximated in the long range limit by the one point correlator $\langle B_{n_1}^l \rangle = E_{l,0}$. From (9.41) we get

$$\boxed{E_{l,0} = \frac{2^{(l+2)/2}}{\sqrt{\pi}}\,\Gamma\left(\frac{2l+3}{2}\right) V_{nn}^l.} \tag{9.45}$$

This result will be used in the calculation of the path integral over $\vec{B}$ in subsection (6.5.3).

9.2.3 The path integral for arbitrary external field

In this section we demonstrate the evaluation of $P_{B,\uparrow} = \left\langle e^{-i[f_2 - f_1]} \right\rangle_B$.

$$P_{B,\uparrow} = N_0 \int \mathcal{D}B^x \int \mathcal{D}B^y \int \mathcal{D}B^z \; e^{\frac{i}{2}\int dy \int dy' \sum\limits_{i=x,y,z} B^i(y) V_t^{-1}(y,y') B^i(y') + i[f_2(x) - f_1(x')]} \tag{9.46}$$

The expectation value looks like o problem given in equation (6.177). This time however the f's have a more complicated structure as they are not proportional to the absolute value B but proportional to $\widehat{B} = \sqrt{(B^z + B_{ex})^2 + (B^x)^2 + (B^y)^2}$. We have in unrotated representation

$$f_\alpha(x) = \mu_B \sum_{\beta=1}^{2} \int dy \; A_{\alpha\beta}(x,y) \sqrt{(B_\beta^x)^2(y) + (B_\beta^y)^2(y) + (B_\beta^z(y) + B_{ex})^2}. \tag{9.47}$$

The constant external field B_{ex} does not have a Keldysh index as it has the same value on both branches. We approximate the path integral in the same way as before assuming that the magnetic field components vary slowly in space and time meaning: $B^i(x) \approx B^i(x')$. That implies

$$\left\langle \widehat{B}(x_1)\widehat{B}(x_2)...\widehat{B}(x_n) \right\rangle_B \approx \left\langle \widehat{B}^n(x) \right\rangle_B. \tag{9.48}$$

With the theorem of section (9.2.1) we obtain

$$P_{B,\uparrow} := \left\langle e^{-i[f_2(x)-f_1(x')]} \right\rangle_B \approx N_0 \int_{-\infty}^{\infty} dB^x \int_{-\infty}^{\infty} dB^y \int_{-\infty}^{\infty} dB^z \; e^{-A\tilde{B}} \; e^{-\frac{1}{2V}B^2} \tag{9.49}$$

where $N_0 = (2\pi V)^{-3/2}$ and $e^{-i[f_2(x)-f_1(x')]} = e^{-A\tilde{B}}$ (summarizing everything in front of $\tilde{B}$ by A). We substitute $B_0 = B^z + B_{ex}$.

$$P_{B,\uparrow} = N_0 \int_{-\infty}^{\infty} dB^x \int_{-\infty}^{\infty} dB^y \int_{-\infty}^{\infty} dB_0 \; e^{-A\tilde{B}} \; e^{-\frac{1}{2V}\left[(B^x)^2+(B^y)^2+(B_0-B_{ex})^2\right]} \tag{9.50}$$

It is $\tilde{B} = \sqrt{(B^x)^2 + (B^y)^2 + (B_0)^2}$.

$$P_{B,\uparrow} = N_0 \; e^{-\frac{1}{2V}B_{ex}^2} \int_{-\infty}^{\infty} dB^x \int_{-\infty}^{\infty} dB^y \int_{-\infty}^{\infty} dB_0 \; e^{-A\tilde{B}} \; e^{-\frac{1}{2V}\left[\tilde{B}^2-2B_0B_{ex}\right]} \tag{9.51}$$

We go over to spherical coordinates ($\tilde{B}$,θ,ϕ). Then we have $B_0 = \tilde{B}\cos\theta$.

$$P_{B,\uparrow} = N_0 \; e^{-\frac{1}{2V}B_{ex}^2} \int_0^{\infty} d\tilde{B} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; \tilde{B}^2 \sin\theta \; e^{-A\tilde{B}} \; e^{-\frac{1}{2V}\left[\tilde{B}^2-2B_{ex}\tilde{B}\cos\theta\right]} \tag{9.52}$$

