Hitchin and Calabi-Yau integrable systems

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Introduction

In the nineteenth century, LIOUVILLE ([Lio55]) discovered that mechanical systems, which have 'enough' independent constants of motion, can be solved by quadrature. This essentially means that solutions to the corresponding differential equations of motion can be solved by integration, algebraic operations and by using known functions. Hence such solutions can be given in a closed form, which is in contrast to arbitrary differential equations, for which solutions can only be approximated. His discovery, which was based on earlier work by JACOBI ([Jac43]) and HAMILTON, was the starting point for the theory of *integrable systems* (more precisely, *completely integrable Hamiltonian systems*). Already at that time it became clear that integrable systems are rare among mechanical systems, for example the two-body problem is integrable but the three-body problem is not.
In the twentieth century, mathematicians developed symplectic geometry, which provides the mathematical framework for (global) classical mechanics. In particular, the phase space of any classical mechanical system is a symplectic manifold. ARNOLD ([Arn78]) realized how to define integrable systems in symplectic geometry (not necessarily coming from classical mechanics), which gives a precise and modern formulation of LIOUVILLE's statement to have 'enough' independent constants of motion. Moreover, he proved a celebrated theorem, the *Arnold-Liouville theorem*, which gives a geometric explanation, why integrable systems can be solved by quadrature.

Even though they are very special and rare, it turned out over the years that integrable systems have interesting and fascinating links to other areas of mathematics, for example representation theory, algebraic geometry and (non-classical) mathematical physics. One relation of such kind was the starting point for our thesis at hand: DIACONESCU et al. ([DDD$^+$06]) discovered that integrable systems play an important role in *geometric transitions* of certain Calabi-Yau threefolds in connection with *large N duality*. This discovery gave rise to a relation between two important classes of integrable systems, namely *Hitchin systems* and *Calabi-Yau integrable systems*. It was further established by DIACONESCU, DONAGI & PANTEV ([DDP07]) but there were still some important cases missing. The aim of the present thesis is to cover these cases as well, hence extending the relation between Hitchin systems and Calabi-Yau integrable systems.

Unlike classical integrable systems, Hitchin systems and Calabi-Yau integrable systems are complex-geometric objects. In complex (algebraic) geometry, one can define integrable systems as follows, which is motivated by the Arnold-Liouville theorem: Let $(\mathbf{M}, \omega)$ be a holomorphic symplectic manifold and $\mathbf{B}$ a complex manifold. An *integrable system* is a holomorphic map $\pi : \mathbf{M} \to \mathbf{B}$ which is generically a proper (polarized) Lagrangian submersion. It then follows (cf. Chapter 2) that the generic fibers are complex tori, which is in analogy with the Arnold-Liouville theorem. Even though this seems to be just an abstraction of the real symplectic case, it is in fact not. ADLER & VAN MOERBEKE ([AvM80a],[AvM80b],[AvMV04]) showed that the phase spaces of many classical integrable systems, for example Euler & Lagrange tops, can be 'complexified' to give integrable systems in the above sense. In fact, they are often even *algebraically completely integrable systems*: The generic fibers are not only arbitrary complex tori but they are even *abelian varieties*. This gave a deep explanation of why some classical integrable systems, e.g. the geodesic flow on ellipsoids, are solvable in terms of elliptic and theta functions.

Calabi-Yau integrable systems. Donagi & Markman ([DM96a]) constructed integrable systems $\mathbf{M}_{CY}(\mathcal{X}) \to \mathbf{B}_{CY}$ for any complete family $\mathcal{X} \to \mathbf{B}_{CY}$ of compact Calabi-Yau threefolds[1]. They are called Calabi-Yau integrable systems. By construction, the fibers of $\mathbf{M}_{CY}(\mathcal{X}) \to \mathbf{B}_{CY}$ are *Griffiths' intermediate Jacobians*, a generalization of the Jacobian of a compact Kähler manifold. These are in particular complex tori. One of the great insights of Donagi & Markman was that these integrable systems are governed by the corresponding *Yukawa* or *Bryant-Griffiths cubics* ([BG83]), which play an important role in mirror symmetry ([CK99]). But not much is known about Calabi-Yau integrable systems.

Hitchin systems. The situation is different for Hitchin systems $\mathbf{M}_{Hit}(\Sigma, G) \to \mathbf{B}_{Hit}(\Sigma, G)$. They are constructed from pairs (Σ, G), consisting of a compact Riemann surface Σ of genus $g \geq 2$ and a reductive complex Lie group G. Hitchin ([Hit87a], [Hit87b]) discovered them for semisimple classical groups $G \subset GL(n, \mathbb{C})$, most prominently $G = SL(2, \mathbb{C})$ ([Hit87a]), which was later extended to general reductive G by work of Faltings ([Fal93]) and Donagi ([Don93]) (also [Sco98] and work of Beilinson & Kazdhan ([BK90])). They have a very rich geometry, for example their total spaces $\mathbf{M}_{Hit}$ carry hyperkähler structures. Moreover, the generic fibers are by now well-understood ([DG02], [DP12], [Sco98]) and turn out to be generalized Prym varieties associated with Σ. Over the years, Hitchin systems gave rise to new developments such as non-abelian Hodge theory ([Sim92]), which has its origin in Hitchin's original paper [Hit87a] and recovers classical Hodge theory for $G = \mathbb{C}^*$. Furthermore, surprising and fascinating links between Hitchin systems and other areas of mathematics and physics have been discovered. Most notably, the role of Hitchin systems in the geometric Langlands program ([DP12]) and relations to quantum field theory ([Don97], [KW07]).

State of the art

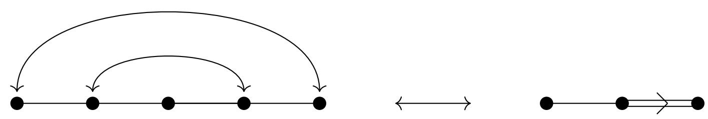
With all the notions at hand, we can now specify the previously mentioned relation between Hitchin systems and Calabi-Yau threefolds due to Diaconescu, Donagi & Pantev ([DDP07]): Let Σ be a compact Riemann surface of genus $g \geq 2$ and G a simple complex Lie group of adjoint type and with simply-laced (resp. ADE-)Dynkin diagram. They constructed a family $\mathcal{X} \to \mathbf{B}_{Hit}(\Sigma, G)$ of *non-compact* (possibly singular) Calabi-Yau threefolds over the corresponding Hitchin base $\mathbf{B}_{Hit}(\Sigma, G)$ and showed that $\mathbf{M}_{CY}(\mathcal{X}) \cong \mathbf{M}_{Hit}(\Sigma, G)$ as integrable systems over an open and Zariski-dense subset $\mathbf{B}^{\circ}_{Hit}(\Sigma, G) \subset \mathbf{B}_{Hit}(\Sigma, G)$. As already mentioned, this relation has its origin in mathematical physics, more precisely geometric transitions and large N duality ([DDD$^+$06]). But also from the perspective of integrable systems, this result is important. It is known that many classical integrable systems, for example the Calogero-Moser systems, can be expressed as (generalized) Hitchin systems. The above result shows that (certain) Hitchin systems, in turn, can be expressed as Calabi-Yau integrable systems. On the other hand, it gives an instance of a Calabi-Yau integrable system, for which the fibers can be well-understood.

[1] This is in fact a simplification, see Chapter 3 for more details.

Main results

For the above reasons, it is not only natural but also important to ask, what happens, if the Dynkin diagram of the simple adjoint complex Lie group G is non-simply-laced, i.e. is a Dynkin diagram of type B_k, C_k, F_4 or G_2 (for short: BCFG-Dynkin diagram)? This question was already raised in the original paper [DDP07] and was later taken up by Kontsevich & Soibelman ([KS14]) in the context of *wall-crossing (structures)*.
The main result of this thesis is an answer to this question. To state it more precisely, we need the simple and elegant idea of *folding* from Lie theory. It is based on the observation that for every BCFG-Dynkin diagram Δ, there exists an ADE-Dynkin diagram Δ_h and a subgroup $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ such that $\Delta = \Delta_h^{\mathbf{C}}$ (see Section 1.2 for details). Figure 1 pictures this in an example.

$\Delta_h = \mathrm{A}_5$
$\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$



$\Delta = \mathrm{B}_3$

Figure 1: Folding of $\Delta_h = \mathrm{A}_5$ to $\Delta_h^{\mathbf{C}} = \Delta = \mathrm{B}_3$.

We can now state the main result of this thesis:

Theorem 0.1. *Let G be a simple adjoint complex Lie group with* non-simply-laced *Dynkin diagram $\Delta = \Delta(G)$. Further let Δ_h be the* ADE-*Dynkin diagram such that $\Delta_h^{\mathbf{C}} = \Delta$ for an appropriate subgroup $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ of the graph automorphisms of Δ_h. Then there exists a family $\pi\colon \mathcal{X} \to \mathbf{B}(\Sigma, G) := \mathbf{B}_{Hit}(\Sigma, G)$ of non-compact (possibly singular) CY3s,* endowed with a $\mathbf{C}$-action *and a Zariski-open and dense subset $\mathbf{B}^\circ(\Sigma, G) \subset \mathbf{B}(\Sigma, G)$ such that there is an isomorphism of integrable systems*

$$\begin{tikzcd} \mathbf{M}_{CY}^{\mathbf{C}}(\mathcal{X}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{M}_{Hit}^\circ(\Sigma, G) \arrow[dl] \\ & \mathbf{B}^\circ(\Sigma, G). & \end{tikzcd} \tag{1}$$

Here $\mathbf{M}_{CY}^{\mathbf{C}}(\mathcal{X}^\circ) \subset \mathbf{M}_{CY}(\mathcal{X}^\circ)$ is determined by the $\mathbf{C}$-invariants in cohomology.

The ADE-case from [DDP07] is the analogous statement but with $\mathbf{C} = 1$. So our result can be seen as an $\mathrm{Aut}(\Delta_h)$-equivariant version of theirs. It would be interesting to see how this result can contribute to geometric transitions and large N duality as in [DDD$^+$06].
There are several crucial steps to obtain Theorem 0.1 and we highlight three of them (keeping the notation):

I) *Construction of the family:* The first step is to construct a family $\mathcal{X} \to \mathbf{B}(\Sigma, G)$ together with a fiber-preserving $\mathbf{C}$-action over the Hitchin base. We achieve this in Section 5.3 by using constructions from [DDP07], [Sze04] and [Slo80b]. Slodowy's work ([Slo80b]) on simple singularities is of great importance for us, since it provides a theory of singularities of type Δ, where Δ is *any* irreducible Dynkin diagram. This includes in particular ADE-singularities, which already played a crucial role in [DDP07].

II) *Invariant volume forms:* The second step is to show that the (non-singular) Calabi-Yau threefolds $X_b = \pi^{-1}(b)$, $b \in \mathbf{B}^\circ(\Sigma, G)$ carry an $\mathrm{Aut}(\Delta_h)$-invariant holomorphic volume form (Proposition 5.36). This requires studying the semi-universal deformations of singularities of type Δ in more detail (Section 1.5), in particular the relative canonical classes and period maps.

III) *Isomorphism of VMHS:* The family $\mathcal{X}^\circ \to \mathbf{B}^\circ := \mathbf{B}^\circ(\Sigma, G)$ and the Hitchin system $\mathbf{M}^\circ_{Hit}(\Sigma, G)$ induce a variation of (mixed) Hodge structures V^{CY} and V^H respectively over the 'good' locus $\mathbf{B}^\circ$. The group $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ naturally acts on V^{CY}. We deduce Theorem 0.1 from an isomorphism (Theorem 5.55)

$$(\mathsf{V}^{CY})^{\mathbf{C}} \cong \mathsf{V}^H$$

(up to a Tate twist). In fact, we first prove an analogous isomorphism in the ADE-case (Theorem 5.15), which yields an alternative proof of [DDP07].

Outline of thesis

We now give a detailed outline of the thesis at hand, which, at the same time, is a guideline. As the reader will notice, only the last chapter follows the ordering of the above steps. The first four chapters are meant as a preparation and background.

Chapter 1: As already mentioned, singularities of type Δ for any irreducible Dynkin diagram Δ are an important ingredient for proving Theorem 0.1. In this chapter we outline their theory, following Brieskorn ([Bri71]), Grothendieck, Slodowy ([Slo80b]) and Yamada ([Yam95]). However, there are aspects that we either present in a different viewpoint or cannot be found in the literature at all:

a) Equivariant cohomology: We look at Slodowy's definition of BCFG-*singularities* (i.e. singularities of type Δ, an irreducible Dynkin diagram of type BCFG) via *equivariant integral cohomology* (Section 1.3.1) and show that its torsion part is independent of Δ. A motivation for these considerations is to find a geometric object, whose cohomology groups yield root systems of type BCFG (see Section 1.3.1 for more details).

b) Derivatives (Section 1.4.4): We take a closer look at the derivatives of the semi-universal deformation of a singularity of type Δ. This is useful to determine, when a threefold X_b, $b \in \mathbf{B}(\Sigma, G)$, as in Step I) is non-singular (cf. Proposition 1.61). Here we also discuss how the threefolds X_b (Remark 1.64) can be locally described in an explicit way.

c) Stratification (Section 1.4.5): Slodowy ([Slo80b]) implicitly gave a stratification of the base[2] $\mathfrak{t}/W$ of the semi-universal deformation of a singularity of type Δ. We clarify its relation to subdiagrams of Δ (cf. Example 1.4.5). Moreover, we study two natural sheaves on $\mathfrak{t}/W$, associated with the semi-universal deformation and the quotient $\mathfrak{t} \to \mathfrak{t}/W$, and relate them to (part of) the above stratification (Section 1.4.6). Even though they already appeared implicitly in [DDP07], we elaborate on them because they are crucial relating Hitchin systems with Calabi-Yau integrable systems.

d) Relative symplectic form (Section 1.5): Yamada ([Yam95]) gave a symplectic-geometric construction of the simultaneous resolution of ADE-singularities. A by-product of this

[2]This notation is no coincidence: $\mathfrak{t}$ and W are a Cartan subalgebra and Weyl group of type Δ respectively.

construction is a relative symplectic form on the simultaneous resolution, which enables him to study a period map. We begin by discussing *relative symplectic reduction* more generally than in [Yam95] (Proposition 1.82). After that we extend most of his results to BCFG-singularities, i.e. take into account graph automorphisms (Proposition 1.91 and Corollary 1.98). Moreover, we make explicit and extend the relation between the relative symplectic form and the Kostant-Kirillov form (Corollary 1.104), which partly already appeared in [Yam95]. All these results are crucial for Step II) above.

Chapter 2: In the first half of this chapter, we begin with proper Lagrangian submersions and then add more and more structure and conditions to end up with the definition of an *polarized integrable system*. All of this is well-known, but we consider it useful to have it in one place. This exposition is in particular suited for our purposes and we focus on the non-singular part of a (polarized) integrable system. Here we made an effort to include integrable systems, whose generic fibers are non-degenerate complex tori, but not necessarily abelian varieties. We define them as *polarized integrable systems* (Section 2).
Although Hamiltonian functions, known from integrable systems in real symplectic geometry, have analogues for polarized integrable systems and appear at least implicitly in our presentation, we do not discuss them any further. The reason being that they are not crucial for our purposes. We refer to [DM96a] and [Hit87a] for Hamiltonian functions of Calabi-Yau integrable systems and Hitchin systems respectively.
In the second half, we discuss in great detail the cubic condition(s) by Donagi & Markman ([DM96a]):

a) We give some supplements to their proof, which we consider useful and could not find elsewhere, for example a discussion of global connecting homomorphisms and fiberwise ones (Lemma 2.29).

b) Abstract Seiberg-Witten differentials: In the closing section of this chapter (Section 2.2.4), we present a sheaf-theoretic viewpoint on the smooth part of an integrable system (with section). This seems to be known, but we could not find it in this form in the literature (the closest can be found in [KS14]). Moreover, it provides a useful perspective on proving Theorem 0.1 and explains the significance of Step III) from above. More specifically, we introduce the notion of an *abstract Seiberg-Witten differential* which links the sheaf-theoretic description to the cubic condition (Proposition 2.36). An abstract Seiberg-Witten differential is a section of a variation of Hodge structures whose derivative with respect to the natural connection satisfies a special property. Our definition is inspired by the Seiberg-Witten differential(s) of Hitchin systems ([Don97], also Remark 2.38) which yield examples of abstract Seiberg-Witten differentials (see Corollary 4.32). Another motivation for this definition is that it allows to abstract some of the arguments from [DM96a] (see the proof of Proposition 2.36).

Chapter 3: Calabi-Yau integrable systems constructed from compact CY3s comprise our first class of examples of polarized integrable systems. As already mentioned, they were discovered by Donagi & Markman ([DM96a]). To show the usefulness of abstract Seiberg-Witten differential, we construct Calabi-Yau integrable systems by employing an abstract Seiberg-Witten differential. Of course, this is not new but we consider it more conceptual to obtain them in this way (Lemma 3.12). Furthermore, we explain in detail how the Bryant-Griffiths or Yukawa cubic is related to these integrable systems (supplementing [DM96a]).

Chapter 4: The second class of examples of polarized integrable systems are Hitchin systems. Here we can only give a brief outline of their construction, starting with the deformation theory

and moduli of G(-Higgs) bundles. We put an emphasis on the governing differential graded Lie algebras but cannot go into detail due to lack of space. However, we highlight some of the relations between infinitesimal deformations of a G-Higgs bundle and its underlying G-bundle.
After that we focus on the generic fibers of Hitchin systems, so-called generalized Prym varieties, following [DG02] and [DP12]. More precisely:

a) We give a few explicit examples and calculations illustrating some aspects of the literature. For example, we explicitly compare the description of the generic $SL(2,\mathbb{C})$-Hitchin fiber due to Hitchin ([Hit87a] with the one of Donagi & Gaitsgory ([DG02]) in Example 4.27.

b) Hodge structures: After a general discussion, we focus on the special case, where G is a simple adjoint or simply-connected complex Lie group. We describe the VHS determined by the corresponding (neutral component of the) Hitchin system over $\mathbf{B}^\circ$ and compare it with another VHS which is used in the literature (Corollary 4.30 and Remark 4.31). Here we give a new point of view (Proposition 4.35) on the Hodge structure corresponding to the generalized Prym varieties by using a Theorem of Zucker ([Zuc79]). This is particularly useful for proving the isomorphism in Step III).

Chapter 5: The last chapter ties together all the above preparation. As the reader might have noticed, Calabi-Yau integrable systems in the sense of Donagi & Markman are constructed from families of *compact* CY3s. However, Theorem 0.1 is a statement about families of *non-compact* CY3s. In fact, it is not difficult to show that CY integrable systems from families of compact CY3s as in Chapter 3 cannot be isomorphic to *any* Hitchin system. We begin this chapter by discussing the differences in more detail and emphasize that there is so far no general theory for 'non-compact CY integrable systems' (see Section 5.2.1 and Remark 5.14). After that we construct and prove the following:

a) Families of surfaces: Let Δ_h be an irreducible ADE-Dynkin diagram. Following basic ideas from [DDP07] and [Sze04], we construct families $\boldsymbol{\sigma} : \mathcal{S}(\Delta_h) \to \boldsymbol{U}_h$ of surfaces over a vector bundle $\boldsymbol{U}_h \to \Sigma$, where Σ is a curve as in Theorem 0.1. These are obtained by gluing the semi-universal deformation of the singularity of type Δ_h. Even though they already appeared in [DDP07], we give a more detailed account with further properties (Proposition 5.5). The latter are important for constructing families of quasi-projective CY3s because we deduce many of their properties from the families of surfaces. Here especially our preparation of Sections 1.4 & 1.5 is useful.
We see that the general construction (referred to as *local construction*, 5.1.1) does *not* yield families of surfaces with an $\mathrm{Aut}(\Delta_h)$-action. Instead, we have to consider special cases (referred to as *global construction*, Section 5.1.2).

b) Families of CY3s: Using a fiber product construction as in [DDP07] and [Sze04], we obtain a family $\mathcal{X}(\Delta_h) \to \mathbf{B}(\Delta_h)$ of quasi-projective Gorenstein CY3s over the Hitchin base $\mathbf{B}(\Delta_h)$ of type Δ_h for any family $\mathcal{S}(\Delta_h) \to \boldsymbol{U}_h$ as in [DDP07]. We relate this family to the ones of [Sze04] by constructing yet another family that fits in-between these two (Lemma 5.9).

c) ADE-case via V(M)HS: Before we turn to the BCFG-case as in Theorem 0.1, we reestablish the ADE-case of [DDP07] by proving that the variations of (mixed) Hodge structures (V(M)HS) induced by the Hitchin system of type Δ_h and of the family $\mathcal{X}(\Delta_h) \to \mathbf{B}(\Delta_h)$ are isomorphic (Theorem 5.15). Our preparations of Section 1.4.6 (especially Proposition 1.76 and Corollary 1.78) as well as Section 4.3.2 are important here. Another ingredient is Saito's theory of *(mixed) Hodge modules* ([Sai88], [Sai90]), which is well-suited to deal

with our situation (and much more complicated ones).
Using this theorem and Proposition 5.5, we reobtain (Corollary 5.16) the main result of [DDP07].

d) Langlands dual group: Although it is known to experts, we give a written account of how the Hitchin systems of the Langlands dual group fit into this picture (in the ADE-case) (Theorem 5.33). This requires *homology* intermediate Jacobians and appeared in the A_1-case in [DDD$^+$06] before.

e) Families with graph automorphisms: Now let Δ be a BCFG-diagram and Δ_h and ADE-Dynkin diagram with $\Delta = \Delta_h^{\mathbf{C}}$ for a subgroup $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ as in Theorem 0.1. In Section 4.3.2 we directly construct families $\mathcal{X} \to \mathbf{B}(\Delta)$ of quasi-projective Gorenstein CY3s over the Hitchin base $\mathbf{B}(\Delta)$ of type BCFG using the global construction of Section 5.1. They admit a natural $\mathbf{C}$-action, which preserves the fibers, and we show that they carry $\mathbf{C}$-invariant volume forms. We relate these families to the ones of Section 5.1 and [DDP07] (Proposition 5.36). These constructions establish Step I) and II) from above.

f) Isomorphism of V(M)HS: Proving the isomorphism of Step III) is more involved than in c) because we have to take into account the $\mathbf{C}$-action. We first prove the fiberwise case (Theorem 5.43), from which we eventually establish Step III) (Theorem 5.55). Then we deduce Theorem 0.1 (Corollary 5.56) from Theorem 5.55, using Proposition 5.5.

g) Equivariant cohomology (Section 5.4.2): In analogy to Section 1.3.1 from Chapter 1, we look at equivariant (integral) cohomology with respect to the action by graph automorphisms. Unfortunately, we do not have a final result here. However, our investigation shows that it might be possible to rephrase Theorem 0.1 purely in terms of orbifolds/orbifold stacks and we plan to pursue this in the future.

h) Monodromy along fibers: We close with another possible approach to realize non-simply-laced Dynkin diagrams. This is an idea from the physics literature and Szendrői gave a mathematical account in [Sze04]. We follow this idea and give analogous constructions as before (Section 5.5). However, we argue that this approach does *not* reproduce the above one (Proposition 5.67). Instead, it might give rise to a *new* (non-compact) CY integrable system.

Finally, we collect some useful definitions and facts about non-degenerate tori as well as variations of (mixed) Hodge structures in the Appendix that we need in Chapter 2 and 5.

Conventions

Throughout our thesis at hand, we work over the complex numbers $\mathbb{C}$. If X is a complex algebraic variety, we do not make a notational distinction between X and its analytification X^{an} to keep the notation cleaner. There are at least two reasons for doing so. Firstly, many of our arguments apply both in the algebraic and analytic category (e.g. by invoking GAGA). However, whenever we considered it necessary, we point out the distinction. One example is in Chapter 5, when we construct the families $\mathcal{X} \to \mathbf{B}$ of threefolds (see Remark 5.4). Secondly, it is often apparent from the context, for example Chapter 2 and Chapter 3 are mainly complex-analytic, whereas most parts of Chapter 1 are algebraic (e.g. semi-universal deformations of singularities of type Δ).

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Chapter 1

Singularities of type Δ

The purpose of this chapter is two-fold. Firstly, it introduces Slodowy's singularities of type Δ ([Slo80b]), where Δ is any irreducible Dynkin diagram. When Δ is simply-laced, i.e. it is a Dynkin diagram of type ADE, they coincide with the famous ADE-singularities. In the non-simply-laced case, when Δ is of type BCFG, i.e. B_k, C_k, F_4 or G_2, one needs the concept of *folding*. Since folding will be crucial in the applications in Chapter 5 as well, we discuss it in some detail.
Based on ideas of Brieskorn ([Bri71]) and Grothendieck, Slodowy constructed semi-universal deformations of singularities of type Δ and simultaneous resolutions in terms of the Lie algebras of the corresponding type. We discuss his result and introduce some facts from the theory of Lie algebras, in particular the adjoint quotient. The latter is in particular important for Hitchin systems, see Chapter 4.
Secondly, parts of this chapter (most importantly Section 1.4.4, 1.4.6 and 1.5) provide the local theory of global constructions in Chapter 4 and Chapter 5.

1.1 ADE-singularities

Given a finite subgroup $\Gamma \subset SL(2,\mathbb{C})$, we can consider the quotient $\mathbb{C}^2/\Gamma$. This is in a natural way an algebraic variety, given as the (maximal) spectrum $\mathrm{Spec}(\mathbb{C}[u,v]^\Gamma)$. It turns out that its only singular point is $\bar{0} \in \mathbb{C}^2/\Gamma$, the image of zero. The singularity can be resolved by a finite sequence of blowups. If we stop the sequence of blowups as soon as the resulting variety is non-singular, then we obtain the *minimal* resolution

$$\pi : \hat{Y} \longrightarrow Y := \mathbb{C}^2/\Gamma.$$

The exceptional divisor $E := \pi^{-1}(\bar{0}) = \bigcup_i E_i$ is a tree of projective lines, $E_i \cong \mathbb{CP}^1$, and we can consider the dual of the intersection graph. The surprising fact is that this is related to ADE- or *simply-laced* Dynkin diagrams.

Theorem 1.1 (du Val [DV64])**.** *There is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \Gamma \subset SL(2,\mathbb{C}) \\ \textit{finite subgroup} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \Delta \textit{ irreducible Dynkin} \\ \textit{diagram of type ADE} \end{array} \right\}. \tag{1.1}$$

It is given by associating to $\Gamma \subset SL(2,\mathbb{C})$ *the dual of the resolution graph* $\hat{Y} \to \mathbb{C}^2/\Gamma$.

Remark 1.2. Even though we do not directly need it, let us mention that there are more sophisticated versions of the above correspondence. For example, there is the so-called *McKay correspondence*, which also takes into account the representation theory of finite subgroups $\Gamma \subset SL(2,\mathbb{C})$ ([McK80]). However, we do need a more refined version due to Brieskorn, see Section 1.1.1.

For this reason, singularities $\mathbb{C}^2/\Gamma$ for a finite subgroup $\Gamma \subset SL(2,\mathbb{C})$ are called ADE-singularities. They actually have many more names (e.g. Kleinian singularities, du Val singularities, simple singularities...). This is reminiscent of the fact that they have many equivalent characterizations ([Dur79]).
The most elementary proof of the correspondence (1.1) is by calculating the resolution graphs of $\hat{Y} \to \mathbb{C}^2/\Gamma$ case-by-case. This is possible because all finite subgroups $\Gamma \subset SL(2,\mathbb{C})$ have already been classified long ago by Felix Klein.

Proposition 1.3 (Klein (1884)). *Let $\Gamma \subset SL(2,\mathbb{C})$ be a finite subgroup. Then Γ is precisely one of the following groups:*

$$\begin{array}{lll}
\textit{cyclic} & \mathrm{A}_k: & \mathbb{Z}_{k+1} = \left\{ \boldsymbol{\zeta} := \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \;\middle|\; \zeta \in \mu_{k+1} \right\} \\[2ex]
\textit{binary dihedral} & \mathrm{D}_k: & \mathbb{D}_{k-2} = \left\langle \boldsymbol{\zeta}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\middle|\; \zeta \in \mu_{2(k-2)} \right\rangle \\[2ex]
\textit{binary tetrahedral} & \mathrm{E}_6: & \mathbb{T} = \left\langle \mathbb{D}_2, \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^7 & \xi^7 \\ \xi^5 & \xi \end{pmatrix} \;\middle|\; \xi \in \mu_8 \textit{ primitive} \right\rangle \\[2ex]
\textit{binary octahedral} & \mathrm{E}_7: & \mathbb{O} = \left\langle \mathbb{T}, \boldsymbol{\xi} \;\middle|\; \xi \in \mu_8 \textit{ primitive} \right\rangle \\[2ex]
\textit{binary icosahedral} & \mathrm{E}_8: & \mathbb{I} = \left\langle -\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}, \frac{1}{\eta^2-\eta^3} \begin{pmatrix} \eta+\eta^4 & 1 \\ 1 & -\eta-\eta^4 \end{pmatrix} \;\middle|\; \eta \in \mu_5 \textit{ primitive} \right\rangle
\end{array}$$

Here $\mu_m \subset \mathbb{C}^*$ is the group of m-th roots of unity. The type of Γ *is its label in this list.*

Needless to say that the type of Γ corresponds precisely to the type of the corresponding Dynkin diagram in (1.1). There is even a relation to the *platonic solids* of the same type. This can be seen from the fact that each finite subgroup $\Gamma \subset SL(2,\mathbb{C})$ can be considered as a subgroup of $SU(2)$. Hence these groups project to finite subgroups of $SO(3)$ via the double covering $SU(2) \to SO(3)$. It turns out that these finite subgroups are isometry groups of the corresponding platonic solids, for details see [Lam86].
Before we come to explicit equations for the quotient singularities $\mathbb{C}^2/\Gamma$, we record some of the relations between the various finite subgroups of $SL(2,\mathbb{C})$. These results will be crucial for extending the correspondence (1.1) to the remaining irreducible Dynkin diagrams. These are the *non-simply-laced* Dynkin diagrams, i.e. of type B_k, C_k, F_4 and G_2. We often refer to such diagrams as BCFG-*Dynkin diagrams.*

Lemma 1.4 ([Slo80b]). *With the above notation, we have the following short exact sequences:*

$$\mathrm{A}_{2k-1}: \; 1 \longrightarrow \mathbb{Z}_{2k} \longrightarrow \mathbb{D}_k \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.2}$$

$$\mathrm{D}_{k+1}: 1 \longrightarrow \mathbb{D}_{k-1} \longrightarrow \mathbb{D}_{2(k-1)} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.3}$$

$$\mathrm{E}_6 : 1 \longrightarrow \mathbb{T} \longrightarrow \mathbb{O} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.4}$$

$$\mathrm{D}_4 : 1 \longrightarrow \mathbb{D}_2 \longrightarrow \mathbb{O} \longrightarrow S_3 \longrightarrow 1. \tag{1.5}$$

To compute the blowups, one realizes $\mathbb{C}^2/\Gamma$ as a hypersurface singularity in $\mathbb{C}^3$ by determining generators and relations for $\mathbb{C}^2[u,v]^\Gamma$. This also shows that these hypersurface singularities carry natural $\mathbb{C}^*$-actions.

Lemma 1.5 (cf. [Slo80b])**.** *Let $\Gamma \subset SL(2,\mathbb{C})$ be a finite group. Then there are independent generators $x, y, z \in \mathbb{C}[u,v]^\Gamma$ with the following relations:*

$$\begin{array}{ll} \mathrm{A}_k : & x^{k+1} - yz = 0, \\ \mathrm{D}_k : & x(x^{k-2} - y^2) - z^2 = 0, \\ \mathrm{E}_6 : & x^4 + y^3 + z^2 = 0, \\ \mathrm{E}_7 : & x^3 y + y^3 + z^2 = 0, \\ \mathrm{E}_8 : & x^5 + y^3 + z^2 = 0. \end{array}$$

Proof. We only give the cases A and D. The exceptional cases are best understood via their relation to platonic solids which we have not developed. This goes already back to Klein ([Kle93], also [Lam86]).
A_k: We can take as generators $x = uv$, $y = u^{k+1}$, $z = v^{k+1}$.
D_k: The generators $x = (uv)^2$, $y = uv(u^{2(k-2)} - v^{2(k-2)})$, $z = u^{2(k-2)} + v^{2(k-2)}$ which are invariant under $\Gamma = \mathbb{D}_{k-2}$ satisfy the relations

$$x(y^2 - 4x^{k-2}) - z^2 = 0.$$

By scaling we obtain the equation given above. □

A look at the above equations immediately shows that they are quasi-homogeneous. Recall that a polynomial $\sum_{I=(i_1,\ldots,i_k)} a_I x^I$ is quasi-homogeneous of weight $(w_1,\ldots,w_k)$ and degree d iff

$$\sum_{j=1}^{k} w_j i_j = d$$

for all $I = (i_1,\ldots,i_k)$ such that $a_I \neq 0$. We have the following list (with $x_1 = x, x_2 = y, x_3 = z$):

	(w_1, w_2, w_3)	d
A_k	$(2, k+1, k+1)$	$2(k+1)$
D_k	$(2, k-2, k-1)$	$2k-2$
E_6	$(3,4,6)$	12
E_7	$(4,6,9)$	18
E_8	$(6,10,15)$	30

(1.6)

In particular, the zero loci of the equations in $\mathbb{C}^3$ carry natural $\mathbb{C}^*$-actions of the corresponding weights. Since the weights are positive, it follows that the singularities are contractible.
There is another way to construct a $\mathbb{C}^*$-action directly on the quotient $\mathbb{C}^2/\Gamma$. To this end we introduce the group

$$C(\Gamma) := C_{GL(2,\mathbb{C})}(\Gamma) \subset GL(2,\mathbb{C}), \tag{1.7}$$

the centralizer of Γ in $GL(2,\mathbb{C})$. It naturally acts on $\mathbb{C}^2/\Gamma$.

Lemma 1.6. *Let $\Gamma \subset SL(2,\mathbb{C})$ be a finite subgroup. Then we have the following cases*

$$\mathrm{A}_k : C(\Gamma) = GL(2,\mathbb{C}) \text{ for } k=1, \quad C(\Gamma) = \mathbb{C}^* \times \mathbb{C}^* \text{ for } k>1,$$

$$\mathrm{D}_k : C(\Gamma) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a^2+b^2 \neq 0 \right\} \text{ for } k=4, \quad C(\Gamma) = \mathbb{C}^* \text{ for } k>4,$$

$$\mathrm{E}_k : C(\Gamma) = \mathbb{C}^*, \quad k = 6,7,8.$$

If Γ is of type D_4, then $C(\Gamma)$ is conjugate to the diagonal matrices in $GL(2,\mathbb{C})$ so that $C(\Gamma) \cong \mathbb{C}^* \times \mathbb{C}^*$.

Remark 1.7. The D_4-case surprisingly appeared incorrectly in several places in the literature (e.g. [Sze04]) and we provide a correction here.

Proof. A_k: Let $\zeta \in \mu_{k+1}$ and compute

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} = \begin{pmatrix} a & b\zeta^2 \\ c\zeta^{-2} & d \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence there is no restriction for $k=1$ and for $k>1$ we need to have $b=c=0$.
D_k: We already know that $C(\mathbb{D}_{k-2}) \subset C(\mathbb{Z}_{k-2})$. Now we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It follows that $a=d$ and $c=-b$. This shows the case $k=4$. Observe that $C(\Gamma)$ is commutative and its matrices are diagonalizable with eigenvalues $a \pm ib$. Hence $C(\Gamma)$ is conjugate to the diagonal matrices. For $k>4$ we additionally have the constraint $b=c=0$.
The cases $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ work similarly. □

Remark 1.8. Clearly, $C(\Gamma)$ naturally acts on $\mathbb{C}^2/\Gamma$ and $\mathbb{C}[u,v]^\Gamma$. The center $Z(\Gamma)$ of Γ is obviously contained in $C(\Gamma)$ and acts trivially on $\mathbb{C}^2/\Gamma$. Therefore it is convenient to define

$$C_\Gamma := C(\Gamma)/Z(\Gamma). \tag{1.8}$$

Since $\mathbb{C}^* \subset C(\Gamma)$ in all the above cases, we can compare it with the quasi-homogeneous structure on the equations from above. It is easy to show that the action via $\mathbb{C}^* \subset C(\Gamma)$ has the following weights (where we consider $\mathbb{C}^2/\Gamma$ again as hypersurface singularity in $\mathbb{C}^3$ as above and $(x_1,x_2,x_3) = (x,y,z)$)

	(w_1,w_2,w_3)
A_k	$(2,k+1,k+1)$
D_k	$(4,2(k-2),2(k-1))$
E_6	$(6,8,12)$
E_7	$(8,12,18)$
E_8	$(12,20,30)$

(1.9)

Note that except for the A_k-case the weights are twice the weights of the previous $\mathbb{C}^*$-action. In some sense, these are the natural weights since they will also show up naturally in the Lie algebraic construction of the ADE-singularities, cf. 1.4.3.

1.1.1 Topology of the minimal resolution

The correspondence (1.1) works via studying the minimal resolutions. As already mentioned, there is a refined version by Brieskorn [Bri68] showing that

$$([E_i] \cdot [E_j])_{ij} = -C(\Delta(\Gamma)),$$

where the left-hand side is the intersection matrix of the exceptional curves (of the minimal resolution) and $C(\Delta(\Gamma))$ is the Cartan matrix of the type of $\Delta(\Gamma)$, the Dynkin diagram corresponding to Γ. We review this result in a way that will suit our purposes later on.

The intersection product of two divisors $D, E \subset X$ on a non-singular complex surface X is defined via (cf. [BHPVdV04])

$$\bullet \cdot \bullet : Div(X) \times Div(X) \to \mathbb{Z}, \quad D \cdot E := ([D], c_1(\mathcal{O}(E))).$$

Here $[D] \in H_2(X, \mathbb{Z}) \cong H^2_c(X, \mathbb{Z})$ stands for the fundamental class of the divisor and $\mathcal{O}(E) \in Pic(X)$ is the line bundle corresponding to the divisor E. Finally, the product $(\bullet, \bullet) : H_2 \otimes_{\mathbb{Z}} H^2 \to \mathbb{Z}$ is the natural pairing between (integral) homology and cohomology.

Proposition 1.9 ([BHPVdV04]). *The intersection product is bilinear and symmetric. If D and E are compact divisors, then $D \cdot E = [D] \cdot [E]$ where the last product is the intersection product in homology (resp. compactly supported cohomology).*

Hence we can unambiguously speak of the intersection product between two exceptional curves. In the case of the minimal resolution $\hat{Y}$ of $Y = \mathbb{C}^2/\Gamma$ the intersection product is closely related to Lie theory. We denote by $\Delta = \Delta(\Gamma)$ and $R = R(\Delta)$ the corresponding Dynkin diagram and root system respectively. Further let $Q = \langle R \rangle_{\mathbb{Z}}$ be the root lattice spanned by R. Fixing a W-invariant inner product $(\bullet, \bullet)$ on $V = Q \otimes_{\mathbb{Z}} \mathbb{R}$ induces

$$\langle \bullet, \bullet \rangle : Q \otimes_{\mathbb{Z}} Q \to \mathbb{Z}, \quad \alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle := (\alpha, \beta^\vee).$$

Here we set $\beta^\vee = 2(\beta, \beta)^{-1}\beta$ as usual. By construction, R is a root system in $(V, (\bullet, \bullet))$. Since Q is free, it can be considered as a lattice in V and we can define the weight lattice $P := \{v \in V \mid \langle v, q \rangle \in \mathbb{Z}\ \forall q \in Q\}$ of R in V.

Proposition 1.10 (Brieskorn). *Let $\hat{Y} \to Y$ be the minimal resolution of Y. Then there is an isomorphism*

$$(H_2(\hat{Y}, \mathbb{Z}), \cdot) \cong (Q, -\langle \bullet, \bullet \rangle)$$

of lattices with non-degenerate symmetric products, where $H_2(\hat{Y}, \mathbb{Z})$ is endowed with the intersection product on homology. In particular, the intersection matrix of the exceptional curves equals the negative of the Cartan matrix. Moreover, $H^2(\hat{Y}, \mathbb{Z}) \cong P$ for the weight lattice P of R.

Proof. By a theorem of Milnor ([Mil68]) $\hat{Y}$ is homotopy equivalent to a bouqet $\bigvee_i^r S^2$ of spheres implying

$$H_0(\hat{Y}, \mathbb{Z}) = 0, \quad H_2(\hat{Y}, \mathbb{Z}) \cong \mathbb{Z}^{\oplus r}$$

and $H_k(\hat{Y}, \mathbb{Z}) = 0$ else. These spheres correspond to the exceptional curves $E_i \cong \mathbb{CP}^1$. In particular, their fundamental classes $[E_i]$ freely generate $H_2(\hat{Y}, \mathbb{Z})$. Under the correspondence (1.1), we fix a bijection $E_i \mapsto \alpha_i$ between the dual of the resolution graph and nodes of the corresponding Dynkin diagram Δ (respectively simple roots of the root system $R(\Delta)$). Since $[E_i] \cdot [E_j] = -\langle \alpha_i, \alpha_j \rangle$, it follows that this bijection extends to an isomorphism

$$(H_2(\hat{Y}, \mathbb{Z}), \cdot) \cong (Q, -\langle \bullet, \bullet \rangle)$$

as claimed. The freeness of $H_2(\hat{Y}, \mathbb{Z})$ implies a natural isomorphism

$$H^2(\hat{Y}, \mathbb{Z}) \cong \mathrm{Hom}(H_2(\hat{Y}, \mathbb{Z}), \mathbb{Z}).$$

Therefore there is a natural map

$$H_2(\hat{Y}, \mathbb{Z}) \to H^2(\hat{Y}, \mathbb{Z}), \quad a \mapsto \langle a, \bullet \rangle.$$

By the above isomorphism, it can be identified with the natural inclusion $Q \hookrightarrow P$ of the root lattice Q into the weight lattice P. It has finite cokernel and therefore becomes an isomorphism over $\mathbb{Q}$ (it is already one for E_8 over the integers). □

Remark 1.11. Let $\mathfrak{g} = \mathfrak{g}(\Delta)$ be the simple Lie algebra corresponding to the Dynkin diagram Δ. Moreover, let G_{ad} and G_{sc} be the adjoint and simply connected complex Lie group respectively with Lie algebra $\mathfrak{g}$. We denote by $(X_{ad}, R_{ad}, X_{ad}^\vee, R_{ad}^\vee)$ and $(X_{sc}, R_{sc}, X_{sc}^\vee, R_{sc}^\vee)$ the respective root data. Then we know (e.g. [Spr09]) that (with the above notation)

$$Q_{ad} \cong Q = Q(\mathfrak{g}), \quad Q \cong Q^\vee, \quad P \cong P^\vee,$$

where the first isomorphism follows from $R(\mathfrak{g}) \cong R(G)$ for any (semi)simple complex Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$. The other two isomorphisms follow from the fact, that we are dealing with simply-laced Dynkin diagrams, so all the roots have the same length. Further we have

$$X_{ad} = Q = X_{sc}^\vee, \quad X_{ad}^\vee = P = X_{sc},$$

$$Q = \mathbf{\Lambda}(G_{sc}) \cong \mathbf{\Lambda}(G_{ad})^\vee, \quad P = \mathbf{\Lambda}(G_{ad}) \cong \mathbf{\Lambda}(G_{sc})^\vee.$$

Here $\mathbf{\Lambda}(G) = \mathrm{Hom}(\mathbb{C}^*, T)$ stands for the cocharacter lattice of the respective groups (with respect to a (fixed) maximal torus, [Spr09]). By definition, Q and P are dual to each other: Since we can consider $P \cong P^\vee$ as a lattice in V, we get a pairing $Q \otimes_{\mathbb{Z}} P \to \mathbb{Z}$ as above (so it is given by $(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle = (\alpha, \beta^\vee)$; note that $\beta^\vee$ is also defined for $\beta \in P$ since we have fixed $(\bullet, \bullet)$). For latter reference we rephrase the statement of Proposition 1.10 as

$$H_2(\hat{Y}, \mathbb{Z}) \cong \mathbf{\Lambda}(G_{sc}), \quad H^2(\hat{Y}, \mathbb{Z}) \cong \mathbf{\Lambda}(G_{ad}), \tag{1.10}$$

and these isomorphisms are compatible with the natural pairings between the respective left- and right-hand side.

1.2 Interlude: Folding in the Lie context

Slodowy gave a natural generalization of ADE-singularities to singularities of any type Δ, where Δ is any irreducible Dynkin diagram (cf. [Slo80b]). It uses a technique, called *folding*, from Lie theory that reduces questions concerning Lie algebras/groups of type BCFG to questions of type ADE ([Spr09]). The basic idea is to use graph automorphisms $\mathrm{Aut}(\Delta)$ of Dynkin diagrams Δ of type ADE. To introduce BCFG-singularities, it would have sufficed to only introduce these graph automorphisms and their action on the Dynkin diagrams. However, we will need their actions in the Lie algebraic context, so that we already discuss this aspect as well. This way, we can also explain the different conventions of folding that can be found in the literature.

1.2.1 Dynkin graph automorphisms

Let Δ be an irreducible Dynkin diagram. In case Δ is of type $\mathrm{A}_{\geq 2}$, $\mathrm{D}_{\geq 2}$ and E_6, it has non-trivial graph automorphisms $\mathrm{Aut}(\Delta) \neq 1$. If we look at the induced action on the corresponding root system $R = R(\Delta)$, it is convenient to restrict to a certain subclass of graph automorphisms. These give better behaved invariants and coinvariants in R.

Definition 1.12. Let Δ be an irreducible Dynkin diagram. A *Dynkin graph automorphism* of Δ is an automorphism $\mathbf{a}$ of the underlying (directed) graph such that α and $\mathbf{a}(\alpha)$ are not direct neighbors of each other for every vertex $\alpha \in \Delta$. We denote by $\mathrm{Aut}_D(\Delta) \subset \mathrm{Aut}(\Delta)$ the subgroup of all Dynkin graph automorphisms.

We can rephrase this condition in terms of the root systems as follows. Let $(R, (V, \langle \bullet, \bullet \rangle))$ be the root system $R = R(\Delta)$ corresponding to Δ (up to isomorphism) and $\mathbf{a} \in \mathrm{Aut}(\Delta)$. Clearly, $\mathbf{a}$ induces an automorphism of V which we also denote by $\mathbf{a} \in GL(V)$. If $e_\alpha \in R$ denotes the vector corresponding to $\alpha \in \Delta$, then $\mathbf{a}(e_\alpha) = e_{\mathbf{a}(\alpha)}$. By definition, we now have for all $\alpha \in R$

$$\langle \mathbf{a}(e_\beta), \mathbf{a}(e_\gamma) \rangle = 0, \quad \forall \beta \neq \gamma \in \mathrm{Aut}_D(\Delta) \cdot \alpha.$$

Moreover, $\mathbf{a}$ is an isometry: $\langle \mathbf{a}(\alpha), \mathbf{a}(\beta) \rangle = \langle \alpha, \beta \rangle$.

Lemma 1.13. *Let Δ be an irreducible Dynkin diagram. Then Δ has trivial Dynkin graph automorphisms except for the following cases*[1]*:*

$$\begin{array}{ll} \mathrm{Aut}_D(\Delta) = \mathbb{Z}/2Z, & \Delta = \mathrm{A}_{2n+1}, n \geq 1, \\ \mathrm{Aut}_D(\Delta) = S_3, & \Delta = \mathrm{D}_4, \\ \mathrm{Aut}_D(\Delta) = \mathbb{Z}/2\mathbb{Z}, & \Delta = \mathrm{D}_n, n \geq 5, \\ \mathrm{Aut}_D(\Delta) = \mathbb{Z}/2\mathbb{Z}, & \Delta = \mathrm{E}_6. \end{array}$$

In fact $\mathrm{Aut}_D(\Delta) = \mathrm{Aut}(\Delta)$ *except for the cases* $\Delta = \mathrm{A}_{2n}$, $n \geq 1$.

Proof. The condition that the automorphisms have to preserve the arrows of the directed graph already implies that $\mathrm{Aut}(\Delta) = 1$ if Δ is not simply-laced.
It is further clear that $\mathrm{Aut}(\Delta) = 1$ for $\Delta = \mathrm{E}_7, \mathrm{E}_8$, $\mathrm{Aut}(\Delta) = \mathbb{Z}/\mathbb{Z}_2$ for $\Delta = \mathrm{A}_{\geq 2}, \mathrm{D}_{\geq 5}, \mathrm{E}_6$ and $\mathrm{Aut}(\Delta) = S_3$ for $\Delta = \mathrm{D}_4$. In case $\Delta = \mathrm{A}_{2n}$, $n \geq 1$, we label the nodes by $\alpha_1, \ldots, \alpha_{2n}$ and let $\mathbf{a} \in \mathrm{Aut}_D(\Delta)$ be the nontrivial graph automorphism. Then $\mathbf{a}(\alpha_n) = \alpha_{n+1}$ so that $\mathbf{a} \notin \mathrm{Aut}_D(\Delta)$.
For the remaining cases we immediately see that $\mathrm{Aut}(\Delta) = \mathrm{Aut}_D(\Delta)$. $\square$

Because of this lemma, we will drop the subindex D from the notation in the cases of interest, namely $\Delta = \mathrm{A}_{2n+1}, \mathrm{D}_k, \mathrm{E}_6$, and just write $\mathrm{Aut}(\Delta)$.