After the integration over ϕ and θ we are left with

$$P_{B,\uparrow} = \frac{2\pi V N_0}{B_{ex}} \; e^{-\frac{1}{2V}B_{ex}^2} \int_0^{\infty} d\tilde{B} \; \tilde{B} \; e^{-A\tilde{B}} \left(e^{-\frac{1}{2V}\left[\tilde{B}^2-2B_{ex}\tilde{B}\right]} - e^{-\frac{1}{2V}\left[\tilde{B}^2+2B_{ex}\tilde{B}\right]} \right). \tag{9.53}$$

We substitute in the first term $B_1 = \tilde{B} - B_{ex}$ and in the second term $B_2 = \tilde{B} + B_{ex}$.

$$\begin{aligned} P_{B,\uparrow} = \frac{1}{B_{ex}\sqrt{2\pi V}} \Bigg(& \int_{-B_{ex}}^{\infty} dB_1 \; (B_1 + B_{ex}) \; e^{-A(B_1+B_{ex})} e^{-\frac{1}{2V}B_1^2} \\ & - \int_{B_{ex}}^{\infty} dB_2 \; (B_1 - B_{ex}) \; e^{-A(B_2-B_{ex})} e^{-\frac{1}{2V_t}B_2^2} \Bigg) \end{aligned} \tag{9.54}$$

Evaluating these four integrals and substituting $J_B = \frac{1}{2}AVA$ and $AB_{ex} = -iB_{ex}t$ one obtains

$$\boxed{\begin{aligned} P_{B,\uparrow} = \frac{e^{J_B}}{2iB_{ex}t} \Bigg(& e^{iB_{ex}t} \; \left(iB_{ex}t + 2J_B\right) \left[1 + \mathrm{Erf}\left(\frac{iB_{ex}t + 2J_B}{2i\sqrt{J_B}}\right)\right] \\ & + e^{-iB_{ex}t} \; \left(iB_{ex}t - 2J_B\right) \left[1 - \mathrm{Erf}\left(\frac{iB_{ex}t - 2J_B}{2i\sqrt{J_B}}\right)\right]. \end{aligned}} \tag{9.55}$$

$P_{B,\downarrow} = P_{B,\uparrow}(-B_{ex})$. $P^*_{B,\uparrow}$ and $P^*_{B,\downarrow}$ are obtained simply by replacing J_B by J^*_B. We see that if $G^>_{\tau,\uparrow}$ and $G^>_{\tau,\downarrow}$ would be equal as in the case without external field the terms containing the Erf would cancel in equation (7.37) and that $P_{B,\uparrow} = \frac{e^{J_B}}{2}\,(1 + 2J_B)$ reproduces the terms without external field. Sending B_{ex} and $V_t \to 0$ we obtain $P_{B,\uparrow} = P_{B,\downarrow} = 1$ which is also correct.

9.3 Calculation of $P_3(E)$ and its contribution to the DOS

We calculate $P_3(E)$ which we define without a factor $e^{\pm iB_{ex}t}$.

$$P_3(E) = \frac{1}{i\pi B_{ex}} \int_{-\infty}^{\infty} dt\ \frac{\mathrm{ReJ_B} + i\mathrm{Im}J_B}{t}\ e^{\mathrm{Re}J + i\mathrm{Im}J + iEt} \tag{9.56}$$

Using $J(-t) = J^*(t)$ we get

$$\begin{aligned} P_3(E) = \frac{1}{i\pi B_{ex}} \int_0^{\infty} dt\ \frac{e^{\mathrm{Re}J}}{t} \Bigg[\mathrm{ReJ_B} \Big(e^{i(\mathrm{Im}J + Et)} - e^{-i(\mathrm{Im}J + Et)} \Big) \\ + i \frac{\mathrm{Im}J_B}{t} \Big(e^{i(\mathrm{Im}J + Et)} + e^{-i(\mathrm{Im}J + Et)} \Big) \Bigg]. \end{aligned} \tag{9.57}$$

It is $u = \mathrm{Re}J(t) = \mathrm{Im}J(t) = -\sqrt{2E_g t}$ for $t > 0$.

$$P_3(E) = \frac{4\sqrt{E_B}}{\pi B_{ex}\sqrt{E_g}} \int_0^{\infty} du\ e^{-u} \Bigg[\cos\Big(\frac{E}{2E_g}\ u^2 - u \Big) + \sin\Big(\frac{E}{2E_g}\ u^2 - u \Big) \Bigg] \tag{9.58}$$

We set $b_1 = \frac{4\sqrt{E_B}}{\pi B_{ex}\sqrt{E_g}}$, $\alpha = \frac{E}{2E_g}$ and use the theorems for $\sin(x)$ and $\cos(x)$ given in (6.224).

$$P_3(E) = b_1 \int_0^{\infty} du\ e^{-u} \Big[\cos\big(\alpha u^2\big)\cos\big(u\big) + \sin\big(\alpha u^2\big)\sin\big(u\big) + \sin\big(\alpha u^2\big)\cos\big(-u\big) - \cos\big(\alpha u^2\big)\sin\big(u\big) \Big] \tag{9.59}$$

Using the integrals (6.226) and (6.226) we find

$$\int^1 d\beta\ I_1(\beta) = \int_0^{\infty} du \frac{e^{-u}}{2} \big[\cos\big(\alpha u^2\big)\sin\big(u\big) - \cos\big(\alpha u^2\big)\cos\big(u\big) \big] \tag{9.60}$$

and

$$\int^1 d\beta\ I_2(\beta) = -\int_0^{\infty} du \frac{e^{-u}}{2} \big[\sin\big(\alpha u^2\big)\sin\big(u\big) + \sin\big(\alpha u^2\big)\cos\big(u\big) \big] \tag{9.61}$$

which implies that $P_3(E) = -2b_1 \int^1 d\beta\ \big[I_1(\beta) + I_2(\beta) \big]$. Thus we get

$$\boxed{P_3(E) = -\frac{4}{\sqrt{\pi} B_{ex}} \left(\frac{E_B}{E} \right)^{1/2}\ e^{-E_g/E}} \tag{9.62}$$

Using formula (6.238) one finds

$$\nu_{P_3} = \nu_1 \int_0^{\epsilon} P_3(E) dE = \nu_1 \frac{8\sqrt{E_g}\sqrt{E_B}}{B_{ex}} \Bigg(\sqrt{\frac{\epsilon}{E_g}}\ \frac{e^{-E_g/\epsilon}}{\sqrt{\pi}} + \mathrm{Erf}\Big(\sqrt{\frac{E_g}{\epsilon}} \Big) - 1 \Bigg). \tag{9.63}$$

9.4 Estimation of E_s and E_t

In this appendix we estimate the characteristic energies E_s and E_t that appear in our calculations. We do this for Palladium at low temperatures ($T \approx 0.1$K). Palladium is an interesting material because it is expected to have a relatively large triplet energy E_t. In SI–units the energies E_s and E_t get an additional $\hbar$ in the denominator:

$$E_s = \frac{(A_0^s)^2}{32\pi\hbar D(\nu_3 \mathrm{a}^2)^2(1 - A_0^s + \sqrt{1 - A_0^s})^2} \qquad E_t = \frac{(A_0^a)^2}{32\pi\hbar D(\nu_3 \mathrm{a}^2)^2(1 - A_0^a + \sqrt{1 - A_0^a})^2}. \tag{9.64}$$

The constants A_0^s and A_0^a are connected with the Landau parameters by

$$A_0^s = \frac{F_0^s}{1 + F_0^s} \qquad A_0^a = \frac{F_0^a}{1 + F_0^a}. \tag{9.65}$$

The Landau parameter appear in the theory of Fermi liquids and take into account the effects of interaction between the particles. F_0^s and F_0^a are the lowest order coefficients representing the s–wave and therefore low energy part of the interaction. The Fermi liquid theory connects the Landau parameters to experimentally measurable quantities. A detailed description of Fermi liquids can be found in [6]. A response function containing F_0^s is for example the compressibility $\mathcal{K}$.

$$\mathcal{K} = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{n^2}\frac{\nu_3}{1 + F_0^s} \tag{9.66}$$

where n is the particle density and ν_3 is the density of states at the Fermi level. The density of states ν_3 for a free gas[5] of particles with effective mass m^* is given by

$$\nu_3 = \frac{m^* k_F}{\pi^2\hbar^2}. \tag{9.67}$$

A quantity that contains F_0^a is the spin susceptibility:

$$\chi = \frac{\mu_0\mu_B^2 m^* k_F}{\pi^2\hbar^2}\frac{1}{1 + F_0^a} = \mu_0\mu_B^2\,\frac{\nu_3}{1 + F_0^a}. \tag{9.68}$$

where $\mu_0 = 1.26 \cdot 10^{-6}\,\frac{\mathrm{mkg}}{\mathrm{s^2A^2}}$ and $\mu_B = \frac{e\hbar}{2mc} = 9.27 \cdot 10^{-24}\,\frac{\mathrm{J}}{\mathrm{T}}$. χ is dimensionless.[6] Looking at (9.66) and (9.68) we see that we need experimental data for the electron density n, the bulk density of states ν_3, the compressibility $\mathcal{K}$ and the spin susceptibility χ in order to get values for the F's.