1.2.2 Folding

We now study invariants of the Dynkin diagrams $\Delta = \mathrm{A}_{2n+1}, \mathrm{D}_k, \mathrm{E}_6$ and the respective root systems under the graph automorphisms. More precisely, we fix a non-trivial automorphism $\mathbf{a} \in \mathrm{Aut}_D(\Delta)$ and look at the invariants under the subgroup $\langle \mathbf{a} \rangle \subset \mathrm{Aut}(\Delta)$ that it generates. Observe that there is a unique choice of $\mathbf{a}$ in the cases $\Delta \neq \mathrm{D}_4$ of order 2, three choices for $\mathbf{a}$ of order 2 respectively two choices of (maximal) order 3 in case $\Delta = \mathrm{D}_4$. In the latter cases, the invariants only depend on the order of $\mathbf{a}$, though.
Let $R = R(\Delta)$ be the root system corresponding to $\Delta = \mathrm{A}_{2n+1}, \mathrm{D}_k, \mathrm{E}_6$ in the Euclidean vector

[1] Even though it is not precise, we neglect the cases D_k for $k \in \{1, 2, 3\}$ since they are covered by the A_k.

space $(V, \langle \bullet, \bullet \rangle)$ and $Q = \langle R \rangle_{\mathbb{Z}} \subset V$ the group that R generates (in particular $Q \otimes_{\mathbb{Z}} \mathbb{R} = V$). Clearly, $\mathbf{a}$ preserves Q, so that it makes sense to consider the invariants $Q^{\mathbf{a}}$. We then define

$$R^{\mathbf{a}} := \left\{ \alpha_O := \sum_{\alpha' \in O(\alpha)} \alpha' \;\middle|\; \alpha \in R \right\} \subset Q^{\mathbf{a}},$$

where $O(\alpha)$ denotes the orbit of $\alpha \in R$ under $\mathbf{a}$. The corresponding coroots $(R^{\mathbf{a}})^\vee \subset V$ are then given by

$$(R^{\mathbf{a}})^\vee := \left\{ \alpha_O^\vee := \frac{1}{|O(\alpha)|} \sum_{\alpha' \in O(\alpha)} \alpha'^\vee \;\middle|\; \alpha \in R^\vee \right\} \subset V$$

Note that this is *not* $(R^\vee)^{\mathbf{a}}$ because we divide by the length of the orbits, cf. Remark 1.16.

Proposition 1.14. *Let Δ be an irreducible Dynkin diagram of type* $\mathrm{A}_{2n+1}, \mathrm{D}_k$ *or* E_6 *and* $\mathbf{a} \in \mathrm{Aut}(\Delta)$ *a non-trivial cyclic graph automorphism. Then $V^{\mathbf{a}} := Q^{\mathbf{a}} \otimes_{\mathbb{Z}} \mathbb{R}$ carries a natural inner product induced from $V = Q \otimes_{\mathbb{Z}} \mathbb{R}$. Moreover, $R^{\mathbf{a}}$ and $R^{\vee,\mathbf{a}}$ are both root systems in $V^{\mathbf{a}}$. The types of the* folded root systems *are given by*

$\mathrm{ord}(\mathbf{a})$	R	$R^{\mathbf{a}}$	$(R^{\mathbf{a}})^\vee$
2	A_{2k+1}	B_k	C_k
3	D_4	G_2	G_2
2	D_4	C_3	B_3
2	$\mathrm{D}_{k+1}, k \geq 4$	C_k	B_k
2	E_6	F_4	F_4

The corresponding Dynkin diagrams will be denoted by $\Delta^{\mathbf{a}} = \Delta(R^{\mathbf{a}})$ and $\Delta_{\mathbf{a}} = \Delta(R^{\vee,\mathbf{a}})$. The latter notation indicates that $R^{\vee,\mathbf{a}}$ is connected to taking coinvariants. This can be made precise in the context of root data, cf. Remark 1.16.

Proof. We first consider $R^{\mathbf{a}} \subset V^{\mathbf{a}}$. It is clear that $V^{\mathbf{a}}$ inherits a Euclidean product from V and that $R^{\mathbf{a}}$ spans $V^{\mathbf{a}}$ by definition. By definition of Dynkin graph automorphisms, we can compute

$$\langle \alpha_O, \alpha_O \rangle := \sum_{\alpha' \in O(\alpha)} \langle \alpha', \alpha' \rangle \neq 0,$$

hence $0 \notin R^{\mathbf{a}}$. By the above definitions we further obtain

$$\langle \alpha_O, \alpha_O^\vee \rangle = 2.$$

It remains to show that the reflections $s_{\alpha_O} : V^{\mathbf{a}} \to V^{\mathbf{a}}$ are well-defined and preserve $R^{\mathbf{a}}$. The well-definedness is immediate. The second claim follows from the equality

$$s_{\alpha_O} = \prod_{\alpha' \in O(\alpha)} s_{\alpha'} \quad \forall \alpha \in R. \tag{1.11}$$

This does imply $s_{\alpha_O}(R^{\mathbf{a}}) = R^{\mathbf{a}}$: Using $\mathbf{a} s_\alpha \mathbf{a}^{-1} = s_{\mathbf{a}(\alpha)}$ we see from (1.11) that

$$s_{\alpha_O}(\beta_O) = \left(\left(\prod_{\alpha' \in O(\alpha)} s_{\alpha'} \right) (\beta) \right)_O \in R^{\mathbf{a}}, \quad \forall \beta \in R.$$

This makes sense because $\mathbf{a}$ acts cyclically. To prove the remaining formula (1.11) we first observe that $\langle \beta_O, \alpha \rangle = \langle \beta_O, \alpha' \rangle$ for every $\alpha' \in O(\alpha)$. The construction of $\alpha_O^\vee$ hence implies that $\langle \beta_O, \alpha_O^\vee \rangle = \langle \beta_O, \alpha'^\vee \rangle$ for each $\alpha' \in O(\alpha)$. Together with the orthogonality of roots in the same orbit, this yields (1.11):

$$\begin{aligned}\prod_{\alpha' \in O(\alpha)} s_{\alpha'}(\beta_O) &= \beta_O - \sum_{\alpha' \in O(\alpha)} \langle \beta_O, \alpha'^\vee \rangle \alpha' \\ &= \beta_O - \langle \beta_O, \alpha_O^\vee \rangle \sum_{\alpha' \in O(\alpha)} \alpha' \\ &= s_{\alpha_O}(\beta_O).\end{aligned}$$

With this at hand, it is now straightforward (since simple roots of R ($R^\vee$) give simple roots in $R^{\mathbf{a}}$ ($R^{\vee,\mathbf{a}}$)) to compute the different types as claimed in the above table. □

Example 1.15. Let us give the two examples related to $\Delta = \mathrm{D}_4$. To fix notation, we realize it in $\mathbb{R}^4$ via the simple roots

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4.$$

In the first case, we take an automorphism $\mathbf{a}$ of order 2, say with $\mathbf{a}(\alpha_3) = \alpha_4$ and $\mathbf{a}(\alpha_j) = \alpha_j$ otherwise. Then simple roots of $R^{\mathbf{a}}$ are given by

$$\alpha_{1,O} = \alpha_1, \quad \alpha_{2,O} = \alpha_2, \quad \alpha_{3,O} = \alpha_3 + \alpha_4.$$

In particular, $\alpha_{1,O}$ and $\alpha_{2,O}$ are short roots, whereas $\alpha_{3,O}$ is a long root. It follows that $\Delta^{\mathbf{a}} = \mathrm{C}_3$. Since we divide by the length of the orbit to define the coroots $\alpha_{i,O}^\vee$, we see that $\Delta_{\mathbf{a}} = \mathrm{B}_3$.
Now choose an automorphism $\mathbf{a}$ of order 3, for example

$$\mathbf{a}(\alpha_1) = \alpha_3, \quad \mathbf{a}(\alpha_3) = \alpha_4, \quad \mathbf{a}(\alpha_4) = \alpha_1, \quad \mathbf{a}(\alpha_2) = \alpha_2.$$

Then $\alpha_{1,O} = \alpha_1 + \alpha_3 + \alpha_4$ is a long and $\alpha_{2,O} = \alpha_2$ a short simple root of $R^{\mathbf{a}}$. Hence $\Delta^{\mathbf{a}} = \mathrm{G}_2$ and dually $\Delta_{\mathbf{a}} = \mathrm{G}_2$.
As mentioned earlier, we see that $R^{\mathbf{a}}$ and $R^{\vee,\mathbf{a}}$ only depend on the order of $\mathbf{a}$. Moreover, the invariants under an automorphism of order 3 gives the invariants under the full automorphism group.

This proposition in particular gives a description of the Weyl group $W^{\mathbf{a}}$ of the folded root system $R^{\mathbf{a}}$ as a subgroup of the Weyl group W of the unfolded root system R,

$$W^{\mathbf{a}} = \{w \in W \mid w\mathbf{a} = \mathbf{a}w\}. \tag{1.12}$$

Let Δ be a non-simply-laced (resp. BCFG-)Dynkin diagram. Proposition 1.14 implies that there is a unique simply-laced or homogeneous Dynkin diagram Δ_h (again using the convention $\mathrm{A}_k = \mathrm{D}_k$, $k \in \{1, 2, 3\}$) that folds to Δ, pictorially

$$\Delta_h \to \Delta = \Delta_h^{\mathbf{a}}.$$

However, this correspondence is not one-to-one because $\Delta_h = \mathrm{D}_4$ folds to both C_3 and G_2. Since two different graph automorphisms of the same order yield the same folded root system and Dynkin diagram, it is sufficient to additionally remember the order of the graph automorphism.

This can also be encoded by defining for each irreducible Dynkin diagram Δ of type BCFG the *associated symmetry group* (cf. [Slo80b], 6.2.)

$$AS(\Delta) := \begin{cases} S_3, & \Delta = \mathrm{G}_2, \\ \mathbb{Z}/2\mathbb{Z}, & \Delta \neq \mathrm{G}_2. \end{cases} \tag{1.13}$$

Then each choice of a non-trivial graph automorphism $\mathbf{a} \in \mathrm{Aut}(\Delta_h)$ which has maximal order in $AS(\Delta)$ gives the same folding result. Hence we obtain a bijection

$$\begin{aligned} \{\Delta \ \text{ of type BCFG}\} &\to \{(\Delta_h, \mathbf{C}) \mid \Delta_h \text{ ADE},\ 1 \neq \mathbf{C} \subset \mathrm{Aut}(\Delta_h)\} \\ \Delta &\mapsto (\Delta_h, AS(\Delta)) \\ \Delta = \Delta_h^{\mathbf{C}} = \Delta_h^{\mathbf{a}} &\leftarrow\!\shortmid (\Delta_h, \mathbf{C}), \end{aligned}$$

where $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ is a (non-trivial) subgroup and $\mathbf{a} \in A$ a non-trivial element of maximal order.

In case $\Delta = \Delta_h$ we set $AS(\Delta) = AS(\Delta_h) = 1$. This might seem counter-intuitive, but it will turn out to be a useful convention. For convenience, we list the non-trivial cases in the following table:

Δ	Δ_h	$AS(\Delta)$
$\mathrm{B}_{\mathrm{k}+1}$	A_{2k+1}	$\mathbb{Z}/2\mathbb{Z}$
C_k	D_{k+1}	$\mathbb{Z}/2\mathbb{Z}$
F_4	E_6	$\mathbb{Z}/2\mathbb{Z}$
G_2	D_4	S_3

(1.14)

We refer back to the introduction, Figure 1, where the case $\Delta = \mathrm{B}_3$, $\Delta_h = \mathrm{A}_5$, is illustrated schematically.

Remark 1.16. Let us relate the above to the approach of [Spr09] which discusses folding of root data $(X, R, X^\vee, R^\vee)$ (in the 'adjoint case', $X = \langle R\rangle_{\mathbb{Z}}$). Recall that by definition there is a non-degenerate pairing $\langle \bullet, \bullet\rangle : X \otimes X^\vee \to \mathbb{Z}$. This pairing realizes R and $R^\vee$ as duals of each other. Springer in [Spr09] uses the following convention for folding root data

$$(X, R, X^\vee, R^\vee) \to (X_{\mathbf{a}}, R_{\mathbf{a}}, X^{\vee,\mathbf{a}}, R^{\vee,\mathbf{a}}).$$

Here $\mathbf{a}$ is a graph automorphism as before and $X_{\mathbf{a}} = X/(1-\mathbf{a})X$ are the *coinvariants*. Note that we do take invariants in $R^\vee$. The pairing descends to a non-degenerate pairing between $X_{\mathbf{a}}$ and $X^{\vee,\mathbf{a}}$. In that sense, taking invariants and coinvariants are dual to each other.
We have actually seen the same thing in our approach above: By using the inner product on $V = X \otimes_{\mathbb{Z}} \mathbb{R}$ (recall that $X = \langle R\rangle_{\mathbb{Z}}$), we have realized both the root system $R \subset X$ and its coroot system $R^\vee$ in V. Observe that we had to divide by the order of $\mathbf{a}$ to define the coroots $(R^{\mathbf{a}})^\vee$. Hence the lengths are interchanged so that

$$R_{\mathbf{a}} \cong (R^{\mathbf{a}})^\vee.$$

In other words, we end up with the same result as for root data under the identification of V with its dual $V^\vee$ via the inner product.

We now come to the $\mathrm{Aut}(\Delta)$-action on the corresponding Lie algebras and Lie groups respectively (following [Spr09]). Recall that every irreducible Dynkin diagram Δ gives rise to a simple Lie algebra $\mathfrak{g} = \mathfrak{g}(\Delta)$ which is unique up to isomorphism. One explicit representative $\mathfrak{g}(\Delta)$ can be constructed by choosing as generators a Cartan-Weyl basis $\{h_\alpha, e_\beta \mid \alpha \in \Delta, \beta \in R\}$ with relations completely determined by the Dynkin diagram, see [Hum78]. It then follows immediately that every graph automorphism $\mathbf{a} \in \mathrm{Aut}(\Delta)$ induces a unique automorphism $\varphi_{\mathbf{a}} \in \mathrm{Aut}(\mathfrak{g}(\Delta))$ satisfying

$$\varphi_{\mathbf{a}}(h_\alpha) = h_{\mathbf{a}(\alpha)}, \quad \varphi_{\mathbf{a}}(e_\beta) = e_{\mathbf{a}(\beta)}, \quad \alpha \in \Delta,\ \beta \in R.$$

Here we have extended $\mathbf{a}$ to an automorphism of the root system R. The assignment $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ is actually a group homomorphism by construction.
More invariantly, let $\mathfrak{g}$ be any Lie algebra with Dynkin diagram Δ and G_{ad} its adjoint group. Then there is a natural exact sequence (e.g. [Slo80b], p. 139)

$$1 \longrightarrow G_{ad} \longrightarrow \mathrm{Aut}(\mathfrak{g}) \xrightarrow{\Pi} \mathrm{Aut}(\Delta) \longrightarrow 1. \tag{1.15}$$

This sequence is in fact split: We have just seen that the choice of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and simple roots α_i (equivalently a Borel subalgebra $\mathfrak{b}$ with $\mathfrak{t} \subset \mathfrak{b}$) gives rise to a group homomorphism $\Phi : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\mathfrak{g})$. It follows from the construction that $\Pi \circ \Phi = 1$. Since the Weyl group permutes bases of the corresponding root system, we see that there are $|W|$ many choices of splittings.
All this immediately lifts to the level of Lie groups[2] G which are simply-connected or of adjoint type with Dynkin diagram Δ: Each Lie algebra automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{g}(\Delta)$, lifts to a unique Lie group automorphism $\phi \in \mathrm{Aut}(G_{sc})$ such that $d\phi = \varphi$ where G_{sc} is the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Since ϕ preserves the center $Z(G_{sc}) \subset G_{sc}$ it follows that ϕ descends to an automorphism of the Lie group $G_{ad} = G_{sc}/Z(G_{sc})$ of adjoint type.

Corollary 1.17 (cf. [Spr09], Chapter 10)**.** *Let Δ be an irreducible Dynkin diagram of type* A_{2k+1}, D_k *or* E_6 *and G the associated complex Lie group which is simply-connected or of adjoint type. Denote by $(X, R, X^\vee, R^\vee)$ its root datum and let* $\mathbf{a} \in \mathrm{Aut}(\Delta)$ *be a non-trivial graph automorphism which is realized by some $\phi = \phi_{\mathbf{a}} \in \mathrm{Aut}(G)$. Then $(G^\phi)^0$, i.e. the connected component of G^ϕ, has root datum $(X_{\mathbf{a}}, R_{\mathbf{a}}, X^{\vee,\mathbf{a}}, R^{\vee,\mathbf{a}})$. In particular, the Lie algebra $\mathfrak{g}^\varphi$ of $(G^\phi)^0$ has type $\Delta_{\mathbf{a}} = \Delta(R^{\vee,\mathbf{a}})$ where $\varphi = d\phi$.*

Note that $(G^\phi)^0$ and $\mathfrak{g}^\varphi$ ($\varphi = \varphi_{\mathbf{a}}$) are of type $\Delta_{\mathbf{a}} = \Delta(R^{\vee,\mathbf{a}})$, not of type $\Delta^{\mathbf{a}}$. The reason for this is that the root system for $\mathfrak{g}^\varphi$ is defined via the *dual* of $\mathfrak{t}^\varphi$. Here $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra that is left invariant by φ (which always exists). But the invariants (in $\mathfrak{g}$ and $\mathfrak{t}$) correspond dually to coinvariants. This is the reason why in Lie theory, one often encounters the dual of our convention (1.14) to fold Dynkin diagrams. However, ours is more natural for singularities, which we now discuss.

1.3 Singularities of type Δ

After this interlude on Lie theory, we are now ready to give the definition of singularities of type Δ which also includes BCFG-Dynkin diagrams.

Definition 1.18 ([Slo80b])**.** Let Δ be an irreducible Dynkin diagram and Δ_h its homogeneous or simply-laced Dynkin diagram. A *singularity of type* Δ, is a tuple (Y, H) where

- $Y = (Y, 0)$ is the germ of a surface singularity of type Δ_h,
- $H \subset \mathrm{Aut}(Y)$ is a subgroup isomorphic to $AS(\Delta)$

such that the lifted action of H to the minimal resolution $\hat{Y} \to Y$ induces the natural $AS(\Delta)$-action on the dual Δ_h of its resolution graph. If Δ is of type BCFG, then (Y, H) is also called BCFG-singularity.

[2] In [Spr09] everything is done algebraically but it immediately translates to the Lie case.

We emphasize that this definition includes ADE-singularities by our convention $AS(\Delta) = 1$ if $\Delta = \Delta_h$ is of type ADE. If (Y, H) is a BCFG-singularity, then the lifted action by $H \cong AS(\Delta) \neq 1$ clearly induces actions on $H_2(\hat{Y}, \mathbb{Z})$ and $H^2(\hat{Y}, \mathbb{Z})$. Together with (1.10) we have that

$$H_2(\hat{Y}, \mathbb{Z})^{AS} = Q^{\mathbf{a}} = \mathbf{\Lambda}(G_{sc})^{\mathbf{a}} = \mathbf{\Lambda}(G_{sc}^{\mathbf{a}}), \tag{1.16}$$

$$H^2(\hat{Y}, \mathbb{Z})^{AS} = P^{\mathbf{a}} = \mathbf{\Lambda}(G_{ad})^{\mathbf{a}} = \mathbf{\Lambda}(G_{ad}^{\mathbf{a}}). \tag{1.17}$$

Here $\mathbf{a} \in AS = AS(\Delta)$ is a non-trivial automorphism of maximal order.
We now give concrete representatives of BCFG-singularities. Let $\Gamma \subset SL(2, \mathbb{C})$ correspond to Δ_h with $\mathrm{Aut}_D(\Delta_h) \neq 1$. By Lemma 1.4 there is another finite subgroup $\Gamma' \subset SL(2, \mathbb{C})$ such that $\Gamma \subset \Gamma'$ is a normal subgroup with quotient isomorphic to $AS(\Delta)$. Clearly, Γ'/Γ acts on $\mathbb{C}^2/\Gamma$.

Proposition 1.19 ([Slo80b]). *Let Δ be an irreducible Dynkin diagram of type* BCFG *and Δ_h the associated simply-laced Dynkin diagram. Then $(\mathbb{C}^2/\Gamma, \Gamma'/\Gamma)$ is a singularity of type Δ. Moreover, the natural $\mathbb{C}^*$- and $AS(\Delta)$-action commute.*

Example 1.20. Let us consider the case $\Delta = \mathrm{B}_{k+1}$, $\Delta_h = \mathrm{A}_{2k+1}$: It is straightforward to see that $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is a non-trivial element in the quotient $\mathbb{D}_k/\mathbb{Z}_{2k} \cong AS(\Delta) = \mathbb{Z}/2\mathbb{Z}$. Taking the generators $x = uv$, $y = u^{k+1}$, $z = v^{k+1}$ as in Lemma 1.5, it is immediate that

$$g \cdot (x, y, z) = (-x, z, y).$$

Comparing with the list (1.6) we see that both actions commute. Moreover, its (lifted) action on the iterated blowups reflects the chain of exceptional curves in its middle points, i.e. it gives the non-trivial graph automorphism on Δ_h.

Remark 1.21. The above proposition in particular implies that each BCFG-singularity carries a natural $\mathbb{C}^* \times AS(\Delta)$-action. In general, this does not work with the $C(\Gamma)$- and C_Γ-action. For example, let Δ be of type B_k so that $\Gamma \subset SL(2, \mathbb{C})$ is of type A_{2k+1}. Hence $C(\Gamma) \cong \mathbb{C}^* \times \mathbb{C}^*$, $AS(\Delta) \cong \mathbb{Z}_2$. For $c = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in C(\Gamma)$ and $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in AS(\Delta)$ we compute

$$g \cdot c \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu y \\ -\lambda x \end{pmatrix} \neq \begin{pmatrix} \lambda y \\ -\mu x \end{pmatrix} = c \cdot g \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Even under the action of Γ, both sides are not equivalent. Clearly, if we restrict to the diagonal matrices $\mathbb{C}^* \subset C(\Gamma)$ then both sides coincide under the Γ-action because $-1 \in \mu_{2m}$, $m \in \mathbb{Z}_+$. This also follows directly from the previous result because the $\mathbb{C}^*$-action is induced from the $C(\Gamma)$-action, cf. Remark 1.8.

There is another point of view on the $\mathrm{Aut}(\Delta_h)$- and $AS(\Delta)$-action. Similarly to the groups $C(\Gamma)$ and C_Γ, we define

$$N(\Gamma) := N_{GL(2,\mathbb{C})}(\Gamma), \quad N_\Gamma = N(\Gamma)/\Gamma,$$

where $N_{GL(2,\mathbb{C})}(\Gamma)$ denotes the normalizer of Γ in $GL(2, \mathbb{C})$. Again, $N(\Gamma)$ and N_Γ naturally act on $Y = \mathbb{C}^2/\Gamma$ and therefore also on its minimal resolution $\hat{Y} \to Y$.

Lemma 1.22 ([Sze04]). *Let $\Gamma \subset SL(2, \mathbb{C})$ be a finite subgroup and Δ_h the corresponding Dynkin diagram. Then the inclusion $C_\Gamma \hookrightarrow N_\Gamma$ fits into an exact sequence:*

$$1 \longrightarrow C_\Gamma \longrightarrow N_\Gamma \xrightarrow{\;p\;} \mathrm{Aut}(\Delta_h) \longrightarrow 1. \tag{1.18}$$

Furthermore, p factors as $p = a \circ i$ where $i : N_\Gamma \to \mathrm{Aut}(\hat{Y})$ is the inclusion and $a : \mathrm{Aut}(\hat{Y}) \to \mathrm{Aut}(\Delta_h)$ is the natural map.

This lemma in particular says that $C(\Gamma)$ and C_Γ do not permute the exceptional divisors of $\hat{Y} \to Y$. Moreover, if $\mathrm{Aut}(\Delta_h) = 1$, then $C_\Gamma = N_\Gamma$.
If (Y, H) is a BCFG-singularity, one might be tempted to think that the quotient Y/H yields a different type of singularity other than ADE. However, the next proposition shows that this is false.

Proposition 1.23 ([Slo80b])**.** *Let $(Y, H) = ((Y, 0), H)$ be singularity of type Δ. Then $(Y/H, 0)$ is an* ADE*-singularity of the following types:*

Δ	$(Y, 0)$	$(Y/H, 0)$
B_{k+1}	A_{2k+1}	D_{k+3}
C_k	D_{k+1}	D_{2k}
F_4	E_6	E_7
G_2	D_4	E_7

Idea of proof. The main idea is the following: As before let $\hat{Y} \to Y$ be the minimal resolution. Then the quotient Y/H is partially resolved by the quotient $Z := \hat{Y}/H$ (which is itself singular). If $\hat{Z} \to Z$ denotes the minimal resolution of Z, then

$$\hat{Z} \to Z \to Y/H$$

is a minimal resolution of Y/H. By a theorem of Artin (in the algebraic setting, [Art66]) and Brieskorn (in the complex-analytic setting, [Bri66]) a normal surface singularity is an ADE-singularity iff the dual of the resolution graph of its minimal resolution is an ADE-Dynkin diagram. Since the quotient Y/H is again normal, it then remains to determine the resolution graph of $\hat{Z} \to Y/H$, cf. [Slo80b], Chapter 6. □

1.3.1 Comment on equivariant cohomology

We give a different viewpoint on BCFG-singularities which is more ‘stacky’ in nature. It is inspired by the (derived) McKay correspondence ([BKR01]). The point here is that we want to obtain geometric objects, in this case the orbifold stacks $[\hat{Y}/AS(\Delta)]$, whose natural cohomology groups, here singular cohomology for orbifold stacks resp. equivariant cohomology, yield root systems of type BCFG. Proposition 1.22 shows that this cannot be achieved via the quotient varieties Y/H.
Let Δ be a BCFG-Dynkin diagram, (Y, H) a singularity of type Δ and $\hat{Y} \to Y$ its minimal resolution. We denote by $H^i_{AS(\Delta)}(\hat{Y}, k)$ the equivariant cohomology groups, $k = \mathbb{Z}, \mathbb{Q}$, of $\hat{Y}$ with respect to the lifted H-action (recall from Proposition 1.23 that H acts non-freely on $\hat{Y}$). If $[\hat{Y}/AS(\Delta)]$ denotes the orbifold (stack) then

$$H^i([\hat{Y}/AS(\Delta)], k) = H^i_{AS(\Delta)}(\hat{Y}, k),$$

where the left-hand side is singular cohomology for orbifold stacks (cf. [Edi13]).
To compute the right-hand side, let us briefly recall the definition of equivariant cohomology: Let G be a finite/discrete group acting on a topological space X. Then there exists a contractible space EG with a free G-action, which is unique up to homotopy. Its quotient $BG = EG/G$ under the G-action is a classifying space for G. By construction $EG \to BG$ is a G-bundle. Therefore we can twist this bundle by X to obtain the G-bundle

$$X \hookrightarrow X \times_G EG = (X \times EG)/G \to BG.$$

The equivariant cohomology groups $H^i_G(X,k)$ ($k=\mathbb{Z},\mathbb{Q},\mathbb{R}$) of X are now defined by

$$H^i_G(X,k) := H^i(X\times_G EG,k)$$

(singular cohomology). Since $X\times_G EG\to BG$ is a fibration, we can apply the Serre spectral sequence to obtain the spectral sequence

$$H^p(BG,\mathcal{H}^q(X,k))\Rightarrow H^{p+q}_G(X,k).$$

Here $\mathcal{H}^q(X,k)$ is the local system on BG determined by the natural representation of $\pi_1(BG)=\pi_0(G)=G$ (recall that G is discrete) on $H^q(X,k)$. Using the fact that $H^p(BG,\mathcal{H}^q(X,k))$ coincides with group cohomology $H^p(G,H^q(X,k))$ with values in the G-module $H^q(X,k)$, we end up with the spectral sequence

$$E_2^{pq}=H^p(G,H^q(X,k))\Rightarrow H^{p+q}_G(X,k). \tag{1.19}$$

We can apply this spectral sequence to the above situation, namely $X=\hat{Y}$, $G=AS(\Delta)$ (of course in the analytic topology). Since $H^i(\hat{Y},k)=0$ for $i\notin\{0,2\}$, the E_2-page looks as follows ($AS=AS(\Delta)$, $H^0(\hat{Y},k)=k$, etc.)

$$\begin{array}{llllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots\\
H^0(AS,H^2) & H^1(AS,H^2) & H^2(AS,H^2) & H^3(AS,H^2) & H^4(AS,H^2) & \cdots\\
0 & 0 & 0 & 0 & 0 & \cdots\\
H^0(AS,k) & H^1(AS,k) & H^2(AS,k) & H^3(AS,k) & H^4(AS,k) & \cdots
\end{array}$$

This already implies that $d_2=0$ and it remains to check d_3 (because $d_k=0$, $k\geq 4$ anyway). Note that AS acts trivially on $H^0=k$. For both $AS=\mathbb{Z}/2\mathbb{Z}$ and $AS=S_3$, we therefore have (cf. [Wei94])

$$H^0(AS,\mathbb{Z})=\mathbb{Z},\quad H^{odd}(AS,\mathbb{Z})=0,\quad H^{even}(AS,\mathbb{Z})_{\mathrm{tf}}=0,$$

where the subscript tf denotes the torsion-free quotient. Over $\mathbb{Q}$ this reduces to $H^0(AS,\mathbb{Q})=\mathbb{Q}$ and $H^{\geq 1}(AS,\mathbb{Q})=0$. Therefore d_3 has to vanish (it either maps from or to 0), so that the spectral sequence (1.19) degenerates for both coefficient rings. However, for $k=\mathbb{Z}$, $E_2^{2k,0}=H^p(AS,\mathbb{Z})$ is non-zero but torsion for $k\geq 1$. More precisely, the even cohomology groups are given by ($k\geq 1$)

$$H^{2k}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z},\quad H^{2k}(S_3,\mathbb{Z})=\begin{cases}\mathbb{Z}/2\mathbb{Z}, & 2k\equiv 2 \bmod 4,\\ \mathbb{Z}/6\mathbb{Z}, & 2k\equiv 0 \bmod 4.\end{cases} \tag{1.20}$$

Hence $H^i_{AS}(\hat{Y},\mathbb{Z})$ is *not* isomorphic to $H^i(\hat{Y},\mathbb{Z})^{AS}$ in general.
Let us analyse this in the case of interest, $i=2$, in more detail. The spectral sequence implies that $H=H^2_{AS}(\hat{Y},\mathbb{Z})$ has the following filtration and graded pieces:

$$0\subsetneq F^2H=F^1H\subsetneq F^0H=H,$$
$$F^0H/F^1H\cong E_2^{0,2}=H^2(\hat{Y},\mathbb{Z})^{AS},\quad F^2H\cong E_2^{2,0}=H^2(AS,\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}.$$

The former is free so that it follows that $F^2H=H_{tors}$ is precisely the torsion subgroup in H and we obtain a (non-canonical) splitting

$$H=H^2_{AS}(\hat{Y},\mathbb{Z})\cong H^2(\hat{Y},\mathbb{Z})^{AS}\oplus\mathbb{Z}/2\mathbb{Z}.$$

Note that the torsion part does neither depend on the singularity nor the group AS. If $k = \mathbb{Q}$, then $H^i(AS, \mathbb{Q}) = 0$ for all $i \neq 0$ so that we obtain the well-known result

$$H^i([\hat{Y}/AS], k) = H^i_{AS}(\hat{Y}, \mathbb{Q}) = H^i(\hat{Y}, \mathbb{Q})^{AS}. \tag{1.21}$$

Hence we see that

$$H^2([\hat{Y}/AS], \mathbb{Z})_{\mathrm{tf}} \subset H^2([\hat{Y}/AS], \mathbb{Q}), \quad AS = AS(\Delta)$$

yields the root system of type Δ in $H^2([\hat{Y}/AS], \mathbb{R}) = H^2(\hat{Y}, \mathbb{R})^{AS}$. In other words, the second (integer) cohomology groups (modulo torsion) of the orbifold $[\hat{Y}/AS]$ give rise to the corresponding BCFG-Dynkin diagrams.

1.3.2 Semi-universal deformations

As we have seen, each singularity of type Δ carries a natural $\mathbb{C}^* \times AS(\Delta)$-action (where $AS(\Delta) = 1$ if Δ is simply-laced). It is well-known that the underlying singularity $(Y, 0)$ of type Δ_h has a semi-universal deformation. In this section, we briefly sketch how all this generalizes to the equivariant setting, i.e. taking into account the aforementioned group actions (following again [Slo80b]).

Let K be a linearly reductive group that acts regularly on an algebraic variety Y (over $\mathbb{C}$). A K-deformation of Y is a deformation of Y in the category of K-varieties. In general deformation theory of arbitrary varieties can be complicated, but we only need to study K-complete intersections Y. That is $Y = f^{-1}(0)$ for a flat K-equivariant morphism $f : V \to W$ where V, W are finite-dimensional $\mathbb{C}$-vector spaces on which K acts linearly.

Proposition 1.24. *Let K be a linearly reductive group and $Y = f^{-1}(0)$, $f : V \to W$, a K-complete intersection with isolated singularities only. Then a semi-universal K-deformation of Y exists and it is semi-universal for any linearly reductive subgroup $K' \subset K$. In particular, each semi-universal K-deformation is a semi-universal deformation of Y.*

Sketch of proof. It is convenient to rephrase the familiar construction ([Loo84]) of a semi-universal deformation Y in a more invariant way (without choosing bases): Let $J \subset \mathbb{C}[Y] \otimes W$ be the Jacobian ideal of f which is a $\mathbb{C}[Y]$-submodule. By assumption the quotient $(\mathbb{C}[Y] \otimes W)/J$ is finite-dimensional and we obtain a natural morphism

$$\phi : \mathbb{C}[V] \otimes W \longrightarrow \mathbb{C}[Y] \otimes W \longrightarrow (\mathbb{C}[Y] \otimes W)/J. \tag{1.22}$$

It is clear that ϕ has a section $s : (\mathbb{C}[Y] \otimes W)/J \to \mathbb{C}[V] \otimes W$. One way to construct such a section would be to choose representatives $b_1, \ldots, b_n \in \mathbb{C}[V] \otimes W$ of a basis of $(\mathbb{C}[Y] \otimes W)/J$. Then we can define

$$F := f + s \in \mathbb{C}[V] \otimes W + \mathbb{C}[U \times V] \otimes W \subset \mathbb{C}[U \times V] \otimes W$$

which corresponds to a morphism $F : U \times V \to W$. Then it turns out that the composition

$$\mathcal{Y} := F^{-1}(0) \hookrightarrow U \times V \xrightarrow{pr_U} U$$

is a semi-universal deformation of Y. To relate this to the more common construction of a semi-universal deformation, one chooses isomorphisms $U \cong \mathbb{C}^r$, $V \cong \mathbb{C}^s$. Then a choice of representatives $b_1, \ldots, b_n \in \mathbb{C}[V] \otimes W \cong \mathbb{C}[x_1, \ldots, x_r]^s$ gives a section s as above and

$$F : \mathbb{C}^{r+n} \to \mathbb{C}^s, \quad (x, u) \mapsto f(x) + \sum_{i=1}^{n} u_i b_i(x)$$

gives rise to a semi-universal deformation.
To incorporate the K-action, we observe that all the modules appearing above are naturally K-modules (which is not immediate for the Jacobian ideal J, cf. [Slo80b], p. 10). Since K is linearly reductive and $\mathbb{C}[V] \otimes W$ locally finite, it follows that (1.22) has K-equivariant sections. Then the argument goes through and gives a semi-universal K-deformation. Observe that by construction it is also semi-universal for any linearly reductive subgroup $K' \subset K$. $\square$

Corollary 1.25. *Each* ADE*-singularity* $(Y, 0)$ *has a semi-universal* $\mathbb{C}^*$*-deformation.*

We now turn to the deformation theory of BCFG-singularities. The appropriate setting for deforming them is the following: Let Y be a variety and $H \subset \mathrm{Aut}(Y)$ be a subgroup of the automorphisms acting on Y. We wish to deform (Y, H), i.e. deforming Y and preserving the H-action. As above we can enhance this adding the action of another group K. For this to make sense, both group actions have to commute. In other words, Y can be considered as a $K \times H$-variety.

Definition 1.26. Let K be an algebraic group and $H \subset \mathrm{Aut}(Y)$ a subgroup. Further let Y be a $K \times H$-variety where H acts naturally. A K-deformation of (Y, H) is a $K \times H$-deformation $\mathcal{Y} \to (U, 0)$ such that H acts trivially on the base U.

Semi-universality is defined in the obvious way.

Corollary 1.27. *Assume additionally that* $K \times H$ *is linearly reductive and that* Y *is a* $K \times H$*-complete intersection with isolated singularities. Let* $\mathcal{Y} \to (U, 0)$ *be a semi-universal* $K \times H$*-deformation which exists by Proposition 1.24. Then a semi-universal* K*-deformation of* (Y, H) *is obtained via the pullback*

$$\mathcal{Y} \times_U U^H \to U^H$$

of $\mathcal{Y} \to U$ *to the fixed point locus* $U^H \subset U$.

Proof. Let $\mathcal{Y}' \to (U', 0)$ be any K-deformation of (Y, H), in particular it is a $K \times H$-deformation. Hence there is a morphism

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ (U', 0) & \xrightarrow{\phi} & U \end{array}$$

such that ψ is $K \times H$-equivariant and $d_0\psi$ is unique. But H acts trivially on the base by assumption and therefore factors over the pullback $\mathcal{Y} \times_U U^H \to U^H$. $\square$

Of course, this immediately applies to BCFG-singularities:

Corollary 1.28. *Each* BCFG*-singularity* (Y, H) *admits a semi-universal* $\mathbb{C}^*$*-deformation.*

1.4 Approach by Brieskorn-Grothendieck-Slodowy

In the previous section, we introduced singularities of type Δ where Δ is any irreducible Dynkin diagram. We have seen that they carry natural $\mathbb{C}^*$-actions and that they admit semi-universal $\mathbb{C}^*$-deformations. The aim of this section is to outline a construction due to Brieskorn, Grothendieck and Slodowy that realizes a singularity of type Δ in the simple Lie algebra $\mathfrak{g}$ of the same type. Additionally, it gives a construction of the semi-universal $\mathbb{C}^*$-deformation via the adjoint quotient $\mathfrak{g} \to \mathfrak{t}/W$ and a simultaneous resolution of it in terms of Lie theory. This approach is well-suited

for relating Hitchin systems, which is Lie-theoretic in nature, with Calabi-Yau integrable systems and to incorporate graph automorphisms.
Yamada ([Yam95]) has given a different construction using symplectic geometry. It is important for us since we need the $AS(\Delta)$-invariance of the relative symplectic form (cf. introduction). We found that this is most naturally seen in his symplectic-geometric construction.

1.4.1 Adjoint quotient

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and W the corresponding Weyl group. The adjoint algebraic group $G = G_{ad}$ corresponding to $\mathfrak{g}$ naturally acts on $\mathfrak{g}$ but the naive quotient $\mathfrak{g}/G$ is not well-behaved (at least not as a variety). To obtain a variety one has to consider the GIT quotient instead,

$$\mathfrak{g} /\!\!/ G := \mathrm{Spec}(\mathbb{C}[\mathfrak{g}]^G).$$

That this is a variety follows from the next

Theorem 1.29 (Chevalley, [Hum78])**.** *The invariants $\mathbb{C}[\mathfrak{g}]^G$ are finitely generated and the restriction morphism $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{t}]$ induces an isomorphism*

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\cong} \mathbb{C}[\mathfrak{t}]^W.$$

Moreover, $\mathbb{C}[\mathfrak{g}]^G$ can be generated by algebraically independent generators $\chi_1, \dots, \chi_r \in \mathbb{C}[\mathfrak{g}]^G$ of degree (with respect to the natural $\mathbb{C}^*$-action) $d_1, \dots, d_r$ where $r = \mathrm{rk}(\mathfrak{g})$. The degrees are independent of the choice of such generators.

The composition $\mathbb{C}[\mathfrak{t}]^W \cong \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{g}]$ gives rise to the *adjoint quotient*

$$\chi : \mathfrak{g} \to \mathfrak{t}/W.$$

It can be described more explicitly: Let $x = x_s + x_n \in \mathfrak{g}$ be the Jordan decomposition. Since all Cartan subalgebras are G-conjugate to each other, there exists an element $t_s \in \mathfrak{t}$ in the G-orbit of x_s. This element is unique up to the action of W and we have

$$\chi(x) = [t_s]_W \in \mathfrak{t}/W.$$

It is important to point out that $\mathfrak{t}/W$ does *not* carry a canonical structure of a vector space. A priori, it is only a cone with weights $d_1, \dots, d_r$ as in the theorem. This follows by choosing algebraically independent generators $f_1, \dots, f_r \in \mathbb{C}[\mathfrak{t}]^W$ and extending them uniquely to $\hat{\chi}_1, \dots, \hat{\chi}_r \in \mathbb{C}[\mathfrak{g}]^G$. Such a choice also yields an isomorphism to a polynomial algebra and therefore $\mathfrak{t}/W \cong \mathbb{C}^r$ which endows $\mathfrak{t}/W$ with a non-canonical vector space structure. Under this isomorphism the adjoint quotient becomes a morphism

$$\hat{\chi} : \mathfrak{g} \to \mathbb{C}^r, \quad x \mapsto (\hat{\chi}_1(x), \dots, \hat{\chi}_r(x)).$$

Before we describe the fibers of the adjoint quotient in a concrete example, we give a convenient description of the fibers in the general case. To do so, define the *nilpotent variety* or *nilpotent cone*

$$N(\mathfrak{g}) := \{n \in \mathfrak{g} \text{ nilpotent}\} \subset \mathfrak{g}. \tag{1.23}$$

It coincides with $\chi^{-1}(\bar{0})$ but can be given the structure of a variety without alluding to the adjoint quotient.

Lemma 1.30 ([Slo80b]). *For each $q(t) = \bar{t} \in \mathfrak{t}/W$ the morphism,*

$$G \times^{Z(t)} N(\mathfrak{z}_{\mathfrak{g}}(t)) \to \mathfrak{g}, \quad g * n \mapsto g \cdot (t + n)$$

is an isomorphism onto $\chi^{-1}(\bar{t})$ where $Z(t) = Z_G(t)$ is the centralizer *of t in G and $\mathfrak{z}_{\mathfrak{g}}(t) =$* Lie$(Z(t))$.

This description is not too surprising because if $x = x_s + x_n$, $y = y_s + y_n \in \chi^{-1}(q(t))$, then x_s and y_s are G-conjugate to t, say $g \cdot x_s = h \cdot y_s = t$ for some $g, h \in G$. Since x_n, y_n commute with x_s, y_s we must have $g \cdot x_n, h \cdot y_n \in N(\mathfrak{z}_{\mathfrak{g}}(t))$. By dividing out the action by $Z(t)$ the above description follows.

Example 1.31. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{t} \subset$ be the diagonal matrices. Then $G = G_{ad} = PSL(2, \mathbb{C})$ and $r = 1$. Clearly, $\chi_1 = \det \in \mathbb{C}[\mathfrak{g}]^G$ and it is a generator according to Theorem 1.29. Its degree is $d_1 = 2$. Then the adjoint quotient is

$$\hat{\chi} : \mathfrak{sl}(2, \mathbb{C}) \to \mathbb{C}, \quad A \mapsto \det(A).$$

Observe that $\det(A)$ is the non-trivial coefficient of the characteristic polynomial $\det(\lambda - A)$. This generalizes to $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$: Consider the G-invariant functions

$$\hat{\chi}_k(A) = \mathrm{tr}(\wedge^k A), \quad i = 2, \ldots, n.$$

They are algebraically independent and since $\mathrm{rk}(\mathfrak{g}) = n - 1$ they generate $\mathbb{C}[\mathfrak{g}]^G$ by Theorem 1.29. Their degrees are $d_k = k + 1$ ($k = 1, \ldots, n - 1$) and their restrictions $\hat{\chi}_{k|\mathfrak{t}} \in \mathbb{C}[\mathfrak{t}]^W$ are the elementary symmetric polynomials e_k of the same degrees. Observe that the $\hat{\chi}_k$ are (up to signs) the coefficients of the characteristic polynomial as before. However, it is more natural to consider them as traces of the irreducible representations of $\mathfrak{sl}(n, \mathbb{C})$ on $\wedge^k \mathbb{C}^n$. The reason for this is that this generalizes to every simple Lie algebra, cf. [Hum78], 23.1.
In this example, the fibers of $\chi : \mathfrak{g} \to \mathfrak{t}/W$ can be understood in an elementary way. Let $\bar{t} = [t_1, \ldots, t_n] \in \mathfrak{t}/W = \mathfrak{t}/S_n$ where we consider $\mathfrak{t} = \{(t_1, \ldots, t_n) \mid \sum_i t_i = 0\} \subset \mathbb{C}^n$. Since

$$\prod_{i=1}^{n} (\lambda - t_i) = \sum_{i=1}^{n} (-1)^i e_i([t_1, \ldots, t_n]) \lambda^{n-i},$$

we see that $\mathfrak{g}_{\bar{t}} := \chi_{red}^{-1}(\bar{t})$ (fibers with reduced structure) consists precisely of the matrices whose eigenvalues are $t_1, \ldots, t_n$. In particular, if $A = A_s + A_n$ is the Jordan decomposition of $A \in \mathfrak{g}_{\bar{t}}$, then A_s lies in the G-orbit of $\mathrm{diag}(t_1, \ldots, t_n)$ (and hence of any representative of $\bar{t}$).
Consequently, we can explicitly describe the orbits that are contained in $\mathfrak{g}_{\bar{t}}$. Let $A \in \mathfrak{sl}(n, \mathbb{C})$ be in Jordan normal form such that $A_s \sim (t_1, \ldots, t_n)$. Assume that there are k pairwise distinct t_i's, denoted by $t'_1, \ldots, t'_k$. Let m_j be the number of t_i with $t_i = t'_j$ so that $\sum_{j=1}^k m_j = n$. Finally, denote by $n_l(t'_j)$ the number of Jordan blocks of size $l \geq 1$ with diagonal entry t'_j. It follows that the orbits in $\mathfrak{g}_{\bar{t}}$ are in bijection with

$$\{(n_l(t'_j))_{j,l} \mid \sum_{\substack{j=1,\ldots,k \\ l=1,\ldots,n}} n_l(t'_j)\, l = n\}.$$

In particular, $\mathfrak{g}_{\bar{t}}$ only contains finitely many orbits and precisely one semisimple orbit, namely the orbit of $\mathrm{diag}(t_1, \ldots, t_n)$.

(Sub)regular elements

Regular and subregular elements of a semisimple Lie algebra $\mathfrak{g}$ (of rank r) are crucial for the construction by Brieskorn and Slodowy. The centralizer $Z_G(x)$ of an element $x \in \mathfrak{g}$ is its isotropy group for the adjoint action,

$$Z_G(x) = \{g \in G \mid g \cdot x = Ad_g(x) = x\}.$$

We are interested in its dimension, so that it is also convenient to consider its infinitesimal version $\mathfrak{z}_\mathfrak{g}(x) = \mathrm{Lie}(Z_G(x)) = \{y \in \mathfrak{g} \mid [x, y] = 0\}$.

Proposition 1.32 ([Hum95]). *If $x \in \mathfrak{g}$ is an element of the semisimple Lie algebra $\mathfrak{g}$, then*

$$r = \mathrm{rk}(\mathfrak{g}) \leq \dim(Z_G(x)) = \dim(\mathfrak{z}_\mathfrak{g}(x)) \leq \dim(\mathfrak{g}).$$

Moreover, the dimension of its G-orbit $O(x) \subset \mathfrak{g}$ is even.

The fact that $\dim O(x) = \dim \mathfrak{g} - r$ is even can be seen by a root space decomposition of $\mathfrak{g}$.

Definition 1.33. An element $x \in \mathfrak{g}$ is called *regular* if

$$\dim Z_G(x) = r = \mathrm{rk}(\mathfrak{g}).$$

Equivalently, x is regular iff its G-orbit has maximal dimension $\dim \mathfrak{g} - r \in 2\mathbb{Z}$. It is called *subregular* if

$$\dim Z_G(x) = r + 2.$$

By Proposition 1.32 it follows that a subregular element has the second lowest centralizer dimension. Clearly, the properties regular and subregular are invariant under the adjoint action so that we can speak about (sub)regular orbits.

Example 1.34. We continue with example 1.31, $g = \mathfrak{sl}_n(\mathbb{C})$. Let $\bar{t} = [t_1, \ldots, t_n] \in \mathfrak{t}/W = \mathbb{C}^n/S_n$ and assume that there are k pairwise distinct t_i's denoted by $t'_1, \ldots, t'_k$ with multiplicities m_j. Then there is precisely one orbit in $\mathfrak{g}_{\bar{t}}$ which is regular. It is determined by $n_{m_j}(t_j) = 1$ so that all other $n_l(t_j) = 0$. In other words, for each t'_j there is precisely one Jordan block.
There are many subregular orbits in $\mathfrak{g}_{\bar{t}}$. Let an orbit O in $\mathfrak{g}_{\bar{t}}$ be determined by $(n_l(t'_j))_{j,l}$. Then O is subregular iff there is precisely one $s \in \{1, \ldots, k\}$ such that $m_s \geq 2$ with

$$n_1(t'_s) = 1, \quad n_{m_s - 1}(t'_s) = 1$$

and $n_{m_j}(t'_j) = 1$ for all $j \neq s$. In particular, a nilpotent orbit $O \subset \mathfrak{g}_{\bar{0}}$ (so that $k = 1$, $m_1 = n$) is subregular iff $n_1(0) = 1$, $n_{m_1 - 1} = n - 1$.