Palladium $Pd^{46}_{106.4}$ has a FCC crystal structure which means that in each unit cell one finds 4 atoms which have each 46 electrons. The lattice constant is 0.39nm according to [9]. From this we estimate the electron density:

$$n = \frac{N}{V} = \frac{4 \cdot 46}{(0.39 \cdot 10^{-9}\mathrm{m})^3} = 3.1 \cdot 10^{30}\frac{1}{\mathrm{m}^3}. \tag{9.69}$$

[5] the word *gas* indicates noninteracting!

[6] $\mu_0\mu_B^2\nu_0 = (\frac{\mathrm{m\cdot kg}}{\mathrm{s^2\cdot A^2}})(\frac{\mathrm{J}}{\mathrm{T}})^2\frac{1}{\mathrm{Jm^3}} = \frac{\mathrm{J}}{\mathrm{m\cdot A^2}}\frac{\mathrm{J^2m^4}}{\mathrm{V^2s^2}}\frac{1}{\mathrm{Jm^3}} = 1$

For the determination of ν_3 we use the specific heat γ which is in the Fermi liquid theory given by

$$\gamma = \frac{c_V}{T} = \frac{m^* k_F k_B^2}{3\hbar^2}. \tag{9.70}$$

γ has the unit $[\frac{\mathrm{J}}{\mathrm{m}^3\cdot\mathrm{K}^2}]$. Using equation (9.67) we have

$$\nu_3 = \frac{3\gamma}{\pi^2 k_B^2}. \tag{9.71}$$

In [58] one finds $\gamma = 9.5\frac{\mathrm{mJ}}{\mathrm{mol}\cdot\mathrm{K}^2}$ at $T = 0.1\mathrm{K}$ which we have to convert into $\frac{\mathrm{J}}{\mathrm{m}^3\cdot\mathrm{K}^2}$. Palladium has a mass density n_M of $12\frac{\mathrm{g}}{\mathrm{cm}^3}$ and weighs $M_{mol} = 106.4\frac{\mathrm{g}}{\mathrm{mol}}$ (see [9]). We obtain

$$\frac{n_M}{M_{mol}} = \frac{12}{106.4}\frac{\mathrm{mol}}{\mathrm{cm}^3} = 112782\frac{\mathrm{mol}}{\mathrm{m}^3}. \tag{9.72}$$

Multiplying the γ given above by this ratio yields $\gamma = 1071\frac{\mathrm{J}}{\mathrm{m}^3\cdot\mathrm{K}^2}$. With the value of the Boltzmann constant $k_B = 1.38\cdot 10^{-23}\frac{\mathrm{J}}{\mathrm{K}}$ the density of states is

$$\nu_3 = 1.7\cdot 10^{48}\frac{1}{\mathrm{Jm}^3}. \tag{9.73}$$

In [57] one finds for Palladium data concerning the compressibility. In the tables the coefficients a and b appear in the formula

$$\frac{\Delta V}{V} = aP + bP^2. \tag{9.74}$$

a is proportional to the compressibility. For Palladium $a_{Pd} = -5.423\cdot 10^7$ and for Copper $a_{Cu} = -7.49\cdot 10^7$. As all this is given in some mysterious units we take from [59] the value for the compressibility of Copper in SI–units. $\mathcal{K}_{Cu} = 7.14\cdot 10^{-12}\frac{\mathrm{m}^2}{\mathrm{N}}$ and obtain for Palladium

$$\mathcal{K}_{Pd} = \frac{a_{Pd}}{a_{Cu}}\mathcal{K}_{Cu} = 5.15\cdot 10^{-12}\frac{\mathrm{m}^2}{\mathrm{N}}. \tag{9.75}$$

Thus we find for $F_0^s = \frac{n^2\mathcal{K}}{\nu_3} - 1$:

$$\boxed{F_0^s = \mathbf{28.6} \qquad \text{and} \qquad A_0^s = \mathbf{0.966}.} \tag{9.76}$$