A very useful characterization of regular elements is the following due to Kostant ([Hum95]).

Theorem 1.35. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and $\chi : \mathfrak{g} \to \mathfrak{t}/W$ the adjoint quotient. The rank of the differential $d\chi_x : T_x\mathfrak{g} \to T_{\chi(x)}(\mathfrak{t}/W)$ at $x \in \mathfrak{g}$ is maximal (i.e. it has rank r) iff it is regular.*

The next result describes the structure of the fibers of the adjoint quotient. We have already seen some of these results in the above example $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

Theorem 1.36 ([Slo80b], Chapter 3). *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and $\chi : \mathfrak{g} \to \mathfrak{t}/W$ its adjoint quotient for $G = G_{ad}$. The morphism χ is flat and its (reduced) fibers $\mathfrak{g}_{\bar{t}} = \chi^{-1}_{red}(\bar{t})$ satisfy:*

i) $\mathfrak{g}_{\bar{t}}$ is irreducible of codimension r in $\mathfrak{g}$ and only contains finitely many G-orbits.

ii) The G-orbit of a (hence any) representative of $\bar{t}$ is the only semisimple G-orbit in $\mathfrak{g}_{\bar{t}}$. It is the only closed orbit in $\mathfrak{g}_{\bar{t}}$ and lies in the closure of any other orbit.

iii) There is precisely one regular orbit in $\mathfrak{g}_{\bar{t}}$. It is precisely the non-singular locus of $\mathfrak{g}_{\bar{t}}$.

1.4.2 Grothendieck's simultaneous resolution

In this section, we briefly review Grothendieck's simultaneous resolution of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$. It is important for us because it induces a simultaneous resolution of the semi-universal deformation of the ADE-singularity of the type of $\mathfrak{g}$. To set the stage we recall:

Definition 1.37. Let $\chi : X \to S$ be a morphism of algebraic varieties. A simultaneous resolution of χ is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$

such that

i) θ is smooth,

ii) p is finite and surjective,

iii) ψ is proper,

iv) for each $t \in T$ the fibers Y_t are a resolution of the reduced fibers $X_{\psi(t)} = f^{-1}(\psi(t))_{red}$, i.e. $\psi_t : Y_t \to X_{\psi(t)}$ is a resolution.

Remark 1.38. This notion can be a bit misleading because ψ is in general only a finite map over the regular locus of f. Therefore it would be more appropriate to call ψ a *simultaneous alteration* but we will stick to Slodowy's notion. Note that one obtains an 'honest' simultaneous resolution after base change along p.

To outline Grothendieck's simultaneous resolution, we need a bit more Lie theory. The set $\mathcal{B} = \{\mathfrak{b} \mid \mathfrak{b} \subset \mathfrak{g} \text{ Borel subalgebra}\}$ carries a natural structure of an algebraic variety. In fact, it is a homogeneous space: G naturally acts on $\mathcal{B}$ by conjugation. If $B \subset G = G_{ad}$ is a Borel subgroup and $\mathfrak{b} \subset \mathfrak{g}$ the corresponding Lie algebra then

$$G/B \to \mathcal{B}, \quad gB \mapsto g \cdot \mathfrak{b}$$

is an isomorphism of varieties. Now define

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}') \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}'\} \tag{1.24}$$

$$\cong \{(x, gB) \in \mathfrak{g} \times G/B \mid x \in g \cdot \mathfrak{b}\}. \tag{1.25}$$

In order to relate this to the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$, we fix a Borel subalgebra $\mathfrak{b}$ such that $\mathfrak{t} \subset \mathfrak{b}$. Besides the natural projection $\psi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ we also have the natural morphism $\theta : \tilde{\mathfrak{g}} \to \mathfrak{t}$, given by

$$\theta(x, \mathfrak{b}') = x \bmod [\mathfrak{b}', \mathfrak{b}'], \tag{1.26}$$

where we use that there are canonical isomorphisms $\mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'] \cong \mathfrak{t}$ ([CG10]). In terms of (1.25) the morphism θ can be written more concretely as

$$\theta(x, gB) = (g^{-1} \cdot x)_s = \mathrm{Ad}(g^{-1})(x_s).$$

Here we have used that $\mathfrak{b}$ uniquely decomposes into $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent subalgebra. Another description of these spaces and maps is particularly useful, when relating $\tilde{\mathfrak{g}}$ to the fibers of χ, cf. Lemma 1.30. Let $T \subset B \subset G = G_{ad}$ be a maximal torus contained in a Borel subgroup.

As usual, we denote $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{b} = \mathrm{Lie}(B)$. The associated fiber bundle[3] $G \times^B \mathfrak{b}$ is isomorphic to $\tilde{\mathfrak{g}}$ via

$$g * b \mapsto (g \cdot b, g \cdot \mathfrak{b}).$$

Under this isomorphism, the morphisms ψ and θ from Theorem 1.39 are given by

$$\psi(g * b) = \mathrm{Ad}(g)(b), \quad \theta(g * b) = b_s,$$

where $b_s \in \mathfrak{t}$ is the semisimple part of $b \in \mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. This is well-defined, because B acts trivially (by conjugation) on $\mathfrak{t}$ (see 4.3. [Slo80b]).

Theorem 1.39 (Grothendieck, [Slo80b]). *The commutative diagram*

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\psi} & \mathfrak{g} \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\chi} \\ \mathfrak{t} & \xrightarrow{q} & \mathfrak{t}/W \end{array} \tag{1.27}$$

is a simultaneous resolution of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$ of a semisimple Lie algebra.

Idea of proof. We confine ourselves to indicate why the above morphism gives a simultaneous resolution of the adjoint quotient. For this one needs Springer's resolution of the nilpotent cone $N(\mathfrak{g}) = \chi^{-1}(\bar{0})$: Decompose $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ as before. Then the morphism

$$G \times^B \mathfrak{n} \to N(\mathfrak{g}), \quad g * n \mapsto g \cdot n \tag{1.28}$$

is a resolution of the singularities of $N(\mathfrak{g})$. This roughly follows from the fact that $G \times^B \mathfrak{n}$ is smooth and that a regular nilpotent element (hence a non-singular point of $N(\mathfrak{g})$) is contained in precisely one Borel subalgebra, see [Slo80b]. For the latter observe that $G \times^B \mathfrak{n}$ is naturally isomorphic to the closed subvariety $\{(x, \mathfrak{b}') \in N(\mathfrak{g}) \times \mathcal{B} \mid x \in \mathfrak{b}'\} \subset N(\mathfrak{g}) \times \mathcal{B}$, cf. (1.25).
Grothendieck gave a relative version of Springer's resolution: It turns out that $\theta_t : \theta^{-1}(t) \to \chi^{-1}(\bar{t})$ is induced by Springer's resolution for $N(\mathfrak{z}_{\mathfrak{g}}(t))$ which makes sense because $\mathfrak{z}_{\mathfrak{g}}(t)$ is itself reductive. This way one concludes that θ_t is a resolution of $\chi^{-1}(\bar{t})$, cf. [Slo80b]. □

Remark 1.40. It can be shown (see [CG10]) that $G \times^B \mathfrak{n} \cong T^*(G/B)$ over G/B. This is a generalization of the minimal resolution of an A_1-singularity Y given by $T^*\mathbb{CP}^1 \to Y$.

The incidence variety $\tilde{\mathfrak{g}}$ has a natural $\mathrm{Aut}(\mathfrak{g})$-action given by

$$\varphi \cdot (x, \mathfrak{b}) = (\varphi(x), \varphi(\mathfrak{b})), \quad \varphi \in \mathrm{Aut}(\mathfrak{g}), \tag{1.29}$$

which is well-defined because $\varphi(\mathfrak{b})$ is again Borel. It was mainly studied by Slodowy in the context of simple singularities, cf. Section 1.4.3. However, we need to consider it directly on $\tilde{\mathfrak{g}}$ and study the equivariance properties of the square (1.27). This will be important for our constructions in the following.
Let us describe this action under $\tilde{\mathfrak{g}} \cong G \times^B \mathfrak{b}$, where B is a fixed Borel subgroup with Borel subalgebra $\mathfrak{b}$. Since G is simple of adjoint type, the morphism $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$, $\phi \mapsto d\phi$, is an isomorphism of groups (cf. Section 1.2). We can therefore define an action of $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(G)$ on $G \times^B \mathfrak{b}$ by

$$\phi \cdot (g * x) := \phi(g) g_0 * (g_0^{-1} \cdot d\phi(x)), \quad \phi \in \mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g}), \tag{1.30}$$

where $g_0 \in G$ is chosen such that $g_0 \cdot \mathfrak{b} = d\phi(\mathfrak{b})$. This choice is irrelevant by the definition of $G \times^B \mathfrak{b}$ and $N_G(B) = B$.

[3] To fix notation (see [Slo80b], Section 3): Let $H \subset G$ be a closed subgroup and F an H-space. Then we denote by $G \times^H F$ the quotient of $G \times F$ under the left action $h \cdot (g, x) = gh^{-1}, h \cdot x)$. It gives a bundle over G/H and we denote the class of (g, x) by $g * x \in G \times^H F$.

Lemma 1.41. *The natural isomorphism $\Psi : \tilde{\mathfrak{g}} \to G \times^B \mathfrak{b}$ is equivariant with respect to the actions (1.29) and (1.30) of* $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(G)$.

Proof. Let $(x, \mathfrak{b}') \in \tilde{\mathfrak{g}}$ and $\phi \in \mathrm{Aut}(G)$ such that $\phi \cdot (x, \mathfrak{b}') = (d\phi(x), d\phi(\mathfrak{b}'))$. If $g \in G$ satisfies $\mathfrak{b}' = g \cdot \mathfrak{b}$, then $\Psi(x, \mathfrak{b}') = g * g^{-1} \cdot x$. Now fix $g_0 \in G$ with $d\phi(\mathfrak{b}) = g_0 \cdot \mathfrak{b}$, then we see that

$$d\phi(\mathfrak{b}') = \phi(g) \cdot d\phi(\mathfrak{b}) = (\phi(g)g_0) \cdot \mathfrak{b}.$$

Hence we can compute

$$\begin{aligned}\Psi(\phi \cdot (x, \mathfrak{b}')) &= \Psi(d\phi(x), \phi(g)g_0 \cdot \mathfrak{b}) \\ &= \phi(g)g_0 * \big((\phi(g)g_0)^{-1} \cdot d\phi(x)\big) \\ &= \phi(g)g_0 * \big(g_0^{-1} \cdot d\phi(g^{-1} \cdot (x))\big) \\ &= \phi \cdot \big(g * (g^{-1} \cdot x)\big) \\ &= \phi \cdot \Psi(x, \mathfrak{b}').\end{aligned}$$

In the third line we have used that $\mathrm{Ad}(\phi(h)) \circ d\phi = d\phi \circ \mathrm{Ad}(h)$ for any $h \in G$. Hence Ψ is $\mathrm{Aut}(\mathfrak{g})$-equivariant. $\square$

Example 1.42. Let us consider two special cases which are the only interesting ones for us (see the next Section 1.4.3): The first is the case when $\phi \in G \hookrightarrow \mathrm{Aut}(\mathfrak{g})$ is inner. Without loss of generality we can then assume that $d\phi = \mathrm{Ad}(g_0)$ respectively $\phi = Int(g_0)$ (conjugation by g_0). Inserting this into (1.30) yields

$$\phi \cdot (g * x) = g_0 g * x.$$

In other words, the action reduces to the natural left G-action on $G \times^B \mathfrak{b}$. Hence, if $\mathfrak{g}$ has no outer automorphisms, then the $\mathrm{Aut}(\mathfrak{g})$-action reduces to this action.
The second special case is when ϕ preserves the Borel subgroup B, i.e. $\phi(B) = B$. Then (without loss of generality) $g_0 = 1$ and

$$\phi \cdot (g * x) = \phi(g) * d\phi(x).$$

For example, this is the case when we split the short exact sequence

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \xrightarrow{\Pi} \mathrm{Aut}(\Delta) \longrightarrow 1$$

e.g. via simple roots corresponding to $\mathfrak{t} \subset \mathfrak{b}$, cf. (1.15, and consider $\phi \in \mathrm{Aut}(\Delta) \subset \mathrm{Aut}(\mathfrak{g})$.

It is clear that $\psi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is $\mathrm{Aut}(\mathfrak{g})$-equivariant. To get a meaningful statement for the other morphisms in (1.27), we further restrict to the second case in the previous example. In particular, we fix a subgroup $A \subset \mathrm{Aut}(\mathfrak{g})$ such that $\Pi_{|A} : A \to \mathrm{Aut}(\Delta)$ is an isomorphism and fixes $T \subset B$. Therefore, such a choice also gives a splitting of the short exact sequence

$$1 \longrightarrow N_G(T) \longrightarrow \mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \longrightarrow \mathrm{Aut}(\Delta) \longrightarrow 1$$

where $\mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \subset \mathrm{Aut}(\mathfrak{g})$ is the subgroup fixing $\mathfrak{t}$ (cf. [Slo80b], 8.8.). Hence A acts on $\mathfrak{t}$ and the previous example implies that θ is A-equivariant. Now A naturally acts on $\mathfrak{t}/W$ and $q : \mathfrak{t} \to \mathfrak{t}/W$ as well as $\chi : \mathfrak{g} \to \mathfrak{t}/W$ are A-equivariant. Hence the square (1.27) is A-equivariant.

Remark 1.43. Observe that $\mathrm{Aut}(\mathfrak{g})/G \cong \mathrm{Aut}(\Delta)$ naturally acts on $\mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W$. Unfortunately, there does not seem to exist a natural $\mathrm{Aut}(\Delta)$-action, in the sense that it does not depend on a splitting as above. However, in our applications below, there exists a natural subgroup $A \subset \mathrm{Aut}(\mathfrak{g})$ such that $A \cong \mathrm{Aut}(\Delta)$ via Π, cf. Lemma 1.51, so that we do have a natural $\mathrm{Aut}(\Delta)$- (or $AS(\Delta)$-)action on $\mathfrak{t}$ in these cases.
Let us briefly mention the case $\mathrm{Aut}(\Delta) = 1$. Then $\mathrm{Aut}(\mathfrak{g}) = G$ and Example 1.42 shows that θ is G-equivariant with respect to the trivial G-action on $\mathfrak{t}$. This fits with the fact that in this case, χ is G-equivariant with respect to the trivial G-action on $\mathfrak{t}/W$.

1.4.3 Slodowy slices

We now have everything in place to give the description of singularities of type Δ due to Brieskorn ([Bri71]) and Slodowy ([Slo80b]). Brieskorn's basic observation is the following: Let $S \subset \mathfrak{g}$ be (the germ of) a transverse slice to the G-orbits through a subregular nilpotent element $x \in \mathfrak{g}$ in a simple Lie algebra $\mathfrak{g}$ with ADE-Dynkin diagram. Then the restriction $\chi_{|S} : S \to \mathfrak{t}/W$ is quasi-homogeneous of degree d_i and weights $(w_1, w_2, w_3, d_1, \dots, d_r)$ where d_i are as in Theorem 1.29 and w_i are the weights of the corresponding singularity, see the Table (1.9). This suffices to

a) identify $(S_{\bar{0}}, x)$ as a singularity of type Δ;

b) show that $\chi_{|S} : S \to \mathfrak{t}/W$ is a semi-universal deformation of the singularity $(S_{\bar{0}}, x)$.

Slodowy has filled this observation with many details, gave a more Lie-theoretic description (in 'good characteristics') and introduced BCFG-singularities.

Instead of presenting the theory in its generality we will restrict our presentation to the approach via *Slodowy slices* $S \subset \mathfrak{g}$. These are special transverse slices $S \subset \mathfrak{g}$ through a subregular nilpotent element $x \in \mathfrak{g}$. Their advantage is that they are defined globally. This means in particular that $S \subset \mathfrak{g}$ is transverse to any G-orbit it meets and the restriction $\chi_{|S} : S \to \mathfrak{t}/W$ is surjective. However, it is in fact important for the theory that all the statements below hold true for *any* germ of a transverse slice through a subregular element $x \in \mathfrak{g}$.

Let $x \in \mathfrak{g}$ be a subregular nilpotent element and (x, h, y) a $\mathfrak{sl}_2$-*triplet* for x. By definition, there is a homomorphism $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ such that $\rho(X) = x$, $\rho(Y) = y$, $\rho(H) = h$ where

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.31}$$

are the standard generators of $\mathfrak{sl}_2$. Such a triplet always exists by the Jacobson-Morozov theorem ([CM93]) since x is nilpotent. We can construct a transverse slice through x as follows: The $\mathfrak{sl}_2$-triplet gives a decomposition of $\mathfrak{g}$ into irreducible $\mathfrak{sl}_2$-modules,

$$\mathfrak{g} = \bigoplus_{j=1}^{s} E_j,$$

with $\dim E_j = n_j + 1$. The action of ad h further decomposes each of the E_j into

$$E_j = \bigoplus_{m \in \mathbb{Z}} E_j(m)$$

with $E_j(m) = \{v \in E_j \mid \text{ad } h(v) = mv\}$ and $\dim E_j(m) = 1$. Since ad $x(E_j(m)) = E_j(m+2)$, we conclude that

$$\text{ad } x(\mathfrak{g}) = \bigoplus_{\substack{1 \le j \le s \\ m \ge -n_j+2}} E_j(m)$$

and a complement to this space in $\mathfrak{g}$ is given by

$$\mathfrak{z}_{\mathfrak{g}}(y) = \ker \text{ad } y = \{v \in \mathfrak{g} \mid \text{ad } y(v) = [y, v] = 0\} = \bigoplus_j E_j(-n_j).$$

Note that the affine tangent space to the G-orbit through x is $x + \text{ad } x(\mathfrak{g})$. We conclude:

Lemma 1.44. *Let x be a nilpotent element and (x,y,h) a $\mathfrak{sl}_2$-triplet for x. Then $S = x + \mathfrak{z}_{\mathfrak{g}}(y)$ is locally around x a transverse slice.*

Transverse slices of this form are called *Slodowy slices*. By constructing a natural $\mathbb{C}^*$-action on a given Slodowy slice S we will be able to conclude that S is a transverse slice everywhere, cf. Corollary 1.47.

$\mathbb{C}^*$-action

As before, let (x,y,h) be a $\mathfrak{sl}_2(\mathbb{C})$-triplet for the subregular nilpotent element x and denote by $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ the corresponding Lie algebra homomorphism. By exponentiating ρ we obtain a group homomorphism $\tilde{\rho} : SL(2,\mathbb{C}) \to G$ and hence a $\mathbb{C}^*$-action

$$\lambda(t)v = \tilde{\rho}(tH) \cdot v = \mathrm{Ad}(\tilde{\rho}(tH))(v) \tag{1.32}$$

on $\mathfrak{g}$. It follows that $\lambda(t)(x) = t^2 x$. So in order to obtain a $\mathbb{C}^*$-action that preserves $S = x + \mathfrak{z}_{\mathfrak{g}}(y)$, we modify it to

$$\mu(t)v := t^2 \lambda(t^{-1})v,$$

where t^2 acts by scalar multiplication. When we decompose $\mathfrak{z}_{\mathfrak{g}}(y) = \bigoplus_j E_j(-n_j)$ as in the previous section and $x + \sum_j a_j e_j$, $e_j \in E_j(-n_j) - \{0\}$, then

$$\mu(t)(x + \sum_j a_j e_j) = x + \sum_j t^{n_j+2} a_j e_j.$$

In particular, the action is algebraic (even though we exponentiated ρ).
Now we describe this $\mathbb{C}^*$-action on $\mathfrak{t}/W$ to study equivariance properties of $\sigma : S \to \mathfrak{t}/W$. To this end let $\hat{\chi}_j$ be independent generators of $\mathbb{C}[\mathfrak{g}]^G$ of degree d_j so that

$$\hat{\chi}_j(\mu(t)v) = \hat{\chi}_j(t^2\lambda(t^{-1})v) = t^{2d_j}\hat{\chi}_j(v)$$

where we have used the G-invariance. To summarize, we formulate:

Proposition 1.45. *Let $\delta : S \to \mathfrak{t}/W$ be the restriction of the adjoint quotient, considered as a morphism $\mathbb{C}^s \to \mathbb{C}^r$ under the natural isomorphism $S \cong \mathbb{C}^s$ and the isomorphism $\mathfrak{t}/W \cong \mathbb{C}^r$ induces by the choice of independent generators $\hat{\chi}_j \in \mathbb{C}[\mathfrak{g}]^G$. Then δ is of quasi-homogeneous type $(2d_1, \ldots, 2d_r; w_1, \ldots, w_s)$ where $d_j = \deg(\hat{\chi}_j)$ and $w_j := n_j + 2$.*

Remark 1.46. We emphasize that δ is *not* $\mathbb{C}^*$-equivariant with respect to the standard $\mathbb{C}^*$-action on $\mathfrak{t}/W$. Also observe that the factor two was forced upon us, because x is nilpotent (also compare with Remark 1.8).
We should mention that our notation differs from the one in [Slo80b]. Slodowy denotes by d_i the degrees multiplied by 2. In [Slo80a] both conventions are mixed, which can cause confusion.

As mentioned earlier the $\mathbb{C}^*$-action has the following application.

Corollary 1.47. *Let $S \subset \mathfrak{g}$ be a Slodowy slice and $\sigma = \chi_{|S} : S \to \mathfrak{t}/W$ the restricted adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$. Then S is transverse to each adjoint orbit it meets. In particular, it only intersects regular and subregular orbits and the singularities of $\sigma^{-1}(\bar{t})$ are precisely the subregular elements.*

Proof. By construction, S intersects each orbit over a neighborhood around $\bar{0} \in \mathfrak{t}/W$. Since the $\mathbb{C}^*$-action has positive weights on both S and $\mathfrak{t}/W$, it also holds in general.
The second statement follows for dimension reasons. $\square$

We end this subsection by reconsidering the $C(\Gamma)$- resp. C_Γ-action on the singularities $\mathbb{C}^2/\Gamma$. An explicit calculation shows the following (cf. Remark 1.8):

Lemma 1.48. *Let $\Gamma \subset SL(2,\mathbb{C})$ correspond to the simple Lie algebra $\mathfrak{g}$ and fix a Cartan $\mathfrak{t} \subset \mathfrak{g}$. Then $C(\Gamma) \subset GL(2,\mathbb{C})$ acts on S via the determinant. More explicitly, fix generators $\hat{\chi}_j \in \mathbb{C}[\mathfrak{t}]^W$ of weights d_j so that $\mathfrak{t}/W \cong \mathbb{C}^r$. Then the induced $C(\Gamma)$-action on $\mathbb{C}^r \cong \mathfrak{t}/W$ is given by*

$$A \cdot (x_1, \dots, d_r) = ((\det A)^{d_1} x_1, \dots, (\det A)^{d_r} x_r).$$

In particular, $\mathbb{C}^* \subset C(\Gamma)$ acts via weights $(2d_1, \dots, 2d_r)$ and coincides with the previous $\mathbb{C}^*$-action.

This explains the factor 2 from another perspective.

Graph automorphisms

There are two ways ([Slo80b]) to realize the graph automorphisms and therefore to establish a theory for singularities of type $\mathrm{B}_r, \mathrm{C}_r, \mathrm{F}_4, \mathrm{G}_2$. The first one is *intrinsic*, namely we start with a transverse (Slodowy) slice $S \subset \mathfrak{g}$ in the simple Lie algebra with BCFG-Dynkin diagram Δ. Recall that

$$S = x + \mathfrak{z}_{\mathfrak{g}}(y)$$

where (x, y, h) is a $\mathfrak{sl}_2$-triplet for the subregular $x \in \mathfrak{g}$. It would be natural to consider a $Z_G(x)$-action on this slice (or any transverse slice), but in general $Z_G(x)$ is not reductive and therefore it might not be possible to choose a $Z_G(x)$-invariant slice S at all. To remedy this, Slodowy considers the action of

$$C(x, h) := Z_G(x) \cap Z_G(h)$$

which is in fact reductive. It is therefore called the *reductive centralizer* of x (with respect to h). It turns out that $C(x, h) = Z_G(x) \cap Z_G(y) = C(x, y)$. Its relevance for BCFG-singularities comes from the following:

Lemma 1.49 ([Slo80b], 7.5.)**.** *Let (x, y, h) be a $\mathfrak{sl}_2$-triplet for a subregular nilpotent $x \in \mathfrak{g}$ as before. Then there exists a subgroup $\mathbf{C} \subset C(x, y) = C(x, h)$ such that $\mathbf{C} \cong AS(\Delta)$.*

Even though the next lemma is somewhat immediate, it will be crucial for latter applications and we add its proof for completeness.

Lemma 1.50. *Let x, y, h and $C(x, h)$ be as before and $S = x + \ker \operatorname{ad}(y)$ the corresponding Slodowy slice. Then the $C(x, h)$- and $\mathbb{C}^*$-action on S commute.*

Proof. Remember that the $\mathbb{C}^*$-action was defined in terms of ordinary scalar multiplication on $\mathfrak{g}$ and via λ which acts on $s \in S$ via

$$\lambda(t)(s) = t^d s$$

in case $\operatorname{ad}(h)(s) = [h, s] = ds$. It is clear that the $C(x, h)$-action, which is given via conjugation, commutes with σ. Let $s \in S$ such that $\operatorname{ad}(h)(s) = ds$. Since $c \cdot h = h$ for any $c \in C(x, h)$ we conclude that

$$[h, c \cdot s] = [c \cdot h, c \cdot s] = c \cdot ([h, s]) = d(c \cdot s).$$

Therefore λ also commutes with the $C(x, h)$-action: $\lambda(t)(c \cdot s) = t^d(c \cdot s) = c \cdot (\lambda(t)(s))$. $\square$

The other approach is *extrinsic* and works with a transverse (Slodowy) slice $S_h \subset \mathfrak{g}_h$ in a homogeneous simple Lie algebra $\mathfrak{g}_h$. More precisely, let Δ be an irreducible Dynkin diagram of type BCFG and Δ_h its homogeneous Dynkin diagram. A Slodowy slice $S_h \subset \mathfrak{g}_h = \mathfrak{g}(\Delta_h)$ for a subregular nilpotent element $x \in \mathfrak{g}_h$ carries a natural action by the *outer centralizer*

$$CA(x,h) := \{\phi \in \mathrm{Aut}(\mathfrak{g}_h) \mid \phi(x) = x, \phi(h) = h\}.$$

Then similar results as before hold.

Lemma 1.51 ([Slo80b], 7.6.). *There exists a subgroup* $\mathbf{CA} \subset CA(x,h)$ *such that* $\mathbf{CA} \cong AS(\Delta) \subset \mathrm{Aut}(\Delta_h)$ *via the morphism* $\Pi : \mathrm{Aut}(\mathfrak{g}_h) \to \mathrm{Aut}(\Delta_h)$ *from (1.15). Further the* $CA(x,h)$*- and* $\mathbb{C}^*$*-action on* S_h *commute.*

Simultaneous resolution

The next statement is actually a corollary to the considerations of Section 1.4.2, since it again applies to any (germ of) transverse slice. However, we will only need it for Slodowy slices.

Corollary 1.52. *Let* $\sigma : S \to \mathfrak{t}/W$ *be the restriction of the adjoint quotient* $\chi : \mathfrak{g} \to \mathfrak{t}/W$ *to a Slodowy slice* $S = x + \ker \mathrm{ad}(y)$*. Moreover, let* $\tilde{S} := \psi^{-1}(S)$*, where* $\psi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ *is as in Theorem 1.39, and denote by* $\tilde{\sigma} : \tilde{S} \to \mathfrak{t}$ *the restriction of* $\tilde{\sigma} : \tilde{\mathfrak{g}} \to \mathfrak{g}$*. Then the commutative diagram*

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d, "\tilde{\sigma}"'] \arrow[rr, bend left, "\psi"] & q^*S \arrow[r] \arrow[d] & S \arrow[d, "\sigma"] \\ \mathfrak{t} \arrow[r, equal] & \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W \end{tikzcd} \tag{1.33}$$

is a simultaneous (minimal) resolution. All these maps are equivariant with respect to the $\mathbf{C}(\mathbf{A})$*-actions.*

Proof. The fact that (1.33) is a simultaneous (minimal) resolution follows from Theorem 1.39 (cf. [Slo80b] for more details). The statement about $\mathbf{C}(\mathbf{A})$-equivariance is a consequence of our considerations in Section 1.4.2. □

Remark 1.53. It is natural to ask if (1.33) is $\mathbb{C}^*$-equivariant as well, when we endow $\tilde{S} \hookrightarrow S \times \mathcal{B}$ with the induced action. Then it is clear that ψ and σ are equivariant, but not so $\tilde{\sigma}$ as follows from the definitions. However, if we choose $\mathfrak{t}$ such that $h \in \mathfrak{t}$, where h is the semisimple element of the $\mathfrak{sl}_2$-triplet (x,y,h), then $\tilde{\sigma}$ is $\mathbb{C}^*$-equivariant if $\mathbb{C}^*$ acts with weight 2 on $\mathfrak{t} \cong \mathbb{C}^r$:

$$\tilde{\sigma}(t \cdot (v, gB)) = t^2 \left(g^{-1} \cdot \lambda(t^{-1})v\right)_s = t^2 \left(g^{-1} \cdot \lambda(t^{-1})v_s\right) = t^2(g^{-1} \cdot v)_s = t^2\theta(v, gB).$$

The second to last equality uses that $h \in \mathfrak{t}$, so that $\tilde{\rho}(tH) \in T = \exp_{\mathfrak{g}}(\mathfrak{t})$ and $\lambda(t)$ acts trivially on $\mathfrak{t}$ (see (1.32)).
Since any two Cartan subalgebras in $\mathfrak{g}$ are conjugate, we assume[4] from now on that $h \in \mathfrak{t}$.

As an application, we consider $H^2(\tilde{S}_0, \mathbb{Z})$. If we already knew that $(S_{\bar{0}}, x)$ is a singularity of type Δ (and so $\tilde{S}_0 \to S_{\bar{0}}$ its minimal resolution), then necessarily $H^2(\tilde{S}_0, \mathbb{Z}) \cong \mathbf{\Lambda}_G$ as in Section 1.1.1. For the moment we only look at the inclusions

$$\tilde{S}_0 \overset{j}{\hookleftarrow} \mathcal{B}_x = \psi^{-1}(x) \overset{i}{\hookrightarrow} \mathcal{B}.$$

For latter reference we formulate:

[4] It is presumably possible to circumvent this by working with (1.26) throughout, but we have not pursued this approach.

Lemma 1.54. *If $\Delta = \Delta_h$ is of type* ADE*, then j^* and i^* are* **CA***-equivariant isomorphisms on second integral cohomology. In case Δ is of type* BCFG*, then $j^* : H^2(\tilde{S}_0, \mathbb{Z}) \to H^2(\mathcal{B}_x, \mathbb{Z})$ is a* **C***-equivariant isomorphism. On the other hand, $i^* : H^2(\mathcal{B}, \mathbb{Z}) \to H^2(\mathcal{B}_x, \mathbb{Z})$ is only an isomorphism onto $H^2(\mathcal{B}_x, \mathbb{Z})^{\mathbf{C}}$.*

Proof. The claims about i^* and j^* are already contained in [Slo80a] except for the **CA**- and **C**-equivariance. But this immediately follows from the equivariance of i and j, which is a consequence of our considerations in Section 1.4.2. □

Now $\mathcal{B}_x$ is a *Dynkin curve* of type Δ ([Ste]), hence a tree of $\mathbb{CP}^1$'s and $\mathcal{B}_x \cong \mathbf{\Lambda}_{G_h}$, where G_h is the adjoint group of type Δ_h. Dynkin curves are the Lie-theoretic description of the corresponding exceptional divisors. Observe that this is still not enough to identify $x \in S_{\bar{0}}$ as a singularity of type Δ_h, because one does not know the self-intersection numbers of the connected components which are rational curves. This is in fact a crucial step in [Slo80b] (apparently going back to Deligne), also cf. [Hin91].

Theorem by Brieskorn-Slodowy

The previous discussion indicated that $(S_{\bar{0}}, x)$ is a singularity of type Δ when $S \subset \mathfrak{g}(\Delta)$. As mentioned earlier, this is true and goes back to Brieskorn ([Bri71]) and Slodowy ([Slo80b]). In the following we summarize their main results which give a complete description of singularities of type Δ together with their semi-universal deformations in the corresponding Lie algebras.

Theorem 1.55 (Brieskorn, Slodowy)**.** *Let Δ be an irreducible Dynkin diagram and $S = x + \ker \mathrm{ad}(y)$ be a Slodowy slice through a subregular nilpotent element x in the Lie algebra $\mathfrak{g} = \mathfrak{g}(\Delta)$. Moreover, let $\sigma = \chi_{|S} : S \to \mathfrak{t}/W$ be the restriction of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$ to S. Then:*

i) $((\sigma^{-1}(\bar{0}), x), \mathbf{C})$ is a singularity of type Δ.

ii) $\sigma : S \to \mathfrak{t}/W$ is a semi-universal $\mathbb{C}^*$-deformation of the singularity $(\sigma^{-1}(\bar{0}), x)$ of type Δ.

For this statement to make sense, we have fixed a subgroup **C** that is isomorphic to $AS(\Delta)$ as in Lemma 1.49.

There is also an extrinsic description of the semi-universal deformation for the BCFG-types corresponding to the extrinisc $AS(\Delta)$-action described above. This is closely related to the construction of a semi-universal deformation of a BCFG-singularity from general principles, cf. Section 1.3.2.

Corollary 1.56 ([Slo80b])**.** *Let Δ be a Dynkin diagram of type* BCFG *and Δ_h its homogeneous Dynkin diagram. Denote by $S_h \subset \mathfrak{g}_h = \mathfrak{g}(\Delta_h)$ a Slodowy slice through a subregular $x \in \mathfrak{g}_h$ and fix a subgroup $\mathbf{CA} \cong AS(\Delta)$ of $CA(x, y)$ as in Lemma 1.50. Finally let $\sigma_h := \chi_{|S_h} : S_h \to \mathfrak{t}_h/W_h$ where $\chi : \mathfrak{g}_h \to \mathfrak{t}_h/W_h$ is an adjoint quotient.*

i) $((\sigma_h^{-1}(\bar{0}), x), \mathbf{CA})$ is a singularity of type Δ.

ii) The $AS(\Delta)$-deformation

$$\sigma_h^{\mathbf{CA}} : S_{h,\mathbf{CA}} := \sigma_h^{-1}((\mathfrak{t}_h/W_h)^{\mathbf{CA}}) \to (\mathfrak{t}_h/W_h)^{\mathbf{CA}}$$

is a semi-universal $\mathbb{C}^*$-deformation of the singularity x of type Δ.

With the notation of Theorem 1.55 and Corollary 1.56, the last statement of the previous corollary yields a commutative diagram The last statement in particular implies that we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "\cong"] \arrow[d, "\sigma"] & S_{h,\mathbf{CA}} \arrow[r, hook] \arrow[d, "\sigma_h^{\mathbf{CA}}"] & S_h \arrow[d, "\sigma_h"] \\ \mathfrak{t}/W \arrow[r, "\cong"] & (\mathfrak{t}_h/W_h)^{\mathbf{CA}} \arrow[r, hook] & \mathfrak{t}_h/W_h \end{tikzcd}$$

Remark 1.57. Corollary 1.56 seems trivial in light of Corollary 1.27. However, there is no straightforward geometric proof of i), cf. [Slo80b], 8.8., Remark 2). After that ii) is immediate. As mentioned earlier, the above two results analogously hold for each germ of a transverse slices through subregular nilpotent elements $x \in \mathfrak{g}$. This is particularly useful for identifying singularities of a specific type. On the other hand, Slodowy slices have the advantage that they are global in nature.

1.4.4 Some remarks on the derivative

We keep the notation of the previous section, in particular $\chi : \mathfrak{g} \to \mathfrak{t}/W$ is the adjoint quotient of a *simple* Lie algebra $\mathfrak{g}$ and $q : \mathfrak{t} \to \mathfrak{t}/W$ the natural quotient. When constructing cameral curves and Calabi-Yau threefolds (Chapter 4, 5), it will be important to understand the derivatives of χ and q in order to make statements about the smoothness/regularity of the curves and threefolds respectively. Since this is a local question, we are led to consider the following fiber products (compare with (4.15), (5.3)): Let $b : U \to \mathfrak{t}/W$ be a morphism, where $U \subset \mathbb{C}$ is a Zariski-open[5] subset. Then we define $\tilde{U}_b$ and X_b by the fiber products

$$\begin{tikzcd} \tilde{U}_b \arrow[r] \arrow[d, "p_b"'] & \mathfrak{t} \arrow[d, "q"] \\ U \arrow[r, "b"] & \mathfrak{t}/W \\ X_b \arrow[r] \arrow[u, "\pi_b"] & S. \arrow[u, "\sigma"'] \end{tikzcd} \tag{1.34}$$

These are both non-singular if b is transversal to q and σ. We are mainly interested in the situations where this is satisfied, so that we need to consider elements $t \in \mathfrak{t}$ and $s \in S$ with

$$\mathrm{rk}(dq_t) = r - 1 = \mathrm{rk}(d\sigma_s), \quad q(t) = \sigma(s).$$

This situation is quite restrictive in the following sense: The transversality condition of b with q implies that the rank of dq may not be less than $r - 1$.

We first compute the rank of the derivative of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$. Since the Slodowy slice is transversal to each orbit it meets, this will also give the result for the restriction $\sigma : S \to \mathfrak{t}/W$.

Proposition 1.58 ([Ric87]). *Let $\chi : \mathfrak{g} \to \mathfrak{t}/W$ be an adjoint quotient of a (semi)simple Lie algebra. Let $x = h + v$ be the Jordan decomposition of $x \in \mathfrak{g}$, i.e. h semisimple and v nilpotent, and*

$$\chi_1 : \mathfrak{g}_1 := [Z_{\mathfrak{g}}(h), Z_{\mathfrak{g}}(h)] \to \mathfrak{t}_1/W_1$$

[5]The same discussion obviously works in the analytic topology as well. Also observe that the analytification of fiber products in the algebraic category are fiber products in the analytic category.

the induced adjoint quotient of the semisimple Lie algebra $\mathfrak{g}_1$. *Then the rank of* $d\chi_x$ *is given by*

$$\mathrm{rk}(d\chi_x) = \dim C(Z_{\mathfrak{g}}(h)) + \mathrm{rk}(d\chi_{1,v}). \tag{1.35}$$

Lemma 1.59. *Let* $x = h$ *be semisimple and* $t \in \mathfrak{t}$ *with* $\chi(h) = q(t)$. *Then* $\mathrm{im}(dq_t) \subset \mathrm{im}(d\chi_h)$ *and*

$$\mathrm{rk}(dq_t) = \dim \bigcap_{\alpha\in R, \alpha(t)=0} \ker\alpha = \dim C(Z_{\mathfrak{g}}(h))$$

where R *are the roots corresponding to* $\mathfrak{t} \subset \mathfrak{g}$.

Proof. Let $\mathfrak{t}(h) \subset \mathfrak{g}$ be a Cartan subalgebra that contains h. Since $\mathfrak{t}(h)$ is conjugate to $\mathfrak{t}$ we may assume that $h \in \mathfrak{t}$ and in fact even $t = h$ by the invariance of χ and q. Then the first claim is obvious because $\chi_{|\mathfrak{t}} = q$.
For the second claim we may assume $t = h$ as before. Then the first equality is proven in [Ste]. For the second equality we claim that in fact

$$\bigcap_{\alpha\in R, \alpha(t)=0} \ker\alpha = C(Z_{\mathfrak{g}}(h)) \tag{1.36}$$

which can be seen as follows: Let $\mathfrak{g} = \mathfrak{t}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$ be the root space decomposition with respect to $\mathfrak{t}$. Then we get

$$Z_{\mathfrak{g}}(h) = \mathfrak{t} \oplus \bigoplus_{\alpha\in R, \alpha(h)=0} \mathfrak{g}_\alpha.$$

Since $[\mathfrak{t}, \mathfrak{g}_\alpha] \neq \{0\}$ and $[h, \mathfrak{g}_\alpha] = 0$ iff $\alpha(h) = 0$, the equality (1.36) follows. □

Corollary 1.60. *If* $x = h$ *is semisimple and* $t \in \mathfrak{t}$ *with* $q(t) = \chi(h)$, *then* $\mathrm{im}(dq_t) = \mathrm{im}(d\chi_h)$.

Proof. By the previous proposition and lemma, it remains to show (since the commutator $[Z_{\mathfrak{g}}(h), Z_{\mathfrak{g}}(h)]$ is semisimple) that $d\chi_0 = 0$ for any semisimple adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$. But this follows from the fact that the degrees d_i of any basis $\hat{\chi}_i$ of G-invariant polynomials are greater or equal 2 because $\mathfrak{g}$ is semisimple ([Bou02]). □

Proposition 1.61. *Let* $b : U \to \mathfrak{t}/W$ *be a morphism from an open* $U \subset \mathbb{C}$ *which is transversal to* $q : \mathfrak{t} \to \mathfrak{t}/W$. *Then it is also transversal to* $\chi : \mathfrak{g} \to \mathfrak{t}/W$ *and* $\sigma : S \to \mathfrak{t}/W$.

Proof. Let $x = h + v \in \mathfrak{g}$ and $t \in \mathfrak{t}$ such that $\chi(x) = q(t)$. The previous corollary together with (1.35) implies that

$$r \geq \mathrm{rk}(d\chi_x) \geq \mathrm{rk}(dq_t) \geq r - 1.$$

If $x = h$ is semisimple, then χ is also transversal to b at x by the previous lemma. So we are left with the case $x = h + v$ and $\mathrm{rk}(d\chi_x) = r - 1 = \mathrm{rk}(dq_t)$. We claim that $v = 0$ and $x = h$ must be semisimple which would conclude the proof. Without loss of generality we assume again that $h \in \mathfrak{t}$. Since $\dim C(Z_{\mathfrak{g}}(h)) = \mathrm{rk}(dq_t) = r-1$ by Lemma 1.59, it follows that $Z_{\mathfrak{g}}(h) = \mathfrak{t}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}$ for a root α with respect to $\mathfrak{t}$. Therefore the derived algebra is

$$[Z_{\mathfrak{g}}(h), Z_{\mathfrak{g}}(h)] = \langle h_\alpha, \mathfrak{g}_{\pm\alpha}\rangle \cong \mathfrak{sl}_2(\mathbb{C}),$$

where h_α generates the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}$. As a consequence v can be considered as a nilpotent element in $\mathfrak{sl}_2(\mathbb{C})$ because $v \in \mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha} \subset Z_{\mathfrak{g}}(h)$. By formula (1.35) and $\mathrm{rk}(dq_t) = r-1$ we must have $\mathrm{rk}(d\chi_{1,v}) = 0$ for the adjoint quotient $\chi_1 = \det : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{t}_1/W_1$. But $d_A\det = (-2a, -c, -b)$ for $A = aH + bX + cY \in \mathfrak{sl}_2(\mathbb{C})$ in the standard basis H, X, Y (cf. (1.31)). Hence we must have $v = 0$, i.e. $x = h$ is semisimple (and subregular). Since S is transversal to the G-orbits it meets, the statement is also true for $\sigma = \chi_{|S}$. □

Corollary 1.62. *The discriminants of* $\chi : \mathfrak{g} \to \mathfrak{t}/W$ *and* $q : \mathfrak{t} \to \mathfrak{t}/W$ *coincide,*

$$\operatorname{discr}(\chi) = \operatorname{discr}(q).$$

It is the image of the reflection hyperplanes $\mathfrak{t}_\alpha \subset \mathfrak{t}$, $\alpha \in R$, *under* q *and is given (set-theoretically) by the vanishing of the section*

$$s_{br} := \prod_{\alpha \in R} \alpha \in \mathbb{C}[\mathfrak{t}]^W. \tag{1.37}$$

Proof. We have just seen that $D := \operatorname{discr}(q) \supset \operatorname{discr}(\chi)$. For the converse inclusion observe that by Theorem 1.35 an element $x \in \mathfrak{g}$ is regular iff $d\chi_x$ is surjective. But if $\bar{t} \in \operatorname{discr}(q) \subset \mathfrak{t}/W$ then the G-orbit $O(t) \subset \chi^{-1}(\bar{t})$ is a semisimple non-regular orbit, hence $\bar{t} \in \operatorname{discr}(\chi)$. It follows immediately that $V(s_{br}) = D$ as sets. □

Note that the vanishing locus $V(s_{br})$ is non-reduced, but we will consider $\operatorname{discr}(\chi)$ and $\operatorname{discr}(q)$ with their reduced structures.

Example 1.63. We consider the simplest example, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Of course, a Slodowy slice S has to be all of $\mathfrak{g}$, $\mathfrak{t} = \mathbb{C} \cong \mathfrak{t}/W$ and

$$\begin{aligned} q : \mathbb{C} \to \mathbb{C}, &\quad z \mapsto z^2, \\ \sigma : S \to \mathbb{C}, &\quad A \mapsto \det(A). \end{aligned}$$

A morphism $b : U \to \mathfrak{t}/W$ is transversal to q iff it has only zeros of multiplicity 1. Therefore $\tilde{U}_b = \{(x, y) \in U \times \mathbb{C} \mid y^2 - b(x) = 0\}$ is non-singular and a branched double covering of U. It branches precisely over the zeros of b.
Identify $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$ via

$$\begin{pmatrix} u & v \\ w & -u \end{pmatrix} \mapsto (u, v, w),$$

so that $\sigma(a, b, c) = -u^2 - vw$. Again using that b has simple zeros only, we see that $X_b = \{((u, v, w), x) \in \mathbb{C}^3 \times U \mid -u^2 - vw - b(x) = 0\}$ is non-singular. Note however, that if $b(x) = 0$, then $\pi_b^{-1}(x)$ is an A_1-singularity.

Remark 1.64. As already mentioned, $\tilde{U}_b$ and X_b are local models for the cameral curves and non-compact Calabi-Yau threefolds respectively, that we encounter in Chapter 4 and 5 respectively. The previous example has shown that it is in principle possible to write down local equations for them. In general, it is cumbersome to obtain explicit equations for X_b using Slodowy slices though. There is a more direct way: We know that $S \to \mathfrak{t}/W$ is a semi-universal deformation of the corresponding singularity. Hence we can also work with an explicit model of a semi-universal deformation as in Section 1.3.2. This easily gives equations for X_b. For example, if we start with an A_k-singularity, then $b = (b_1, \dots, b_k) : U \to \mathbb{C}^k$ and

$$X_b \cong \{(x, u, v, w) \in \mathbb{C}^4 \mid uv - v^k - \sum_{i=1}^{k} b_i(x) v^{(k-1)-i} = 0\}.$$

On the other hand, Slodowy slices provide a very useful tool to express properties of X_b, and especially the fibers of $\pi_b : X_b \to U$, in Lie-theoretic terms as we see in the next section.

1.4.5 Stratification of $\mathfrak{t}/W$

It is well-known (e.g. [Pfl01]) that the orbit space $\mathfrak{t}/W$ carries a natural stratification. We study it here in some detail because it gives a convenient way to make statements about the fibers of $\sigma : S \to \mathfrak{t}/W$. To avoid confusion, let us fix what we mean by that: If X is a connected complex-analytic space, then a *stratification* $\{X_i\}_{i\in I}$ is a decomposition

$$X = \bigsqcup_{i\in I} X_i$$

which is locally finite and

i) each stratum $X_i \subset X$ is a locally closed connected submanifold;

ii) the closure $\bar{X}_j$, $j \in I$, is a union of strata (*frontier condition*).

In particular, we do not consider more refined versions involving Whitney regularity etc.
An orbit space can be stratified by its orbit types. In the case at hand this goes as follows: For a given $t \in \mathfrak{t}$ denote by $W_t = \{w \in W \mid w \cdot t = t\}$ its isotropy group. Moreover, for any subgroup $W' \subset W$ define

$$\begin{aligned} \mathfrak{t}_{W'} &:= \{t \in \mathfrak{t} \mid W_t = W'\} \\ \mathfrak{t}_{(W')} &:= \{t \in \mathfrak{t} \mid W_t \sim W'\}, \end{aligned} \tag{1.38}$$

where $W_t \sim W'$ means that W_t is conjugate in W to W'.

Example 1.65. The subgroups $W' = \langle s_\alpha\rangle$, $\alpha \in R$, give

$$\mathfrak{t}_{\langle s_\alpha\rangle} = \mathfrak{t}_\alpha - (\bigcup_{\beta\neq\alpha} \mathfrak{t}_\alpha \ \cap \mathfrak{t}_\beta),$$

for the hyperplanes $\mathfrak{t}_\alpha = \ker\alpha$ (considering $\alpha \in \mathfrak{t}^*$). As we assume $\mathfrak{g}$ to be simple, there are at most two conjugacy classes $(\langle s_\alpha\rangle)$ depending on the length of α.