The value of A_0^s is close to 1 as expected for metals. A_0^a we determine from the magnetic susceptibility. In [60] a magnetic susceptibility of

$$\chi_m(T=0) = 710\cdot 10^{-6}\frac{emu}{\mathrm{mol}} \tag{9.77}$$

was measured for Palladium. The unit emu is used for various quantities and has each time a different meaning. In the case of magnetic susceptibilities it translates into

$$\frac{emu}{\mathrm{mol}} = \frac{\mathrm{cm}^3}{\mathrm{mol}}. \tag{9.78}$$

The standard dimensionless susceptibility χ is defined by $B = (1+4\pi\chi)H$. In experiments the common susceptibilities are however the susceptibility per gram $\chi_g = \frac{\chi}{\rho}$ and the susceptibility

per mole $\chi_m = \chi_g M_{mol} = \chi V_{mol}$ where M_{mol} is the molecular weight in gram per mole and V_{mol} is the volume of a mole. Furthermore one has to pay attention that susceptibilities quoted in the context of *emu* need a factor 4π: $4\pi\chi[emu] = \chi[\mathrm{SI}-\mathrm{units}]$. The volume of one mole of Palladium is $V_{mol} = \frac{M_{mol}}{n_M} = 8.9\frac{\mathrm{cm}^3}{\mathrm{mol}}$. Thus we get

$$\chi = 4\pi\frac{\chi_m}{V_{mol}} = 1\cdot 10^{-3}. \tag{9.79}$$

We can finally isolate F_0^a in equation (9.68): $F_0^a = \frac{\mu_0\mu_B^2\nu_3}{\chi} - 1$ and obtain

$$\boxed{F_0^a = -\mathbf{0.8} \qquad \text{and} \qquad A_0^a = -\mathbf{4}.} \tag{9.80}$$

One can see that increasing the magnetic susceptibility or reducing the density of states let F_0^a converge to -1 which means that the parameter A_0^a goes to $-\infty$. For the ratio of E_t/E_s we find

$$\frac{E_t}{E_s} \approx 0.015 \qquad \text{or} \qquad R := \sqrt{\frac{E_t}{E_s}} \approx 0.12. \tag{9.81}$$

R is the parameter that appears in our results for the DOS in the various situations. Our plots we did with a conservative value of $\mathbf{R = 0.05}$ because we did not want to overestimate the effect of the triplet.

Finally let us determine the absolute value of E_s for a wire with transverse extension $a = 10\,\mathrm{nm}$. In order to get a value for the diffusion constant we use the relation $D = \frac{\sigma}{e^2\nu_3}$. It is $\sigma_{Pd} = 9.5\cdot 10^6\frac{1}{\Omega\mathrm{m}}$ which is the value for clean Palladium. Thus $D = 2.2\cdot 10^{-4}\frac{\mathrm{m}^2}{\mathrm{s}}$. [7] We obtain for E_s (with $a = 10\,\mathrm{nm}$):

$$E_s = \mathbf{5\cdot 10^{-28}\,J} \tag{9.82}$$

which is translated into temperature and voltage

$$T_s = 3.3\cdot 10^{-5}K \approx \mathbf{0.03}\,\mathrm{mK} \qquad V_s = 3.1\cdot 10^{-9}V = \mathbf{3.1}\,\mathrm{nV}. \tag{9.83}$$

E_s strongly depends on the transverse extension a. For a wire with $a = 20\mathrm{nm}$ the temperature is reduced by a factor 16: $T_s|_{a=20} \approx 0.002\mathrm{mK}$. So in order to increase the energy scale E_s one needs to fabricate samples with small cross section a^2. Decreasing the diffusion constant and lowering the DOS also help.

Let us compare our absolute energy scale to the one derived for gated wires by [19]. The characteristic energy E_0 that determines the magnitude of the correction of the DOS

$$\frac{\delta\nu_1}{\nu_1} = -\frac{2}{\sqrt{\pi}}\sqrt{\frac{E_0}{eV}} \tag{9.84}$$

is given by

$$E_0 = \frac{\hbar}{D^*(R_K C_0)^2} \tag{9.85}$$

where $R_K = 25800\,\Omega$ and C_0 is the capacitance per unit length and D^* is the field diffusion constant. Taking the values form [1] where the effect was actually observed in the perturbative

[7] Due to impurities the expected diffusion constant D in experiments could be an order of magnitude lower.

regime we find $E_0 \approx 2,7 \cdot 10^{-28}$J. So E_s is of order E_0 if the transverse extensions are fabricated small enough. The triplet effect we predict is of the order of some percent thus one order of magnitude smaller.