The subsets $\mathfrak{t}_{(W_t)}/W \subset \mathfrak{t}/W$, $t \in \mathfrak{t}$, define a stratification of $\mathfrak{t}/W$,

$$\mathfrak{t}/W = \bigsqcup \mathfrak{t}_{(W_t)}/W. \tag{1.39}$$

Indeed, it is certainly locally finite because there are only finitely many strata. Moreover, it can be shown that

$$\mathfrak{t}_{(W_t)}/W \cong \mathfrak{t}_{W_t}/\Gamma(W_t), \quad \Gamma(W_t) := N_W(W_t)/W_t,$$

and $\Gamma(W_t)$ acts freely on $\mathfrak{t}_{W_t}$. Hence $\mathfrak{t}_{(W_t)}/W \subset \mathfrak{t}/W$ is a non-singular subvariety and $\mathfrak{t}_{(W_t)} \to \mathfrak{t}_{(W_t)}/W$ is a covering. The subvariety is connected because $\mathfrak{t}_{W_t}$ is: W_t is generated by simple reflections s_α, $\alpha \in \Delta_t \subset \Delta$, for an appropriate subset $\Delta_t \subset \Delta$ of simple roots. Clearly, $\cap_{\alpha\in\Delta_t}\mathfrak{t}_\alpha$ is connected because it is a (complex) vector space. Therefore

$$\mathfrak{t}_{W_t} = \bigcap_{\alpha\in\Delta_t} \mathfrak{t}_\alpha - \bigcup_{\beta\neq\alpha}\left(\mathfrak{t}_\beta \cap \bigcap_{\alpha\in\Delta_t} \mathfrak{t}_\alpha\right)$$

must be connected because it is the complement of a hyperplane arrangement in a complex vector space. Finally, it follows from general principles that the frontier condition is satisfied ([Pfl01]).

Subgroups of W that are generated by simple reflections are called *parabolic subgroups*. In particular, such subgroups are themselves Weyl groups and can be described more explicitly: For $t \in \mathfrak{t}$ define

$$R_t := \{\alpha \in R \mid \alpha(t) = 0\} \subset R.$$

Then this is a root subsystem of R with Weyl group W_t. There are simple roots Δ of R such that there exists $\Delta_t \subset \Delta$ with $R_t = R(\Delta_t)$. In analogy with W_t, such root systems are called *parabolic (root) subsystems*. In particular, the Dynkin diagram of R_t, also denoted by Δ_t, is a subdiagram of the Dynkin diagram Δ of R. Observe that each parabolic subgroup of W is the isotropy group W_t of some $t \in \mathfrak{t}$.

Example 1.66. Let Δ be of type A_3 and $\mathfrak{t}$ the corresponding Cartan with root system R. If $t \in \mathfrak{t}$ then there are simple roots $\alpha_1, \alpha_2, \alpha_3$ such that $t \in \cap_{i \in J} \mathfrak{t}_{\alpha_j}$ for some $J \subset \{1, 2, 3\}$. Hence we have three cases:

$|J| = 1$: one conjugacy class $(\langle s_{\alpha_1} \rangle)$.

$|J| = 2$: two conjugacy classes (W_1), (W_2) where $W_1 := \langle s_{\alpha_1}, s_{\alpha_2} \rangle$, $W_2 := \langle s_{\alpha_1}, s_{\alpha_3} \rangle$. Observe that W_1 is of type A_2 whereas W_2 is of type $\mathrm{A}_1 \times \mathrm{A}_1$.

$|J| = 3$: just the full group.

This example shows that parabolic Weyl subgroups are not the same as subgroups that are Weyl groups. Indeed, consider the subgroup $\langle s_{\alpha_1} s_{\alpha_3} \rangle \cong S_2$. Seen as an abstract group it is a Weyl group but it is not a Weyl subgroup because it is not generated by *simple* reflections. However, it gives a Weyl subgroup for the folded root system which is of type B_2.

Let $I \subset \Delta$ be a subset of a choice of simple roots Δ of R. Then we denote by $R_I \subset R$ and $W_I \subset W$ the corresponding parabolic subsystem and subgroup respectively. By abusing notation, we write Δ_I for the Dynkin diagram of R_I as well as for $I \subset \Delta$ itself. Since the above strata of $\mathfrak{t}/W$ are labelled by the conjugacy classes (W_t), $t \in \mathfrak{t}$, it is interesting to know the W-conjugacy classes of parabolic subgroups. This can be reduced to the study of W-conjugacy classes of parabolic subgroups of the form W_I where $I \subset \Delta$ is a subset of *fixed* simple roots Δ. The reason for this is that W acts simply transitively on bases of R.

Lemma 1.67 ([Kan01])**.** *Let Δ be simple roots of R and $\Delta_I, \Delta_J \subset \Delta$ subsets. Then the following are equivalent:*

i) R_I and R_J are W-equivalent, i.e. there exists $w \in W$ such that $w \cdot R_I = R_J$;

ii) Δ_I and Δ_J are W-equivalent;

iii) W_I and W_J are W-conjugate.

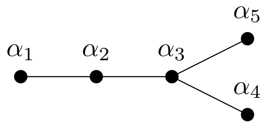
Remark 1.68. Let $\Delta_I \neq \Delta_J$ be subsets of Δ. If there exists $w \in W$ such that $w \cdot \Delta_I = \Delta_J$, then $w \cdot \Delta \neq \Delta$ because otherwise $w = 1$ and $\Delta_I = \Delta_J$. So even though W cannot permute all of Δ, it can permute subsets of it.

In particular, we see from the lemma that the natural map

$$\{(W_I) \mid I \subset \Delta\} \to \{\text{type of } \Delta_I\}$$

is well-defined. Here we mean by *type of* Δ_I the type of the Dynkin diagram of Δ_I. However, this map is in general not injective. Indeed, it can happen that Δ_I and Δ_J of the same type might not be W-conjugate to each other as the next example shows:

Example 1.69. This example is taken from [Kan01]. Consider the Dynkin diagram D_5:



Set $I = \{\alpha_1, \alpha_3\}$, $J = \{\alpha_4, \alpha_5\}$. Hence W_I and W_J are of type $\mathrm{A}_1 \times \mathrm{A}_1$. However, using Coexter elements of appropriate parabolic subsystems one can show that W_I and W_J cannot be conjugate to each other.

The basic observation that leads to a relation between the above stratification on $\mathfrak{t}/W$ and the types of the singularities of the semi-universal deformation $\sigma : S \to \mathfrak{t}/W$ is the following:

Lemma 1.70 ([Slo80b]). *Let $t \in \mathfrak{t}$ and denote by $Z(t) = Z_G(t)$ be its centralizer in the adjoint group G. Then $Z(t)$ is reductive and its semisimple commutator has the type of R_t and W_t.*

Fix $t \in \mathfrak{t}$ and let

$$\Delta(t) := \Delta(R_t) = \Delta_1(t) \cup \dots \Delta_m(t)$$

be the decomposition of the Dynkin diagram of $Z(t)$ into irreducible components. The singularities of $\sigma^{-1}(\bar{t})$ are precisely the intersection points of S with the subregular orbits lying in $\chi^{-1}(\bar{t})$. These in turn correspond to the subregular nilpotent $Z(t)$-orbits of types Δ_j. It is not hard to show that there is an injection $\Delta_j \mapsto y_j$ of the irreducible components Δ_j of the Dynkin diagram Δ_t to singularities y_j in $\sigma^{-1}(\bar{t})$ of type Δ_j (cf. [Slo80b], 6.5). Hence it remains to see how often S intersects a subregular nilpotent $Z(t)$-orbit of type Δ_j in order to obtain a complete description of the singularities of $\sigma^{-1}(\bar{t})$.
To give Slodowy's answer (cf. [Slo80b], 6.6), we make the following definition: Let $\Delta' \subset \Delta$ be a Dynkin subdiagram of an irreducible Dynkin diagram Δ and

$$\Delta' = \Delta'_1 \cup \dots \Delta'_m$$

its decomposition into irreducible components. Then define

$$\begin{array}{ll} n(\Delta'_j) := 2(3) & \Delta \text{ of type BCF(G), } \Delta'_j \subset \Delta \text{ of type ADE,} \\ & \text{whose roots are long with respect to}^{6}\ \Delta. \\ n(\Delta'_j) := 1 & \text{else, i.e. } \Delta \text{ of type ADE or } \Delta \text{ of type BCFG} \\ & \text{and } \Delta'_j \subset \Delta \text{ of type BCFG as well.} \end{array}$$

Definition 1.71. Let Y be a complex surface, $\Delta' \subset \Delta$ a Dynkin subdiagram of an irreducible Dynkin diagram Δ. Decompose Δ' into its irreducible components,

$$\Delta' = \Delta'_1 \cup \dots \cup \Delta'_m.$$

Then Y has *singularity configuration of type* Δ' relative to Δ (for short: *singularity configuration* $\Delta' \subset \Delta$) if it has precisely $n(\Delta'_j)$ singularities of type Δ'_j, $j = 1, \dots, m$.

Proposition 1.72 ([Slo80b], 6.6.). *Let $\mathfrak{g}$ be a simple Lie algebra with Dynkin diagram Δ, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and $\sigma : S \to \mathfrak{t}/W$ the restriction of the adjoint quotient χ to a Slodowy slice. Decompose the Dynkin diagram $\Delta(t)$ of the reductive group $Z_G(t)$ ($G = G_{ad}$) of an element $t \in \mathfrak{t}$ into its irreducible components,*

$$\Delta(t) = \Delta(R_t) = \Delta_1(t) \cup \dots \Delta_m(t) \subset \Delta.$$

Then the fiber $\sigma^{-1}(\bar{t})$ has singularity configuration $\Delta(t) \subset \Delta$. In particular, it has precisely $n(\Delta_j(t))$ singularities of type $\Delta_j(t)$.

[6] This makes sense because R_t is a root subsystem of R.

Corollary 1.73. *With the notation of Proposition 1.72, assume that $\bar{t} \in \mathfrak{t}_{(W')}/W \subset \mathfrak{t}/W$ for a parabolic Weyl subgroup $W' \subset W$. Then the singularity configuration of $\sigma^{-1}(\bar{t})$ coincides with the Dynkin type of W'.*

Remark 1.74. Example 1.69 shows that the stratification (1.39) by conjugacy classes of parabolic subgroups is in general strictly finer than the 'stratification' by singularity configurations. We put here stratification in quotation marks because the subsets of $\mathfrak{t}/W$ labelled by the singularity type are in general not connected.

Example 1.75. Fix an irreducible Dynkin diagram Δ (again also considered as simple roots). Let $\bar{t} \in \mathfrak{t}_{(W')}/W$, $W' = \langle s_\alpha \rangle$ for a fixed simple root $\alpha \in \Delta$. Hence $\sigma^{-1}(\bar{t})$ has the following possible singularities:

a) A_1-singularity: α is a short root in Δ.

b) $\mathrm{A}_1 \times \mathrm{A}_1$-singularity: α is a long root where Δ is of type $\mathrm{B}_{\geq 2}$, $\mathrm{C}_{\geq 2}$ or F_4.

c) $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$-singularity: α is a long root in $\Delta = \mathrm{G}_2$.

As an application, let $b : U \to \mathfrak{t}/W$ be transversal to q and hence to σ. Then the fibers of $\pi_b : X_b \to U$ constructed as in (1.34) have at most the above singularity configurations.

The situation in the previous example is the most important for us in our later applications. More precisely, we mainly deal with the following open subsets

$$\mathfrak{t}^1 := \mathfrak{t} - \bigcup_{\alpha \neq \beta} \mathfrak{t}_\alpha \cap \mathfrak{t}_\beta \subset \mathfrak{t}, \tag{1.40}$$

$$\mathfrak{t}^1/W \subset \mathfrak{t}/W, \tag{1.41}$$

of $\mathfrak{t}$ and $\mathfrak{t}/W$ respectively. In other words,

$$\mathfrak{t}^1/W = \begin{cases} \mathfrak{t}^\circ/W \sqcup \mathfrak{t}_{(W_s)}/W, & \Delta \text{ of type ADE}, \\ \mathfrak{t}^\circ \sqcup \mathfrak{t}_{(W_s)}/W \sqcup \mathfrak{t}_{(W_l)}/W, & \Delta \text{ of type BCFG}, \end{cases} \tag{1.42}$$

where we denote by $\mathfrak{t}^\circ = \mathfrak{t}^{reg}$ the regular locus in $\mathfrak{t}$. Here $W_s = \langle s_\alpha \rangle$ for any short (simple) root α and $W_l = \langle s_\beta \rangle$ for any (simple) root β (cf. Example 1.65). Yet another point of view is that

$$\mathfrak{t}^1/W = \mathfrak{t}^\circ/W \cup \mathrm{discr}^{sm}(q), \tag{1.43}$$

i.e. the open stratum together with the smooth part of the discriminant. It is clear that both $\mathfrak{t}^1$ and $\mathfrak{t}^1/W$ are left invariant under the respective $\mathbb{C}^*$-actions. Additionally, $\mathfrak{t}^1$ and $\mathfrak{t}^1/W$ are naturally stratified, e.g. the stratification of $\mathfrak{t}^1/W$ is given by (1.42).
Let us comment how these stratifications behave under folding. In later applications, it allows to go deeper in the stratification of Cartans $\mathfrak{t}_h$, which can be folded. Hence it is worthwhile to mention this explicitly. Let $\mathfrak{t}$ be of type Δ, an irreducible Dynkin diagram of type BCFG. Denote by $\mathfrak{t}_h$ a Cartan of type Δ_h and $AS = AS(\Delta) \subset \mathrm{Aut}(\Delta_h)$, so that $\mathfrak{t}_h^{AS} \cong \mathfrak{t}$. Moreover, the Weyl groups W and W_h of type Δ and Δ_h respectively are related via

$$W \cong \left\langle s_{\alpha_O} = \prod_{\beta \in O(\alpha)} s_\beta \mid \alpha \in R_h \right\rangle \subset W_h.$$

This implies that under the isomorphism $\mathfrak{t} \cong \mathfrak{t}_h^{AS}$, the open part $\mathfrak{t}^1$ (of $\mathfrak{t}$) lies deeper in the stratification of $\mathfrak{t}_h$, more precisely

$$\mathfrak{t}^1 \cong (\mathfrak{t}_h)^{AS} - \bigcup_{\alpha_O \neq \beta_O} \mathfrak{t}_{h,\alpha_O} \cap \mathfrak{t}_{h,\beta_O}.$$

Note that this is not $\mathfrak{t}_h^1 \cap \mathfrak{t}_h^{AS}$ because it also contains parts of deeper strata (e.g. consider the orbits of length ≥ 2 that correspond to long roots, see Section 1.2).

1.4.6 Sheaves associated with Slodowy slices

We now come to two sheaves that are naturally associated with $\sigma : S \to \mathfrak{t}/W$ and $q : \mathfrak{t} \to \mathfrak{t}/W$. The first one is the higher direct image $R^2\sigma_*\mathbb{Z}$ and the second one the equivariant direct image $q_*^W\mathbf{\Lambda}$, which we also denote by $(q_*\mathbf{\Lambda})^W$. Here we endow the constant sheaf $\mathbf{\Lambda}_{\mathfrak{t}}$ on $\mathfrak{t}$ with its natural W-equivariant structure (i.e. via the natural W-action on $\mathbf{\Lambda}$). For the category $Sh^W(\mathfrak{t}^\circ)$ of W-equivariant (abelian) sheaves, we can consider the functors

$$\begin{tikzcd} Sh^W(\mathfrak{t}^\circ) \arrow[r, bend right, "(q^\circ)_*^W"'] & Sh(\mathfrak{t}^\circ/W). \arrow[l, bend right, "(q^\circ)^*"'] \end{tikzcd} \tag{1.44}$$

It is known ([Ive86], IV.9), that they restrict to an equivalence between W-equivariant sheaves on $\mathfrak{t}^\circ$, whose underlying abelian sheaves are local systems, and local systems on $\mathfrak{t}^\circ/W$.
Since this relation between $R^2\sigma_*\mathbb{Z}$ and $q_*^W\mathbf{\Lambda}$ (at least over $\mathfrak{t}^\circ/W$ and $\mathfrak{t}^1/W$) is central to the connection between Hitchin to Calabi-Yau integrable systems (Chapter 5, especially Proposition 5.18), it is important to study it at some length. The first result is somewhat implicitly contained in [Slo80a], but we give its proof for completeness.

Proposition 1.76. *Let $S \subset \mathfrak{g}$ be a Slodowy slice in a simple complex Lie algebra $\mathfrak{g}$ of type Δ and let $(\Delta_h, AS(\Delta))$ be the associated pair. Further denote by $\mathbf{\Lambda}$ and $\mathbf{\Lambda}_h$ the corresponding coweight lattices of type Δ and Δ_h respectively (Remark 1.11). Then there are isomorphisms of local systems*[7]

$$R^2\sigma_*^\circ\mathbb{Z} \cong (q_*^\circ\mathbf{\Lambda}_h)^W, \quad (R^2\sigma_*^\circ\mathbb{Z})^{\mathbf{C}} \cong (q_*^\circ\mathbf{\Lambda})^W, \tag{1.45}$$

where $W = W(\Delta)$ is the Weyl group of type Δ and $\mathbf{C} \cong AS(\Delta)$ as before.

Remark 1.77. The $\mathbf{C}$-action on $R^2\sigma_*\mathbb{Z}$ in this proposition is defined on the presheaf level using the fact that $\mathbf{C}$ acts trivially on the base. Of course, if $\Delta = \Delta_h$ is of type ADE, so that $\mathbf{C} = 1$, then (1.45) reduces to one statement.

Proof. One way to see this is as follows: $\theta : \tilde{S} \to \mathfrak{t}$ is C^∞-trivial and $\sigma^\circ : S^\circ \to \mathfrak{t}^\circ/W$ at least locally C^∞-trivial (see [Slo80a]). Moreover, the restriction of the simultaneous resolution,

$$\begin{tikzcd} \tilde{S}^\circ \arrow[r] \arrow[d, "\tilde{\sigma}^\circ"'] & S^\circ \arrow[d, "\sigma^\circ"] \\ \mathfrak{t}^\circ \arrow[r] & \mathfrak{t}^\circ/W \end{tikzcd}$$

[7] Here and in the following we use $(q^\circ)_*^W\mathbf{\Lambda}$ and $((q^\circ)_*\mathbf{\Lambda})^W$ interchangeably.

to the regular loci is in fact *cartesian.* From these facts, it immediately follows that $R^2\sigma^\circ_*\mathbb{Z}$ is a local system with stalk $\mathbf{\Lambda}_h$. For the moment we consider the case $\Delta = \Delta_h$. Then by base change[8] and the cartesian property, we see that

$$q^{\circ,*}R^2\sigma^\circ_*\mathbb{Z} \cong R^2\theta^\circ_*\mathbb{Z} \cong \underline{\mathbf{\Lambda}_h}_{\mathfrak{t}^\circ}.$$

Therefore one obtains the structure of a W-module on $\mathbf{\Lambda}_h$. Now recall that $W = W(\Delta_h)$ is the monodromy group of the singularity Y of type Δ_h. From Picard-Lefschetz theory it is known (e.g. [Slo80a]) that the action of the monodromy group on $H^2(\hat{Y}, \mathbb{Z}) = \mathbf{\Lambda}_h$ coincides with the natural action of W on $\mathbf{\Lambda}_h$. Then the above mentioned equivalence induced from (1.44) implies that $R^2\sigma^\circ_*\mathbb{Z} \cong (q^\circ_*\mathbf{\Lambda}_h)^W$.
If Δ is of type BCFG and $\mathbf{C} \neq \{1\}$, then the typical stalk of $R^2\sigma^\circ_*\mathbb{Z}$ is still $\mathbf{\Lambda}_h$. By an analogous argument as before (also see [Slo80a], 4.6.), we see that $R^2\sigma^\circ_*\mathbb{Z}$ has monodromy W, which we consider as the subgroup $W_h^{\mathbf{C}} \subset W_h$. Then the $\mathbf{C}$-action on $R^2\sigma_*\mathbb{Z}$ restricts to an action on the local system $R^2\sigma^\circ_*\mathbb{Z}$. By construction, this $\mathbf{C}$-action coincides with the stalkwise $\mathbf{C}$-action which is well-defined because the W-action commutes with the $\mathbf{C}$-action. Since $\mathbf{\Lambda}_h^{\mathbf{C}} = \mathbf{\Lambda}$, we conclude that $(R^2\sigma^\circ_*\mathbb{Z})^{\mathbf{C}} \cong (q^\circ_*\mathbf{\Lambda})^W$. □

The above isomorphism can be extended over the strata $\mathfrak{t}_{(s_\alpha)}/W$ to all of $\mathfrak{t}^1/W$.

Corollary 1.78. *Keep the notation of Proposition 1.76. Consider the restrictions $\sigma^1 : S^1 \to \mathfrak{t}^1/W$ and $q^1 : \mathfrak{t}^1 \to \mathfrak{t}^1/W$ of σ and q to the respective loci over $\mathfrak{t}^1/W \subset \mathfrak{t}/W$. Then there are isomorphisms*

$$R^2\sigma^1_*\mathbb{Z} \cong (q^1_*\mathbf{\Lambda}_h)^W, \quad (R^2\sigma^1_*\mathbb{Z})^{\mathbf{C}} \cong (q^1_*\mathbf{\Lambda})^W. \tag{1.46}$$

In particular, these sheaves are constructible with respect to the natural stratification (1.42).

Proof. To save notation we just write $\sigma = \sigma^1$ and $q = q^1$ during this proof. As a start we restrict to the case $\mathbf{C} = \{1\}$ first, so that $\mathbf{\Lambda} = \mathbf{\Lambda}_h$. Denote by $j : \mathfrak{t}^\circ/W \hookrightarrow \mathfrak{t}^1/W$ the inclusion of the open stratum $\mathfrak{t}^\circ/W$ in $\mathfrak{t}^1/W$ and by $D \subset \mathfrak{t}^1/W$ its complement. Further we define the shortcuts $\mathcal{F}_0 := (q_*\mathbf{\Lambda})^W_{|\mathfrak{t}^1/W}$, $\mathcal{F}_1 := R^2\sigma_*\mathbb{Z}_{|\mathfrak{t}^1/W}$. We claim that the adjunction morphisms

$$a_i : \mathcal{F}_i \to j_*j^*\mathcal{F}_i, \quad i = 0, 1, \tag{1.47}$$

are in fact isomorphisms. Then (1.46) follows from Corollary 1.76 since $j^*\mathcal{F}_0 \cong (q^\circ_*\mathbf{\Lambda})^W$ and analogously $j^*\mathcal{F}_1 \cong R^2\sigma^\circ_*\mathbb{Z}$. Clearly, a_i is an isomorphism away from D, so we only need to show it at a point $d \in D$. Let $\alpha \in R^+$ be a positive root representing the W-orbit corresponding to d. Since D is smooth, we can choose a disk C_d intersecting D transversely in d (with no other intersection points). Restricting to C_d, it suffices to show that $a_{i|C_d}$ is an isomorphism in d.

For $\mathcal{F}_0$, the claim actually follows directly since $j_*(q^\circ_*\mathbf{\Lambda})^W = (q^1_*\tilde{j}_*\mathbf{\Lambda})^W = (q^1_*\mathbf{\Lambda})^W$ for the inclusion $\tilde{j} : \mathfrak{t}^\circ \hookrightarrow \mathfrak{t}^1$. However, we also pursue the approach above, to give a local description of the restriction $q^{-1}(C_d) =: \tilde{C}_d \to C_d$ of q as well. By construction, it is a branched W-Galois covering which is simply ramified over d. More precisely, let $q^{-1}(d) = \{d_1, \ldots, d_s\}$ be the set of ramification points, where $s = |W|/2$. We further choose $\beta_i \in W \cdot \alpha \cap R^+$, such that $d_i \in \mathfrak{t}_{\beta_i}$, and assume $\beta_1 = \alpha$. With this notation, we can write (assuming C_d to be small enough)

$$q^{-1}(C_d) = \coprod_{i=1}^{|W|/2} \tilde{C}_i \to C_d, \quad d_i \in \tilde{C}_i, \tag{1.48}$$

[8]To be precise, we need base change for locally trivial fibrations here (since the maps are *not* proper but locally trivial). This version can be proven as proper base change since $R^kf_*\mathcal{F}_y \cong H^k(f^{-1}(y), \mathcal{F})$ still holds for a locally trivial map $f : X \to Y$ (and non-singular Y).

where $\tilde{C}_i$ are connected disks. Now let $q_i : \tilde{C}_i \to C_d$ be the corresponding restrictions of q (in appropriate coordinates it is given by $z \mapsto z^2$), so that we are reduced to the A_1-case. If $\rho_i = s^\vee_{\beta_i}$ denote the induced reflections on $\mathbf{\Lambda}$, we obtain

$$\begin{aligned}(q_*\mathbf{\Lambda})^W_{|C_d} &\cong \left(\bigoplus_i (q_{i,*}\mathbf{\Lambda})^{\rho_i}_{|C_d}\right)^W \\ &\cong (q_{1,*}\mathbf{\Lambda})^{\rho_1}_{|C_d}.\end{aligned}$$

The second isomorphism is non-canonical, since we have to choose an element of the W-orbit $W \cdot \alpha$. However, it implies that the adjunction morphism is an isomorphism at d since it is true for the single summands: The restrictions

$$(q_{i,*}\mathbf{\Lambda})^{\rho_i}(C_d) \to (q_{i,*}\mathbf{\Lambda})^{\rho_i}(C_d^*) \tag{1.49}$$

are isomorphisms for C_d small enough. Indeed, we may assume that C_d is the unit disc C_1 and $q_\beta : C_1 \to C_1, z \mapsto z^2$. Since C_1 and the punctured disk C_1^* are connected, it follows that

$$(q_{i,*}\mathbf{\Lambda})^{\rho_\beta}(C_1) = \mathbf{\Lambda}^{\rho_i} = (q_{\beta,*}\mathbf{\Lambda})^{\rho_i}(C_1^*).$$

Taking the limit over a neighborhood basis of d (e.g. of arbitrarily small disks centered at d), we conclude that (1.47) is an isomorphism for $i = 0$. As a by-product, we see that $(q_*\mathbf{\Lambda})^W_d \cong \mathbf{\Lambda}^{\rho_\alpha}$ (again non-canonically).

We can argue similarly for $\mathcal{F}_1$. The only essential difference is that in order to conclude (1.47), we need to invoke the monodromy group of Kleinian singularities. Let $\sigma_d : S_{|\sigma^{-1}(C_d)} \to C_d$ be the restriction of $\sigma : S \to \mathfrak{t}/W$ and fix $t_0 \in C_d$. The Picard-Lefschetz transformation $r_\alpha : \mathbf{\Lambda} \to \mathbf{\Lambda}$ (recall $\mathbf{\Lambda} \cong H^2(S_{t_0}, \mathbb{Z})$), which corresponds to the monodromy around d, is $r_\alpha = \rho_\alpha = s^\vee_\alpha$ (cf. Proposition 1.76). Hence the cycle $c_\alpha \in H^2(S_{t_0}, \mathbb{Z})$ corresponding to α is a vanishing cycle ([AGZV12]) so that $(R^2\sigma_{d,*}\mathbb{Z})_d \cong \mathbf{\Lambda}^{\rho_\alpha}$. Therefore both sides of (1.47) ($i = 1$) have stalks $\mathbf{\Lambda}^{\rho_\alpha}$ and a_1 is an isomorphism.

Now we treat the case $\mathbf{C} \neq \{1\}$. The argument for $\mathcal{F}_0$ also applies to $\mathcal{F}_0^{\mathbf{C}}$. For $\mathcal{F}_1^{\mathbf{C}}$ we consider $W = \{w \in W_h \mid \mathbf{c}w = w\mathbf{c}\ \forall \mathbf{c} \in \mathbf{C}\} \subset W_h$ as a subgroup of the unfolded Weyl group W_h and $R = R_h^{\mathbf{C}} \subset R_h$, $\mathbf{\Lambda} = \mathbf{\Lambda}_h^{\mathbf{C}} \subset \mathbf{\Lambda}_h$. Without loss of generality, we may assume that d corresponds to a root $\beta = \sum_i \beta_i$, where β_i are simple roots of R_h that are orthogonal to each other. Hence the monodromy reflection is given by $\rho_\beta = \prod_i \rho_{\beta_i} \in W \subset W_h$[9]. It follows that

$$\mathbf{\Lambda}^{\rho_\beta} = ((\mathbf{\Lambda}_h)^{\mathbf{C}})^{\rho_\beta} = (\mathbf{\Lambda}_h^{\rho_\beta})^{\mathbf{C}}.$$

We know that $\mathcal{F}_{1,d} = \mathbf{\Lambda}_h^{\rho_\beta}$ from above. Since the $\mathbf{C}$-action on S is continuous, $\mathbf{C}$ acts on the stalk $\mathbf{\Lambda}_h^{\rho_\beta}$ in its natural way (because it acts by its natural action on $\mathbf{\Lambda}_h$ over $\mathfrak{t}^\circ/W$). This implies that $\mathcal{F}_{1,d}^{\mathbf{C}} = (\mathbf{\Lambda}_h^{\rho_\beta})^{\mathbf{C}} = \mathbf{\Lambda}^{\rho_\beta}$. But the latter is also the stalk of $j_*j^*(\mathcal{F}_1^{\mathbf{C}})$ at d, showing that the the adjunction morphism $a_1 : \mathcal{F}_1^{\mathbf{C}} \to j_*j^*\mathcal{F}_1^{\mathbf{C}}$ is an isomorphism at d, hence all over $\mathfrak{t}^1/W$.

It remains to study constructibility over $(\mathfrak{t}^1 - \mathfrak{t}^\circ)/W$, which is sufficient to show for the sheaf $\mathcal{F}_0 = (q_*^1\mathbf{\Lambda}_h)^W$. Constructibility follows immediately from the fact that $q : \mathfrak{t} \to \mathfrak{t}/W$ is a covering, when restricted to $\mathfrak{t}^1 - \mathfrak{t}^\circ$. Indeed, set $\mathfrak{t}_{\langle\alpha\rangle} = \mathfrak{t}_{\langle s_\alpha\rangle}$ and $\mathfrak{t}_{(\alpha)} = \mathfrak{t}_{(\langle s_\alpha\rangle)}$ (cf. (1.38)). We know that $q_{\langle\alpha\rangle} : \mathfrak{t}_{\langle\alpha\rangle} \to \mathfrak{t}_{\langle\alpha\rangle}/\Gamma(\rho_\alpha)$ is a covering with covering group

$$\Gamma(\alpha) := \Gamma(\langle\rho_\alpha\rangle) = N_W(\langle\rho_\alpha\rangle)/\langle\rho_\alpha\rangle.$$

[9]Strictly speaking, we would have to write $W^\vee \subset W_h^\vee$ here. But since $W \cong W^\vee$ naturally, we save notation and omit the superscript $^\vee$.

We assume for the moment, that all the roots have the same length. Then[10]

$$q^{-1}(\mathfrak{t}_{(\alpha)}/W) = \mathfrak{t}_{(\alpha)} = \coprod_{\beta \in R^+} \mathfrak{t}_{\langle\beta\rangle}, \tag{1.50}$$

where R^+ is a choice of positive roots containing α (cf. Example 1.65). Of course, W permutes the $\mathfrak{t}_{\langle\beta\rangle}$. This implies that $q_{(\alpha)} : \mathfrak{t}_{(\alpha)} \to \mathfrak{t}_{(\alpha)}/W$ is a covering as well. Hence

$$R^2\sigma_*\mathbb{Z}_{|\mathfrak{t}_{(\alpha)}/W} \cong (q_*\mathbf{\Lambda})^W_{|\mathfrak{t}_{(\alpha)}/W} = (q_{(\alpha),*}\mathbf{\Lambda})^W \cong (q_{\langle\alpha\rangle,*}\mathbf{\Lambda}^{\rho_\alpha})^{\Gamma(\alpha)}$$

are local systems. The last isomorphism is constructed as follows: Let $U \subset \mathfrak{t}_{(\alpha)}/W$ be an open subset. Then define

$$\begin{aligned} \varphi_U : (q_{(\alpha),*}\mathbf{\Lambda}_h)^W(U) &\to (q_{\langle\alpha\rangle,*}\mathbf{\Lambda}_h^{\rho_\alpha})(U), \\ f &\mapsto f_{|q^{-1}_{\langle\alpha\rangle}(U)}, \end{aligned}$$

i.e. we restrict onto the factor $\mathfrak{t}_{\langle\alpha\rangle}$ in (1.50). It is not hard to see that this defines a well-defined sheaf morphism $\varphi : (q_{(\alpha),*}\mathbf{\Lambda})^W \to (q_{\langle\alpha\rangle,*}\mathbf{\Lambda}^{\rho_\alpha})^{\Gamma(\alpha)}$. It is an isomorphism because it is injective and both sheaves have stalk $\mathbf{\Lambda}^{\rho_\alpha}$. Note that this description is again non-canonical because we have chosen α. The case of two lengths works similar by distinguishing whether α is a short or a long root (cf. (1.42)). □

Example 1.79. Let us consider the A_2-case and let $\alpha \in R$ be a root. Then $W = S_3$ and the normalizer $N_W(\langle s_\alpha\rangle)$ has order 2. Therefore $\Gamma(\alpha) = 1$, so that $q_\alpha : \mathfrak{t}_{\langle\alpha\rangle} \to \mathfrak{t}_{\langle\alpha\rangle}/\Gamma(\alpha) \cong \mathfrak{t}_{\langle\alpha\rangle}/W$ is an isomorphism. In particular,

$$(q_{\alpha*}\mathbf{\Lambda}^{\rho_\alpha})^{\Gamma(\alpha)} \cong \underline{\mathbf{\Lambda}}^{\rho_\alpha}_{\mathfrak{t}_{(\alpha)}/W} \cong R^2\sigma_*\mathbb{Z}_{|\mathfrak{t}_{(\alpha)}/W}$$

is the constant sheaf. Note that $\mathfrak{t}_{(\alpha)}/W \cong \mathbb{C}^*$ and the fibers of σ over this locus have A_1-singularities and $\sigma^{-1}(\bar{0})$ is an A_2-singularity.
For A_k, $k \geq 3$, this is no longer true, because then $|\Gamma(\alpha)| = |N_W(\langle\alpha\rangle)/\langle\alpha\rangle| = (k-1)! \neq 1$.

Remark 1.80. By Slodowy's result, we have seen that the natural stratification on $\mathfrak{t}/W$ essentially coincides with the stratification by the type of the singularity configuration. It seems plausible that over these strata, σ is in fact topologically locally trivial. This holds true at least on a (Zariski-)dense open subset in the strata (cf. [Ver76]). Corollary 1.76 gives evidence that this is in fact true on all of the strata. It would be interesting to study deeper strata, for which the language of perverse sheaves seems to be most appropriate (also see Section 5.2).

The isomorphism of Corollary 1.76 can be lifted to variations of (mixed) Hodge structures. Intuitively, this is not too surprising because the cohomology of the fibers of σ is quite simple. However, we still give a proof following ideas from [DDP07].

Lemma 1.81. *$R^2\sigma^\circ_*\mathbb{Z}$ underlies a variation of pure Hodge structures of weight 2 of Tate type. It is isomorphic to $(q^\circ_*\mathbf{\Lambda}_h)^W(-1)$, i.e. the local system $(q^\circ_*\mathbf{\Lambda}_h)^W$ with a Tate twist. Furthermore, if $\mathbf{C} \neq \{1\}$ then $(R^2\sigma^\circ_*\mathbb{Z})^{\mathbf{C}} \cong (q^\circ_*\mathbf{\Lambda})^W(-1)$.*

Proof. Let $\hat{Y} := \tilde{S}_0 \cong \widehat{\mathbb{C}^2/\Gamma}$ be the minimal resolution of $Y := \mathbb{C}^2/\Gamma$, where $\Gamma \subset SL(2,\mathbb{C})$ corresponds to $\mathfrak{g}$. We claim that $H^2(\hat{Y},\mathbb{Z}) \cong \mathbf{\Lambda}(-1)$ as pure Hodge structures so that $H^2(\hat{Y},\mathbb{Z})$

[10] Note that this is different from the situation in (1.48) because we consider the whole $\mathfrak{t}_{\langle\alpha\rangle}$.

is of weight 2 and Tate type. We have already seen that the underlying groups are isomorphic. To see the claim on the Hodge structures, consider the compactifications

$$\begin{array}{ccc} \hat{Y} & \hookrightarrow & \hat{Z} := \widehat{\mathbb{CP}^2} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \mathbb{CP}^2/\Gamma \end{array}$$

of Y and $\hat{Y}$ respectively. Here Γ acts on $\mathbb{CP}^2$ via the natural inclusion $SL(2,\mathbb{C}) \hookrightarrow SL(3,\mathbb{C})$. Since Z has only isolated orbifold surface singularities (this is a direct computation, also see [Sai87], [RŞ15]), it follows that the divisor $D := \hat{Z} - \hat{Y}$ consists of trees of $\mathbb{CP}^1$'s. Moreover, $\hat{Y}$ only has even cohomology, so that the relative cohomology sequence yields the exact sequence (with $\mathbb{Z}$-coefficients)

$$0 \longrightarrow H^2(\hat{Z},\hat{Y}) \longrightarrow H^2(\hat{Z}) \longrightarrow H^2(\hat{Y}) \longrightarrow H^3(\hat{Z},\hat{Y}).$$

Now $H^3(\hat{Z},\hat{Y})$ is naturally isomorphic to the local cohomology group $H^3_D(\hat{Z})$ and Lefschetz duality ([PS08]) implies

$$H^3(\hat{Z},\hat{Y}) \cong H^3_D(\hat{Z}) \cong H^{BM}_{4-3}(D) = H^{BM}_1(D) = H_1(D)$$

for Borel-Moore homology. Note that the last equality uses that D is compact. However, $H_1(D) = 0$ by the above observation, so that we have a surjection

$$H^2(\hat{Z}) \twoheadrightarrow H^2(\hat{Y}).$$

All the above morphisms are compatible with mixed Hodge structures ([PS08]). Consequently, Deligne's mixed Hodge structure on $H^2(\hat{Y})$ is pure of weight 2 and of Tate type, in particular $H^2(\hat{Y},\mathbb{Z}) \cong \mathbf{\Lambda}(-1)$.

Since the family $\theta : \tilde{S} \to \mathfrak{t}$ is topologically trivial, $R^2\theta_*\mathbb{Z}$ carries the structure of a $\mathbb{Z}$-variation of mixed Hodge structures $\mathsf{V} = (\mathsf{V}_{\mathbb{Z}}, \mathbb{W}_\bullet, \mathcal{F}^\bullet)$ (cf. [BEZ14]). By construction it satisfies

$$(\mathsf{V}_{\mathbb{Z}}, \mathbb{W}_\bullet, \mathcal{F}^\bullet)_t \cong (H^2(\tilde{S}_t,\mathbb{Z}), W^t_\bullet, F^\bullet_t), \quad t \in \mathfrak{t},$$

as mixed Hodge structures. On the right hand side we have Deligne's mixed Hodge structure $(H^2(\tilde{S}_t,\mathbb{Z}), W^t_\bullet, F^\bullet_t)$, which makes sense because each fiber $\tilde{S}_t$ is algebraic. Since the weight filtration $\mathbb{W}$ is locally constant, the above considerations imply that it has to be zero. The same argument shows that the Hodge filtration is trivial,

$$0 = \mathcal{F}^1 \subset \mathcal{F}^2 = \mathsf{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{t}}.$$

In other words, we obtain an isomorphism of $\mathbb{Z}$-VMHS:

$$\mathsf{V} \cong \underline{\mathbf{\Lambda}}_{\mathfrak{t}}(-1).$$

Now we can conclude the final statement from the commutative diagram

$$\begin{array}{ccccc} \tilde{S}_{|\mathfrak{t}^\circ} & \xrightarrow{\cong} & q^*S_{|\mathfrak{t}^\circ} & \longrightarrow & S_{|\mathfrak{t}^\circ/W} \\ \downarrow{\scriptstyle \theta^\circ} & & \downarrow & & \downarrow{\scriptstyle \sigma^\circ} \\ \mathfrak{t}^\circ & = & \mathfrak{t}^\circ & \xrightarrow{q^\circ} & \mathfrak{t}^\circ/W. \end{array}$$

Here we need again that $R^2\sigma_*^\circ\mathbb{Z}$ carries the structure of a natural VMHS. But since $S_{\bar{t}} \cong \tilde{S}_t$ as algebraic varieties (using the fact that θ is a simultaneous (minimal) resolution), the claim follows from the previous observations. Note however that $R^2\sigma_*^\circ\mathbb{Z}$ is no longer constant, but isomorphic to $(q_*^\circ\mathbf{\Lambda})^W$ by Proposition 1.76 proving the second claim.
Finally, it remains to consider the case $\mathbf{C} \neq \{1\}$. Since $\mathbf{C}$ acts by algebraic automorphisms, $(R^2\sigma_*^\circ\mathbb{Z})^{\mathbf{C}}$ is a sub-V(M)HS of $R^2\sigma_*^\circ\mathbb{Z}$. The rest is now again a consequence of Proposition 1.76. □

1.5 The relative symplectic and Kostant-Kirillov form

We now turn to the symplectic geometry of the adjoint quotient and Grothendieck's simultaneous resolution. Of particular interest for us is the relative symplectic form on Grothendieck's simultaneous resolution due to Yamada ([Yam95]). This will be useful in constructing $AS(\Delta)$-invariant volume forms in Chapter 5, which is crucial for establishing a relation between BCFG-Hitchin systems and Calabi-Yau integrable systems. This relative symplectic form is related to the Kostant-Kirillov form, as we will show later in this section. Even though the latter is partially known, we investigate this relation in more detail and further extend it.
Kronheimer gave similar constructions as Yamada ([Kro89]) via hyperkähler geometry. But it is not clear to us, how they are precisely related.

1.5.1 Relative symplectic reduction

The following has been considered in [Yam95] in a more special case. Since it is interesting in itself we give here a brief account of what holds in more generality. Let H be a complex Lie group acting on a complex manifold M[11]. We denote by X_ξ the vector field on M associated with $\xi \in \mathfrak{h} = \mathrm{Lie}(H)$. For example, if H acts on $\mathfrak{h}$ by the adjoint action, then $X_\xi(\eta) = [\xi, \eta]$ under the natural trivialization $TH \cong H \times \mathfrak{h}$. Similarly, if H acts by the coadjoint action on $\mathfrak{h}^*$, then $X_\xi : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by

$$X_\xi(\lambda)(\eta) = -\lambda([\eta, \xi]). \tag{1.51}$$

In case M and N are H-manifolds and $f : M \to N$ is a H-equivariant map, then $df(X_\xi) = f^*Y_\xi$ as sections of the pullback f^*TN.
Now assume that (M, ω) is symplectic and H not only acts symplectically, but even in a Hamiltonian way. Then there exists a moment map $\mu : M \to \mathfrak{h}^*$ which is in particular H-equivariant. If f_ξ is the Hamiltonian function, such that $\omega(X_\xi, -) = df_\xi(-)$, then it satisfies

$$\langle d\mu(p), \xi\rangle = f_\xi(p) \quad \forall p \in M. \tag{1.52}$$

By the H-equivariance the level sets $\mu^{-1}(\lambda)$ are preserved by $H_\lambda \subset H$, the isotropy group for the coadjoint action. In case $\lambda \in \mathfrak{h}^*$ is a regular value, $\mu^{-1}(\lambda)$ is a submanifold. If further H_λ acts properly and freely on $\mu^{-1}(\lambda)$ then the famous Marsden-Weinstein theorem ([MW74]) says that

$$(\mu^{-1}(\lambda)/H_\lambda, \omega_\lambda)$$

is a symplectic manifold. Here ω_λ is the restriction of ω to $\mu^{-1}(\lambda)$.

We are interested in the following situation: Assume that the Hamiltonian H-action on (M, ω)

[11] Manifolds and Lie groups will be complex throughout this section unless otherwise stated.

preserves the fibers of the moment map μ. Then there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\mu"'] & M/H \arrow[d, "\hat{\mu}"] \\ & \mathfrak{h}^*. \end{tikzcd}$$

It is natural to ask under which conditions there is a relative 2-form $\hat{\omega} \in \Omega^2_{\hat{\mu}}(\mathfrak{h}^*)$ such that

$$\hat{\omega}_{|\mu^{-1}(\lambda)} = \omega_\lambda$$

on $\hat{\mu}^{-1}(\lambda) = \mu^{-1}(\lambda)/H$. Such a situation we call *relative symplectic reduction.*

We first make some more or less obvious observations. For convenience, we restrict to the case where μ is a submersion. This already implies that H_p is finite for all $p \in M$ ([LM87]). Then it is a strong restriction that H preserves the fibers of the moment map μ: Indeed, all the X_ξ, $\xi \in \mathfrak{h}$, are tangent to the fibers of μ so that $d\mu(X_\xi)(p) = Y_\xi \circ \mu(p) = Y_\xi = 0$ for all $p \in \mu^{-1}(\lambda)$. Hence by (1.51)

$$\lambda([\xi, \eta]) = 0, \quad \forall \lambda \in \mathfrak{h}^*, \forall \xi, \eta \in \mathfrak{h}.$$

Since $\lambda \in \mathfrak{h}^*$ is arbitrary, this implies that $[\mathfrak{h}, \mathfrak{h}] = 0$ and hence $\mathfrak{h}$ and H have to be abelian. Observe that this argument also works, even if we only assume that the H-action preserves the fibers of μ over a neighborhood of $0 \in \mathfrak{h}^*$. This leads us to the following version of relative symplectic reduction. Part of it appeared in [Yam95], but since we could not follow all of the arguments there we give our own proof.

Proposition 1.82 (Relative symplectic reduction)**.** *Let (M, ω) be a (holomorphic) symplectic manifold and let H be an* abelian *Lie group that acts freely and properly on M. Then we have the following diagram*

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[dr, "\mu"'] & M/H \arrow[d, "\hat{\mu}"] \\ & \mathfrak{h}^*. \end{tikzcd} \tag{1.53}$$

If ω' denotes the image of ω in $\Gamma^2(M, \Omega^2_\mu)$, then there exists a unique relative symplectic form $\hat{\omega} \in \Gamma^2(M/H, \Omega^2_{\hat{\mu}})$ such that

$$\pi^* \hat{\omega} = \omega'.$$

The restriction of $\hat{\omega}$ to the fiber $\hat{\mu}^{-1}(\lambda)$, $\lambda \in \mathfrak{t}^*$, is the symplectic form coming from symplectic reduction.

Proof. By assumption, the diagram (1.53) exists and μ as well as $\hat{\mu}$ are submersions. The former holds true because a moment map is a submersion iff the corresponding Hamiltonian action has finite stabilizers only ([LM87]). Therefore the relative cotangent sheaves Ω^1_μ and $\Omega^1_{\hat{\mu}}$ are locally free (and finitely generated) and we have an inclusion $\pi^*\Omega^1_{\hat{\mu}} \xrightarrow{d\pi^*} \Omega^1_\mu$. By local freeness and finiteness, this gives an inclusion

$$\pi^*\Omega^2_{\hat{\mu}} \xrightarrow{\wedge^2 d\pi^*} \Omega^2_\mu \ .$$

A section $s \in \Gamma(M, \Omega^2_\mu)$ lies in $\Gamma(M, \pi^*\Omega^2_{\hat{\mu}})$ iff $s(v, -) = 0$ where v is any local section of the relative tangent sheaf T_π (i.e. which are tangent to the fibers). In the situation at hand, the relative tangent sheaf is globally spanned by the vector fields $\tilde{\xi} \in \Gamma(M, T_\pi)$ of the group action for $\xi \in \mathfrak{h}$. Since the image $\omega' \in \Gamma(M, \Omega^2_\mu)$ of ω is H-invariant by construction and

$$\Gamma(M, \Omega^2_{\hat{\mu}}) \xrightarrow{\pi^*} \Gamma(M, \pi^*\Omega^2_{\hat{\mu}})^H$$

is an isomorphism, it remains to prove that $\omega' \in \Gamma(M, \pi^*\Omega_{\hat{\mu}})$. This is equivalent to showing that

$$\omega'(\tilde{\xi}, \tilde{\eta}) = 0 \quad \forall \xi, \eta \in \mathfrak{h}.$$

However, the property (1.52) of a moment map implies

$$\begin{aligned} \omega(\tilde{\xi}, -) &= d\langle \mu(-), \xi \rangle \\ &= d(\mu^*(\langle -, \xi \rangle)) \\ &= \mu^* d(\langle -, \xi \rangle) \in \mu^*\Omega^1_{\mathfrak{h}^*}. \end{aligned}$$

By definition of the relative cotangent sheaves Ω^k_μ, it follows that $\omega'(\tilde{\xi}, v) = 0$ for any local vector field v on M. Hence we have proven that

$$\omega' \in \Gamma(M, \pi^*\Omega^2_{\hat{\mu}})^H \cong \Gamma(M/H, \Omega^2_{\hat{\mu}}).$$

The last statement of the proposition immediately follows by restricting to the fibers of $\hat{\mu} : M/H \to \mathfrak{h}^*$. □

Remark 1.83. One can weaken the assumptions of the previous proposition in several ways. For example, the assumption that H is abelian can be relaxed as follows (still assuming that μ is a submersion): For a Lie subgroup $H' \subset H$ consider

$$\mathfrak{h}^*_{H'} = \{\lambda \in \mathfrak{h}^* \mid H_\lambda = H'\}. \tag{1.54}$$

Then $M' := \mu^{-1}(\mathfrak{h}^*_{H'})$ is a submanifold, giving the commutative diagram

$$\begin{tikzcd} M' \arrow[r, "\pi'"] \arrow[dr, "\mu'"'] & M'/H' \arrow[d, "\hat{\mu}'"] \\ & \mathfrak{h}^*_{H'}. \end{tikzcd}$$

In case H' acts freely and properly on M' similar results hold as before. In some sense this gives a stratified version of the above. But since we will not need it later on, we do not pursue it here any further.