Bibliography

[1] F. Pierre, H. Pothier, P. Joyez, Norman O. Birge, D. Esteve and M.H. Devoret, Phys. Rev. Lett. **86**, 1590 (2001).

[2] P. W. Anderson, Phys. Rev. **102**, 1008 (1958).

[3] L. D. Landau, J. Phys. Moscow **10**, 25 (1946).

[4] L. D. Landau, Sov. Phys. JETP **3**, 920 (1956).

[5] L. D. Landau, Sov. Phys. JETP **5**, 101 (1957).

[6] David Pines, Philippe Nozieres, *Theory of Quantum Liquids* (Redwood City, Calif.: Addison-Wesley, 1989).

[7] B. L. Altshuler and A. G. Aronov, Solid State Commun. **39**, 115 (1979).

[8] B. L. Altshuler and A. G. Aronov, Zh. Eksp. Teor. Fiz. [Sov. Phys. JETP **50**, 968 (1979)].

[9] Neil W. Ashcroft, N. David Mermin, *Solid state physics* (College ed.. - Fort Worth : Saunders College, 1976).

[10] P. Chaudari and H. U. Habermeier, Solid State Commun. **34**, 687 (1980).

[11] N. Giordano, W. Gilson and D. E. Prober, Phys. Rev. Lett. **43**, 725 (1979).

[12] N. Giordano, Phys. Rev. B. **22**, 5635 (1980).

[13] A. E. White. et al., Phys. Rev. Lett. **48**, 1752 (1982).

[14] J. T. Masden and N. Giordano, Physica **107B**, 3 (1981).

[15] J. T. Masden and N. Giordano, Phys. Rev. Lett. **49**, 819 (1982).

[16] Y. Imry and Z. Ovadyahu, Phys. Rev. Lett. **49**, 841 (1982).

[17] A. Kamenev and A. Andreev, Phys. Rev. B **60**, 2218 (1999).

[18] J. Rollbühler and H. Grabert, Phys. Rev. Lett. **87**, 126804 (2001).

[19] J. Rollbühler, Dissertation, Albert–Ludwigs–Universität, Freiburg (2002).

[20] B. L. Altshuler and A. G. Aronov and A. Yu. Zyuzin, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 15 (1982)[JETP Lett. **35**, 16 (1982)].

[21] B. L. Altshuler and A. G. Aronov and A. Yu. Zyuzin, Pis'ma Zh. Eksp. Teor. Fiz. **84**, 1525 (1983)[Sov. Phys. JETP **57**, 889 (1983)].

[22] B. L. Altshuler and A. G. Aronov, Solid State Commun. **46**, 429 (1983).

[23] B. L. Altshuler and A. G. Aronov, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 145 (1983) [JETP Lett. **37**, 175 (1983)].

[24] B. L. Altshuler, A. G. Aronov in *Electron–Electron Interactions in Disordered Systems*, published by A. L. Efros, M. Pollak (North–Holland, Amsterdam, 1985).

[25] R. Raimondi, C. Castellani and C. Di Castro, Phys. Rev. B **42**, 4724 (1990).

[26] S. Doniach, E. H. Sondheimer, *Green's Functions for Solid State Physicists* (Imperial College Press, London, 1998).

[27] A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, (Dover, New York, 1975).

[28] G. D. Mahan, *Many–Particle Physics* (Plenum Press, New York, 1993).

[29] C. Di Castro, R. Raimondi, *Disordered Electron Systems* (cond–mat/0402203).

[30] P. A. Lee, T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[31] J. W. Negele, H. Orland, *Quantum Many–Particle Systems* (Addison–Wesley, 1987).

[32] B. F. Hatfield, *Quantum field theory of point particles and strings* (Addsion–Wesley, 1992).

[33] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[34] C. Theis, Dissertation, Albert–Ludwigs–Universität, Freiburg (2004).

[35] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[36] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, 1997).

[37] J. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961).

[38] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[39] A. A. Abrikosov, I. E. Dzyaloshinski, A. I. Larkin and P. A. Lee, Phys. Rev. B **22**, 5142 (1980).

[40] E. Akkermans and G. Montambaux, *Physique des electrons et des photons* (Savoir Actuels, EDP Sciences/CNRS Edition, 2004).