We end this section with another observation that will be needed later on. Let H be an abelian Lie group and K a connected Lie group. Assume that they act on a (holomorphic) symplectic manifold (M, ω) in a Hamiltonian way and that their actions commute. We denote by $\mu_H : M \to \mathfrak{h}^*$ and $\mu_K : M \to \mathfrak{k}^*$ the corresponding moment maps. If μ_K is constant on the H-orbits, we obtain the following commutative diagram

$$\begin{tikzcd} M \arrow[dr, "\pi"] \arrow[drr, bend left, "\mu_G"] \arrow[ddr, bend right, "\mu_H"'] & & \\ & M/H \arrow[r, "\hat{\mu}_K"] \arrow[d, "\hat{\mu}_H"'] & \mathfrak{k}^* \\ & \mathfrak{h}^*. & \end{tikzcd}$$

It turns out ([LM87], Theorem 6.2) that there is an induced Hamiltonian K-action on the reduced symplectic manifold $(\hat{\mu}_H^{-1}(\lambda), \hat{\omega}(\lambda))$, $\lambda \in \mathfrak{h}^*$, such that

$$\mu_{K,\lambda} = \hat{\mu}_{K|\hat{\mu}_H^{-1}(\lambda)} : \hat{\mu}_H^{-1}(\lambda) = \mu_H^{-1}(\lambda)/H \to \mathfrak{k}^* \tag{1.55}$$

is a moment map for this action.

1.5.2 Yamada's construction

Let $\mathfrak{g}$ be a simple complex Lie algebra and $G = G_{ad}$ its adjoint group. Further, fix a maximal torus $T \subset G$ together with a Borel subgroup $B \supset T$ and denote by $N \subset B$ the nilradical of B. The corresponding Lie algebras are denoted by $\mathfrak{t}$, $\mathfrak{b}$ and $\mathfrak{n}$ respectively. Then we have $B = T \ltimes N$ and $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. Yamada studied the following special case of the previous section with

$$(M, \omega) = (T^*(G/N), \omega_{can}), \quad H = T, \quad K = G,$$

where ω_{can} is the canonical symplectic form. Let us briefly describe the corresponding actions. The homogeneous space G/N carries two natural actions

$$G \curvearrowright G/N, \quad G/N \curvearrowleft T.$$

Here the first action is simply given by multiplication from the left and the second one by multiplication from the right,

$$g \cdot \overline{h} = \overline{gh}, \quad \overline{g} \cdot t = \overline{gt}, \quad g, h \in G, t \in T.$$

We denote by L_g and R_t the induced actions on the cotangent bundle $T^*(G/N)$. They leave the Liouville 1-form θ on $T^*(G/N)$ invariant and therefore the canonical symplectic form $\omega = d\theta$ is left invariant as well. It follows that one obtains two moment maps for these (Hamiltonian) actions (Proposition 2.1 in [Yam95])

$$\mu_G : T^*(G/N) \to \mathfrak{g}^*, \quad \mu_T : T^*(G/N) \to \mathfrak{t}^*.$$

Clearly, T acts properly on $T^*(G/N)$ and freely because it does so on G/N. Moreover, both moment maps are T-invariant, so that they descend to maps $\hat{\mu}_G$ and $\hat{\mu}_T$ from $T^*(G/N)/T$ to $\mathfrak{g}^*$ and $\mathfrak{t}^*$ respectively (also cf. (1.5.1)). The next theorem summarizes some of the key results from [Yam95]. It gives in particular a symplectic-geometric construction of Grothendieck's simultaneous resolution.

Theorem 1.84 (Yamada). *Let $M = T^*(G/N)$ be as before. Then the above maps fit into the following commutative diagram*

$$\begin{tikzcd}
M \arrow[dr, "\pi_T"] \arrow[drr, bend left, "\mu_G"] \arrow[ddr, bend right, "\mu_T"'] & & \\
& M/T \arrow[r, "\hat{\mu}_G"] \arrow[d, "\hat{\mu}_T"] & \mathfrak{g}^* \arrow[d, "\chi^*"] \\
& \mathfrak{t}^* \arrow[r, "q^*"] & \mathfrak{t}^*/W.
\end{tikzcd} \tag{1.56}$$

The morphisms χ^* and q^* are the coadjoint quotient and the ordinary quotient respectively. After identifying $\mathfrak{g}^* = \mathfrak{g}$ and $\mathfrak{t}^* = \mathfrak{t}$ via the Killing form, the square is isomorphic to Grothendieck's simultaneous resolution (cf. Section 1.4.2).

Sketch of proof. The basic idea is the observation that

$$T^*(G/N) \cong G \times^N \mathfrak{n}^\perp = G \times^N \mathfrak{b},$$

which holds for general homogeneous spaces. It follows that $(G \times^N \mathfrak{b})/T \cong G \times^B \mathfrak{b} \cong \tilde{\mathfrak{g}}$ holomorphically.

Then it can be shown that $\hat{\mu}_T$ and $\hat{\mu}_G$ correspond to $\theta : \tilde{\mathfrak{g}} \to \mathfrak{t}$ and $\psi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ respectively (Lemma 2.4 in [Yam95]). □

Remark 1.85.

a) Yamada's isomorphism holds in the complex-analytic category. It seems plausible that it is even algebraic. However, all arguments involving (relative) symplectic reduction are complex-analytic in nature, especially taking quotients. This is the reason why we mainly work in the complex-analytic category in this section.

b) Note that the theorem (together with Slodowy's results) implies that $\hat{\mu}_T$ is in fact a C^∞-trivial fiber bundle with typical fiber $T^*(G/B)$ (which is the fiber over $0 \in \mathfrak{t}^*$, cf. 1.4.2). This can also be seen directly by using a compact real form in G, cf. [Yam95], Section 3.

Proposition 1.82 immediately implies:

Corollary 1.86. *Grothendieck's simultaneous resolution $\theta : \tilde{\mathfrak{g}} \cong G \times^B \mathfrak{b} \to \mathfrak{t}$ carries a relative symplectic form $\hat{\omega} \in \Omega^2_\theta(\tilde{\mathfrak{g}})$ coming from the canonical symplectic form on $T^*(G/N)$ and Yamada's isomorphism.*

1.5.3 Relation to the Kostant-Kirillov form

The diagram (1.56) suggests that there is a relation between the Kostant-Kirillov form and the relative symplectic form. Before giving such a relation, we briefly recall the Kostant-Kirillov form (see [CG10] for more details). It is most frequently considered on coadjoint orbits $O_\lambda = G \cdot \lambda$ for $\lambda \in \mathfrak{g}^*$ where $\mathfrak{g}$ is a complex Lie algebra. In that situation $O_\lambda \cong G/G_\lambda$ and the form

$$\omega_{KK,\lambda} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad (\xi, \eta) \mapsto \lambda([\xi, \eta]),$$

the *Kostant-Kirillov form*, descends to G/G_λ to give a symplectic form $\omega = \omega_{KK}$. This can be seen by observing that

$$\mathrm{Ann}(\omega_\lambda) = \{\xi \in \mathfrak{g} \,|\, \omega_\lambda(\xi, \eta) = 0 \,\forall \eta \in \mathfrak{g}\} = \mathfrak{g}_\lambda,$$

which follows from (1.51). Hence ω_{KK} is non-degenerate along O_λ. The tangent space to the orbit O_λ at $g \cdot \lambda$ is

$$T_{g\cdot\lambda} O_\lambda = \mathrm{span}_\mathbb{C}\{(g \cdot \lambda)([\xi, -]) \in \mathfrak{g}^* \,|\, \xi \in \mathfrak{g}\}. \tag{1.57}$$

Together with the Jacobi identity, it implies the closedness of ω_{KK} along each orbit O_λ. Hence ω_{KK} restricts to a symplectic form along each coadjoint orbit O_λ.
If $\mathfrak{g}$ is semisimple, we can use the Killing form $(-,-)$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Note that this isomorphism is G-equivariant with respect to the adjoint and coadjoint action (the adjoint with respect to the Killing form satisfies $\mathrm{Ad}_g^* = \mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$). By the above reasoning the 2-form

$$\omega_{KK} \in \Omega^2(\mathfrak{g}, \mathbb{C}), \quad \omega_{KK,\xi}(\eta, \eta') = (\xi, [\eta, \eta'])$$

yields a symplectic form on the adjoint orbits in $\mathfrak{g}$. When identifying $T_\xi(G\cdot\xi) \cong [\xi, \mathfrak{g}]$ (cf. (1.57)), this symplectic form is given by

$$\omega_{KK,\xi}([\xi, \eta], [\xi, \eta']) = (\xi, [\eta, \eta']).$$

Remark 1.87. Maybe a word of warning is in order at this point: The form $\omega_{KK} \in \Omega^2(\mathfrak{g}, \mathbb{C})$ is non-degenerate (away from 0), if $\mathfrak{g}$ is semisimple. Indeed, in that case the Killing form is non-degenerate and $\mathfrak{g}$ is perfect, i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. However, it is not closed even in the simplest case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In fact, it turns out that if e, f, h denote the standard basis for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $x \in \mathfrak{g}$, then $d\omega_{KK,x}(e, f, h) = 24$, i.e. $d\omega_{KK} \neq 0$ on all of $\mathfrak{g}$. However, after restricting to an adjoint orbit, the description of the tangent space to an adjoint orbit together with the Jacobi identity imply the closedness as mentioned before.

We now want to consider the relative case as before. Namely, let $\hat{\nu} \in \Gamma(\mathfrak{g}, \Omega^2_\chi)$ be the image of the Kostant-Kirillov form $\nu = \omega_{KK}$ where $\chi : \mathfrak{g} \to \mathfrak{t}/W$ is the adjoint quotient . If $\bar{t} \in (\mathfrak{t}/W)^\circ$ is the image of a regular $t \in \mathfrak{t}^\circ$, then the fiber $\chi^{-1}(\bar{t})$ is precisely the regular orbit $O_t = G \cdot t$ (Section 1.4.1). Hence the 2-form $\hat{\nu}(\bar{t}) = \hat{\nu}_{|\chi^{-1}(\bar{t})}$ is a symplectic form on such a fiber of the adjoint quotient.
In general, $\chi^{-1}(\bar{t})$ is a finite union of adjoint orbits so that $(\chi^{-1}(\bar{t}), \hat{\nu}(\bar{t}))$ is a (singular) stratified variety with symplectic non-singular strata. In fact, it is more natural to consider the fibers of the adjoint quotient as Poisson varieties via the Poisson structure coming from the Kostant-Kirillov form. Then the adjoint orbits in a fiber, endowed with the restriction of ω_{KK}, are just the symplectic leaves of this Poisson structure. This is the usual context in which ω_{KK} is considered. However, this viewpoint is only of minor relevance for our purposes.
Observe that the above implies that $\hat{\nu}$ restricts to a relative symplectic form on the adjoint quotient $\chi^{reg} : \mathfrak{g}^{reg} \to \mathfrak{t}/W$ restricted to the regular locus $\mathfrak{g}^{reg}$ which is open and dense in $\mathfrak{g}$. Note that each fiber of χ^{reg} is a (single) regular orbit (cf. Theorem 1.36). We denote the restriction $\hat{\nu}_{|\mathfrak{g}^{reg}}$ again by $\hat{\nu} \in \Gamma(\mathfrak{g}^{reg}, \Omega^2_\chi)$. It is then natural to relate this relative symplectic form with the relative symplectic form $\hat{\omega} \in \Gamma(G \times^B \mathfrak{b}, \Omega^2_\theta)$ from above, restricted to $(G \times^B \mathfrak{b})^{reg} = \psi^{-1}(\mathfrak{g}^{reg})$. Note that this is just $G \times^B \mathfrak{b}^{reg}$. Then the diagram

$$\begin{tikzcd}
G \times^B \mathfrak{b}^{reg} \arrow[r, "\psi'^{reg}", "\cong"'] \arrow[rr, bend left, "\psi^{reg}"] \arrow[d, "\theta"'] & q^*\mathfrak{g}^{reg} \arrow[r] \arrow[d] & \mathfrak{g}^{reg} \arrow[d, "\chi"] \\
\mathfrak{t} \arrow[r, equal] & \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W
\end{tikzcd}$$

is *cartesian*. Indeed, the right square is cartesian by definition. The left square is cartesian because $\psi'^{reg} : G \times^B \mathfrak{b}^{reg} \to \mathfrak{g}^{reg}$ is an isomorphism. This follows from the fact that $\psi' : G \times^B \mathfrak{b} \to \mathfrak{g}$ is an honest simultaneous resolution. Hence the differential of ψ^{reg} induces a natural isomorphism

$$(\psi^{reg})^* \Omega^2_\chi \cong \Omega^2_\theta \tag{1.58}$$

over $G \times^B \mathfrak{b}^{reg}$, since χ and θ are submersions over this locus (so that the relative canonical sheaves coincide with the sheaves of relative differentials of top degree). Moreover, the morphisms θ and χ are still surjective and ψ gives a G-equivariant moment map $\psi_t : \theta^{-1}(t) \to \mathfrak{g}$ on each fiber, where we identify $\mathfrak{g} = \mathfrak{g}^*$ (cf. (1.55)). This turns out to be crucial to give the following comparison.

Proposition 1.88. *Let $\hat{\omega} \in \Gamma(G \times^B \mathfrak{b}^{reg}, \Omega^2_\theta)$ be the relative symplectic form of Section 1.5 and $\hat{\nu} \in \Gamma(\mathfrak{g}^{reg}, \Omega^2_\chi)$ the image of the Kostant-Kirillov form restricted to $\mathfrak{g}^{reg}$, then* [12]

$$\psi^* \hat{\nu} = \hat{\omega} \in \Gamma(G \times^B \mathfrak{b}^{reg}, \Omega^2_\theta) \tag{1.59}$$

under the isomorphism (1.58).

[12] Just to be precise: In (1.59) we denote by $\psi^*\hat{\nu}$ the pullback of $\hat{\nu}$ as a section. The combination with (1.58) then yields a (well-defined) relative 2-form.

Proof. Since we restrict to the regular locus, $\psi_t^{reg} : (G \times^B \mathfrak{b}^{reg})_t \to \mathfrak{g}_{\bar{t}}^{reg}$ is an isomorphism for any $t \in \mathfrak{t}$ with $p(t) = \bar{t}$. But $\mathfrak{g}_{\bar{t}}^{reg}$ is is a regular orbit so that $(G \times^B \mathfrak{b}^{reg})_t$ is a single G-orbit by the G-equivariance of ψ_t^{reg} as well. In particular, G acts transitively. Then Lemma 1.90 below implies that ψ_t^{reg} is in fact a symplectomorphism,

$$(\psi_t^{reg})^* \hat{\nu}(\bar{t}) = \hat{\omega}(t) \in \Gamma((G \times^B \mathfrak{b}^{reg})_t, \Omega_\theta^2).$$

So (1.59) holds true when restricted to the fibers of θ. Since relative forms only see the vertical tangent directions, the claim follows. $\square$

Remark 1.89. When we restrict to the Slodowy slice $S \subset \mathfrak{g}$ and $\tilde{S} \subset G \times^B \mathfrak{b}$, we will see that this statement actually extends over the non-regular locus in S and $\tilde{S}$ respectively.

The next lemma seems to be well-known but we give a proof for completeness.

Lemma 1.90. *Let $\mu : M \to \mathfrak{g}^*$ be a moment map for a* transitive *G-action on a (holomorphic) symplectic manifold (M, ω). If $\mu(p) = \lambda \in \mathfrak{g}^*$ for $p \in M$, then*

$$\mu : (M, \omega) \to (O_\lambda, \omega_{KK|O_\lambda})$$

is a symplectomorphism.

Proof. This statement makes sense because G acts transitively, so that μ maps onto the G-orbit O_λ for $\lambda = \mu(p)$, $p \in M$. Hence μ factorizes through $O_\lambda \hookrightarrow M$.
Since G acts transitively, it follows that $X_\xi(p)$, $\xi \in \mathfrak{g}$, spans T_pM. Using the properties of the moment map $\mu : M \to \mathfrak{g}$, we can compute for $\xi, \eta \in \mathfrak{g}$:

$$\omega_p(X_\xi, X_\eta) = df_{\xi,p}(X_\eta) = \{f_\xi, f_\eta\}(p) = f_{[\xi,\eta]}(p) = (\mu(p), [\xi, \eta])$$

Recalling that $d\mu_p(X_\xi) = \tilde{X}_\xi(\mu(p)) = [\xi, \mu(p)]$, where $\tilde{X}_\xi$ is the vector field associated with the adjoint action on $\mathfrak{g}$, and by the fact that μ maps onto an orbit, we obtain

$$(\mu(p), [\xi, \eta]) = \omega_{KK,\mu(p)}([\mu(p), \xi], [\mu(p), \eta]) = (\mu^* \omega_{KK})_p(X_\xi, X_\eta).$$

Hence μ is a symplectomorphism in this case. $\square$

Period maps

The relative symplectic form $\hat{\omega} \in \Gamma(M/T, \Omega_{\hat{\mu}}^2)$ obtained from Proposition 1.82 for $M = T^*(G/N)$, $\hat{\mu} = \hat{\mu}_T$, gives a period map as follows. As $\mathfrak{t}^*$ is simply connected, the local system $R^2\hat{\mu}_*\mathbb{C}$ is in fact trivial (also see Remark 1.85) with fiber $H^2(T^*(G/B), \mathbb{C})$. Therefore we can define the period map

$$P_{\hat{\omega}} : \mathfrak{t}^* \to H^2(T^*(G/B), \mathbb{C}),$$
$$P_{\hat{\omega}}(\lambda) = \Phi_\lambda([\hat{\omega}(\lambda)]),$$

where Φ_λ is the parallel transport from λ to 0. The projection $\pi : T^*(G/B) \to G/B$ is a homotopy equivalence, so that

$$\pi^* : H^2(T^*(G/B), \mathbb{C}) \to H^2(G/B, \mathbb{C})$$

is an isomorphism. In this way, we can consider the period map as $P_{\hat{\omega}} : \mathfrak{t}^* \to H^2(G/B, \mathbb{C})$.

Proposition 1.91. *Let G be a simple adjoint complex Lie group and let Δ be the Dynkin diagram corresponding to $T \subset B$. Then the period map*

$$P_{\hat{\omega}} : \mathfrak{t}^* \to H^2(G/B, \mathbb{C})$$

is a W- and $\mathrm{Aut}(\Delta)$*-equivariant isomorphism.*

Remark 1.92. This proposition is a refined version of Corollary 3.3 in [Yam95]. More precisely, the version in [Yam95] only considers the W-action whereas we also include the $\mathrm{Aut}(\Delta)$-action.

Here we let $\mathrm{Aut}(\Delta)$ act on G/B via a splitting of the short exact sequence (1.42) compatible with $T \subset B$, cf. Example 1.42. This induces an action on forms (resp. cohomology) via

$$a \cdot \beta := (\varphi_a^{-1})^* \beta, \quad a \in \mathrm{Aut}(\Delta) \subset \mathrm{Aut}(G),$$

for the action map $\varphi_a : G/B \to G/B$. The W-action is more elaborate because there is no natural (non-trivial) W-action on G/B. To get a W-action on $H^2(G/B, \mathbb{C})$, we consider the projection (cf. [CG10], Chapter 3)

$$p : G/T \to G/B,$$

which is a C^∞-locally trivial fiber bundle with contractible fiber. The long exact sequence of homotopy groups implies that $\pi_*(G/T) \cong \pi_*(G/B)$ via p_*. Since G/T and G/B are CW-complexes we can apply Whitehead's theorem ([Hat02]) to conclude that p is in fact a homotopy equivalence, in particular

$$p^* : H^2(G/B, \mathbb{C}) \xrightarrow{\cong} H^2(G/T, \mathbb{C}).$$

Observe that G/T has a natural right W-action

$$G/T \curvearrowleft W, \quad gT \cdot w := R_w(gT) := gnT,$$

where $w = \overline{n} \in W = N(T)/T$. We take this action to finally get a left W-action on $H^2(G/B, \mathbb{C}) \cong H^2(G/T, \mathbb{C})$. Explicitly, it is given by

$$H^2(G/B, \mathbb{C}) \curvearrowleft W, \quad w \cdot \beta = R_w^* p^*(\beta)$$

for the natural projection $p : G/T \to G/B$.

We now outline an important result due to Borel and Hirzebruch, which is the main step in proving Proposition 1.91. It gives an isomorphism $\mathfrak{t}^* \to H^2(G/B, \mathbb{C})$ which is constructed by explicitly defining a 2-form $\Omega(\lambda) \in \Omega^2(G/B)$ as follows. Every left invariant 1-form $\alpha \in \Omega^1_l(G)$ is determined by its action on $\mathfrak{g} = T_eG$. To construct left invariant 1-forms it is therefore useful to write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

and we choose Cartan-Weyl generators ([Hum78]) $\{h_\alpha, e_\beta \mid \alpha \in \Delta, \beta \in R\}$ of $\mathfrak{g}$ with $\mathfrak{t} = \langle h_\alpha \mid \alpha \in \Delta \rangle_{\mathbb{C}}$ and $\mathfrak{g}_\beta = \langle e_\beta \rangle_{\mathbb{C}}$, $\beta \in R$. Then each $\lambda \in \mathfrak{t}^*$ defines a left-invariant 1-form $\tilde{\lambda} \in \Omega^1_l(G)$ by extending $\lambda : \mathfrak{t} \to \mathbb{C}$ to all of $\mathfrak{g}$ through extension by zero. Further we get a left-invariant 1-form $\theta_\alpha \in \Omega^1_l(G)$ via the dual element $e^*_{-\alpha} \in \mathfrak{g}^*$. The next lemma, except for the part about graph automorphisms, is essentially contained in Section 3 of [Yam95] but the statements missed a factor $\frac{1}{2}$. So we also give a proof of its first part.

Lemma 1.93. *The left-invariant* 1*-forms* $\tilde{\lambda}$*,* $\theta_\alpha \in \Omega^1_l(G)$ *are related as follows:*

$$d\tilde{\lambda} = \frac{1}{2} \sum_{\alpha \in R^+} (\lambda|\alpha)\ \theta_{-\alpha} \wedge \theta_\alpha. \tag{1.60}$$

Additionally, there exists a unique 2*-form* $\Omega(\lambda) \in \Omega^2(G/B)$ *such that*

$$\pi_B^* \Omega(\lambda) = d\tilde{\lambda}$$

for each $\lambda \in \mathfrak{t}^*$ *and where* $\pi_B : G \to G/B$ *is the natural projection. It satisfies for each* $a \in \mathrm{Aut}(\Delta)$

$$a \cdot \Omega(\lambda) = \Omega(a \cdot \lambda), \tag{1.61}$$

where $\mathrm{Aut}(\Delta)$ *acts in the natural way on* $\mathfrak{t}^*$ *and* $\mathfrak{g}^*$*.*

Proof. Recall the Maurer-Cartan equation

$$d\omega_{MC} + \frac{1}{2}[\omega_{MC} \wedge \omega_{MC}] = 0,$$

where ω_{MC} is the unique left-invariant 1-form satisfying $\omega_{MC,e}(v) = v$ for all $v \in \mathfrak{g}$. Hence it follows immediately that

$$\tilde{\lambda} = \lambda \circ \omega_{MC}, \quad \theta_\alpha = e^*_{-\alpha} \circ \omega_{MC},$$

where, by abuse of notation, we consider $\lambda : \mathfrak{g} \to \mathbb{C}$ through extension by zero. Now let $v, w \in \mathfrak{g}$ and compute

$$\begin{aligned} d\tilde{\lambda}(v, w) &= -\frac{1}{2}\lambda([\omega_{MC} \wedge \omega_{MC}])(v, w) \\ &= -\lambda([\omega_{MC}(v), \omega_{MC}(w)]). \end{aligned}$$

If $v = e_\beta$, $w = e_\gamma$ for $\beta, \gamma \in \Delta$, we obtain

$$-\lambda([e_\beta, e_\gamma]) = \lambda([e_\gamma, e_\beta]) = \begin{cases} -\lambda(h_\beta) & \text{if } \gamma = -\beta \\ 0 & \text{else.} \end{cases}$$

Let us check the right hand side of (1.60):

$$\begin{aligned} &\frac{1}{2} \sum_{\alpha \in R^+} (\lambda|\alpha)\ \theta_{-\alpha} \wedge \theta_\alpha(e_\beta, e_\gamma) \\ &= \frac{1}{2}(-(\lambda|\beta) - (\lambda|\beta)) \\ &= -(\lambda|\beta) \\ &= -\lambda(h_\beta). \end{aligned}$$

Hence both sides coincide on basis elements of $\mathfrak{g}$, which proves the claimed formula.
For the second claim we have to prove that $\theta_{-\alpha} \wedge \theta_\alpha$, $\alpha \in R^+$, vanishes at $e \in G$ on $\mathfrak{b} \subset \mathfrak{g}$. Let $\beta, \gamma \in R^+$ and e_β, $e_\gamma \in \mathfrak{g}$ be the corresponding basis elements. Then

$$\theta_{-\alpha} \wedge \theta_\alpha(e_\beta, e_\gamma) = \delta_{-\alpha,\beta}\delta_{\alpha,\gamma} - \delta_{-\alpha,\gamma}\delta_{\alpha,\beta} = 0,$$

because $-\alpha \in R^-$. In total, $d\tilde{\lambda}$ uniquely descends to G/B to a 2-form $\Omega(\lambda) \in \Omega^2(G/B)$. The last statement follows from the next elementary lemma:

$$\pi_B^*(a \cdot \Omega(\lambda)) = (a^{-1})^*(d\tilde{\lambda}) = d(\widetilde{\lambda \circ a^{-1}}) = d(\widetilde{a \cdot \lambda}) = \pi_B^* \Omega(a \cdot \lambda).$$

□

Lemma 1.94. *Let $\varphi \in \mathrm{Aut}(G)$ be an automorphism of G. If $\tilde{\lambda} \in \Omega^1_l(G)$ denotes the left-invariant 1-form corresponding to $\lambda \in \mathfrak{g}^*$, then*

$$\varphi^* \tilde{\lambda} = \widetilde{\lambda \circ \varphi_*}$$

for the induced automorphism $\varphi_* \in \mathrm{Aut}(\mathfrak{g})$.

Proof. This is a straightforward computation: Let $v \in T_g G$, then

$$\begin{aligned} \varphi^* \tilde{\lambda}_g(v) &= \lambda \circ dL_{\varphi(g)^{-1}} \circ d\varphi_g(v) \\ &= \lambda \circ \varphi_* \circ dL_{g^{-1}|g}(v) \\ &= (\widetilde{\lambda \circ \varphi_*})(v). \end{aligned}$$

□

The next proposition gives a complete description of $H^2(G/B, \mathbb{C})$ as already announced.

Proposition 1.95 ([BH58])**.** *Let $B \subset G$ and $\mathfrak{t}^*$ be as before. Then*

$$\mathfrak{t}^* \to H^2(G/B, \mathbb{C}), \quad \lambda \mapsto [\Omega(\lambda)],$$

is a W- and $\mathrm{Aut}(\Delta)$-equivariant isomorphism where $\Omega(\lambda) \in \Omega^2(G/B)$ is as in Lemma 1.93.

Proof. The isomorphism statement and the W-equivariance is due to Borel and Hirzebruch [BH58]. The $\mathrm{Aut}(\Delta)$-equivariance, which was not considered in loc. cit., follows immediately from the last statement in Lemma 1.93. □

Proof of Proposition 1.91. It is shown in [Yam95], Section 3, that

$$P_{\hat{\omega}}(\lambda) = \Phi_\lambda([\hat{\omega}(\lambda)]) = [\pi^* \Omega(\lambda)] \in H^2(T^*(G/B), \mathbb{C})$$

for the projection $\pi : T^*(G/B) \to G/B$. Hence $P_{\hat{\omega}} : \mathfrak{t}^* \to H^2(G/B, \mathbb{C})$ is a W- and $\mathrm{Aut}(\Delta)$-equivariant isomorphism by Proposition 1.95. □

Restrictions of the period map

We want to apply the previous construction of the period map $P_{\hat{\omega}} : \mathfrak{t}^* \to H^2(G/B, \mathbb{C})$ to obtain a period map

$$P : \mathfrak{t} \to H^2(\tilde{S}_0, \mathbb{C})$$

for the simultanenous resolution $\theta : \tilde{S} \to \mathfrak{t}$ of the $\sigma : S \to \mathfrak{t}/W$, cf. Section 1.4.2. Here $S = x + \mathfrak{z}_\mathfrak{g}(y) \subset \mathfrak{g}$ is a Slodowy slice through a subregular nilpotent $x \in \mathfrak{g}$ as before. As in the previous section, we fix a maximal torus $T \subset G$ together with a Borel subgroup $T \subset B \subset G$.

The first step is to use the isomorphism of Theorem 1.84 to realize this diagram as a 'subsquare' of the square in (1.56). We will from now on identify the square in (1.56) with Grothendieck's simultaneous resolution. Note that we identify $\mathfrak{g} = \mathfrak{g}^*$ by means of the Killing form. Hence we obtain a diagram as follows

$$\begin{tikzcd} S^* \arrow[dr, "\pi_T"] \arrow[drr, bend left, "\mu_G"] \arrow[ddr, bend right, "\mu_T"'] & & \\ & \tilde{S} \arrow[r, "\hat{\mu}_G"] \arrow[d, "\hat{\mu}_T"] & S \arrow[d, "\chi"] \\ & \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W. \end{tikzcd} \tag{1.62}$$

Here $S^* = \pi_T^{-1}(\tilde{S}) \subset M = T^*(G/N)$ and, by abuse of notation, we denote the restrictions of the maps by the same symbols. This is justified because the restriction is compatible with the symplectic and Hamiltonian structure.

Proposition 1.96 ([Yam95]). *The submanifold $S^* \subset M$ is in fact a symplectic submanifold which is T-invariant such that the restricted T-action is again Hamiltonian. Furthermore, the relative symplectic form $\hat{\omega} \in \Gamma(M/T, \Omega_{\hat{\mu}_T})$ restricts to a relative symplectic form $\hat{\omega}_{\tilde{S}} \in \Gamma(\tilde{S}, \Omega^2_{\hat{\mu}_T})$.*

In particular, this gives a completely symplectic-geometric description of the simultaneous resolution $\tilde{S} = S^*/T$. It also reproduces the well-known fact that the fibers $\tilde{S}_t$ have canonical trivial bundle, because the restrictions $\hat{\omega}_{\tilde{S}}(t)$ of the relative symplectic form $\hat{\omega}_{\tilde{S}}$ give a nowhere-vanishing top-degree form.

Remark 1.97. It is in general *not* true that S^* is invariant under the G-action. Therefore $\mu_G : S^* \to \mathfrak{g}^*$ in (1.62) cannot have an interpretation as moment map.

As the map $\tilde{\sigma} : \tilde{S} \to \mathfrak{t}$ is C^∞-trivial as well, hence $R^2\tilde{\sigma}_*\mathbb{C} \cong H^2(\tilde{S}_0, \mathbb{C})_{\mathfrak{t}}$ for the trivial local sytem on $\mathfrak{t}$ with stalk $H^2(\tilde{S}_0, \mathbb{C})$, we can define a period map

$$P_{\tilde{S}} : \mathfrak{t} \to H^2(\tilde{S}_0, \mathbb{C}), \quad t \mapsto \Phi_t([\hat{\omega}_{\tilde{S}}(t)])$$

(again Φ_t is parallel transport from t to 0). It fits into the commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "P_{\tilde{S}}"] \arrow[dr, "P_{\hat{\omega}}"'] & H^2(\tilde{S}_0, \mathbb{C}) \\ & H^2(G/B, \mathbb{C}) \arrow[u] \end{tikzcd}$$

by construction. For the vertical arrow we refer to Lemma 1.54. There we have seen that it is a $\mathbf{CA}$-equivariant isomorphism in case $\Delta = \Delta_h$ is homogeneous and an isomorphism onto $H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}}$ in case Δ is of type BCFG. As a corollary of Proposition 1.91, we obtain a stronger statement as Theorem 5.3 in [Yam95], which also includes the BCFG-case and the $\mathbf{CA}$-equivariance.

Corollary 1.98. *If $\Delta = \Delta_h$ is of type* ADE*, then the period map $P_{\tilde{S}} : \mathfrak{t} \to H^2(\tilde{S}_0, \mathbb{C})$ is a W- and $\mathbf{CA}$-equivariant isomorphism. When Δ is of type* BCFG*, then it is a W-equivariant isomorphism onto $H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}}$.*

Remark 1.99. Let us comment a bit more on this result. In [Yam95] only the simply-laced case was considered, even though the non-simply-laced case is mentioned (see Remark 3 in Section 5). We also incorporated the non-simply-laced case, as well as the equivariance under $\mathbf{CA}$ in the simply-laced case (recall that $\mathbf{CA} \cong AS \subset \mathrm{Aut}(\Delta_h)$). Moreover, all this is compatible in the following sense: Let Δ be a Dynkin diagram of type BCFG and Δ_h the corresponding homogeneous Dynkin diagram. The simultaneous resolutions $\tilde{S} \to \mathfrak{t}$ and $\tilde{S}_h \to \mathfrak{t}_h$ of type Δ and Δ_h respectively are then isomorphic over $\mathfrak{t}$ and $\mathfrak{t}_h^{AS}$ (Corollary 1.56). By the $\mathbf{CA}$-equivariance of $P_{\tilde{S}_h} : \mathfrak{t}_h \to H^2(\tilde{S}_{h,0}, \mathbb{C})$, we obtain the commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "P_{\tilde{S}}"] \arrow[d, "\cong"'] & H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}} \arrow[d, "\cong"] \\ \mathfrak{t}_h^A \arrow[r, "P_{\tilde{S}_h}"] & H^2(\tilde{S}_{h,0}, \mathbb{C})^{\mathbf{CA}}. \end{tikzcd}$$

So also from the perspective of period maps, both the intrinsic and the extrinsic approach to BCFG-singularities, as described in Section 1.4.3, are equivalent in a natural way.

Yamada's description of Grothendieck's simultaneous resolution also includes the natural $\mathbb{C}^*$-actions. Let us briefly describe them on $M = T^*(G/N)$ and M/T. Let $\tilde{L}_g$ denote the G-action on $T^*(G/N)$, i.e.

$$\tilde{L}_g(\alpha_p) = \alpha_p \circ dL_{g^{-1}|g\cdot p}$$

(L_g the action from the left induced by multiplication) for $\alpha_p \in T^*_h(G/N)$, $p \in G$. If (x, y, h) is a $\mathfrak{sl}_2(\mathbb{C})$-triplet (not necessarily for a nilpotent *subregular* x), then the $\mathbb{C}^*$-action on M is given by

$$\lambda \cdot \alpha = \lambda^2 \tilde{L}_{\exp(\lambda h)}(\alpha).$$

This action obviously commutes with the T-action and therefore descends to M/T. The isomorphism to Grothendieck's simultaneous resolution is then $\mathbb{C}^*$-equivariant by construction[13].

Lemma 1.100 (Yamada)**.** *The relative symplectic structure $\hat{\omega} \in \Gamma(M/T, \Omega_{\hat{\mu}_T})$, $M = T^*(G/N)$, is of weight 2 with respect to the $\mathbb{C}^*$-action,*

$$\lambda^* \hat{\omega} = \lambda^2\, \hat{\omega}, \quad \forall \lambda \in \mathbb{C}^*.$$

1.5.4 Slodowy slices and the Kostant-Kirillov form

Since a Slodowy slice S intersects each orbit O it meets tranversely, the intersection $S \cap O \subset \mathfrak{g}$ is non-singular. It is not obvious, why the symplectic form $\omega = \omega_{KK}$ restricts to $S \cap O$ to give a symplectic form.

Lemma 1.101 ([GG02])**.** *Let $O = G \cdot \xi$ be an orbit in $\mathfrak{g}$ under the adjoint action, $\nu = \omega_{KK}$ the Kostant-Kirillov form and S a Slodowy slice. Its restriction $\nu_{S\cap O}$ to the submanifold $S \cap O$ is still symplectic.*

Sketch of proof. We follow the proof from [GG02]. Let $x \in \mathfrak{g}$ be a subregular nilpotent element and (x, y, h) an $\mathfrak{sl}_2$-triplet for x so that $S = x + \mathfrak{z}_{\mathfrak{g}}(y) = x + \ker \mathrm{ad}(y)$ is a Slodowy slice. Let further $O = O_\xi$ be an orbit through $\xi \in \mathfrak{g}$ so that

$$T_\xi(S \cap O) \cong \ker \mathrm{ad}(y) \cap \mathrm{im}\, \mathrm{ad}(\xi).$$

Then one first proves

$$\mathrm{Ann}(\nu_O) \subset \mathrm{im}\,(\mathrm{ad}(\xi) \circ \mathrm{ad}(y))$$

and in the second step that $\mathrm{im}\,(\mathrm{ad}(\xi) \circ \mathrm{ad}(y)) \cap \ker \mathrm{ad}(y) = 0$, i.e. $\mathrm{Ann}(\nu_{S\cap O}) = 0$. This shows that $\nu_{S\cap O}$ is again symplectic. □

Corollary 1.102. *The simultaneous resolution $\psi : (\tilde{S}, \hat{\omega}) \to S$ is a simultaneous symplectic resolution when $S^{reg} = S^{reg}_\sigma$ is endowed with the relative Kostant-Kirillov form $\hat{\nu}$.*

Proof. The first statement follows from Proposition 1.88, since $\hat{\omega}$ and $\hat{\nu}$ are obtained from $G \times^B \mathfrak{b}$ and $\mathfrak{g}^{reg}$ respectively via restriction. The second claim is now obvious because the fibers are (complex) surfaces. □

Remark 1.103. This result is probably well-known to experts but we could not locate it in the literature. Since we are in (relative) dimension 2, a symplectic form is the same as a holomorphic volume form. So equivalently, ψ is a simultaneous crepant resolution. In particular, we reobtain the statement that $\psi : \tilde{S} \to S$ is a simultaneous minimal resolution.

[13]Yamada actually 'exponentiates' this action, because he wants to relate $\hat{\omega}$ to Saito's primitive form. Since we do not need this aspect, we work with the above $\mathbb{C}^*$-action. The proof of the next Lemma still works. Moreover, this modified $\mathbb{C}^*$-action coincides with Slodowy's under Yamada's isomorphism.

Since σ is a Gorenstein morphism (its fibers are complete intersections, in particular Gorenstein), the relative dualizing/canonical sheaf K_σ is an invertible sheaf on S, hence reflexive. It coincides with $\Omega^2_{\sigma^{reg}}$ over S^{reg} . The codimension of $S - S^{reg}$ is at least 2, so that $\hat{\nu}$ can be extended to a global section of K_σ. We also denote it by $\hat{\nu} \in \Gamma(S, K_\sigma)$. For the same reason, we must have $i_*\Omega^2_{\sigma^{reg}} \cong i_*K_{\sigma^{reg}} \cong K_\sigma$. As $\hat{\nu}$ is nowhere vanishing on S^{reg}, it follows that $K_\sigma \cong i_*\mathcal{O}_{S^{reg}} \cong \mathcal{O}_S$. Clearly, $K_{\tilde{\sigma}} \cong \mathcal{O}_{\tilde{S}}$ so that we conclude

$$\psi^* K_\sigma \cong \mathcal{O}_{\tilde{S}} \cong K_{\tilde{\sigma}}$$

as $\mathcal{O}_{\tilde{S}}$-modules. In fact, there is another way to obtain this isomorphism: The differential of ψ^{reg} gives a morphism

$$\Phi^{reg} : \psi^* K_{\sigma^{reg}} = \psi^* \Omega^2_{\sigma^{reg}} \longrightarrow K_{\tilde{\sigma}^{reg}}.$$

This is in fact an isomorphism because $\Phi^{reg}(\psi^*\hat{\nu}) = \hat{\omega}$ (see Proposition 1.88) and these sections trivialize the corresponding line bundles over $\tilde{S}^{reg}$.

Corollary 1.104. *The isomorphism Φ^{reg} extends to an isomorphism*

$$\Phi : \psi^* K_\sigma \longrightarrow K_{\tilde{\sigma}},$$

such that $\Phi(\psi^*\hat{\nu}) = \hat{\omega} \in \Gamma(\tilde{S}, K_{\tilde{\sigma}})$. In particular, $\hat{\nu}$ is a nowhere vanishing global section of K_σ. It is further $\mathbb{C}^*$-equivariant, i.e.

$$\lambda^*\hat{\nu} = \lambda^2\,\hat{\nu}.$$

Moreover, $\hat{\omega}$ and $\hat{\nu}$ are **C***-invariant in case $\mathfrak{g}$ is of type* BCFG *and* **CA***-invariant in case $\mathfrak{g} = \mathfrak{g}_h$ is of type* ADE.

To make sense of the last statements, we have to give equivariant structures on K_σ and $K_{\tilde{\sigma}}$. But this follows from the fact that any automorphism $a : S \to S$ gives rise to a natural base change isomorphism

$$\Phi_a : a^* K_\sigma \longrightarrow K_\sigma,$$

which is in particular compatible with compositions (and similarly for $K_{\tilde{\sigma}}$).

Proof. Since K_σ and $K_{\tilde{\sigma}} = \Omega^2_{\tilde{\sigma}}$ are reflexive, it suffices to prove that $\operatorname{codim}_{\tilde{S}}\tilde{T} \geq 2$ for $\tilde{T} := \tilde{S} - \tilde{S}^{reg}$. The irreducible components of $\tilde{T}$ of highest dimension lie over the hypersurfaces $\mathfrak{t}_\alpha - \cap_{\beta\neq\alpha}\mathfrak{t}_\beta \cap \mathfrak{t}_\alpha$. If t lies in such a hypersurface, then $\tilde{\sigma}^{-1}(t) \cap \tilde{T}$ consists of the exceptional divisor of $\psi_t : \tilde{S}_t \to S_{\bar{t}}$. Hence these irreducible components have dimension $(r-1)+1 = r$ which is of codimension 2 because $\dim \tilde{S} = r+2$. Therefore Φ^{reg} extends to an isomorphism $\Phi : \psi^*K_\sigma \to K_{\tilde{\sigma}}$. Over $\tilde{S}^{reg}$ we have already seen that

$$(\psi^{reg})^*\hat{\nu} = \hat{\omega} \in \Gamma(\tilde{S}^{reg}, K_{\tilde{\sigma}}) = \Gamma(\tilde{S}^{reg}, \Omega^2_{\tilde{\sigma}})$$

under Φ^{reg}. Using again that $K_{\tilde{\sigma}}$ is reflexive it follows that $\Phi(\psi^*\hat{\nu}) = \hat{\omega}$. This also shows that $\hat{\nu}$ is nowhere vanishing because $\hat{\omega}$ is and ψ is surjective.
The fact that $\psi^{reg} : \tilde{S}^{reg} \to S^{reg}$ is a submersion together with Lemma 1.100 imply that $\lambda^*\hat{\nu} = \lambda^2\,\hat{\nu}$ on S^{reg}. Now apply the previous argument to conclude that it holds an all of S.
The Kostant-Kirillov form $\nu = \omega_{KK}$ is invariant under the natural $\operatorname{Aut}(\mathfrak{g})$-action:

$$\begin{aligned}(\varphi^*\nu)_\xi(\eta, \eta') &= \nu_{\varphi(\xi)}(\varphi(\eta), \varphi(\eta')) \\ &= (\varphi(\xi), \varphi([\eta, \eta'])) \\ &= (\xi, [\eta, \eta']) \\ &= \nu_\xi(\eta, \eta') \quad \forall \xi, \eta, \eta' \in \mathfrak{g}.\end{aligned}$$

Since σ is equivariant for the $\mathbf{C}(\mathbf{A})$-action, the same holds true for $\hat{\nu}^{reg}$. Using the first part and the codimension argument, we see that $\hat{\nu}$ and hence $\hat{\omega}$ are $\mathbf{C}(\boldsymbol{A})$-equivariant. □

Remark 1.105.

a) The first statement is actually well known, because we can always find a generator for $K_{X/B}$ if $X \to B$ is (the germ of) the semi-universal deformation of a locally complete intersection ([Loo84]). However, here we have an explicit global generator that fits in the Lie-theoretic description for singularities of type Δ. This is particular useful because it ensures the invariance under $\mathbf{C}(\mathbf{A})$, which we do not know how to prove otherwise. It will become important in Chapter 5.

b) The relation of the relative symplectic form $\hat{\omega}$ to the Kostant-Kirillov form is useful to study its $\mathbf{C}(\boldsymbol{A})$-invariance. On the other hand, the $\mathbb{C}^*$-equivariance of $\hat{\nu}$ can be deduced from the same property of $\hat{\omega}$

c) As mentioned earlier, we worked in the complex-analytic category throughout this section. However, the Kostant-Kirillov form is clearly algebraic and so is $\hat{\nu}$ and $\psi^*\hat{\nu}$ on S and $\tilde{S}$ respectively. Under Yamada's isomorphism (Theorem 1.84), the latter coincides with $\hat{\omega}$. Since we do not know that Yamada's isomorphism is algebraic (Remark 1.85), we cannot conclude the same for $\hat{\omega}$. But we can conclude that the algebraic section $\psi^*\hat{\nu}$ of ψ^*K_σ has the same algebraic properties as $\hat{\omega}$: For example, it is nowhere vanishing and has the same equivariance properties with respect to the $\mathbf{C}(\mathbf{A})$- and $\mathbb{C}^*$-actions as $\hat{\omega}$. Since the isomorphism $\Phi : \psi^*K_\sigma \to K_{\tilde{\sigma}}$ is algebraic (because it is induced from the algebraic morphism $\psi : \tilde{S} \to S$), we denote the algebraic section $\Phi(\psi^*\hat{\nu}) \in \Gamma(\tilde{S}, K_{\tilde{\sigma}})$ by $\hat{\omega}$ in the following.
Of course, we could work with the latter section from the beginning. But some of its properties can be easily deduced from those of $\hat{\omega}$ (e.g. the $\mathbb{C}^*$-equivariance). The advantage is that $\hat{\omega}$ has these properties by construction. Moreover, this approach gives us additional information about the corresponding period maps (cf. Remark 1.99).

Chapter 2

Polarized integrable systems

This chapter gives a concise introduction to integrable systems in complex geometry (e.g. [DM96b], [Fre99][1]). As already mentioned in the introduction, there are at least two motivations to consider them: The first one is the work of Adler and van Moerbeke ([AvM80a], [AvM80b], or the textbook account [AvMV04]) showing that many integrable systems from real symplectic geometry (more precisely completely integrable Hamiltonian system) can be 'complexified'. The other one is the Arnold-Liouville theorem ([Arn78]), which can be seen as the statement that any completely integrable Hamiltonian system yields a (proper) Lagrangian submersion.

We take the latter motivation as our starting point to introduce integrable systems in complex geometry. Hence we begin with proper Lagrangian holomorphic submersions and add more and more conditions to arrive at a definition of polarized integrable systems. The first observation is, as in the real case, that the (connected components of the) fibers of a proper Lagrangian submersion are tori. However, there is a feature which is not present in real geometry: Complex tori have moduli which is essential for the *cubic condition* of Donagi and Markman ([DM96a]), that we later discuss. We further incorporate polarizations to distinguish between the cases where the fibers are abelian varieties or not. This is important because we deal with both cases in Chapter 3 and Chapter 4 respectively

2.1 Lagrangian torus fibrations

We will first restrict our attention to proper submersions $\pi : (\mathcal{M}, \omega) \to B$ with Lagrangian fibers where $(\mathcal{M}, \omega)$ is a holomorphic symplectic manifold and B a complex manifold[2]. As a short-hand we will call such maps *proper Lagrangian submersions*. It will turn out that this already puts restrictions on the fibers of π.
Let $\pi : (\mathcal{M}, \omega) \to B$ be a proper Lagrangian submersion. We assume without loss of generality (and without mentioning it explicitly in the following) that $\mathcal{M}$ and B are connected and that π is surjective. Since π is a submersion, it gives rise to the exact tangent sequence

$$0 \longrightarrow \ker d\pi \longrightarrow T\mathcal{M} \longrightarrow \pi^*TB \longrightarrow 0. \tag{2.1}$$

[1] For completeness, we add some useful references for the real symplectic sirutation as well: [Arn78], [GS90] [LM87].

[2] Note that in this situation π is already a flat morphism.

The sections of $\ker d\pi$ can be thought of as the vector fields which are tangent to the fibers of π. It is therefore natural to call
$$\mathcal{V} = \mathcal{V}_\pi := \pi_* \ker d\pi$$
the *vertical sheaf* of π on B. If π were a general submersion, then this sheaf might be ill-behaved. However, the Lagrangian property gives strong restrictions on $\mathcal{V}$.

Proposition 2.1. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a proper Lagrangian submersion between a holomorphic symplectic manifold and a complex manifold.*

i) Let $\mathcal{V} = \pi_* \ker d\pi$ be the vertical sheaf of π. Then we have
$$\mathcal{V} \cong T^*B \otimes \pi_*\mathcal{O}_\mathcal{M}$$
as $\mathcal{O}_B$-modules. This $\mathcal{O}_B$-module is not only coherent but in fact locally free, the vertical bundle *of π.*

ii) The connected components of the fibers of π are affine tori, i.e. torsors for complex tori.

Proof. i) Consider again the short exact sequence (2.1) of $\mathcal{O}_\mathcal{M}$-modules, which implies that
$$\pi^*T^*B \cong (T\mathcal{M}/\ker d\pi)^* \cong (\ker d\pi)^\perp,$$
the annihilator of $\ker d\pi$. Using the symplectic form ω we have a natural isomorphism
$$(\ker d\pi)^\perp \cong (\ker d\pi)^{\perp_\omega} = \{X \in T\mathcal{M} \mid \omega(X, Y) = 0 \quad \forall\, Y \in \ker d\pi\} \subset T\mathcal{M}.$$
But the fibers are Lagrangian so that $(\ker d\pi)^{\perp_\omega} = \ker d\pi$. In total we obtain a natural isomorphism $\ker d\pi \cong \pi^*T^*B$. Applying π_* together with the projection formula yields
$$\mathcal{V} \cong T^*B \otimes \pi_*\mathcal{O}_\mathcal{M}$$
as $\mathcal{O}_B$-modules. Since T^*B is locally free, it remains to prove the same for $\pi_*\mathcal{O}_M$. By Grauert's base change theorem [BS76], this follows if we can prove that the function
$$b \mapsto h^0(M_b, \mathcal{O}_{M_b})$$
is constant. But this follows because the fibers are compact and they all have the same number of connected components. Indeed, the direct image $\pi_*\mathbb{Z}$ is a local system and therefore its rank is locally constant, hence constant because B is connected.

ii) Fix a point $b_0 \in B$ and a connected component T_0 of M_{b_0}. By the results from *i)* it follows that the tangent bundle of T_0 is trivial with typical fiber isomorphic to $T^*_{b_0}B$. Trivializing sections can be constructed as follows: Choose a chart $U \cong \mathbb{C}^n$ around b_0 so that π can be considered as a map
$$h = (h_1, \ldots, h_n) : \pi^{-1}(U) \to \mathbb{C}^n.$$
The Hamiltonian vector fields $X_i := X_{h_i}$ restricted to $T_0 \subset \pi^{-1}(U)$ give a trivialization of the tangent bundle of T_0 because π is a submersion. Since $[X_i, X_j] = 0$, they define an action of $\mathbb{C}^n$ on T_0 after fixing a point $t_0 \in T_0$ via integration:
$$(\zeta_1, \ldots, \zeta_n) \cdot t_0 = \Phi^1_{\zeta_1} \circ \cdots \circ \Phi^n_{\zeta_n}(t_0).$$
Here $\Phi^i_\zeta(\cdot)$ is the flow of the vector field X_i. The same argument as in the real case shows that the action is transitive and the isotropy group of t_0 is a full sublattice of $\mathbb{C}^n$. It follows that T_0 is an affine torus (e.g. [LM87], [DM96b]).

□

This proposition motivates the following definition.

Definition 2.2. A *Lagrangian torus fibration* $\pi : (\mathcal{M}, \omega) \to B$ is a proper Lagrangian submersion with *connected* fibers.

Corollary 2.3. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a Lagrangian torus fibration. Then ω gives a natural isomorphism*

$$T^*B \to \mathcal{V}, \quad \alpha \mapsto v_\alpha,$$

such that $i(v_\alpha)\omega = \pi^*\alpha$.

Let $\pi : \mathcal{M} \to B$ be a Lagrangian torus fibration. The second part of the previous Proposition 2.1 has a relative version in the sense that the bundle $\mathcal{V}$ acts in a fiber-preserving way on $\mathcal{M}$. Note that the sheaf $T\mathcal{M}$ carries a Lie bracket and therefore the $\mathcal{O}_B$-module $\pi_*T\mathcal{M}$ carries a Lie bracket as well. The next lemma states that this Lie bracket restricts to a Lie bracket on $\mathcal{V}$. Therefore $\mathcal{V} \cong T^*B$ has the structure of a bundle of (finite-dimensional) Lie algebras.

Lemma 2.4. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a Lagrangian torus fibration and $\mathcal{V}$ its vertical bundle. Then for any two (local) sections v, w of $\mathcal{V}$, we have*

$$[v, w] = 0.$$

In particular, $\mathcal{V} \cong T^*B$ acts in a fiber-preserving way on $\mathcal{M}$.

Proof. Let $v, w \in \mathcal{V}(U)$ be a section. After restricting $U \subset B$ if necessary, we may assume that v, w are symplectic vector fields. Indeed, if $U \subset B$ is small enough, we have

$$v = X_{\pi^* f}, \quad w = X_{\pi^* g}$$

for functions $f, g \in \mathcal{O}_B(U)$. This follows for example by expressing h in coordinates as in the proof of Proposition 2.1. Hence we can conclude

$$[v, w] = X_{\omega(v,w)} = 0,$$

because π is a Lagrangian fibration.
Let $v \in \mathcal{V}(U)$ be a local section. As in the proof of Proposition 2.1 ii) (cf. Remark 2.5), we can define an action via

$$\exp(v) \cdot x := \Phi_1^v(x), \quad x \in \pi^{-1}(U) \subset \mathcal{M}.$$

Here $\exp : (\mathcal{V}, [.,.]) \to (\mathcal{V}, +)$ is the fiberwise exponential map. Since $(\mathcal{V}, [.,.])$ is abelian, it is an isomorphism of the underlying abelian groups. Note that the above formula does define an action because

$$\Phi_\zeta^{v+w} = \Phi_\zeta^v \circ \Phi_\zeta^w = \Phi_\zeta^w \circ \Phi_\zeta^v$$

due to $[v, w] = 0$. This action preserves the fibers, since the flows Φ_ζ^v are vertical, i.e. $\pi \circ \Phi_\zeta^v = \pi$. □

Remark 2.5. It might seem that the action in the previous lemma restricted to a fiber does not coincide with the action defined in the proof of Proposition 2.1 ii). However, the former can be thought of a basis-independent definition of the latter. Indeed, for any $\zeta, \xi \in \mathbb{C}$ we have

$$\Phi_\zeta^{\xi v + w} = \Phi_{\xi\zeta}^v \circ \Phi_\zeta^w.$$

Expressing $v(b) = \sum_i v_i(b) X_i$ for fixed $b \in B$, we see that the actions do coincide.
For latter reference, we fix the notation for the vector fields of the group action. Let $v \in \mathcal{V}(U)$

be a local section considered as a section of the bundle of Lie algebras. Then the vector field $X_v \in T\mathcal{M}(\pi^{-1}(U))$ corresponding to the (fiber-wise) action is defined by[3]

$$X_v(p) := \frac{\partial}{\partial\zeta}_{|\zeta=0} \exp(-\zeta v)\cdot p. \tag{2.2}$$

Observe that $X_v = -v$, if we consider v as a vector field on $\mathcal{M}_{|U}$. Further set $X_\alpha := X_{v_\alpha}$.

Lemma 2.6. *Let $\alpha \in T^*B(U)$ be a local section and $\phi_\alpha : \mathcal{M}_{|U} \to \mathcal{M}_{|U}$ the corresponding action map. Then the formula*

$$\phi_\alpha^*\omega = \omega + d(\pi^*\alpha)$$

holds.

Proof. By definition of v_α we have $i(v_\alpha)\omega = d(\pi^*\alpha)$. Hence Cartan's formula yields

$$\mathcal{L}_{X_\alpha}\omega = d(i(X_\alpha)\omega) = d(\pi^*\alpha).$$

Using the definition of the Lie derivative and the flow property, we arrive at the claimed formula[4]:

$$\begin{aligned}(\Phi_1^{X_\alpha})^*\omega - (\Phi_0^{X_\alpha})^*\omega &= \int_0^1 \frac{\partial}{\partial\zeta}(\Phi_t^{X_\alpha})^*\omega \; dt \\ &= \int_0^1 (\Phi_t^{X_\alpha})^*(\mathcal{L}_{X_\alpha}\omega)\; dt \\ &= d(\pi^*\alpha).\end{aligned}$$

Note that in the last step we have used that the flows are vertical.

□

The lemma can be used to see that every Lagrangian torus fibration $\pi : \mathcal{M} \to B$ has local *Lagrangian* sections.

Corollary 2.7. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a Lagrangian torus fibration. Then π has local Lagrangian sections. Moreover, if π has a global section and ω is exact or $H^2(B, \mathbb{C}) = 0$, then π also has a global Lagrangian section.*

Proof. It is clear that π has local sections s (because we have the holomorphic implicit function theorem). Let $U \subset B$ be contractible so that $s^*\omega = d\alpha$ for a 1-form on U. Then $\phi_{-\alpha} \circ s : U \to \mathcal{M}_{|U}$ is a Lagrangian section by Lemma 2.6:

$$(\phi_{-\alpha} \circ s)^*\omega = s^*(\omega - d(\pi^*\alpha)) = d\alpha - s^*\pi^*d\alpha = 0.$$

The second claim can be shown analogously. □

Next we want to consider the kernel $\Lambda \subset T^*B$ of the action of T^*B on $\mathcal{M}$, which implicitly already occurred in the proof of Proposition 2.1. Explicitly, it is given by

$$\begin{aligned}\Lambda :&= \{\alpha \in T_b^*B,\ b \in B \mid \alpha \cdot x = x\ \forall x \in M_b\} &(2.3)\\ &= \{\alpha \in T_b^*B,\ b \in B \mid \alpha \cdot x_0 = x_0 \text{ for one } x_0 \in M_b\} \subset T^*B. &(2.4)\end{aligned}$$

This is a submanifold which admits local sections $U \to \Lambda_{|U} \subset T^*U$, and has further special properties:

[3] Recall here that the minus is necessary so that $v \to X_v$ becomes a Lie algebra homomorphism.

[4] Here we integrate over the path $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = t$.

Lemma 2.8. *Let (T^*B, η) be the cotangent bundle of B endowed with its natural symplectic structure. Then the submanifold $\Lambda \subset (T^*B, \eta)$ is Lagrangian and intersects each fiber of $T^*B \to B$ in a lattice.*

Proof. Let α be a local section of Λ so that $\phi_\alpha = id$. In particular, we obtain by Lemma 2.6:

$$\omega = \phi_\alpha^* \omega = \omega + \pi^* d\alpha.$$

Therefore α is a closed differential form, or alternatively a Lagrangian section of (T^*B, η). It follows that $\Lambda \subset T^*B$ is a Lagrangian submanifold. The fact that $\Lambda_b \subset T_b^*B$ is a lattice follows from Proposition 2.1 ii) and its proof. $\square$

Clearly, Λ is also a locally trivial bundle of abelian groups over B which in turn can be considered as a local system (by considering its sections).
The previous lemma in particular implies that

$$(T^*B/\Lambda, \hat{\eta}) \to B$$

is itself a Lagrangian torus fibration. Here we endow the left-hand side with the symplectic structure $\hat{\eta}$ induced by the canonical symplectic form η. It turns out that for $U \subset B$ sufficiently small enough, the Lagrangian torus fibration $T^*U/\Lambda \to U$ is a local model for the original Lagrangian torus fibration.

Proposition 2.9. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a Lagrangian torus fibration and $T^*B/\Lambda \to B$ the associated Lagrangian torus fibration with section. Then both are isomorphic locally on B. Moreover, if $\mathcal{M} \to B$ has a global Lagrangian section, they are globally isomorphic.*

Proof. As we have seen earlier, we may choose a local Lagrangian section $s : U \to \mathcal{M}_{|U}$ of π. It defines a fiber-preserving morphism

$$\phi = \phi_s : T^*U \to \mathcal{M}_{|U}, \quad \alpha \mapsto \alpha \cdot (s \circ p(\alpha))$$

where $p : T^*U \to U$ is the natural projection. This morphism is surjective because the fiberwise action is transitive and maps $\Lambda_{|U}$ to $s(U)$. If we show that ϕ is symplectic, then it follows by construction that it induces a symplectic isomorphism $T^*U/\Lambda \cong \mathcal{M}_{|U}$. This also implies the second claim.
Since we can always translate along the fibers symplectically, we only need to show that ϕ is symplectic at an element $\alpha \in T^*U$ which lies in the zero section (or in Λ). Let $\pi' : T^*U \to U$ be the projection and $b = \pi'(\alpha)$, $x = \phi(\alpha) = s(b)$. By splitting the relative tangent sequence (shrink U if necessary) for π and π', the differential $d\phi_\alpha$ can be considered as a map

$$d\phi_\alpha : \ker d\pi'_\alpha \oplus T_bU \to \ker d\pi_x \oplus T_bU.$$

The map ϕ is fiber-preserving and therefore respects the two summands. Since π and π' are Lagrangian, it hence suffices to compute

$$\phi^*\omega(X, \beta), \quad X \in T_bU, \ \beta \in \ker d\pi'. \tag{2.5}$$

We claim that (dropping α from the notation)

$$d\phi(X) = ds \circ d\pi(X), \quad d\phi(\beta) = -X_\beta(s(b)).$$

The first equation is immediate because X is tangent to the zero section. The second is also readily obtained:

$$d\phi(\beta) = \frac{\partial}{\partial\zeta}_{|\zeta=0} \exp(\zeta)\cdot s(b) = -X_\beta(s(b)),$$

cf. Remark (2.2). Plugging these equations into 2.5 and using Corollary 2.3 we find

$$\phi^*\omega(X,\beta) = -\omega(ds_b(X), X_\beta(s(b))) = \pi^*\beta(ds_b(X)) = \eta(X,\beta).$$

The last equation can be deduced by writing η in standard Darboux coordinates for T^*U as $\eta = \sum_i dq_i \wedge dp_i$ where $q = (q_i)_i$ are local coordinates on B. In total we have shown that ϕ is symplectic, $\phi^*\omega = \eta$, which concludes the proof. □

To simplify notions we make the following convention:

Definition 2.10. A *family of complex tori* is a torus fibration $\pi : \mathcal{M} \to B$ (not necessarily Lagrangian) with a global section $s : B \to \mathcal{M}$.

In particular, a family of complex tori over B is an abelian group 'scheme' (in quotation marks because we are not in the algebraic category) over B.

Corollary 2.11. *Let $\pi : \mathcal{M} \to B$ and $\Lambda \subset T^*B$ be as above. Then $\mathcal{M}$ is a torsor for T^*B/Λ over B. Its isomorphism class is determined by an element $\psi \in H^1(B, T^*_{cl}B/\Lambda)$ where $T^*_{cl}B$ is the sheaf of closed (holomorphic) 1-forms on B.*

Proof. The first claim is immediate because the action of $T^*B \cong \mathcal{V}$ on $\mathcal{M}$ over B descends to an action of T^*B/Λ. This action endows $\mathcal{M}$ with the structure of a T^*B/Λ-torsor by Proposition 2.9. Hence its isomorphism class is determined by an element $\psi \in H^1(B, T^*B/\Lambda)$. However, this class actually comes from $H^1(B, T^*_{cl}B/\Lambda)$, since the local models are glued by translating along the fibers by *closed* 1-forms. More precisely, let $\mathfrak{U} := \{U_i\}$ be a covering of B such that $\varphi_i : T^*U_i/\Lambda_i \xrightarrow{\cong} \mathcal{M}_{U_i}$ via a local Lagrangian section $s_i : U_i \to \mathcal{M}_{U_i}$ as in Proposition 2.9. This defines a cocycle $\psi_{ij} := \varphi_i^{-1}\varphi_j$ over U_{ij}, pictorially we have over U_{ij}

$$\begin{array}{ccc} T^*U_i/\Lambda_i & \xrightarrow{\varphi_i} & \mathcal{M}_{U_{ij}} \\ {\scriptstyle\psi_{ij}}\uparrow & & \uparrow{\scriptstyle id} \\ T^*U_j/\Lambda_j & \xrightarrow{\varphi_j} & \mathcal{M}_{U_{ij}}. \end{array} \tag{2.6}$$

Clearly, ψ_{ij} is symplectic by construction. Since it maps the zero section to a Lagrangian section, it can be considered as translation by a section of $T^*_{cl}U_{ij}/\Lambda_{ij}$ also denoted by ψ_{ij}. It follows that the isomorphism class of $\mathcal{M} \to B$ is determined by the cohomology class $\psi \in H^1(B, T^*_{cl}B/\Lambda)$ of the Čech cocycle $\psi_{ij} \in \check{Z}^1(\mathfrak{U}, T^*_{cl}B/\Lambda)$ □

The bundle T^*B/Λ is related to another natural torus fibration associated with $\pi : \mathcal{M} \to B$. To this end, we recall that every complex torus T is naturally isomorphic to its Albanese variety

$$T \cong Alb(T) := H^0(T, \Omega^1_T)^*/H_1(T, \mathbb{Z}).$$

The right hand side also has a relative version

$$Alb(\mathcal{M}/B) := (\pi_*\Omega^1_{\mathcal{M}/B})^*/\mathcal{H}_1(\mathcal{M}/B)$$

which can be constructed in more generality, cf. Chapter 3. Since $\mathcal{V} = \pi_*T_{\mathcal{M}/B} = (\pi_*\Omega^1_{\mathcal{M}/B})^*$ it follows that the isomorphism $\mathcal{V} \cong T^*B$ (coming from the symplectic structure) induces an isomorphism $Alb(\mathcal{M}/B) \cong T^*B/\Lambda$.

2.1.1 Polarizations

Let $\pi : (\mathcal{M}, \omega) \to B$ be a proper Lagrangian fibration. Then we have seen that the connected components of the fibers of π are isomorphic to complex tori. Moreover, if the fibers are connected, then $\mathcal{M}$ can be glued from the local models $T^*U/\Lambda \to U \subset B$. We now add some additional structure which is the last ingredient for integrable systems.

Definition 2.12. Let $\pi : \mathcal{M} \to B$ be a torus fibration over a connected base B. A *relative polarization of index k* is a global section ρ of $R^2\pi_*\mathbb{Z}$ such that each pair (M_b, ρ_b), $b \in B$, is a non-degenerate complex torus of index k. We then call $(\pi : \mathcal{M} \to B, \rho)$ a *polarized torus fibration of index k*. If there exists at least one polarization on $\pi : \mathcal{M} \to B$, we simply call it a *polarizable torus fibration* (*of index k*).

Remark 2.13. As in the case of a single complex torus, we will drop the index k of a polarization if $k = 0$. A polarization of index $k = 0$ induces a polarization on the corresponding $\mathbb{Z}$-variation of Hodge structures V in the usual sense (Appendix A.2). If $k \geq 1$, then we obtain at least a non-degenerate pairing on the $\mathbb{Z}$-VHS, a *polarization of index k*. It satisfies at least the first of Riemann's bilinear relations (cf. Appendix A.1). This is in fact enough for our considerations. Most importantly, Lemma A.7 still holds true.

Observe that if B is not connected, then the same definition works but one has to label the indices by the connected components of B.

Definition 2.14. A *polarized integrable system of index k* is a (surjective) holomorphic map $\pi : (\mathbf{M}, \omega) \to \mathbf{B}$ with the following property: There is a Zariski-open dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ such that the restriction

$$\pi^\circ : \mathbf{M}^\circ \to \mathbf{B}^\circ, \quad \mathbf{M}^\circ = \pi^{-1}(\mathbf{B}^\circ),$$

is a polarized Lagrangian torus fibration of index k.
An *algebraically completely integrable system* (ACIS) is a polarized integrable system of index 0.

The notion of a *polarizable integrable system* is analogously defined.

Remark 2.15.

a) Note that the definition does not specify the Zariski-open subset $\mathbf{B}^\circ \subset \mathbf{B}$. The maximal choice would be $\mathbf{B}^\circ = \mathbf{B} - \Delta(\pi)$ where $\Delta(\pi)$ is the discriminant of π. However, it is often difficult in concrete examples (e.g. Hitchin systems) to determine $\Delta(\pi)$ explicitly and instead easier to give *some* Zariski-open dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ over which π is smooth.

b) Integrable systems are usually defined without the assumption that the fibers are connected. And this is what usually happens in nature, e.g. Hitchin systems have disconnected fibers in general. Then each connected component of a generic fiber is a torsor for a complex torus. However, we mainly deal with integrable systems that have connected fibers. Since the case of connected fibers is conceptually more transparent, we therefore decided for this slightly restrictive definition.

c) In many cases, ACIS are even principally polarized. However, it is an important feature, e.g. of Hitchin systems, that they are not principally polarized ACIS ([DP12]).

The action-angle variables of a completely integrable Hamiltonian system (in symplectic geometry over $\mathbb{R}$) have analogues for an integrable system $\pi : (\mathbf{M}, \omega) \to \mathbf{B}$. Let $U \subset \mathbf{B}^\circ$ be an open and simply-connected subset such that

$$(\mathbf{M}^\circ_{|U}, \omega) \cong (T^*U/\Lambda, \hat{\eta})$$

for the descended canonical symplectic structure $\hat{\eta}$. Since $\mathbf{M}^\circ_{|U}$ is polarized, so is T^*U/Λ. We may therefore choose a symplectic basis $(\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g)$ of Λ over U with respect to the induced polarization. Now there are two ways to proceed[5]: The first one is to consider these elements as closed 1-forms on U. Hence there are functions $u_i, t_j \in \mathcal{O}(U)$ such that

$$\lambda_i = du_i, \quad \mu_j = dt_j.$$

By construction, we obtain a pair of action variables (after possibly shrinking U)

$$\begin{aligned} u: &\quad b \mapsto (u_1(b), \dots, u_g(b)), \\ t: &\quad b \mapsto (t_1(b), \dots, t_g(b)). \end{aligned}$$

The second viewpoint (e.g. [DM96a], [GS90]) is to consider the λ_i and μ_j as elements of the fiber homologies $\mathcal{H}_1(T^*U/\Lambda, \mathbb{Z})(U) = \Lambda(U)$. Then we can integrate $\hat{\eta}$ along these cycles to obtain closed holomorphic 1-forms. Again there are functions $z_i, w_j \in \mathcal{O}(U)$ such that

$$dz_i = \int_{\lambda_i} \hat{\eta}, \quad dw_j = \int_{\mu_j} \hat{\eta}.$$

Observe that the respective right-hand sides do define 1-forms on the base, because the fibers are Lagrangian with respect to $\hat{\eta}$. More explicitly, let X be a holomorphic vector field on U. Then we have

$$\left(\int_{\lambda_i} \hat{\eta} \right)_b (X) = \int_0^1 \hat{\eta}(s\, \lambda_i(b), \tilde{X})\; ds, \tag{2.7}$$

where $\tilde{X}$ is any (local) holomorphic lift of X and where $\lambda_i \in \Lambda(U)$ is considered as a vertical vector field (which is constant along the torus fibers). Note that the right-hand side is well-defined because every contribution along the Lagrangian fibers vanishes. Therefore we obtain another pair of action variables via the z_i and w_j respectively.
Almost tautologically, this yields the same results as the previous approach: Recall from Corollary 2.3 that under $T^*U \cong \mathcal{V}_{|U}$, λ_i corresponds to a section v_{λ_i} such that $\hat{\eta}(v_{\lambda_i}, -) = \pi^*\lambda_i(-)$. Hence (2.7) together with $d\pi(\tilde{X}) = X$ becomes

$$\left(\int_{\lambda_i} \hat{\eta} \right)_b (X) = \lambda_{i,b}(X).$$

Thus the two approaches produce precisely the same pairs of action variables.

Before we come to the construction of the angle variables, we briefly discuss the relation between a pair $(u = (u_i), t = (t_j))$ of action variables as constructed above. Observe that $\lambda_i = du_i$ is in particular a section of the vertical bundle, hence a (constant) holomorphic vector field along the fibers. Its dual α_i is therefore a section of $\hat{\pi}_*\Omega^1_{(T^*U/\Lambda)/U}(U)$ where $\hat{\pi} : T^*U/\Lambda \to U$ is the projection. Then we can conclude

$$\int_{\lambda_i} \alpha_j = (\alpha_j, \lambda_i) = \delta_{ij},$$

where $(\bullet, \bullet)$ is the natural duality pairing. Therefore the period matrix is obtained via

$$p_{kl}(b) = \int_{\gamma_k(b)} \alpha_l(b) = (\alpha_l(b), \gamma_k(b)).$$

This immediately gives us:

[5]Even though the first approach is straightforward it is rarely used in the literature, presumably because it is clear to experts that both coincide.

Corollary 2.16. *Let (u,t) be a pair of action variables constructed from a symplectic basis (λ_i, μ_j) of $\Lambda \subset T^*U$. Then for each $b \in U$ the relation*

$$du_i = \sum_{j=1}^{g} p_{ij}(b) dt_j$$

holds, where p_{ij} is the period map with respect to $\{\lambda_i, \mu_j\}$.

Remark 2.17. This is clearly different from the situation over $\mathbb{R}$ because in that case any basis of fiber homologies yields action variables. In the complex case this is 'too much' and we have to choose a Lagrangian sublattices to obtain action variables.

Now let us fix action variables $(u_1, \ldots, u_g) : U \to \mathbb{C}^g$ as above. As in the proof of Proposition 2.1 the corresponding Hamiltonian vector fields X_{u_j} give affine coordinates φ_j for the fibers M_b, $b \in U$, via their flows. The coordinates

$$(u_i, \varphi_j) : U \to \mathbb{C}^n \times \mathbb{C}^n/\Lambda$$

are then called *action-angle coordinates*. Observe that the lattice $\Lambda \subset \mathbb{C}^n$ is just a coordinate description of the lattice $\Lambda \subset T^*B$ of (2.3).

Corollary 2.18. *Let $\pi : \mathbf{M} \to \mathbf{B}$ be a principally polarized integrable system. Then there exists a Zariski-open dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ which carries a natural integral affine-symplectic structure.*

An integral affine-symplectic structure on B is an atlas of coordinates whose transition functions lie in $Sp_{2n}(\mathbb{Z}) \rtimes \mathbb{C}^n$, i.e. the integral affine-symplectic transformations.

Proof. Let $\mathbf{B}^\circ \subset \mathbf{B}$ be any Zariski-open subset such that $\pi^\circ : \mathbf{M}^\circ \to \mathbf{B}^\circ$ is smooth. Then we only need to show that the action coordinates on $\mathbf{B}^\circ$ transform in this way. But this follows immediately because any two choices of symplectic bases are related by a transformation $Sp_{2g}(\mathbb{Z})$. □

Remark 2.19. If the integrable system is not principally polarized, then one can find at least coordinates on a Zariski-open dense subset $\mathbf{B}^\circ \subset \mathbf{B}$ whose transition functions are in $Sp_{2g}^D(\mathbb{Z}) \rtimes \mathbb{C}^g$. Here $D = (d_1, \ldots, d_g)$ stands for the type of the polarization (cf. Appendix A.1) and $Sp_{2g}^D(\mathbb{Z})$ are the matrices stabilizing

$$Q_D = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

the 'standard symplectic structure of type D'. The last corollary is in fact only one ingredient of a richer structure on the base of a Lagrangian torus fibration - the structure of a *special Kähler manifold*, which is a topic in its own right ([Fre99]).

2.2 Cubic condition

It is a natural question to ask if a polarized torus fibration has the structure of an integrable system. This question was first answered by Donagi-Markman in [DM96a] in terms of the cubic condition, which is a condition on the derivative of the period map.
Let $\pi : \mathcal{M} \to B$ be a polarized torus fibration of index k and $\mathcal{V} \to B$ its vertical bundle. There

are at least two constraints to be an integrable system. The obvious one is $\dim B = 2\dim\mathcal{M}$. The other one is the existence of an isomorphism[6]

$$\iota : \mathcal{V}^* \to TB, \tag{2.8}$$

see Corollary 2.3. If $\pi : \mathcal{M} \to B$ carries a Lagrangian structure, then such an isomorphism is induced by the symplectic form, see Proposition 2.1. Hence we refine the previous question to:

Question. When does a polarized torus fibration $\pi : \mathcal{M} \to B$ of index k allow a Lagrangian structure which induces a given isomorphism $\iota : \mathcal{V}^* \to TB$?

To state an answer to this question, we briefly need to recall the period map of a polarized torus fibration $\pi : \mathcal{M} \to B$ of relative dimension g and index k. The period domain for such complex tori is given by

$$\mathcal{H}_{g,k} = \{F \in Gr^g(\mathbb{C}^{2g}) \mid v_F Q v_F^t = 0, \quad \mathrm{ind}(iv_F Q\overline{v}_F^t) = k\}. \tag{2.9}$$

Here $Q = Q_{1_g}$ is the standard symplectic form[7], cf. Remark 2.19, and we have chosen a matrix representative $v_F \in Mat(g \times 2g, \mathbb{C})$ for the subspace F. These conditions are independent of the chosen representative and simply state that the subspace F is isotropic with respect to Q and the form $z \mapsto izQ\overline{z}^t$ on F is non-degenerate with index k. Clearly, the case $k = 0$ is (isomorphic to) Siegel's upper half-space $\mathcal{H}_{g,0} = \mathcal{H}_g$ which is the period domain for abelian varieties.
If B is simply connected, then we can unambiguously define the period map

$$\mathcal{P} : B \to \mathcal{H}_{g,k}.$$

It is defined by fixing a base point $0 \in B$ and parallel-transporting the subspace $H^{1,0}(M_0, \mathbb{C})$ to 0. Therefore it can be considered as a g-dimensional subspace of $H^1(M_0, \mathbb{C}) \cong \mathbb{C}^{2g}$. In case B is not simply connected, one has to quotient by the monodromy group Γ which gives a well-defined period map

$$\tilde{\mathcal{P}} : B \to \mathcal{A}_{g,k} = \Gamma\backslash\mathcal{H}_{g,k}.$$

Locally in B there is always a lift $\mathcal{P} : U \to \mathcal{H}_{g,k}$ of $\tilde{\mathcal{P}} : U \to \mathcal{A}_{g,k}$. Since we will be mainly interested in the derivative of the period map $\tilde{\mathcal{P}}$, it is therefore sufficient to look at the derivative $d\mathcal{P}$ of such local lifts.

Lemma 2.20 ([CMSP03])**.** *Let $\pi : \mathcal{M} \to B$ be a torus fibration (not necessarily non-degenerate) and $0 \in B$. Then the derivative of the (local) period map is a map*

$$d\mathcal{P}_0 : T_0B \to \mathrm{Hom}(H^0(M_0, \Omega^1_{M_0}), H^1(M_0, \mathcal{O}_{M_0})).$$

It can be described as follows: $d\mathcal{P}_0(u)$ is cup product with the Kodaira-Spencer class $\kappa_0(u)$, followed by the map on cohomology induced by the contraction $T_{M_0} \otimes \Omega^1_{M_0} \to \mathcal{O}_{M_0}$,

$$d\mathcal{P}_0(u)(\alpha) = \kappa_0(u) \lrcorner \alpha.$$

This lemma specializes as follows to our situation: Let $\pi : \mathcal{M} \to B$ be polarized torus fibration and fix $0 \in B$. Then π is, locally on B, naturally isomorphic to the associated Albanese family $Alb(\mathcal{M}/B)$. In particular, $M_0 \cong H^0(M_0, \Omega^1_{M_0})^*/H_1(M_0, \mathbb{Z})$ and setting $V_0 = H^0(M_0, \Omega^1_{M_0})$, we see that

$$H^1(M_0, T_{M_0}) \cong V_0 \otimes H^1(M_0, \mathcal{O}_{M_0}) \cong V_0 \otimes V_0.$$

[6]Note that this condition already implies $\dim B = 2\dim\mathcal{M}$.

[7]To be precise, one would have to consider Q_D instead of Q in (2.9) to also incorporate the type of the polarizations. However, all types give isomorphic period domains.

Here we have used the polarization coming form the variation of Hodge structures for the last isomorphism. Since we deform $M := M_0$ to nearby non-degenerate tori, the symmetry condition is preserved, so that we have for the Kodaira-Spencer map

$$\kappa : T_0 B \to \mathrm{Sym}^2(V_0) \subset V_0 \otimes V_0.$$

Together with Lemma 2.20, this implies that the derivative of the period map can be considered as a map

$$d\mathcal{P}_0 : T_0 B \to \mathrm{Sym}^2 V_0.$$

Indeed, a priori this is only a map with values in $\mathrm{Hom}(H^{1,0}(M), H^{0,1}(M)) = H^{1,0}(M)^* \otimes H^{0,1}(M)$ and by identifying $H^{1,0}(M)^* = H^{0,1}(M)$ via the polarization Q, it takes values in $\otimes^2 \mathcal{V}_b$. The fact that it is symmetric, i.e. maps to $\mathrm{Sym}^2 \mathcal{V}_b$, follows from

$$\begin{aligned} Q(d\mathcal{P}_b(u)(\alpha), \beta) &= -Q(\alpha, d\mathcal{P}_b(u)(\beta)) \\ &= Q(d\mathcal{P}_b(u)(\beta), \alpha) \qquad u \in T_b B, \quad \alpha, \beta \in H^{1,0}(M). \end{aligned}$$

Here the first equality follows from flatness of Q and the second one by the skew-symmetry of Q on $H^1(M, \mathbb{C})$.
The same argumentation also works if we replace the local Kodaira-Spencer map $\kappa_0 : T_0 B \to H^1(M_0, T_{M_0})$ with the global Kodaira-Spencer map

$$\kappa : TB \to R^1\pi_*(\pi^*\mathcal{V}) \cong \mathcal{V} \otimes R^1\pi_*\mathcal{O}_{\mathcal{M}} \cong \mathcal{V} \otimes \mathcal{V}.$$

As in the pointwise case, we have used the (relative) polarization which induces an isomorphism $R^1\pi_*\mathcal{O}_{\mathcal{M}} \cong \mathcal{V}$, cf. Lemma A.7 and Remark A.8. Therefore the derivative is a map $d\mathcal{P} : TB \to \mathrm{Sym}^2(\mathcal{V})$. We will say more about the relation between the local and the global Kodaira-Spencer map below.

2.2.1 Local cubic condition

We now come back to our initial question, so fix an isomorphism $\iota : \mathcal{V}^* \to TB$. Composing with the derivative of the period map, we obtain an element

$$d\mathcal{P} \circ \iota \in H^0(B, \mathcal{V} \otimes \mathrm{Sym}^2(\mathcal{V})).$$

The existence of a Lagrangian structure imposes further symmetry conditions. We first present a weaker version, the local cubic condition, which only makes a statement about the existence of an almost symplectic form (i.e. a non-degenerate but (possibly) not closed) making the fibers of π maximally isotropic. We call such a structure an *almost Lagrangian structure* on π.

Theorem 2.21 ([DM96a])**.** *Let $\pi : \mathcal{M} \to B$ be a polarized torus fibration with vertical bundle $\mathcal{V} \to B$. Assume that we are given an isomorphism $\iota : \mathcal{V}^* \to TB$. Then there exist* local *non-degenerate 2-forms $\omega \in \Omega^2_{\mathcal{M}}(U)$, such that π is maximally isotropic with respect to ω and induces ι over U iff*

$$d\mathcal{P} \circ \iota \in H^0(B, \mathrm{Sym}^3(\mathcal{V})).$$

In case π has a global (zero) section z, there exists a unique global *non-degenerate 2-form with the additional property that z is maximally isotropic.*

Before we give a detailed proof (following [DM96a]) of this theorem we give an application.

Corollary 2.22. *Let $\pi : (\mathcal{M}, \omega) \to B$ be a polarized (almost) Lagrangian torus fibration and $\sigma : \mathcal{N} := Alb(\mathcal{M}/B) \to B$ the associated polarized torus fibration with section. Then π carries a canonical almost Lagrangian structure.*

Proof. We have already seen that $\mathcal{M}$ and $\mathcal{N}$ are locally isomorphic. Under the natural isomorphism $\mathcal{V}_\sigma \cong \mathcal{V}_\pi$, we can therefore identify the derivatives of the period maps, i.e.

$$d\mathcal{P}_\sigma = d\mathcal{P}_\pi : TB \to \mathrm{Sym}^2(\mathcal{V}_\pi).$$

Since $d\mathcal{P}_\pi$ satisfies the cubic condition (σ carries an almost Lagrangian structure), so does $d\mathcal{P}_\sigma$. Hence $\pi : \mathcal{N} \to B$ carries a unique almost Lagrangian structure inducing the isomorphism $\mathcal{V}_\sigma^* \cong \mathcal{V}_\pi^* \cong TB$ and making the zero section almost Lagrangian. $\square$

Remark 2.23. Since $\pi : \mathcal{M} \to B$ has in general no section, there is no straightforward way to pullback the (almost) symplectic form from $\mathcal{M}$ to $\mathcal{N}$. Also note that each (almost) Lagrangian torus fibration gives rise to a cubic. In particular, this works for the smooth part of a polarized integrable system (cf. [Bal06], [BD14]).

Proof of Theorem 2.21. We mainly follow the proof of [DM96a]. However, we will use this opportunity to give detailed proofs to the key steps below.
Consider the tangent sequence for $\pi : \mathcal{M} \to B$,

$$0 \longrightarrow \pi^*\mathcal{V} \longrightarrow T\mathcal{M} \xrightarrow{d\pi} \pi^*TB \longrightarrow 0.$$

By Lemma 2.25 the subsheaf $\mathcal{F} = \ker \wedge^2 d\pi \subset \Lambda^2 T\mathcal{M}$ sits inside the two exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \Lambda^2 T\mathcal{M} \xrightarrow{\wedge^2 d\pi} \pi^*\Lambda^2 TB \longrightarrow 0, \tag{2.10}$$

$$0 \longrightarrow \pi^*\Lambda^2\mathcal{V} \longrightarrow \mathcal{F} \longrightarrow \pi^*(\mathcal{V} \otimes TB) \longrightarrow 0. \tag{2.11}$$

The exact sequence (2.10) tells us that the fibers of π are Lagrangian with respect to a 2-vector $\psi \in H^0(\mathcal{M}, \Lambda^2 T\mathcal{M})$ iff it is a section of $\mathcal{F}$. The second exact sequence (2.11) describes how 2-vectors in $\mathcal{F}$ induce morphisms $\pi^*\mathcal{V}^* \to \pi^*TB$. Hence we need to check if $\iota \in H^0(B, \mathcal{V} \otimes TB) \subset H^0(\mathcal{M}, \pi^*(\mathcal{V} \otimes TB))$ lies in the image of $H^0(\mathcal{M}, \mathcal{F})$. This happens locally in B iff ι is in the kernel of the connecting homomorphism $\delta : \mathcal{V} \otimes TB \to R^1\pi_*(\pi^*\Lambda^2\mathcal{V})$. The polarization gives an isomorphism $R^1\pi_*\mathcal{O}_\mathcal{M} \cong (\pi_*\Omega^1_{\mathcal{M}/B})^* = \mathcal{V}$, cf. Lemma A.7. Combined with base change, we can therefore consider δ as a map

$$\delta : \mathcal{V} \otimes TB \to \Lambda^2\mathcal{V} \otimes \mathcal{V}.$$

By Proposition 2.24, it factorizes as follows

$$\begin{tikzcd} & & \mathcal{V} \otimes TB \arrow[d, "id\otimes d\mathcal{P}"'] \arrow[dr, "\delta"] & \\ 0 \arrow[r] & \mathrm{Sym}^3\mathcal{V} \arrow[r] & \mathcal{V} \otimes \mathrm{Sym}^2\mathcal{V} \arrow[r, "\beta"] & \Lambda^2\mathcal{V} \otimes \mathcal{V}, \end{tikzcd} \tag{2.12}$$

where the lower line is exact. We conclude that $\delta(\iota) = 0$ iff [8] $id \otimes d\mathcal{P}(\iota) \in \mathrm{Sym}^3\mathcal{V} \subset \mathcal{V} \otimes \mathrm{Sym}^2\mathcal{V}$ which yields the local statement of the claim.
To prove the global statement, let $\psi_i \in H^0(\pi^{-1}(U_i), \Lambda^2 T\mathcal{M})$, $i = 1, 2$, be two local 2-vectors

[8] It is easy to see that $id \otimes d\mathcal{P}(\iota)$ corresponds to $d\mathcal{P} \circ \iota$ under the natural isomorphism $Hom(\mathcal{V}^*, \mathrm{Sym}^2\mathcal{V}) \cong \mathcal{V} \otimes \mathrm{Sym}^2\mathcal{V}$.

with the above properties with $U_1 \cap U_2 \neq \emptyset$ and $s : B \to \mathcal{M}$ a global section. Along $s(B)$ we can split $T\mathcal{M}_{|s(B)} \cong (\pi^*TB \oplus \pi^*\mathcal{V})_{|s(B)}$ holomorphically. Therefore we obtain two local sections $s^*(\psi_{i|s(U_i)}) \in H^0(U_i, \Lambda^2\mathcal{V})$. Now define

$$\psi_i' := \psi_i - \pi^* s^*(\psi_{i|s(U_i)}) \in H^0(\pi^{-1}(U_i), \Lambda^2 T\mathcal{M}).$$

We claim that $\psi_1' - \psi_2' = 0$ on $U_1 \cap U_2$. Indeed, $\psi_1' - \psi_2'$ is a local section of $\pi^*\Lambda^2\mathcal{V}$. Since $\pi^*\mathcal{V}$ is trivial along the fibers, $\psi_1' - \psi_2'$ is already determined on $s(U_1 \cap U_2)$. But $(\psi_1' - \psi_2')_{|s(U_1 \cap U_2)} = 0$ by definition, which shows the uniqueness statement as well.
Finally, we remark that even though we worked with 2-vectors throughout, they uniquely correspond to non-degenerate 2-forms. This follows because they induce ι and are therefore non-degenerate everywhere. $\square$

2.2.2 Supplements to the proof of Theorem 2.21

In this subsection we elaborate more on one of the key steps of the proof of Theorem 2.21. More precisely, the aim of this subsection is to prove the following proposition which is certainly well-known to experts.

Proposition 2.24. *Let $\pi : \mathcal{M} \to B$ be a polarized torus fibration. Then the subsheaf $\mathcal{F} := \ker \wedge^2 d\pi \subset \Lambda^2 T\mathcal{M}$ fits into the two exact sequences*

$$0 \longrightarrow \mathcal{F} \longrightarrow \Lambda^2 T\mathcal{M} \longrightarrow \pi^*\Lambda^2 TB \longrightarrow 0 \tag{2.13}$$

$$0 \longrightarrow \pi^*\Lambda^2\mathcal{V} \longrightarrow \mathcal{F} \longrightarrow \pi^*(\mathcal{V} \otimes TB) \longrightarrow 0. \tag{2.14}$$

The first connecting homomorphism δ of the induced long exact sequence

$$0 \longrightarrow \Lambda^2\mathcal{V} \longrightarrow \pi_*T\mathcal{M} \to \mathcal{V} \otimes TB \xrightarrow{\delta} R^1\pi_*(\pi^*\Lambda^2\mathcal{V}) \longrightarrow \cdots \tag{2.15}$$

is a map $\delta : \mathcal{V} \otimes TB \to \mathcal{V} \otimes \Lambda^2\mathcal{V}$ that factorizes as

$$\delta = \beta \circ (d\mathcal{P} \otimes \mathrm{id})$$

where $\beta : \mathrm{Sym}^2\mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \Lambda^2\mathcal{V}$ is a map in the exact Koszul complex of holomorphic vector bundles

$$0 \longrightarrow \mathrm{Sym}^3\mathcal{V} \xrightarrow{\alpha} \mathrm{Sym}^2\mathcal{V} \otimes \mathcal{V} \xrightarrow{\beta} \mathcal{V} \otimes \Lambda^2\mathcal{V} \longrightarrow \Lambda^3\mathcal{V} \longrightarrow 0.$$

We give a concrete fiberwise description of the map β, which we will need later on. Let $(\sum_{i,j} b_{ij} v_i \otimes v_j) \otimes v \in \mathrm{Sym}^2\mathcal{V}_b \otimes \mathcal{V}_b \subset \otimes^3\mathcal{V}_b$, i.e. $b_{ij} = b_{ji}$, then β is given by

$$\beta((\sum_{i,j} b_{ij} v_i \otimes v_j) \otimes v) = \sum_{i,j} b_{ij} v_i \otimes (v_j \wedge v) \in \mathcal{V}_b \otimes \Lambda^2\mathcal{V}_b.$$

This map is clearly the restriction of the natural map $\otimes^3\mathcal{V}_b \to \mathcal{V}_b \otimes (\Lambda^2\mathcal{V}_b)$ that we also denote by β. On arbitrary open sets $U \subset B$ the same formula gives a map of presheaves because, for example, $U \mapsto \mathcal{V}(U) \otimes \Lambda^2\mathcal{V}(U)$ is only the presheaf underlying $\mathcal{V} \otimes \Lambda^2\mathcal{V}$. However, this uniquely determines β by the universal property of sheafification.
We now give the proof of the proposition in several lemmata that hold in general. The first part of the proposition is contained in

Lemma 2.25. *Let X be a complex manifold and*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence of locally free $\mathcal{O}_X$-modules, i.e. holomorphic vector bundles over X. Then the subsheaf $F := \ker \wedge^2 g \subset \Lambda^2 B$ is sitting inside the following exact sequences

$$0 \longrightarrow F \longrightarrow \Lambda^2 B \longrightarrow \Lambda^2 C \longrightarrow 0,$$

$$0 \longrightarrow \Lambda^2 A \longrightarrow F \longrightarrow A \otimes C \longrightarrow 0.$$

Remark 2.26. Observe that Λ^2 is not an exact functor. It is true that it preserves injectivity and surjectivity for $\mathcal{O}_X$-modules but does not preserve exactness in the middle. Let us examine this for free R-modules (R a commutative unital ring). So let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence of free R-modules. By freeness, we can split

$$B = \mathrm{im} f \oplus B' = \ker g \oplus B' \cong \ker g \oplus C$$

and therefore

$$\Lambda^2 B \cong \Lambda^2 A \oplus \Lambda^2 C \oplus (A \otimes C)$$

(recall that $\oplus_l \Lambda^l V \otimes \Lambda^{k-l} W \cong \Lambda^k (V \oplus W)$ is simply induced by $v \otimes w \mapsto v \wedge w$). Now one can show that $F := \Lambda^2 A \oplus (A \otimes C) \subset \Lambda^2 B$ is spanned by

$$\{f(a) \wedge b \mid a \in A, b \in B\}$$

which is precisely $\ker \wedge^2 g$. But since f is not necessarily surjective, we have in general that $\mathrm{im}(\wedge^2 f) \subsetneq \ker(\wedge^2 g)$, i.e. the induced sequence

$$0 \longrightarrow \Lambda^2 A \longrightarrow \Lambda^2 B \longrightarrow \Lambda^2 C \longrightarrow 0$$

is in general not exact in the middle. However, F gives exact sequences

$$0 \longrightarrow F \longrightarrow \Lambda^2 B \xrightarrow{\wedge^2 g} \Lambda^2 C \longrightarrow 0, \tag{2.16}$$

$$0 \longrightarrow \Lambda^2 A \xrightarrow{\wedge^2 f} F \longrightarrow A \otimes C \longrightarrow 0. \tag{2.17}$$

The two non-trivial maps without a label are inclusion and projection respectively.

Proof of the Lemma. Clearly, $F = \ker \wedge^2 g$ is a sheaf because $\wedge^2 g$ is a sheaf homomorphism. Therefore the first exact sequence is simply given by inclusion and $\wedge^2 g$.
For the second exact sequence, consider the quotient sequence

$$0 \longrightarrow \Lambda^2 A \xrightarrow{\wedge^2 f} F \longrightarrow F/(\Lambda^2 A) \longrightarrow 0.$$

We have to show that $F/(\Lambda^2 A)$ is isomorphic to $A \otimes C$ (as holomorphic vector bundles; this is clear for smooth bundles). To this end, let $h : C \to B$ be a local splitting of the original sequence, i.e. $g \circ h = \mathrm{id}$ which exists at least locally. We locally define a map $\psi : A \otimes C \to F/(\Lambda^2 A)$ via

$$a \otimes c \mapsto [f(a) \wedge h(c)]$$

(where $\psi(a \otimes c)$ corresponds to the antisymmetrization $[f(a) \otimes h(c) - h(c) \otimes f(a)]$). This locally defined map is an isomorphism because the splitting gives the local isomorphism $B \cong A \oplus C$ and therefore $F \cong \Lambda^2 A \oplus (A \otimes C)$. With respect to this splitting, the above map is precisely $F/(\Lambda^2 A) \cong A \otimes C$. This argument clearly does not work globally because the original sequence is in general not split. However, the map above is globally well-defined: Let $h, \tilde{h}$ be two local splittings, so that

$$g \circ h = \mathrm{id} = g \circ \tilde{h}$$

locally. In particular, $g \circ (h - \tilde{h}) = 0$ implying $\operatorname{im}(h - \tilde{h}) \subseteq \ker g = f(\Lambda^2 A)$. So if ψ and $\tilde{\psi}$ are the induced maps from above, we conclude

$$\psi(a \otimes c) - \tilde{\psi}(a \otimes c) = [f(a) \wedge h(c)] - [f(a) \wedge \tilde{h}(c)] = [f(a) \wedge (h(c) - \tilde{h}(c))] = 0.$$

It follows that ψ is a globally defined isomorphism which yields the second exact sequence. □

For the second part of Proposition 2.24, the factorization of the connecting homomorphism δ, we examine at first the tangent sequence of π restricted to the fibers M_b and the induced long exact sequence in cohomology thereof. So in what follows, we have the bundles[9]

$$A = \pi^* \mathcal{V}_{|M_b} = \underline{\mathcal{V}_b}, \quad B = T\mathcal{M}_{|M_b}, \quad C = \pi^* TB_{|M_b} = \underline{T_b B} \tag{2.18}$$

in mind and not the whole tangent sequence.