[41] G. Zala, B. N. Narozhny, I. L. Aleiner, Phys. Rev. B **64**, 214204 (2001).

[42] P. Schwab, and R. Raimondi, Ann. Physik (Leipzig), **12**, 471-516 (2003), cond-mat/0310012.

[43] J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

[44] G. Eilenberger, Z. Phys. **214**, 195 (1968).

[45] R. Raimondi, C. Castellani and C. Di Castro, Phys. Rev. B **42**, 4724 (1990).

[46] H. Fukuyama, Surf. Sci. **113**, 489 (1982).

[47] G. Bergmann, Z. Phys. B – Condensed Matter **48**, 5–15 (1982).

[48] G. Bergmann, Phys. Rev. B **28**, 2914 (1983).

[49] C. Chamon, A. W. Ludwig, C. Nayak, Phys. Rev. B **60**, 2239 (1999).

[50] M. V. Feigel'man, A. I. Larkin, M. A. Skortsov, Phys. Rev. B **61**, 12 361 (2000).

[51] I. S. Gradsteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1995).

[52] M. H. Devoret, D. Esteve, H. Grabert, G. L Ingold, H. Pothier, and C. Urbina, Phys. Rev. Lett. **64**, 1824 (1990).

[53] S. M. Girvin, L. I. Glazman, M. Jonson, D. R. Penn, and M. D. Stiles, Phys. Rev. Lett. **64**, 3183 (1990).

[54] G. L. Ingold, H. Grabert, Europhys. Lett. **14**, 371 (1991)

[55] *Single Charge Tunneling*, Vol. 294 of *NATO ASI Series B*, hrsg. von H. Grabert and M. H. Devoret (Plenum, New York, 1992).

[56] *Handbook of Mathematical Functions*, hrsg. von M. Abramowitz und I. A. Stegun (Dover, New York, 1970).

[57] Gschneider, Solid State Physics **16**, 275 (1964).

[58] G. Chouteau, R. Fourneaux, K. Gobrecht and R. Tournier, Phys. Rev. L **20**, 193 (1968).

[59] C. Gerthsen, H. Vogel, *Physik* (Springer, 1997).

[60] Boyd W. Yeals, John A. Rayne, Physical Review **136**, 442 (1964).

Lebenslauf

Persönliche Daten

Name:	Wolfgang Körner
Geburtstdatum:	29.9.1976
Geburtsort:	Ulm
Familienstand:	ledig

Schulausbildung

1983–1987:	Grundschule Dornstadt
1987–1996:	Schubart Gymnasium Ulm

Zivildienst

8/1996–1/1997:	Altenzentrum Dornstadt
1/1997–9/1997:	AWO Ulm

Universitätsausbildung

10/1997–9/1999:	Beginn des Studiums der Physik an der Georg–August–Universität Göttingen mit Abschluss Vordiplom
10/1999–8/2002:	Hauptstudium der Physik an der Universität Hamburg mit Abschluss Diplom
4/2003–8/2006:	Doktorarbeit in der Arbeitsgruppe von Prof. Dr. Grabert an der Albert–Ludwigs–Universität Freiburg

Auslandsaufenthalt

9/2002–3/2003:	Freiwilligenarbeit als Mathematik– und Physiklehrer an 2 Gymnasien in Kpalime (Togo) und Mitarbeit an AIDS–Aufklärungsprojekten für die Bevölkerung

Acknowledgement

First and foremost, I am grateful to Prof. Dr. Grabert for letting me join his research group and giving me the possibility to get insight into the actual development of theoretical mesoscopic physics. I appreciated his continuous support and his never–ending optimism.

I thank Jörg Rollbühler for being my humorous and patient supervisor during the first months. We had many helpful and motivating discussions throughout my whole Ph.D thesis.

I also want to thank Peter Schwab who provided me an important insight about diffusive perturbation theory and the quasiclassical Green's function.

I gratefully acknowledge the help of my father and Henning in proofreading this manuscript and appreciate their suggestions to improve the transparency of the presentation.

I also would like to thank all my colleagues who worked with me during the last years. In particular I would like to mention Andrei Komnik, Daniel Urban, Thomas Schmidt, Lars Kecke and Fabrizio Dolcini.

Last but not least I thank my mother, my sister, Achim, Rainer and my roommates for their support and simply for being there.

If someone feels that he or she is missing on this list please help yourself and take a pencil. There is enough space left.