Lemma 2.27. *Let X be a complex manifold and*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.19}$$

an exact sequence of holomorphic vector bundles as well as

$$0 \longrightarrow \Lambda^2 A \longrightarrow F \longrightarrow A \otimes C \longrightarrow 0$$

the induced exact sequence, where $F = \ker \wedge^2 g$. Then the connecting homomorphism

$$\cdots \longrightarrow H^0(X, A \otimes C) \xrightarrow{\delta} H^1(X, \Lambda^2 A) \longrightarrow \cdots$$

factorizes as $\delta = \wedge \circ c_\tau$, where

$$c_\tau = \mathrm{ev} \circ \cup \tau : H^0(X, A \otimes C) \to H^1(X, A \otimes A)$$

is the map induced by cup product with the extension class $\tau \in H^1(X, \mathrm{Hom}(C, A))$ together with evaluation

$$\mathrm{ev} : \mathrm{Hom}(C, A) \times C \to A,$$

and the map $\wedge : H^1(X, A \otimes A) \to H^1(X, \Lambda^2 A)$ induced by $A \otimes A \to \Lambda^2 A$.

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of Stein open sets, so that $H^p(\mathcal{U}, \mathcal{E}) \cong H^p(X, \mathcal{E})$ for each locally free $\mathcal{O}_X$-module $\mathcal{E}$ (cf. Serre). By refining $\mathcal{U}$ if necessary, we find local holomorphic splittings

$$h_i : C_{|U_i} \to B_{|U_i}$$

[9] An underlined vector space V here stands for the trivial bundle over M_b.

of holomorphic vector bundles. Then the extension class $\tau \in H^1(X, \mathrm{Hom}(C, A))$ is represented in Čech cohomology by the 1-cocycle $\{(h_j - h_i)_{|U_{ij}}\}_{ij}$. The splittings h_i in turn give splittings

$$\hat{h}_i : A \otimes C_{|U_i} \to F_{|U_i}, \quad a \otimes c \mapsto a \wedge h_i(c)$$

just as in the construction of the map $F \to A \otimes C$, cf. Remark 2.26. Hence the connecting homomorphism $\delta : H^0(X, A \otimes C) \to H^1(X, \Lambda^2 A)$ is given in terms of representatives in Čech cohomology by

$$\{a_i \otimes c_i\}_i \mapsto (\partial\{\hat{h}(a_i \otimes c_i)\})_{ij} = \{(a_j \wedge h_j(c_j) - a_i \wedge h_i(c_i))_{|U_{ij}}\}_{ij}$$

for $\partial\{a_i \otimes c_i\}_{ij} = 0$, i.e. $(a_i \otimes c_i)_{|U_{ij}} = (a_j \otimes c_j)_{|U_{ij}}$.
On the other hand, the map $c_\tau : H^0(X, A \otimes C) \to H^1(X, A \otimes A)$ is represented in Čech cohomology as the composition

$$\begin{aligned} \alpha = \{a_i \otimes c_i\} &\mapsto \{(a_j \otimes c_j \otimes h_j - a_i \otimes c_i \otimes h_i)_{|U_{ij}}\}_{ij} = \alpha \cup \tau \\ &\mapsto \{(a_j \otimes h_j(c_j) - a_i \otimes h_i(c_i))_{|U_{ij}}\}_{ij}, \end{aligned} \tag{2.20}$$

again for $\partial\alpha = 0$, and $\wedge : H^1(X, A \otimes A) \mapsto H^1(X, \Lambda^2 A)$ is represented by

$$\{(a \otimes a')_{ij}\} \mapsto \{(a \wedge a')_{ij}\}.$$

Hence the composition $\wedge \circ c_\tau$ is precisely the connecting homomorphism δ as claimed. □

We can express the connecting homomorphism in familiar terms for the choices (2.18), i.e. the fiberwise restriction of the tangent sequence. More concretely, we claim that in this case, the fiberwise connecting homomorphism factors precisely as stated in Proposition 2.24.

Corollary 2.28. *Let $M = M_b$ be a fiber of a family of polarized abelian varieties $\pi : \mathcal{M} \to B$. Then the connecting homomorphism δ in cohomology of the short exact sequence*

$$0 \longrightarrow \Lambda^2 \underline{\mathcal{V}_b} \longrightarrow \mathcal{F}_{|M} \longrightarrow \underline{T_b B} \longrightarrow 0$$

factors as $\delta = \beta \circ (\mathrm{id} \otimes d\mathcal{P}_b)$ for the map β of Proposition 2.24 and the derivative of the period map $d\mathcal{P}_b : T_b B \to \mathrm{Sym}^2 \mathcal{V}_b$ in b.

Proof. In this case, the initial exact sequence is

$$0 \to \underline{\mathcal{V}_b} \to T\mathcal{M}_{|M} \to \underline{T_b B} \to 0$$

over the fiber $M = M_b$. The Kodaira-Spencer map $\kappa_b : T_b B \to H^1(M, \underline{\mathcal{V}_b})$ is precisely cup product with the extension class $\tau \in H^1(M, \mathrm{Hom}(\underline{T_b B}, \underline{\mathcal{V}_b}))$ of this sequence and contraction on the $T_b B$-part. Keeping the notation as in the previous proof, this can be expressed for representatives in Čech cohomology as

$$H^0(M, \underline{T_b B}) = T_b B \ni \{u_i\} \mapsto \{(h_j(u_j) - h_i(u_i))_{|U_{ij}}\}.$$

Here h_i are again local splittings which correspond in this case to local lifts of (germs of) holomorphic vector fields around $b \in B$. Comparing with the description of c_τ in (2.20), we see that

$$c_\tau = \mathrm{id} \otimes \kappa_b : \mathcal{V}_b \otimes T_b B \to \mathcal{V}_b \otimes H^1(M, \underline{\mathcal{V}_b}) = H^1(M, \underline{\mathcal{V}_b} \otimes \underline{\mathcal{V}_b}).$$

This can also expressed under the isomorphisms $H^1(M, \underline{\mathcal{V}_b}) \cong H^{0,1}(M) \otimes \mathcal{V}_b$ etc. For simplicity, we assume that the extension class τ is of the form $\tau = \beta \otimes \phi \in H^{0,1}(M) \otimes \mathrm{Hom}(T_bB, \mathcal{V}_b)$ (in general one clearly has a finite sum over such elements). Then $\kappa_b(u) = \beta \otimes \phi(u)$ and $c_\tau : \mathcal{V}_b \otimes T_bB \to \mathcal{V}_b \otimes H^{0,1}(M) \otimes \mathcal{V}_b$ can be expressed as

$$v \otimes u \mapsto v \otimes \beta \otimes \phi(u)$$

and we see again that $c_\tau = \mathrm{id} \otimes \kappa_b$.

We claim that in the case of a fibration of abelian varieties, the Kodaira-Spencer map κ_b can be identified with the derivative $d\mathcal{P}_b : T_bB \to \mathrm{Sym}^2\mathcal{V}_b \subset \otimes^2\mathcal{V}_b$ of the period map $\mathcal{P}$ under the isomorphism $\mathcal{V}_b = H^{1,0}(M)^* \cong H^{0,1}(M)$. Without using this isomorphism, the latter is actually given by $d\mathcal{P}_b : T_bB \to \mathrm{Hom}(H^{1,0}(M), H^{0,1}(M))$,

$$d\mathcal{P}_b(u)(\omega) = \omega(\kappa(u)) \in H^{0,1}(M),$$

see Lemma 2.20. Choose a basis $v_1, \dots, v_g \in \mathcal{V}_b = H^{1,0}(M)^*$, the dual basis $\alpha_1, \dots, \alpha_g \in H^{1,0}(M)$ and let $\beta_1, \dots, \beta_g \in H^{0,1}(M)$ correspond to the v_j under the polarization Q. Then we can express

$$\kappa(u) = \sum_{i,j} a_{ij}\, v_i \otimes \beta_j \in H^1(M, \underline{\mathcal{V}_b}) = H^{0,1}(M) \otimes \mathcal{V}_b.$$

This implies $\alpha_k(\kappa(u)) = \sum_j a_{kj}\, \eta_j$ so that $d\mathcal{P}_b(u)$ and $\kappa(u)$ are both given by

$$\sum_{i,j} a_{ij}\, v_i \otimes v_j \in \mathcal{V}_b \otimes \mathcal{V}_b$$

under the ismorphism $H^{1,0}(M)^* \cong H^{0,1}(M)$. Note that this expression is actually in $\mathrm{Sym}^2\mathcal{V}_b$ because $d\mathcal{P}_b(u)$ is (which is in turn a consequence of the fact that we consider a family of polarized manifolds). This yields the first half of the statement, i.e. $c_\tau = \mathrm{id} \otimes d\mathcal{P}_b$.

The map $\wedge : H^1(M, \underline{\mathcal{V}_b} \otimes \underline{\mathcal{V}_b}) \to H^1(M, \Lambda^2\underline{\mathcal{V}_b})$ is induced from

$$\eta \otimes v \otimes v' \mapsto \eta \otimes (v \wedge v')$$

for $\eta \otimes v \otimes v' \in H^{0,1}(M) \otimes \mathcal{V}_b \otimes \mathcal{V}_b$. Hence we compute with the notation from above, without the identification $H^{1,0}(M)^* = H^{0,1}$, that

$$\wedge \circ (\mathrm{id} \otimes d\mathcal{P}_b)(v \otimes u) = \wedge(v \otimes (\sum_{i,j} a_{ij}\, v_i \otimes \eta_j)) = \sum_{i,j} a_{ij}\, (v \wedge v_i) \otimes \eta_j.$$

But this corresponds precisely to $\beta \circ (\mathrm{id} \otimes d\mathcal{P}_b)(v \otimes u)$ under the isomorphism $H^{1,0}(M)^* \cong H^{0,1}(M)$. □

The connecting homomorphism $\delta : \mathcal{V} \otimes TB \to R^1\pi_*(\pi^*\Lambda^2\mathcal{V}) \cong \mathcal{V} \otimes \Lambda^2\mathcal{V}$ (considered as a vector bundle homomorphism) restricted to the fibers over b gives[10]

$$\delta_{|b} : \mathcal{V}_{|b} \otimes T_bB \to \mathcal{V}_{|b} \otimes \Lambda^2\mathcal{V}_{|b} = H^1(M_b, \underline{\mathcal{V}_{|b}}).$$

Therefore it remains to show that $\delta_{|b}$ can be identified with the fiberwise connecting homomorphism δ from Corollary 2.28. Even though this is not difficult, we prove it for completeness.

[10]In order to avoid confusions, we use here and in the next subsection the following convention: For any locally free $\mathcal{O}_{\mathcal{M}}$-module E we will denote by E_b its stalk at b and $E_{|b} = E_b \otimes \kappa(b)$ for its fiber at b. An analogous notation will be used for germs (of morphisms between locally free $\mathcal{O}_{\mathcal{M}}$-modules).

Fiberwise connecting homomorphisms

Let $\pi : \mathcal{M} \to B$ be a proper surjective submersion with connected fibers and let

$$0 \longrightarrow E \xrightarrow{a} F \xrightarrow{b} \pi^* G \to 0 \tag{2.21}$$

be an exact sequence of locally free $\mathcal{O}_{\mathcal{M}}$-modules on $\mathcal{M}$ (i.e. holomorphic vector bundles). We want to relate the connecting homomorphism $\delta : \pi_*(\pi^* G) = G \to R^1\pi_* E$ with the fiberwise connecting homomorphism, namely $\delta_{|b} : \pi^* G_{|M_b} = G_{|b} \to H^1(M_b, E_{|M_b})$ of the restricted sequence

$$0 \longrightarrow E_{|M_b} \xrightarrow{e} F_{|M_b} \xrightarrow{f} \pi^* G_{|M_b} \longrightarrow 0. \tag{2.22}$$

To do so, take a *smooth* splitting $h : \pi^* G \to F$ of (2.21). The following diagram gives a relation between these connecting homomorphisms

$$\begin{tikzcd} H^1(\pi^{-1}(U), E) \arrow[r, "stalk"] & R^1\pi_*(E)_b \arrow[r, "r_b"] & H^1(M_b, E_{|M_b}) \\ G(U) \arrow[u, "\delta_U"] \arrow[r, "stalk"] & G_b \arrow[u, "\delta_b"] \arrow[r, "ev_b"] & G_{|b} \arrow[u, "\delta_{|b}"'] \end{tikzcd} \tag{2.23}$$

for $b \in U \subset B$. Here *stalk* are simply the corresponding stalk maps, ev_b is the evaluation map at b (i.e. it evaluates a germ of a section in b) and δ_b is the map on stalks induced by δ_U. The latter map is defined via

$$\delta_U(s) = [\bar{\partial}^F h(\pi^* s)], \quad s \in G(U),$$

where the brackets stand for the equivalence class in cohomology, which is independent of the chosen splitting h. Note that this class does lie in $H^1(\pi^{-1}(U), E)$ because s and f are holomorphic and h is a splitting,

$$f(\bar{\partial}^F h(\pi^* s)) = \bar{\partial}^{\pi^* G}(\pi^* s) = \pi^*(\bar{\partial}^G s) = 0.$$

Finally, r_b is defined by restriction

$$r_b([\psi]_b) = i^*_{M_b}\psi, \quad [\psi]_b \in R^1\pi_*(E)_b.$$

This is well-defined as we see in the next proof.

Lemma 2.29. *The diagram (2.23) is commutative.*

Proof. The left square is commutative by definition. For the right square, recall that the restriction maps of the presheaf $U \mapsto H^1(\pi^{-1}(U), E)$ are given by

$$i^*_{VU} : H^1(\pi^{-1}(U), E) \to H^1(\pi^{-1}(V), E), \quad i_{VU} : V \hookrightarrow U,$$

which also shows the well-definedness of r_b. It remains to prove that

$$r_b \circ \delta_b([s]) = i^*_{M_b}(\delta_U(s)) = \delta_{|b} \circ ev_b([s]) = \delta_{|b}(s(b)).$$

To this end, let $g \in G_{|b}$ and $s \in G(U)$ be any local holomorphic section satisfying $s(b) = g$ (for U small enough). Furthermore, let $V = \pi^{-1}(U) \subset \mathcal{M}$ be an open neighborhood of M_b and $\phi \in F(V)$ any smooth lift of $\pi^* s$, in particular

$$i^*_{M_b} f(\phi) = i^*_{M_b}(\pi^* s) = g$$

where we consider g as a constant section of the trivial bundle $\pi^*G_{|M_b} = M_b \times G_{|b}$. Then we compute ($i_b = i_{M_b}$)

$$i_b^*(\delta_U(s)) = [i_b^*(\bar\partial^F \phi)] = [\bar\partial^{i_b^* F}(i_b^*\phi)] = \delta_b(g).$$

The last equality follows because $i_b^*\phi$ is a (smooth) lift of the constant section g. Hence also the left square is commutative. □

The commutativity of diagram (2.23) can also be stated as the expected fact that the fiberwise connecting homomorphism $\delta_{|b}$ is a 'linearization' of the connecting homomorphism δ_b on stalks which is a morphism of $\mathcal{O}_{B,b}$-modules.

Example 2.30. Maybe the most common application of this result is the Kodaira-Spencer map. In that case, we have the exact sequence

$$0 \longrightarrow T_{\mathcal{M}/B} \longrightarrow T\mathcal{M} \longrightarrow \pi^*TB \longrightarrow 0$$

and $\delta_{|b} = \kappa_b$ is the Kodaira-Spencer map (for the fiber M_b). The above lemma shows us that the connecting homomorphism $\kappa : TB \to R^1\pi_*(T_{\mathcal{M}/B})$ can be seen as a global version of the Kodaira-Spencer map.

End of proof of Proposition 2.24. So far we have seen that the fiberwise morphims $\delta_{|b} : \mathcal{V}_{|b} \otimes T_bB \to \mathcal{V}_{|b} \otimes \Lambda^2\mathcal{V}_{|b}$ factorizes as

$$\delta_{|b} = \beta_{|b} \circ (d\mathcal{P}_{|b} \otimes id).$$

Moreover, $\delta_{|b}$ is the linearization of the morphism on stalks δ_b (with $E = \pi^*\mathcal{V}, \quad F = T\mathcal{M}, \quad G = TB$). By the equivalence between the category of holomorphic vector bundles on M_b and locally free $\mathcal{O}_{M_b}$-modules, these two maps determine each other uniquely. On the other hand, all δ_b uniquely determine δ so that $\delta = \beta \circ (d\mathcal{P} \otimes id)$ as claimed. □

2.2.3 Global cubic condition

The local cubic condition of the previous section only gave a condition for the existence of an almost Lagrangian structure. We now discuss the global cubic condition which gives a necessary and sufficient condition for the existence of a Lagrangian structure, i.e. of a *closed* non-degenerate 2-forms making the fibers Lagrangian.

Theorem 2.31 (Global cubic condition, [DM96a]). *Let $\pi : \mathcal{M} \to B$ be a family of polarized complex tori and $\iota : \mathcal{V} \to T^*B$ an isomorphism. Assume that there exists a sublattice $\mathcal{L} \subset (R^1\pi_*\mathbb{Z})^\vee$ which is Lagrangian with respect to the polarization that is mapped to a Lagrangian sublattice $\iota(\mathcal{L}) \subset T^*B$ with respect to the canonical symplectic structure. Then there exists a Lagrangian structure on π inducing ι iff*

$$d\mathcal{P} \circ \iota \in H^0(B, \mathrm{Sym}^3(\mathcal{V})). \tag{2.24}$$

Proof. The main idea is the following: We have the natural embedding $\mathcal{H}_1(\mathcal{M}/B) \hookrightarrow \mathcal{V} = (\pi_*\Omega^1_{\mathcal{M}/B})^*$ given by integration. In each fiber $\mathcal{H}_1(\mathcal{M}/B)_b$ is a sublattice of $(\pi_*\Omega^1_{\mathcal{M}/B})^*_b = H^0(M_b, \Omega^1_{M_b})^*$ (cf. relative Albanese variety). If any local section of $\mathcal{H}_1(\mathcal{M}/B)$ is mapped to a local closed 1-form under ι, then the canonical symplectic form on T^*B descends to

$$T^*B/\iota(\mathcal{H}_1(\mathcal{M}/B)) \cong \mathcal{V}/(\mathcal{H}_1(\mathcal{M}/B)) \cong \mathcal{M}.$$

To link this to the cubic condition, we may work locally. So fix $0 \in B$ and choose a basis $\alpha_1, \dots, \alpha_g$ of the Lagrangian lattice $\mathcal{L}_0$ and $\beta_1, \dots, \beta_g$ its dual basis with respect to the polarization $\langle \bullet, \bullet \rangle$, i.e. $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Observe that the sublattice $\mathcal{L}'_0$ spanned by the β_j is Lagrangian as well. By parallel translation, we can extend it to a trivialization of $\mathcal{H}_1(\mathcal{M}/B)$ in a neighborhood $U \subset B$ of 0. We further assume that $\mathcal{V}_{|U} \cong \underline{V_0}$ where $V_0 := H^0(M_0, \Omega^1_{M_0})$ (holomorphic trivialization) and similarly for $T^*B_{|U}$. In terms of this trivialization, we can express the (local) period map $\mathcal{P} : U \to D$ via $(p_{ij}) : U \to \mathrm{Sym}^2 V_0$. Then we obtain lattices

$$\Lambda_b := \left\{ \sum_i m_i \alpha_i + \sum_j n_j \sum_k p_{jk}(b) \beta_k \;\middle|\; m_i, n_j \in \mathbb{Z} \right\} \subset V_0.$$

In the light of the Albanese construction, this gives a natural isomorphism

$$M_0 \cong V_0/\Lambda_0.$$

We claim that $\iota(\Lambda) \subset T^*U$ consists of Lagrangian sections (i.e. closed 1-forms) iff the cubic condition (2.24) is satisfied. This would finish the proof by the idea mentioned at the beginning. It suffices to show that the sections

$$b \mapsto m\iota(\alpha_i) + n \sum_k p_{jk}(b)\iota(\beta_k)$$

(given in the trivialization) are closed 1-forms. Since $\iota(\alpha_i)$ spans a Lagrangian sublattice by assumption, so does $\iota(\beta_j)$. In other words, these are (locally defined) closed 1-forms. Therefore it remains to prove that

$$b \mapsto \hat{\beta}_k(b) := \sum_j p_{jk}(b)\iota(\beta_k)$$

are closed. But this is equivalent to the equality of mixed partials, hence to (2.24). Indeed, at a point $b \in U$ we may suppose that $\iota(\beta_j) = dt_j$ for some local coordinates t_j. Then $\hat{\beta}_k$ is closed (at b) iff

$$\begin{aligned} d\hat{\beta}_k \left(\frac{\partial}{\partial t_l}, \frac{\partial}{\partial t_m} \right) &= \sum_k dp_{jk} \wedge dt_k \left(\frac{\partial}{\partial t_l}, \frac{\partial}{\partial t_m} \right) \\ &= \frac{\partial p_{jl}}{\partial t_l} - \frac{\partial p_{jm}}{\partial t_m} = 0 \quad \forall l, m, \end{aligned}$$

which is true iff (2.24) is satisfied. □

The canonical symplectic structure η on T^*B is closed but the descended symplectic structure $\hat{\eta}$ on $T^*B/\iota(\mathcal{L})$, as in the previous theorem, might not be. Indeed, it is not automatic that a 'potential' τ with $d\tau = \eta$ descends to the quotient $T^*B/\iota(\mathcal{L})$.

2.2.4 Sheaf-theoretic description

Given an algebraically completely integrable system $\pi : \mathbf{M} \to \mathbf{B}$, it yields a polarizable $\mathbb{Z}$-VHS of weight 1 and -1 over a Zariski-dense open subset $\mathbf{B}^\circ \subset \mathbf{B}$. In this section, we briefly discuss the inverse which is particularly suited for our later applications. Moreover, we give a criterion when a (polarized) VHS of weight ± 1 gives rise to an ACIS. This approach is very natural but we could not find it in this form in the literature. The closest account, that we could found, is contained in [KS14] .

Let $\mathrm{VHS}^p_{\mathbb{Z}}(B,\pm 1)$ be the category of polarizable $\mathbb{Z}$-VHS on the complex manifold B and $\mathrm{AVF}^p(B)$ the category of families of abelian varieties over B (in particular, they have a global section and admit a global polarization). Then we have duality functors

$$(.)^* : \mathrm{VHS}^p_{\mathbb{Z}}(B,\pm 1) \to \mathrm{VHS}^p_{\mathbb{Z}}(B,\mp 1), \quad \mathsf{V} \mapsto \mathsf{V}^* = \mathrm{Hom}_{\mathrm{VHS}}(\mathsf{V},\mathbb{Z}_B(0)),$$
$$\widehat{(.)} : \mathrm{AVF}^p(B) \to \mathrm{AVF}^p(B), \quad \pi \mapsto \hat{\pi} := Jac(\pi).$$

Here $\mathbb{Z}_B(0)$ denotes the constant $\mathbb{Z}$-VHS of weight 0 and $Jac(\pi)$ is the Jacobian torus fibration assoicated with π (cf. Chapter 3). There are several ways to relate the two categories $\mathrm{VHS}^p(\mathbb{Z})(B,\pm 1)$ and $\mathrm{AVF}^p(B)$. To go from the latter to the former, we define the functors

$$\mathsf{V} : \mathrm{AVF}^p \to \mathrm{VHS}^p_{\mathbb{Z}}(B,1), \quad \mathsf{V}(\pi) := (R^1\pi_*\mathbb{Z}, \mathcal{F}^\bullet\mathcal{H}^1(\pi,\mathbb{C})),$$
$$\mathsf{V}^\dagger : \mathrm{AVF}^p \to \mathrm{VHS}^p_{\mathbb{Z}}(B,-1), \quad \mathsf{V}^\dagger(\pi) := (.)^* \circ \mathsf{V}(\pi).$$

Each polarization on $\pi : \mathcal{M} \to B$ clearly induces one on $\mathsf{V}(\pi)$ and $\mathsf{V}^\dagger(\pi)$ so that V and $\mathsf{V}^\dagger$ are well-defined. We can also go the other way round by defining the functors

$$\mathcal{J} : \mathrm{VHS}^p_{\mathbb{Z}}(B,1) \to \mathrm{AVF}^p(B), \quad \mathcal{J}(\mathsf{V}) = \mathsf{V}_{\mathcal{O}}/(\mathcal{F}^1\mathsf{V}_{\mathcal{O}} + \mathsf{V}_{\mathbb{Z}}),$$
$$\mathcal{A} : \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \to \mathrm{AVF}^p(B), \quad \mathcal{A}(\mathsf{W}) := \mathsf{W}_{\mathcal{O}}/(\mathcal{F}^{-1}\mathsf{W}_{\mathcal{O}} + \mathsf{W}_{\mathbb{Z}}).$$

The relations between these functors are summarized in the following

Proposition 2.32. *Let B be a complex manifold. Then the following diagram is commutative*

$$\begin{tikzcd} \mathrm{VHS}^p_{\mathbb{Z}}(B,1) \arrow[r, "\mathcal{J}"] \arrow[d, "(.)^*"'] & \mathrm{AVF}^p(B) \arrow[d, "\widehat{(.)}"] \\ \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \arrow[r, "\mathcal{A}"] & \mathrm{AVF}^p(B). \end{tikzcd} \tag{2.25}$$

Moreover, $\mathcal{A}$ and $\mathsf{V}^\dagger$ yield an equivalence between $\mathrm{VHS}^p_{\mathbb{Z}}(B,-1)$ and $\mathrm{AVF}^p(B)$ whereas

$$\mathcal{J} \circ \mathsf{V} \simeq \widehat{(.)}, \quad \mathsf{V} \circ \mathcal{J} \simeq (-1) \circ (.)^*. \tag{2.26}$$

Here $(-1) : \mathrm{VHS}^p_{\mathbb{Z}}(B,-1) \to \mathrm{VHS}^p_{\mathbb{Z}}(B,1)$ is Tate twist.

Proof. The diagram (2.25) is commutative because it commutes fiberwise by definition of the dual torus. To see the claimed equivalence, observe that $\mathcal{A} \circ \mathsf{V}^\dagger(\pi) = Alb(\pi)$, the Albanese fibration associated with π. But $Alb(\pi)$ is naturally isomorphic to $\pi \in \mathrm{AVF}^p(B)$ so that $\mathcal{A} \circ \mathsf{V}^\dagger \simeq \mathrm{id}_{\mathrm{AVF}^p}$. Conversely, the dual of the VHS of weight 1 associated to $\mathcal{A}(\mathsf{W})$ is isomorphic to W itself. Hence we also have $\mathsf{V}^\dagger \circ \mathcal{A} \simeq \mathrm{id}_{\mathrm{VHS}^p}$.
The first relation in (2.26) follows by definition of the dual torus fibration. For the second relation denote $\pi : \mathcal{J}(\mathsf{V}) \to B$. Of course, if $A_b = \pi^{-1}(b)$ for $b \in B$, then $H^1(A_b,\mathbb{Z}) = \mathsf{V}^\vee_{\mathbb{Z},b}$. This implies[11]

$$\mathsf{V}(\pi) = (R^1\pi_*\mathbb{Z}, \mathcal{F}^\bullet\mathcal{H}^1(\pi,\mathbb{C})) \cong (\mathsf{V}^\vee_{\mathbb{Z}}, \mathcal{F}^\bullet\mathsf{V}^*_{\mathcal{O}})(-1) = \mathsf{V}^*(-1)$$

and therefore $\mathsf{V} \circ \mathcal{J} \simeq (-1) \circ (.)^*$. □

Remark 2.33.

[11] See (A.1) for the notation $\mathcal{H}^k(\pi,\mathbb{C})$.

a) There is a straightforward way to generalize the previous proposition to families of non-degenerate complex tori. The only difference is that one has to weaken the notion of a polarization on a VHS of weight ± 1 to a non-degenerate pairing which is allowed to have non-zero index.

b) Although VHS of weight -1 behave better in relation with families of abelian varieties/non-degenerate complex tori, we often work with VHS of weight 1 and the functor $\mathcal{J}$. The reason being that many of the families of abelian varieties/non-degenerate complex tori, that we consider, are induced from other families of varieties. And for the latter it is more natural to consider the induced VHS of positive weights.

There is an immediate analogue for isogenous abelian varieties on the side of VHS.

Definition 2.34. Let V, V' be two $\mathbb{Z}$-VHS of weight $k \in \mathbb{Z}$ on a complex manifold B. We say that V and V' are *isogenous* to each other, $\mathsf{V} \simeq \mathsf{V}'$, if there is an isomorphism $\mathsf{V} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathsf{V}' \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-VHS.

Observe that two isogenous $\mathbb{Z}$-VHS V, V' have isomorphic associated filtered holomorphic bundles $(\mathsf{V} \otimes \mathcal{O}_B, \mathcal{F}^\bullet) \cong (\mathsf{V}' \otimes \mathcal{O}_B, \mathcal{F}'^\bullet)$. In particular, all statements that are independent of the underlying lattices $\mathsf{V}_{\mathbb{Z}}$, $\mathsf{V}'_{\mathbb{Z}}$ (essentially everything that involves analysis) hold true for whole isogeny classes of VHS. Further specializing to $\mathbb{Z}$-VHS of weight ± 1, we see that isogenous $\mathbb{Z}$-VHS of weight ± 1 give rise to isogenous families of abelian varieties and vice versa.

Example 2.35. Let (V, Q) be a polarized $\mathbb{Z}$-VHS of weight 1 over B. Its (Tate-twisted) dual $\mathsf{V}^\vee(-1)$ is a $\mathbb{Z}$-VHS of weight 1 and the polarization $Q : \mathsf{V} \to \mathsf{V}^\vee(-1)$ is an isogeny. Under the functors $\mathcal{A}$ and $\mathcal{J}$ this corresponds to the fact that a family of abelian varieties is isogenous to its dual. Since there exist abelian varieties A such that its dual $\hat{A}$ is not isomorphic to itself, the polarization $Q : \mathsf{V} \to \mathsf{V}^\vee(-1)$ is in general not an isomorphism. For example, this is the case for (the neutral component of) Hitchin systems ([DP12]).

The next result gives a criterion when a VHS yields a Lagrangian torus fibration.

Proposition 2.36. *Let* V *be a* $\mathbb{Z}$*-VHS of weight* 1 *over* B *and* Q *a polarization on* $\mathsf{V}_R = \mathsf{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ *for* $R = \mathbb{Z}$, $\mathbb{Q}$ *or* $\mathbb{C}$. *Assume there is a global section* $\lambda \in \Gamma(B, \mathsf{V}_{\mathcal{O}})$ *such that*

$$\phi_\lambda : TB \to \mathcal{F}^1\mathsf{V}, \quad X \mapsto \nabla^{GM}_X \lambda, \tag{2.27}$$

is an isomorphism. Further let $\iota : \mathcal{V} \to T^*B$ *be the isomorphism induced by (2.27) and the polarization* Q, *where* $\mathcal{V} = \mathcal{V}_\pi$ *is the vertical bundle of*

$$\pi : \mathcal{J}(\mathsf{V}) = \mathsf{V}_{\mathcal{O}}/(\mathcal{F}^1 + \mathsf{V}_{\mathbb{Z}}) \to B.$$

Then π *carries a unique Lagrangian structure which makes the zero section Lagrangian and induces* ι. *Up to symplectomorphisms, it is independent of* Q. *Moreover, the same results hold true for* $\mathcal{J}(\mathsf{V}') \to B$ *where* V' *is any VHS in the isogeny class of* V, *in particular for* $\mathsf{V}' = \mathsf{V}^\vee(-1)$.

Proof. We begin by recalling how $\iota : \mathcal{V} \to T^*B$ is constructed. To this end, observe that the polarization Q induces an isomorphism

$$\phi_Q : \mathcal{V}_\pi = \mathsf{V}_{\mathcal{O}}/\mathcal{F}^1 \to (\mathcal{F}^1)^*.$$

Then ι is simply the composition $\iota = \phi_\lambda^\vee \circ \phi_Q$. These isomorphisms further induce isomorphisms (denoted by the same symbols)[12]

$$\mathcal{J}(V) \xrightarrow{\phi_Q} (\mathcal{F}^1)^*/\phi_Q(\mathsf{V}_{\mathbb{Z}}) \xrightarrow{\phi_\lambda^\vee} T^*B/\Lambda, \quad \Lambda := \phi_\lambda^*(\phi_Q(\mathsf{V}_{\mathbb{Z}})).$$

[12] If Q is not defined over $\mathbb{Z}$, then $\phi_Q(\mathsf{V}_{\mathbb{Z}})$ is *not* contained in $\mathsf{V}_{\mathbb{Z}}^\vee = Hom(\mathsf{V}_{\mathbb{Z}}, \mathbb{Z})$. In any case, $\phi_Q(\mathsf{V}_{\mathbb{Z}})$ is a local system of lattices.

If we can show that $\Lambda \subset T^*B$ is Lagrangian, then the canonical symplectic structure η on T^*B descends to a symplectic structure $\hat{\eta}$ on T^*B/Λ. The induced symplectic structure on $\mathcal{J}(\mathsf{V})$ will then satisfy all the claimed properties, in particular π will be Lagrangian.
To show that $\Lambda \subset T^*B$ is Lagrangian, we have to prove that the image of $\mathsf{V}_{\mathbb{Z}}$ in T^*B under ι consists of closed (local) 1-forms. If γ is a local section of $\phi_Q(\mathsf{V}_{\mathbb{Z}}) \subset (\mathcal{F}^1)^*$, then its image is the local 1-from

$$\phi_\lambda^\vee(\gamma)(X) = \langle \gamma, \nabla_X \lambda \rangle, \quad X \in TB,$$

where the brackets are the duality pairing between $(\mathcal{F}^1)^*$ and $\mathcal{F}^1$. Its closedness can be shown similarly as in the proof of Theorem 3 in [DM96a]: Since $\mathsf{V}_{\mathbb{Z}}^\vee$ is a local system and ∇ is flat, we can represent γ around $b \in U \subset B$ as some fixed element $\gamma_0 \in \phi_\lambda^\vee(\mathsf{V}_{\mathbb{Z}})_b$, ∇ as d and λ as a map $f : U \to \mathsf{V}_0$. In particular, $v = \nabla_X \lambda \in \mathsf{V}$ is represented by $df(X)$ where $X \in TU$. It then follows that $g : U \to \mathbb{C}$, $g(b) = \langle \gamma_0, f(b) \rangle$, satisfies

$$\begin{aligned} dg(X) =& \frac{d}{dt}_{|t=0} g(\alpha(t)) \\ =& \langle \gamma_0, df(X) \rangle \\ =& \phi_\lambda^\vee(\gamma)(v). \end{aligned}$$

Here α is a curve representing the tangent vector X. Hence $\phi_\lambda^\vee(\gamma)$ is locally exact and therefore closed.
Now let Q' be another polarization like Q. Then the previous construction can be performed for Q' as well and we denote by ω and ω' the corresponding Lagrangian structures. Morever, it follows that there is an automorphism $\psi : T^*B \to T^*B$ such that $\psi(\Lambda) = \Lambda' = \phi_\lambda^\vee \circ \phi_{Q'}(\mathsf{V}_{\mathbb{Z}})$. It induces a symplectomorphism $\psi : (T^*B/\Lambda, \hat{\eta}) \to (T^*B/\Lambda', \hat{\eta}')$. Since ω and ω' on $\mathcal{J}(\mathsf{V})$ are pull backs of $\hat{\eta}$ and $\hat{\eta}'$ respectively, it follows that ω and ω' are symplectomorphic to each other.
The last statement follows immediately, because if $\mathsf{V} \simeq \mathsf{V}'$ are isogenous, then V' admits a section $\lambda \in \Gamma(B, \mathsf{V}'_{\mathcal{O}})$ with the same properties as well. □

Definition 2.37. A section $\lambda \in \Gamma(B, \mathsf{V}_{\mathcal{O}})$, such that

$$TB \to \mathcal{F}^1\mathsf{V}, \quad X \mapsto \nabla_X^{GM} \lambda,$$

is an isomorphism as above, will be called an *abstract Seiberg-Witten differential.*

Remark 2.38. This definition is motivated by Seiberg-Witten differentials[13] of Hitchin systems, cf. Corollary 4.32. It seems likely that integrable systems with an abstract Seiberg-Witten differential are in fact exact. At least this is true for Hitchin and Calabi-Yau integrable systems, which admit (abstract) Seiberg-Witten differentials.

Proposition 2.36 shows that the existence of an abstract Seiberg-Witten differential is a strong restriction on a VHS of weight 1. To illustrate this from another viewpoint, we check directly that the cubic condition for the (local) period map $\mathcal{P}$ of $\mathcal{J}(\mathsf{V}) \to B$ is satisfied if V admits an abstract Seiberg-Witten differential. To this end, recall that we can write $d\mathcal{P}_b(v)(\alpha, \beta) = Q(\alpha, \nabla_v \beta)$ for $\alpha, \beta \in \mathcal{F}^1$. Since $Q(\mathcal{F}^1, \mathcal{F}^1) = 0$ and $\nabla Q = 0$, we conclude

$$Q(\alpha, \nabla_v \beta) = Q(\beta, \nabla_v \alpha). \tag{2.28}$$

[13]Seiberg-Witten differentials are often considered for *meromorphic* Hitchin systems, because these naturally occur in physics ([Don97]). In these cases they are meromorphic differentials. The Seiberg-Witten differentials, that we consider, are always holomorphic and are the analogues of the meromorphic ones for holomorphic Hitchin systems (cf. [HHP10]).

By the property that $X \mapsto \nabla_X \lambda$ is an isomorphism, this can be written as

$$d\mathcal{P}(X, Y, Z) = Q(\nabla_X \lambda, \nabla_Y \nabla_Z \lambda) = d\mathcal{P}(Z, Y, X)$$

with $\alpha = \nabla_X \lambda$, $\beta = \nabla_Z \lambda$, $v = Y$. For the last equality we have employed (2.28). The symmetry in Y and Z can be seen by using flatness of ∇ together with $Q(\mathcal{F}^1, \mathcal{F}^1) = 0$:

$$Q(\nabla_X \lambda, \nabla_Y \nabla_Z \lambda) - Q(\nabla_X \lambda, \nabla_Z \nabla_Y \lambda) = Q(\nabla_X \lambda, \nabla_{[Y,Z]} \lambda) = 0.$$

Hence the cubic condition is satisfied.

Chapter 3

Calabi-Yau integrable systems

We come to our first example of a polarized integrable system, the Calabi-Yau integrable system. It was first constructed by Donagi and Markman ([DM96a]) and can be associated to every complete family $\pi : \mathcal{X} \to B$ of compact Calabi-Yau threefolds (cCY3s). Even though cCY3s have been extensively studied, not much seems to be known about Calabi-Yau integrable systems. For example, their fibers are difficult to describe more explicitly. This is in contrast to Hitchin systems as we will see in the next chapter. At least we give here the construction of Calabi-Yau integrable systems mainly following [DM96a]. The only deviation is that we show the existence of a Lagrangian structure by using our approach from Section 2.2.4.

3.1 Intermediate Jacobians of compact Calabi-Yau threefolds

To each compact Kähler manifold one can associate two complex tori in a natural way, the Jacobian and the Albanese torus. These are special cases of Griffiths' intermediate Jacobians, which in some sense interpolate between these two tori.

Definition 3.1. Let X be a compact Kähler manifold of dimension n. For any $1 \leq p \leq n$ Griffiths' p-th intermediate Jacobian is defined as

$$\begin{aligned} J^p(X) :&= J^p H^{2p-1}(X,\mathbb{C}) = H^{2p-1}(X,\mathbb{C})/(H^{2p-1}(X,\mathbb{Z}) + F^p H^{2p-1}(X,\mathbb{C})) \\ &= \overline{F^p H^{2p-1}(X,\mathbb{C})}/H^{2p-1}(X,\mathbb{Z}). \end{aligned}$$

Remark 3.2. Apart from Griffiths' intermediate Jacobians there are also Weil's intermediate Jacobians. They have the advantage that they are even abelian varieties in contrast to Griffiths' intermediate Jacobians. However, their drawback is that they do not vary holomorphically in families which does hold for Griffiths' intermediate Jacobians.

Lemma 3.3. *Let X be a compact Kähler manifold of dimension n and $1 \leq p \leq n$. Then its intermediate Jacobian $J^p(X)$ is a complex torus of dimension $\frac{1}{2}h^{2p-1}(X,\mathbb{C})$. Moreover, there is a canonical isomorphism*

$$J^p(X) \cong F^p H^{2p-1}(X,\mathbb{C})^*/\lambda(H_{2p-1}(X,\mathbb{Z}))$$

where $\lambda : H_{2p-1}(X,\mathbb{Z}) \to F^p H^{2p-1}(X,\mathbb{C})^*$ is the natural map $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$ combined with Poincaré duality.

Proof. See [BL99]. □

Before we specialize to the case $\dim_{\mathbb{C}} X = 3$, we give two classical examples that we already mentioned.

Example 3.4. Let X be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$. We consider $J^p(X)$ in the two extreme cases.

$p = 1$: By definition and the Dolbeault isomorphism, we have

$$J^1(X) = H^1(X, \mathbb{C}) / \left(F^1 H^1 + H^1(X, \mathbb{Z}) \right) \cong H^1(X, \mathcal{O}_X) / H^1(X, \mathbb{Z}).$$

But this is precisely the Jacobian $\mathrm{Jac}(X)$ of X.

$p = n$: The Hodge $*$-operator yields $F^n = H^{n,n-1} \cong H^{1,0}$. Using the description of Lemma 3.3, it follows that

$$J^n(X) \cong (H^{1,0}(X))^* / H_1(X, \mathbb{Z}) = (H^0(X, \Omega_X^1))^* / H_1(X, \mathbb{Z}),$$

the Albanese torus $Alb(X)$ of X.

Before studying intermediate Jacobians of threefolds in more detail, we consider the family case. Let $\pi \colon \mathcal{X} \to B$ be a family of compact Kähler manifolds of dimension n and $(R^{2p-1}\pi_*\mathbb{Z}, \mathcal{F}^\bullet)$ its VHS of weight $2p-1$ for $1 \le p \le n$. Since $\mathcal{F}^p$ is a holomorphic subbundle of $\mathcal{H}^{2p-1}(\mathcal{X}/B)$ it follows that (see A.1 for the notation)

$$\mathcal{J}^p(\mathcal{X}/B) = \mathcal{H}^{2p-1}(\mathcal{X}/B) / \left(\mathcal{F}^p \mathcal{H}^{2p-1} + R^{2p-1}\pi_*\mathbb{Z} \right) \to B$$

is a holomorphic family of complex tori over B. We will see (at least in the three-dimensional case) that it is a polarized family of complex tori if the X_b's are even projective.

Polarizations We now turn to polarizations on intermediate Jacobians. Everything what follows also holds in more generality (see [BL99]). But since we are mainly interested in the case of compact (Calabi-Yau) threefolds, we restrict to the case $\dim_{\mathbb{C}} X = 3$. In that case the discussion becomes more transparent without losing contact to the general idea. Note that a compact Kähler threefold X has three intermediate Jacobians, $J^1(X) = \mathrm{Jac}(X)$, $J^2(X)$ and $J^3(X) = Alb(X)$. We will call $J^2(X)$ *the* intermediate Jacobian of X.

Lemma 3.5. *Let X be a* projective *threefold and $J := J^2(X)$ its intermediate Jacobian. Then there is a natural isomorphism*

$$H^{1,0}(J) = F^2 H^3(X, \mathbb{C}), \quad H^1(J, \mathbb{C}) = H^3(X, \mathbb{C}).$$

Besides J is a non-degenerate complex torus of index $h^{0,1} + h^{0,3}$ in a natural way.

Proof. Consider $J^2(X) = F^2 H^3(X, \mathbb{C}) / \lambda(H_3(X, \mathbb{Z}))$, where $\lambda : H_3(X, \mathbb{Z}) \to F^2 H^3(X, \mathbb{C})^*$ is the natural map. By identifying the holomorphic 1-forms on J with the dual of the tangent space $T_0 J$ (i.e. the Lie algebra), we immediately obtain $H^{1,0}(J) = F^2 H^3(X, \mathbb{C})$ and therefore

$$H^1(J, \mathbb{C}) = F^2 H^3(X, \mathbb{C}) \oplus \overline{F^2 H^3(X, \mathbb{C})}.$$

Since the second claim works more generally, we first consider the general case and then specialize. So assume for the moment that $\dim_{\mathbb{C}} X = n$ is arbitrary, $1 \le p \le n$ and $k = 2p - 1$. On $T_0 J^{2p-1}(X) \cong \overline{F^p H^{2p-1}(X, \mathbb{C})}$ we can define the bilinear form

$$E(\alpha, \beta) = \epsilon(k) \int_X \alpha \wedge \beta \wedge \omega^{n-2k},$$

where $\epsilon(k) = \frac{1}{2}k(k-1)$, as well as the sesquilinear form[1]

$$H(\alpha, \beta) = -iE(\alpha, \overline{\beta}).$$

Comparing with the natural polarization Q on $H^k(X, \mathbb{C})$ ([PS08]), we see that

$$E = (-1)^r Q_{|L^r H^k \times L^r H^k}$$

where L is the Lefschetz operator. By the Lefschetz decomposition, it suffices to compute the indices of H on the subspaces

$$L^r H_{prim}^{p-a-1-r,p+a-r}, \quad 0 \leq a \leq p-1.$$

Recall that the Weil operator C acts on this space via multiplication by $i^{-(2a+1)} = (-1)^{a+1}i$. For a nonzero element $\alpha \in L^r H_{prim}^{p-a-1-r,p+a-r}$ we can now compute

$$\begin{aligned} H(\alpha, \alpha) &= -iE(\alpha, \overline{\alpha}) \\ &= -i(-1)^r Q(\alpha, \overline{\alpha}) \\ &= (-1)^{r+1}(-1)^{a+1} Q(C\alpha, \overline{\alpha}) \\ &= (-1)^{r+a} Q(C\alpha, \overline{\alpha}) = \begin{cases} < 0 & r+a \equiv 1 \bmod 2 \\ > 0 & r+a \equiv 0 \bmod 2. \end{cases} \end{aligned} \tag{3.1}$$

Let us now specialize to our case in question, i.e. $n = 3$ and $p = 2$ and

$$\overline{F^2 H^3(X, \mathbb{C})} = L^1 H_{prim}^{0,1} \oplus H_{prim}^{1,2} \oplus H_{prim}^{0,3}. \tag{3.2}$$

We therefore have the cases

$$\begin{aligned} L^1 H_{prim}^{0,1} &: r = 1, a = 0 \Rightarrow H < 0, \\ H_{prim}^{1,2} &: r = 0, a = 0 \Rightarrow H > 0, \\ H_{prim}^{0,3} &: r = 0, a = 1 \Rightarrow H < 0. \end{aligned}$$

It follows that H has index $h^{0,1} + h^{0,3}$.
The condition that X is projective ensures that H is integer-valued on the lattice $H^3(X, \mathbb{Z})$. Hence (J^2, H) is a non-degenerate complex torus of index $h^{0,1} + h^{0,3}$. □

Remark 3.6. Observe that a priori $H_1(J^2, \mathbb{Z}) = H^3(X, \mathbb{Z})$. But since the Poincaré pairing is unimodular in this case (X is compact), we obtain an isomorphism $H^3(X, \mathbb{Z}) \cong H_3(X, \mathbb{Z})$ so that $H^1(J^2, \mathbb{Z}) \cong H^3(X, \mathbb{Z})$. Put differently, the intermediate Jacobian $J^2(X)$ of a threefold satisfies

$$J^2(X) = \widehat{J^2(X)} = J_2(X).$$

Here $J_2(X)$ stands for the homology intermediate Jacobian, which is analogously defined as $J^2(X)$ via homology. In the non-compact case, one has to distinguish between the two in general, see Chapter 5.

[1] One word on the chosen sign: It guarantees that we have index $h^{0,1} + h^{0,3}$ and not the 'dual' index $h^{1,2}$. The former can be easily computed for compact Calabi-Yau threefolds, cf. below.

Corollary 3.7. *Let $\pi : \mathcal{X} \to B$ be a family of projective threefolds over a connected base B. Then the intermediate Jacobian fibration $\mathcal{J}^2(\mathcal{X}/B) \to B$ is a family of non-degenerate self-dual complex tori of index $h^{0,1} + h^{0,3}$ in a natural way. Its polarizable $\mathbb{Z}$-VHS $\mathsf{V}(\mathcal{J}^2) = (\mathsf{V}_{\mathbb{Z}}(\mathcal{J}^2), \mathcal{F}^\bullet\mathsf{V}(\mathcal{J}^2))$ of weight 1 is given by*

$$\mathsf{V}_{\mathbb{Z}}(\mathcal{J}^2) = R^3\pi_*\mathbb{Z}, \quad \mathcal{F}^1\mathsf{V}_{\mathcal{O}}(\mathcal{J}^2) = \mathcal{F}^2\mathcal{H}^3(\mathcal{X}/B).$$

Proof. By assumption, there exists a relative polarization for π. Hence everything above in the previous proof readily generalizes to the relative situation. □

We restrict further to the case where X is a compact Calabi-Yau threefold. Since there does not seem to be a uniform definition of compact Calabi-Yaus, let us fix the definition that we will use.

Definition 3.8. A compact Calabi-Yau threefold (cCY3) is a compact Kähler manifold X such that

$$K_X \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0.$$

Remark 3.9. This definition excludes complex tori of dimension 3. Also note that with this definition, all cCY3s are even *projective.* As we will see, this ensures that the intermediate Jacobian of a cCY3 is a non-degenerate complex torus in a natural way.

Corollary 3.10. *Let X be a compact CY3. Then its intermediate Jacobian $J^2(X)$ is a non-degenerate complex torus of index 1 and dimension $h^{1,2} + 1$.*

Proof. This is immediate: $h^{1,0} = 0$ by definition and $h^{3,0} = 1$ by definition and compactness. □

Hence the intermediate Jacobian $J^2(X)$ of a cCY3 X cannot be an abelian variety, even though X is projective itself.

3.2 The Calabi-Yau integrable system

Each family $\pi : \mathcal{X} \to B$ of compact Calabi-Yau threefolds gives rise to a family $\mathcal{J}^2(\mathcal{X}/B) := \mathcal{J} \to B$ of non-degenerate complex tori of index 1. Hence we are in the situation of Section 2.2. The fibers of $\mathcal{J}^2$ have complex dimension $h^{2,1} + h^{3,0}$, so that we have the constraint

$$\dim_{\mathbb{C}} B = h^{2,1} + h^{3,0}.$$

Since a *complete* family $\pi : \mathcal{X} \to B$ has $\dim_{\mathbb{C}} B = h^{2,1}$, Donagi and Markman ([DM96a]) considered its associated gauged family defined by the fiber product

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xrightarrow{\sigma} & \mathcal{X} \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{B} & \xrightarrow{\rho} & B, \end{array} \tag{3.3}$$

where $\tilde{B} \to B$ is the $\mathbb{C}^*$-bundle underlying the pushforward $\pi_*\omega_{\mathcal{X}/B}$. Each fiber $\tilde{B}_b$, $b \in B$, is given by $H^0(X_b, K_{X_b})\backslash\{0\}$, i.e. all possible trivializations of the canonical bundle K_X of $X = X_b$ so that $\dim_{\mathbb{C}} \tilde{B} = h^{2,1} + h^{3,0}$. Hence the associated family $\tilde{\mathcal{J}}^2 \to \tilde{B}$ of intermediate

Jacobians satisfies at least the dimension constraint. The base change diagram 3.3 yields natural isomorphisms

$$F^p\mathcal{H}^3(\tilde{\mathcal{X}},\mathbb{C}) \cong \rho^* F^p\mathcal{H}^3(\mathcal{X},\mathbb{C}), \quad p = 0,\dots,3, \tag{3.4}$$

$$\tilde{\mathcal{J}}^2 = \mathcal{J}^2(\tilde{\mathcal{X}}/\tilde{B}) \cong \rho^*\mathcal{J}^2(\mathcal{X}/B). \tag{3.5}$$

In particular, the vertical bundle $\tilde{\mathcal{V}} \to \tilde{B}$ of $\tilde{\mathcal{J}}^2 \to \tilde{B}$ is naturally isomorphic to $\rho^* F^2\mathcal{H}^3(\mathcal{X},\mathbb{C})^*$.

Lemma 3.11. *The Gauß-Manin connection ∇^{GM} on $F^2\mathcal{H}^3(\mathcal{X},\mathbb{C})$ induces an isomorphism of bundles*

$$T\tilde{B} \xrightarrow{\cong} \rho^* F^2\mathcal{H}^3(\mathcal{X},\mathbb{C}), \quad v \mapsto (\rho^*\nabla^{GM})_v \underline{s},$$

where $\underline{s} : \tilde{B} \to \rho^* F^2\mathcal{H}^3$ is the tautological section.

Proof. From now on, we denote $F^2\mathcal{H}^3 = F^2\mathcal{H}^3(\mathcal{X},\mathbb{C})$ and so on. We prove the claim by establishing a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_{\tilde{B}/B} \arrow[r] \arrow[d, "\cong"', "\alpha"] & T\tilde{B} \arrow[r, "d\rho"] \arrow[d, "\beta"] & \rho^* TB \arrow[r] \arrow[d, "\cong"', "\gamma"] & 0 \\
0 \arrow[r] & \rho^* F^3\mathcal{H}^3 \arrow[r] & \rho^* F^2\mathcal{H}^3 \arrow[r, "q"] & \rho^*(F^2\mathcal{H}^3/F^3\mathcal{H}^3) \arrow[r] & 0.
\end{tikzcd}$$

The upper row is the exact sequence of the relative tangent bundle and the lower row is the usual exact sequence of a quotient (pulled back via ρ which is exact). We construct the isomorphisms α and γ first and then β. By the commutativity of the diagram, the latter is then automatically an isomorphism.
To construct α, we observe that $\tilde{B}$ is canonically isomorphic to $F^3\mathcal{H}^3$ (minus the zero section) simply by sending an element $s \in \tilde{B}_b = H^0(X_b, \omega_{X_b})$ to its cohomology class in $H^{3,0}(X_b) = F^3\mathcal{H}^3_b$. Since $T_{E/M} \cong p^*E$ canonically for any vector bundle $p : E \to M$ over a (complex) manifold M, we obtain an isomorphism $\alpha : T_{\tilde{B}/B} \to \rho^* F^3\mathcal{H}^3$.
Recall that the family $\mathcal{X} \to B$ is complete, which in particular means that we have the following isomorphism

$$T_b B \xrightarrow{\kappa_b} H^1(X_b, T_{X_b}) \xrightarrow{\cong} H^1(X_b, \Omega^2_{X_b}).$$

The latter isomorphism is induced by the non-canonical isomorphism $T_{X_b} \to \Omega^2_X$ (using the Calabi-Yau condition). All this can be extended globally over $\tilde{B}$, where 'gauge fixing is already built in' by construction. Indeed, the fiberwise map

$$\lrcorner : \rho^* R^1\pi_* T_{\mathcal{X}/B,s} \to \rho^*(F^2\mathcal{H}^3/F^3\mathcal{H}^3)_s, \quad v \mapsto s \lrcorner v,$$

yields an isomorphism $\lrcorner$ on bundle level. Since the (global) Kodaira-Spencer map κ also induces an isomorphism

$$\rho^*\kappa : \rho^* TB \xrightarrow{\cong} \rho^* R\pi^1_* T_{\mathcal{X}/B},$$

the composition $\gamma := \lrcorner \circ \rho^*\kappa$ is an isomorphism.
It remains to construct the map $\beta : T\tilde{B} \to \rho^* F^2\mathcal{H}^3$ in a compatible fashion. Let $\underline{s} : \tilde{B} \to \rho^* F^3\mathcal{H}^3$ be the tautological section, $\underline{s}(s) = s$ We claim that

$$\beta(v) = (\rho^*\nabla^{GM})_v \underline{s} \in \rho^* F^2 H^3_s, \quad v \in T_s\tilde{B}$$

makes the diagram commutative. Note that β does map to $\rho^* F^2\mathcal{H}^3$ by Griffiths transversality. First, we prove commutativity of the left square: Let $\nabla^{GM} = d + \omega$ in a local trivialization in which $\psi : U \to F^3\mathcal{H}^3$ trivializes $F^3\mathcal{H}^3$. Then $\underline{s}(\lambda\psi) = \lambda\psi$, $\lambda \in \mathbb{C}^*$ on $\rho^{-1}(U)$. If $v \in T_{\tilde{B}/B,s}$, $s = \lambda\psi$, is represented by $\lambda(s + tv) = s + tv \in \rho^* F^3\mathcal{H}^3_s$, we compute

$$\rho^*\nabla^{GM}_v \underline{s}(s) = (d + \rho^*\omega)_v\lambda = \frac{d}{dt}_{|t=0}\lambda(t) = v.$$

This directly shows the commutativity of the left square, when we identify $T_{\tilde{B}/B} \cong \rho^* F^3\mathcal{H}^3$ in the canonical way.
There are two different descriptions of the derivative of period maps, which we briefly recall below (also see [CMSP03]). One is via the Gauß-Manin connection and the other by the Kodaira-Spencer map. They imply that

$$q \circ \beta = d\tilde{\mathcal{P}}^3(.)(\underline{s}) = \gamma \circ d\rho,$$

which is precisely the commutativity of the right square. Note here that $\tilde{\kappa} = \rho^*\kappa \circ d\rho$ up to a canonical isomorphism, where $\tilde{\kappa}$ is the Kodaira-Spencer map of the gauged family. □

Corollary 3.12. *Let $\pi : \mathcal{X} \to B$ be a complete family of compact Calabi-Yau threefolds and $\tilde{B} \to B$ be the $\mathbb{C}^*$-bundle corresponding to $\pi_*\omega_{\mathcal{X}/B}$. Then the intermediate Jacobian fibration $\tilde{\mathcal{J}}^2 \to \tilde{B}$ of the pullback family $\tilde{\mathcal{X}} = \mathcal{X} \times_B \tilde{B}$ carries the structure of a polarized integrable system of index* 1*, called* Calabi-Yau integrable system*.*

In the introduction we denoted such an integrable system by $\mathbf{M}_{CY} \to \tilde{B}$ (without the base change $\tilde{B} \to B$ though). Note that it is a Lagrangian fibration (with section) over *all* of $\tilde{B}$. This is not true in the non-compact examples in Chapter 5.

Proof. The previous Lemma 3.11 shows that the tautological section $\underline{s}$ is an abstract Seiberg-Witten differential for the VHS $\mathsf{V}(\tilde{\mathcal{J}}^2) \cong \rho^*\mathsf{V}(\mathcal{J}^2)$ of weight 1. Now the claim follows from Corollary 2.36. □

In the following we want to examine the corresponding cubic in more detail (cf. [DM96a]). To this end we have to study the (derivatives of the) local period maps $\mathcal{Q} : (B, 0) \to \mathcal{E}$ and $\mathcal{P} : (B, 0) \to \mathcal{D}$ of $\mathcal{X}$ and $\mathcal{J}^2$ respectively, as well as their pullbacks. The respective period domains are

$$\begin{aligned} \mathcal{D} &= \mathrm{Gr}(h^{1,0}(J_0), H^1(J_0, \mathbb{C})), \\ \mathcal{E} &= \mathrm{Fl}(f^\bullet(X_0), H^3(X_0, \mathbb{C})), \end{aligned}$$

where $f^\bullet(X_0) = (f^0(X_0), \ldots, f^3(X_0))$ for $f^p(X_0) = \dim F^p H^3(X_0, \mathbb{C})$. The derivative of $\mathcal{Q}$ is a map (see [CMSP03])

$$d\mathcal{Q}_b : T_bB \to \bigoplus_{p=1}^{3} \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p) = \bigoplus_{p+q=3} \mathrm{Hom}(H^{p,q}(X_b), H^{p-1,q+1}(X_b)),$$

where $F^p := F^p H^3(X_b, \mathbb{C})$, $p + q = 3$. The single components

$$T_bB \to \mathrm{Hom}(H^{p,q}(X_b,), H^{p-1,q+1}(X_b))$$

have two well-known descriptions. The first is given by cup-product and contraction with the Kodaira-Spencer class (Lemma 2.20)

$$v \mapsto (\eta \mapsto \kappa_b(v) \lrcorner \eta)\,.$$

The second is obtained via the Gauß-Manin connection $\nabla = \nabla^{GM}$ as $v \mapsto \overline{\nabla}^p_v$, where

$$\overline{\nabla}^p_v(\eta) = (\nabla_v \hat{\eta})(b) \bmod F^p H^3(X_b, \mathbb{C}).$$

Here $\hat{\eta}$ is a section such that $\hat{\eta}(b) = \eta$ and one can check that this is well-defined ([CMSP03]). Since we consider complete families, the components can also be seen as maps

$$H^1(T_{X_b}) \to \mathrm{Hom}(H^{p,q}(X_b,), H^{p-1,q+1}(X_b)).$$

The maps $\overline{\nabla}^p$ can be iterated to yield a map

$$c_b := \overline{\nabla}^1 \overline{\nabla}^2 \overline{\nabla}^3 : \otimes^3 T_b B \to \mathrm{Hom}(H^{3,0}(X_b), H^{0,3}(X_b))$$

Fixing a non-zero element $s \in H^{3,0}(X_b)$, which is unique up to a scalar for a compact Calabi-Yau threefold[2], c_b maps to $\mathbb{C}$,

$$(u \otimes v \otimes w) \mapsto Q(s, \overline{\nabla}^1_u \overline{\nabla}^2_v \overline{\nabla}^3_w s).$$

Seen this way, $c_b : \otimes^3 T_b B \to \mathbb{C}$ is even a cubic, i.e. $c_b \in \mathrm{Sym}^3(T^*_b B)$. This can be seen as in Section 2.2.4. It is called the *Yukawa* or *Bryant-Griffiths cubic* ([BG83]).

Remark 3.13. The subbundle $F^2\mathcal{H}^3 \subset \mathcal{H}^3$ is totally isotropic with respect to the polarization Q ([CMSP03]), i.e. $Q(\mathcal{F}^2\mathcal{H}^3, \mathcal{F}^2\mathcal{H}^3) = 0$. Furthermore the polarization is flat with respect to ∇ so that

$$(Q\nabla\eta, \xi) + Q(\eta, \nabla\xi) = 0.$$

This yields the following equality for the Yukawa cubic:

$$c^s_b(u, v, w) = Q(s, \overline{\nabla}^1_u \overline{\nabla}^2_v \overline{\nabla}^3_w s) = -Q(\overline{\nabla}^3_u s, \overline{\nabla}^2_v \overline{\nabla}^3_w s). \tag{3.6}$$

By Corollary 3.7, $\mathcal{P} = \mathcal{Q}^2$ for the 'F^2H^3-component' $\mathcal{Q}^2$ of $\mathcal{Q}$, in particular $d\mathcal{P} = d\mathcal{Q}^2$. This has restrictions on the values of $d\mathcal{P}$, as depicted in the next diagram:

$$\begin{tikzcd}
\overline{\nabla}^2 = d\mathcal{Q}^2_b : T_b B \arrow[r] & \mathrm{Hom}(H^{2,1}(X_b), H^{1,2}(X_b)) \arrow[d, hook] \\
d\mathcal{P}_b : T_b B \arrow[r] & \mathrm{Hom}(H^{1,0}(J(X_b), \mathbb{C}), H^{0,1}(J(X_b), \mathbb{C})).
\end{tikzcd}$$

The above space is not visible if we only use the Hodge filtration on $H^3(X_b, \mathbb{C}) = H^1(J(X_b), \mathbb{C})$ coming from the Jacobian $J(X_b)$, cf. Lemma 3.5. Also recall that $d\mathcal{Q}^2$ a priori maps to the lower space which can be reduced to the above space by using Griffiths transversality.
As the family $\mathcal{X} \to B$ is complete, the local Kodaira-Spencer map $\kappa_b : T_b B \to H^1(T_{X_b})$ is an isomorphism. Since $T_{X_b} \cong \Omega^2_{X_b}$ after the choice of a non-zero section $s \in H^{3,0}(X_b)$[3], it follows that $T_b B \cong H^1(\Omega^2_{X_b}) = H^{2,1}(X_b)$, so that $d\mathcal{P}_b$ can be considered as a map

$$d\mathcal{P}_b : H^{2,1} \to \mathrm{Hom}(H^{2,1}, H^{1,2}).$$

But we also know that it takes values in the symmetric homomorphisms so that $d\mathcal{P}_b \in H^{1,2} \otimes \mathrm{Sym}^2(H^{1,2})$ under the isomorphism $H^{2,1} = (H^{1,2})^*$. But even more is true.

[2] We often fix such a choice for convenience and sometimes write c^s_b. But both maps are often denoted by c_b.
[3] Here we make the same choice as for the Yukawa cubic c^s_b.

Lemma 3.14. *Under the isomorphisms $T_bB \cong H^{1,2}(X_b)$ (induced by the Kodaira-Spencer map and a non-zero section $s \in H^{3,0}$) and $H^{1,2}(X_b)^* \cong H^{2,1}(X_b)$ (via the polarization), the derivative $d\mathcal{P}_b = d\mathcal{Q}_b^2$ satisfies*

$$d\mathcal{P}_b = -c_b^s \in \mathrm{Sym}^3(H^{1,2}(X_b))$$

for the Yukawa cubic c_b^s.

Proof. First of all, we observe that for a fixed $s \in H^{3,0}$, the map

$$T_bB \to H^{2,1}, \quad w \mapsto \overline{\nabla}^3_w s$$

is an isomorphism. This follows because this map is the isomorphism γ of Lemma 3.11 restricted to $\rho^*TB_s = T_bB$ (here we use again the different characterizations of the derivatives of the period map). Since $d\mathcal{P}_b = \overline{\nabla}^2$, we see that

$$d\mathcal{P}_b(v)(\overline{\nabla}^3_w) = \overline{\nabla}^2_v\overline{\nabla}^3_w s \in H^{1,2}.$$

Applying the duality between $H^{1,2}$ and $H^{2,1}$, the derivative $d\mathcal{P}_b$ yields a map

$$u \otimes v \otimes w \mapsto Q(\overline{\nabla}^3_u s, \overline{\nabla}^2_v\overline{\nabla}^3_u s).$$

By equality (3.6) this is precisely the cubic $-c_b^s$ (up to a sign). □

The cubic c_b^s can in general cannot be extended on all of the vertical bundle $\mathcal{V}(\mathcal{J})$ of the original intermediate Jacobian fibration $\mathcal{J} \to B$ because it depends on the choice of s. On $\tilde{\mathcal{V}}(\tilde{\mathcal{J}})$ this is possible by using the tautological section $\underline{s}$ on $\tilde{B}$. Hence one obtains a cubic $c_{\underline{s}} \in \Gamma(\tilde{B}, \mathrm{Sym}^3\tilde{\mathcal{V}})$, such that

$$d\tilde{\mathcal{P}} = -c_{\underline{s}}$$

for the period map $\tilde{\mathcal{P}}$ of the gauged family. In other words, the cubic associated with the Calabi-Yau integrable system coincides with the (gauged) Yukawa or Bryant-Griffiths cubic.

Remark 3.15. The base $\tilde{B}$ of a Calabi-Yau integrable system is not only special Kähler but carries the richer structure of a *projective special Kähler manifold* ([Fre99]).

Chapter 4

Hitchin system

Hitchin systems comprise the other important class of integrable systems that are relevant for us. They were first discovered by Hitchin ([Hit87a], [Hit87b]) and have been extensively studied since then (e.g. [Fal93], [Don95], [DG02], [DP12][1]). Its construction is somewhat more involved than the construction of Calabi-Yau integrable systems from the previous chapter. However, they have in common that their construction is deformation-theoretic in nature: Calabi-Yau integrable systems are constructed via *complete* families of compact Calabi-Yau threefolds, e.g. the Kuranishi family of a compact CY3. Hitchin systems are in turn constructed via the moduli space of so-called *G-Higgs bundles* over a fixed compact Riemann surface. Let us briefly recall:

Definition 4.1. Let Σ be a compact Riemann surface and G a complex Lie group. A *G-Higgs bundle* is a pair (E, θ) consisting of a holomorphic G-bundle E over Σ and a section

$$\theta \in H^0(\Sigma, \mathrm{ad}(E) \otimes K_\Sigma) \subset A^{1,0}(\Sigma, \mathrm{ad}(E)),$$

a *Higgs field*, where $\mathrm{ad}(E)$ denotes the adjoint bundle associated with E.
A morphism $f : (E, \theta) \to (F, \psi)$ of G-Higgs bundles over Σ is a bundle morphisms $f : P \to Q$ such that $\mathrm{Ad}(f)(\theta) = \psi$.

In contrast to Calabi-Yau integrable systems, we will see that much more is known about Hitchin systems. For example, the generic fibers are by now well-studied ([DG02], [DP12]).

Since it is so important for the construction of Hitchin systems, we begin this chapter by briefly collecting some facts about the moduli of G(-Higgs)-bundles over a fixed compact connected Riemann surface Σ. It is convenient to first start with the case of G-bundles, even though it has a lot of similarities with the case of G-Higgs bundles (see Remark 4.10 though). As always in deformation theory it is useful to go in two steps:
First, we begin with the infinitesimal or formal deformations of G(-Higgs)-bundles. Since the deformations functors

$$\mathrm{Def}_G : \mathbf{Art}_\mathbb{C} \to \mathbf{Sets}, \quad \mathrm{Def}_{G-\mathsf{Higgs}} : \mathbf{Art}_\mathbb{C} \to \mathbf{Sets}$$

are governed by certain *differential graded Lie-algebras* (DGLAs), we confine ourselves to describe these only. Our presentation is very brief because these aspects are not crucial for further developments. However, we want to showcase here, how this approach makes the relationship

[1] For developments in other directions (especially *meromorphic* Hitchin systems), see the excellent survey [Dal16], which we found useful as well.

between infinitesimal deformations of a G-Higgs bundle (E, θ) and its underlying G-bundle E very transparent.
Second, we treat, even more briefly, the moduli spaces of G- and G-Higgs bundles over the same fixed curve Σ. We do not go into the details of their construction. At least it is good to know different methods how they can be constructed:

a) *Kuranishi theory*: For G-bundles this is due to Ramanathan ([Ram75]) and for G-Higgs bundles to Fujiki ([Fuj91]).

b) *Gauge-theoretic*: The gauge-theoretic construction goes back to Atiyah-Bott ([AB83]) for (vector) bundles and to Hitchin's original paper ([Hit87a]). These approaches use (hyper)kähler reduction starting from an infinite-dimensional setting but which yield the finite-dimensional moduli spaces (also cf. [Fuj91]) endowing them with a hyperkähler structure.

c) *GIT*: In the case of G-bundles, Ramanathan gave a detailed GIT-construction of the moduli space ([Ram96b]). For G-Higgs bundles it goes back to Nitsure (for $G = GL(n)$, [Nit91]) and Simpson ([Sim94a]).

d) *Faltings*: Faltings in [Fal93] gave another algebraic construction of the moduli space of G(-Higgs) bundles without using GIT-methods.

Each method has its advantages and gives new insights into the moduli spaces (see Remark 4.13). The first two give moduli spaces in the complex-analytic category, whereas the last two are in the algebraic category. In fact, it can be shown that these moduli spaces behave well under analytification (see especially Proposition 5.5 in [Sim94a] and [Ram96b], [Ram96a]), i.e. the analytification of the algebraic moduli spaces give the analytic moduli spaces. Since we only want to provide an overview, we are therefore not very rigorous in distinguishing between the complex-analytic and algebraic category.
However, we have to point out one subtlety: G-bundles in the algebraic category are assumed to be isotrivial, i.e. locally trivial in the étale topology ([Ram96b]). There are two special cases though. Firstly, if the base is a curve Σ (over $\mathbb{C}$) and G is a connected reductive group (so our case), then G-bundles are even Zariski-locally trivial ([Ste65]). Even though we deal with this situation, one still has to keep in mind that higher dimensional bases occur when one works with families of G-bundles over Σ. Secondly, $GL(n, \mathbb{C})$-bundles are Zariski-locally trivial for any base because this essentially reduces to the vector bundle case.

After that, we turn to Hitchin systems and the study of generic fibers. For the latter, we give a detailed account of the case where G is a (semi)simple complex Lie group that is either of adjoint type or simply connected. These are the most important cases that we need in the next chapter. Some of these results are contained in [DP12] and we supplement the discussion with extended proofs and a few simple examples. Moreover, we put a much larger emphasis on the corresponding variations of Hodge structures than is usually done in the literature. This point of view is crucial for our constructions in the next chapter.

Throughout this chapter, we fix a *connected* reductive complex Lie group G (alternatively, an affine reductive algebraic group G over $\mathbb{C}$) and a compact connected Riemann surface Σ of genus $g \geq 2$ (alternatively, an irreducible projective curve). Even though we only need the semisimple case later on, it is useful to have the example $G = GL(n, \mathbb{C})$ at hand in the beginning. It provides a bridge between the conceptually simpler case of (Higgs) vector bundles and general G(-Higgs)-bundles.

4.1 Infinitesimal deformations

Let E be a holomorphic G-bundle over Σ and $\mathrm{ad}(E)$ its adjoint bundle. We denote by

$$\mathrm{Def}_E : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets},$$
$$\mathrm{Def}_E(A) = \{\mathcal{E} \to \Sigma \times \mathrm{Spec}(A) \text{ deformation of } E\} \,/\, \sim$$

the deformation functor associated with E. Observe that the Dolbeault resolution of $\mathrm{ad}(E)$ defines a DGLA L_E. This means that we have a triple $(L, [.,.], d)$ consisting of

i) a $\mathbb{Z}$-graded $\mathbb{C}$-vector space $L_E = \bigoplus_{i\in\mathbb{Z}} L^i$,

ii) a bracket $[.,.] : L^i \otimes_{\mathbb{C}} L^j \to L^{i+j}$ which is graded-anticommutative, $[x, y] = (-1)^{ij+1}[y, x]$ for $x \in L^i$, $y \in L^j$, and satisfies a graded Jacobi identity,

iii) a differential $d : L \to L$ such that $d[x, y] = [dx, y] + (-1)^i[x, dy]$ for $x \in L^i$.

Indeed, define

$$L_E := \bigoplus_{i\in\mathbb{Z}} L^i_E := \bigoplus_{i\in\mathbb{Z}} A^{0,i}(\Sigma, \mathrm{ad}(E)) \tag{4.1}$$

where $A^{0,i} = 0$ if $i < 0$ by definition. Then $(L_E, [.,.], \bar\partial^E)$ is a DGLA, where $[.,.]$ is the natural bracket and $\bar\partial^E$ the Dolbeault differential. Observe that there are only two non-zero summands in (4.1) for dimension reasons.

From general principles, one knows that L_E gives rise to a deformation functor $\mathrm{Def}_{L_E} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Sets}$. This is done by associating the corresponding solutions of the *Maurer-Cartan equation* modulo gauge equivalence (cf. the overview [Man09][2]). It is known (e.g. [Sim97]) that the DGLA L_E governs infinitesimal deformations of the G-bundle E, i.e.

$$\mathrm{Def}_E \simeq \mathrm{Def}_{L_E}$$

as deformation functors. Again from the general theory, it follows that the infinitesimal automorphisms of E are isomorphic to $H^0(L_E) = H^0(\Sigma, \mathrm{ad}(E))$ and the infinitesimal deformations to $H^1(L_E) = H^1(\Sigma, \mathrm{ad}(E))$. Moreover, the obstructions are contained in $H^2(L_E) = H^2(\Sigma, \mathrm{ad}(E))$ which trivially vanish here.

A similar discussion applies to infinitesimal deformations of a given G-Higgs bundle (E, θ). To this end, define the complex

$$\mathcal{K}^\bullet = \mathcal{K}^\bullet(E, \theta) : 0 \longrightarrow \mathrm{ad}(E) \xrightarrow{\mathrm{ad}\theta} \mathrm{ad}(E) \otimes K_\Sigma \longrightarrow 0 \tag{4.2}$$

where we put $\mathrm{ad}(E)$ in degree 0. As for a G-bundle its Dolbeault resolution,[3] defines a DGLA $L_{(E,\theta)}$ with graded pieces

$$L^i_{(E,\theta)} := \bigoplus_{p+q=i} A^{0,p}(\Sigma, \mathcal{K}^q), \quad k \in \mathbb{Z}. \tag{4.3}$$

Of course, these pieces are zero for $i \neq 0, 1$. It is known ([Sim94a], [Mar12]) that

$$\mathrm{Def}_{(E,\theta)} \simeq \mathrm{Def}_{L_{(E,\theta)}}.$$

[2] A historical remark is here in order: The philosophy that DGLAs control deformations goes back to Quillen, Deligne, Drinfeld and Kontsevich. One of the first written accounts can be found in [GM88] and since then the theory has been developed by many mathematicians. See [Man09] for more references

[3] More precisely, the natural Cartan-Eilenberg resolution.

One of the features of the approach to deformations theory via DGLAs is, that morphisms of DGLAs give morphisms of the corresponding deformation functors. It can be used in our case to conveniently describe the relation between infinitesimal deformations of a G-Higgs bundle (E,θ) and deformations of its underlying G-bundle. Consider the exact sequence (cf. [BR94]) of complexes

$$0 \longrightarrow \mathrm{ad}(E)\otimes K_\Sigma[-1] \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathrm{ad}(E)[0] \longrightarrow 0. \tag{4.4}$$

It gives a long exact sequence in (hyper)cohomology (where we drop Σ from the notation)

$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{K}^\bullet) \xrightarrow{\gamma^0} H^0(\mathrm{ad}(E)) \xrightarrow{\delta^0} H^0(\mathrm{ad}(E)\otimes K_\Sigma) \\ \quad \longrightarrow H^1(\mathcal{K}^\bullet) \xrightarrow{\gamma^1} H^1(\mathrm{ad}(E)) \xrightarrow{\delta^1} H^1(\mathrm{ad}(E)\otimes K_\Sigma) \longrightarrow H^2(\mathcal{K}^\bullet) \longrightarrow 0. \end{array} \tag{4.5}$$

The maps γ^0 and γ^1 can be identified with the maps that are induced from the forgetful map of functors $\mathrm{Def}_{(E,\theta)} \to \mathrm{Def}_E$. Further, using the Dolbeault resolution from above it is not hard to see that the connecting homomorphisms δ^i are given by $h^i(\mathrm{ad}(\theta))$, i.e. the homomorphism on cohomology induced by $\mathrm{ad}(\theta)$. In particular, if $\theta = 0$ then the infinitesimal deformations of the G-Higgs bundle $(E,\theta) = (E,0)$ are determined by the exact sequence

$$0 \longrightarrow H^0(\mathrm{ad}(E)\otimes K_\Sigma) \longrightarrow H^1(\mathcal{K}^\bullet) \xrightarrow{\gamma^1} H^1(\mathrm{ad}(E)) \longrightarrow 0. \tag{4.6}$$

In fact, this naturally splits because every infinitesimal deformation of E gives an infinitesimal deformation of $(E,0)$. Hence infinitesimal deformations of $(E,0)$ are, as expected, just infinitesimal deformations of the underlying bundle E and deforming the zero Higgs field. Of course, this is false in general but see (4.8) for the semisimple case.

4.2 Moduli spaces

For both G- and G-Higgs bundles, it is necessary to impose *stability conditions* in order to obtain well-behaved moduli spaces. The aim of this section is to give the definitions of the corresponding stability conditions without going into the construction methods of the moduli spaces listed above. At least we describe local neighborhoods of a given point in the moduli spaces.

G-bundles

Before we can define stability, recall that there is a one-to-one correspondence between reductions $E_H \to \Sigma$ of the structure group to a subgroup $H \subset G$ and sections $\sigma : \Sigma \to E/H$. Here E/H is the quotient space under the fiberwise H-action. It is an H-bundle over E so that $\sigma^*(E/H)$ is an H-bundle over Σ. Conversely, every H-reduction is of this form. Moreover, if G acts on a (vector) space M we denote by $E(M)$ the associated bundle, in particular $E(\mathfrak{g}) = \mathrm{ad}(E)$.

Definition 4.2. A G-bundle $E \to \Sigma$ is called *semistable* if

$$\deg(\sigma^* E(\mathfrak{g}/\mathfrak{p})) \geq 0 \tag{4.7}$$

for every reduction $\sigma : \Sigma \to E/P$ of E to a maximal parabolic subgroup $P \subset G$. It is *stable* if the inequality (4.7) is strict for each such reduction. Finally, E is called *regularly stable* if E is stable with $\mathrm{Aut}(E) = Z(G)$.

Remark 4.3.

a) Observe that regularly stable bundles have as few automorphisms as possible. For vector bundles this coincides with simple vector bundles (i.e. having multiples of the identity as only automorphisms).

b) The notions stable and semistable have their origin in GIT. In the GIT-constructions of the moduli spaces, they coincide with the corresponding notions from the general theory. Therefore the points of the moduli space of semistable bundles actually correspond to S-*equivalence classes* of such bundles. One can equivalently work with isomorphism classes of polystable bundles instead (cf. [Ram96b]).

Example 4.4. Consider the case $G = GL(V) \cong GL(n, \mathbb{C})$, $\dim_{\mathbb{C}} V = n$. Then a maximal parabolic subgroup $P \subset G$ is just the stabilizer of a minimal flag $(V_1 \subseteq V_2 = V)$. Hence a maximal parabolic subgroup is (in an appropriate basis of V) of the form

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \,\middle|\, A \in GL(r, \mathbb{C}), D \in GL(n-r, \mathbb{C}) \right\} \subset GL(n, \mathbb{C}).$$

If one considers G-bundles as holomorphic vector bundles V, then such P-reductions correspond to holomorphic subbundles $W \subset V$. It then follows that (4.7) is equivalent to the well-known stability condition

$$\frac{\deg W}{\mathrm{rk} W} \leq \frac{\deg V}{\mathrm{rk} V}$$

for holomorphic vector bundles.

Theorem 4.5 ([Ram75])**.** *Let Σ be a compact Riemann surface of genus $g \geq 2$ and G a connected reductive complex Lie group. Then the moduli space $\mathbf{Bun}(\Sigma, G)$ of* semistable *G-bundles carries a natural structure of normal complex Hausdorff space of dimension*

$$\dim_{\mathbb{C}} \mathbf{Bun}(\Sigma, G) = \dim G(g-1) + \dim Z(G).$$

It contains the (regularly) stable loci $\mathbf{Bun}^{(r)s}(\Sigma, G)$ as non-empty open subsets. In general, it has singular points but the locus $\mathbf{Bun}_d^{rs}(\Sigma, G)$ is non-singular.
Finally, the connected components of $\mathbf{Bun}(\Sigma, G)$ are the moduli spaces $\mathbf{Bun}_d(\Sigma, G)$ of semistable G-bundles of fixed degree $d \in \pi_1(G)$ giving

$$\mathbf{Bun}(\Sigma, G) = \coprod_{d \in \pi_1(G)} \mathbf{Bun}_d(\Sigma, G).$$

Remark 4.6. The algebraic constructions (e.g. [Ram96b]) show that $\mathbf{Bun}(\Sigma, G)$ is a normal projective variety.

We briefly discuss the local structure of $\mathbf{Bun}_d^s(\Sigma, G)$ (following [Ram75], [Sim94b]). Let $[E]$ be the isomorphism class of a stable G-bundle E. From the previous section, we know that $H^1(L_E) = H^1(\Sigma, \mathrm{ad}(E))$ are the infinitesimal deformations of E. Together with Luna's étale slice theorem, it follows that the GIT-quotient

$$H^1(\Sigma, \mathrm{ad}(E)) /\!\!/ \mathrm{Aut}(E)$$

(around 0) is isomorphic to an analytic (or étale) neighborhood of $[E]$ in $\mathbf{Bun}_d^s$ where E is any representative of $[E]$. Note that $\mathrm{Aut}(E)$ naturally acts on the space $H^1(\Sigma, \mathrm{ad}(E))$ of infinitesimal

deformations of E by the adjoint action. Moreover, $Z(G) \subset \mathrm{Aut}(E)$ acts trivially on the base ([Ram75]) so that one can in fact quotient by $\mathrm{Aut}(E)/Z(G)$ only. This group turns out to be finite, showing that $\mathbf{Bun}_d^s$ is an orbifold. It further implies that the subset $\mathbf{Bun}_d^{rs} \subset \mathbf{Bun}_d^s$ is open and non-singular because $\mathrm{Aut}(E)/Z(G) = 1$ for E regularly stable.
We conclude by computing $\dim_{\mathbb{C}} \mathbf{Bun}_d^{rs}$ which equals $\dim_{\mathbb{C}} H^1(\Sigma, \mathrm{ad}(E))$ for any regularly stable E. For this, we need to know that $H^0(\Sigma, \mathrm{ad}(E)) \cong \mathrm{Lie}(Z(G))$ if E is stable ([Ram75]). Then one can compute via Riemann-Roch

$$\begin{aligned} h^1(\Sigma, \mathrm{ad}(E)) &= h^0(\Sigma, \mathrm{ad}(E)) - \deg \mathrm{ad}(E) - \mathrm{rk}(E)(1-g) \\ &= \dim(Z(G)) + \dim(G)(g-1). \end{aligned}$$

Here we have used the isomorphism $\mathrm{ad}(E) \cong \mathrm{ad}(E)^*$ induced by a non-degenerate bilinear form on $\mathfrak{g}$, which exists since G is reductive. It will be convenient to fix such a choice. If G is semisimple, it is natural to choose the Killing form.

G-Higgs bundles

Stability of G-Higgs bundles is similarly defined as for G-bundles but we have to include additionally the Higgs field. Let (E, θ) be a G-Higgs bundle and $H \subset G$ a subgroup. An *H-reduction of* (E, θ) is an H-reduction $\sigma : \Sigma \to E/H$ of E such that θ vanishes under the natural map

$$\mathrm{ad}(E) \otimes K_\Sigma \longrightarrow (\mathrm{ad}(E)/\mathrm{ad}(\sigma^*(E/H))) \otimes K_\Sigma.$$

Note that this condition guarantees that θ descends to a Higgs field θ_H on the H-reduction $E_H = \sigma^*(E/H)$, i.e. (E_H, θ_H) is an H-Higgs bundle.

Definition 4.7. A G-Higgs bundle (E, θ) is called *semistable* if

$$\deg(E(\mathfrak{g}/\mathfrak{p})) \geq 0$$

for any reduction of (E, θ) to a maximal parabolic subgroup $P \subset G$.

A (regularly) stable G-Higgs bundle is defined in complete analogy with the notions for G-bundles. In particular, for a regularly stable G-Higgs bundle (E, θ), the Higgs field is generically regular, i.e. $\theta(x)$ is in a local trivialization a regular element of $\mathfrak{g}$ for generic $x \in \Sigma$. This is independent of the chosen trivialization.

Example 4.8.

a) Of course, each stable G-bundle E gives rise to a stable G-Higgs bundle (E, θ) for *any* Higgs field θ.

b) Hitchin's famous example ([Hit87a]) shows that there are stable G-Higgs bundles (E, θ) such that E is unstable: Let $G = SL(2, \mathbb{C})$ and consider $E = L \oplus L^{-1}$ for a spin bundle $L^2 = K_\Sigma$. Then the pair $(L \oplus L^{-1}), \theta)$ with

$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where $1 \in H^0(\Sigma, L^{-2}K) \cong \mathbb{C}$, is a stable G-Higgs bundle. But E is obviously not stable as a G-bundle.

Theorem 4.9 ([Hit87a],[Nit91],[Sim94a])**.** *Let G be a connected reductive complex Lie group and Σ a compact Riemann surface of genus $g \geq 2$. Then the moduli space* $\mathbf{Higgs}(\Sigma, G)$ *of* semistable *G-Higgs bundles is a complex analytic space of dimension*

$$\dim_{\mathbb{C}} \mathbf{Higgs}(\Sigma, G) = 2(\dim(Z(G)) + \dim(G)(g-1)).$$

Again the loci $\mathbf{Higgs}^{(r)s}(\Sigma, G)$ *of (regularly) stable bundles are non-empty open subsets and* $\mathbf{Higgs}^{rs}(\Sigma, G)$ *is non-singular.*

Remark 4.10.

a) As in the case of G-bundles, the connected components of $\mathbf{Higgs}(\Sigma, G)$ are labelled by the topological type of the underlying G-bundles. More precisely, we have a decomposition

$$\mathbf{Higgs}(\Sigma, G) = \coprod_{d \in \pi_1(G)} \mathbf{Higgs}_d(\Sigma, G)$$

in connected components where $\mathbf{Higgs}_d(\Sigma, G)$ are the moduli spaces of semistable G-Higgs bundles of fixed degree $d \in \pi_1(G)$. To prove this is more elaborate as in the case of G-bundles. As far as we can see, it was first establised in [DP12] for the case of a connected reductive complex Lie group G. But it is even true when G is not necessarily connected, [GO14].

b) Again there is also an algebraic construction due to Nitsure ([Nit91]) and Simpson ([Sim94a]). It shows that $\mathbf{Higgs}(\Sigma, G)$ is a *quasi-projective* variety. Observe that this differs from the bundle case.

c) By the considerations of G-bundles we know that

$$T^*_{[E]}\mathbf{Bun}^s \cong H^1(\Sigma, \mathrm{ad}(E))^* \cong H^0(\Sigma, \mathrm{ad}(E)^* \otimes K_\Sigma) \cong H^0(\Sigma, \mathrm{ad}(E) \otimes K_\Sigma)$$

for the isomorphism class $[E] \in \mathbf{Bun}^{rs}$ of a regularly stable bundle E. Note that for the last isomorphism we have used again that G is reductive. In fact this shows that we have a natural open immersion

$$T^*\mathbf{Bun}^{rs} \subset \mathbf{Higgs}^{rs}.$$

Example 4.11. Let us consider the simplest reductive example, $G = \mathbb{C}^*$. Even though it is 'too simple', it gives some insight into the general theory. Since all $\mathbb{C}^*$-bundles are trivially stable, we obtain

$$\mathbf{Bun}(\Sigma, \mathbb{C}^*) = \mathrm{Pic}(\Sigma) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}_d(\Sigma)$$

which fits nicely with the general theory because $\mathbb{Z} = \pi_1(\mathbb{C}^*)$. Clearly, a $\mathbb{C}^*$-Higgs bundle is simply a pair (L, α) where $\alpha \in H^0(\Sigma, K)$ is a holomorphic differential so that

$$\mathbf{Higgs}(\Sigma, \mathbb{C}^*) = \mathrm{Pic}(\Sigma) \times H^0(\Sigma, K) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}(\Sigma) \times H^0(\Sigma, K).$$

Finally, 4.10c) specializes further: By Serre duality, we have

$$T^*\mathrm{Jac}(\Sigma) \cong \mathrm{Jac}(\Sigma) \times H^0(\Sigma, K) \to \mathrm{Jac}(\Sigma)$$

and hence $\mathbf{Higgs}(\Sigma, \mathbb{C}^*) \cong T^*\mathrm{Pic}(\Sigma)$.

In analogy with G-bundles, the GIT-quotient

$$H^1(\Sigma, \mathcal{K}^\bullet(E,\theta)) /\!\!/ \mathrm{Aut}(E,\theta)$$

is isomorphic to an analytic (or étale) neighborhood of $[E,\theta]$ in $\mathbf{Higgs}^s$. This again proves the claims about the regularly stable locus (note that $\mathbf{Higgs}^{rs} \neq \emptyset$ (Remark 4.10c)), in fact $\mathbf{Higgs}^{rs} - T^*\mathbf{Bun}^{rs} \neq \emptyset$ (Example 4.8b))).

Let us take up the discussion of infinitesimal deformations of Section 4.1 in the special case where (E,θ) is a G-Higgs bundle with underlying *stable* G-bundle E. Since E is stable, we have $H^0(\Sigma, \mathrm{ad}(E)) = \mathrm{Lie}(Z(G))$. This in particular implies that the connecting homomorphism $\delta_0 = h^0(\mathrm{ad}(\theta))$ of (4.5) is zero. Hence the infinitesimal deformations $H^1(\mathcal{K}^\bullet)$ of (E,θ) are determined by the short exact sequence

$$0 \longrightarrow H^0(\mathrm{ad}(E)\otimes K_\Sigma) \longrightarrow H^1(\mathcal{K}^\bullet) \xrightarrow{\gamma^1} \ker\delta^1 \longrightarrow 0. \tag{4.8}$$

This is very similar to the case when $\theta = 0$ and E is arbitrary (cf. (4.6)). If G is semisimple, then (4.8) specializes to (4.6): In that case $Z(G)$ is finite so that (again assuming E to be stable)

$$H^0(\mathrm{ad}(E)\otimes K_\Sigma) \cong H^1(\mathrm{ad}(E))$$

by using Serre duality and the Killing form. It is then immediate that

$$\dim_{\mathbb{C}} H^1(\mathcal{K}^\bullet) = 2\dim_{\mathbb{C}} H^1(\mathrm{ad}(E)) = 2\ \dim(G)(g-1).$$

Of course, this is just $\dim \mathbf{Higgs}^{rs} = \dim \mathbf{Higgs}$.

Remark 4.10c) suggests that all of $\mathbf{Higgs}^{rs}$ is symplectic which turns out to be true.

Theorem 4.12 ([BR94],[Fal93],[Hit87a])**.** *The smooth locus of* $\mathbf{Higgs}(\Sigma, G)$ *carries a symplectic structure which coincides with the canonical symplectic structure on* $T^*\mathbf{Bun}^{rs}(\Sigma,G) \subset \mathbf{Higgs}^{rs}(\Sigma,G)$.

Following [BR94] we can describe the symplectic form at some $[E,\theta] \in \mathbf{Higgs}^{rs}$ when G is semisimple. The latter condition guarantees that the tangent space to $[E,\theta]$ is isomorphic to $H^1(\mathcal{K}^\bullet(E,\theta))$. Then it turns out that Grothendieck-Serre duality (or Serre duality for hypercohomology) yields a non-degenerate pairing on $H^1(\mathcal{K}^\bullet(E,\theta))$. To see that this is not automatic and uses the properties of $\mathcal{K}^\bullet$, we briefly recall its construction in the special case of complexes of length 2,

$$\mathcal{C}^\bullet:\ 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow 0.$$

The 'Serre dual' complex $\check{\mathcal{C}}^\bullet$ of $\mathcal{C}^\bullet$ is defined as

$$\check{\mathcal{C}}^\bullet:\ 0 \longrightarrow C^1\otimes K_\Sigma \longrightarrow C^0\otimes K_\Sigma \longrightarrow 0.$$

Contraction tr induces a morphism $\mathcal{C}^\bullet \otimes \check{\mathcal{C}}^\bullet \to K_\Sigma[-1]$ of complexes and combined with cup product gives the pairing

$$H^i(\mathcal{C}^\bullet)\otimes H^{2-i}(\check{\mathcal{C}}^\bullet) \xrightarrow{\cup} H^2(\mathcal{C}^\bullet\otimes\check{\mathcal{C}}^\bullet) \xrightarrow{tr} H^2(K_\Sigma[-1]) \cong H^1(K_\Sigma) \cong \mathbb{C}.$$

Grothendieck-Serre duality states that this pairing is non-degenerate. In our cases at hand we observe that $\mathcal{C}^\bullet = \mathcal{K}^\bullet$ is 'self-dual', $\mathcal{K}^\bullet \cong \check{\mathcal{K}}^\bullet$ (again using that G is reductive). Note that this is clearly false for $\mathcal{C}^\bullet = \mathrm{ad}(E)[0]$. As claimed we obtain a non-degenerate skewsymmetric form

$$H^1(\mathcal{K}^\bullet(E,\theta)) \otimes H^1(\mathcal{K}^\bullet(E,\theta)) \to \mathbb{C},$$

in other words a symplectic form on the tangent space $T_{[E,\theta]}\mathbf{Higgs}^{rs}$.

Remark 4.13. As mentioned earlier, the smooth locus of $\mathbf{Higgs}^{sm}$ carries a natural hyperkähler structure (cf. [Hit87a], [Fuj91]). This is best seen by the gauge-theoretic construction of the moduli space which also gives an alternative description of the holomorphic symplectic form. The hyperkähler structure is an essential feature for *non-abelian Hodge theory* which gives an extension of 'abelian Hodge theory' ($G = \mathbb{C}^*$), i.e. the classical Hodge decomposition theorem, to non-abelian groups G (cf. [Sim92]). Note that if $\mathbf{Higgs}^{sm}$ were compact, then one could already conclude that $\mathbf{Higgs}^{sm}$ carries a hyperkähler structure by using the holomorphic symplectic structure and Yau's theorem (Calabi's conjecture).

4.3 Hitchin map

The Hitchin map is the remaining ingredient for the construction of the Hitchin system. From now on, we restrict to the case where G is a (semi)simple complex Lie group G because this is the case that we will need later on. However, everything works in the reductive case as well.

In order to construct the Hitchin map, recall that the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{t}/W$ is $\mathbb{C}^*$-equivariant with respect to the two natural $\mathbb{C}^*$-actions on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and the cone $\mathfrak{t}/W$ respectively. All the weights on $\mathfrak{g}$ are 1 whereas they are $d_1, \ldots, d_r$ on $\mathfrak{t}/W$. Here $d_j = \deg(\chi_j)$ for algebraically independent generators $\chi_1, \ldots, \chi_r \in \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W$ as in Section 1.4.1. Such a choice gives an isomorphism $\mathfrak{t}/W \cong \mathbb{C}^r$ and hence a vector space structure on $\mathfrak{t}/W$.

The $\mathbb{C}^*$-equivariance enables us to glue the adjoint quotient over the Riemann surface Σ and this clearly works for the quotient map $q : \mathfrak{t} \to \mathfrak{t}/W$ as well. Setting[4]

$$\tilde{\boldsymbol{U}} := K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t} \xrightarrow{\tilde{u}} \Sigma \xleftarrow{u} \boldsymbol{U} := K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W,$$

we therefore obtain the diagram

$$\begin{array}{ccc} & & \tilde{\boldsymbol{U}} \\ & & \downarrow{\scriptstyle \boldsymbol{q}} \\ K_\Sigma \otimes_{\mathbb{C}} \mathfrak{g} \cong K_\Sigma \times_{\mathbb{C}^*} \mathfrak{g} & \xrightarrow{\boldsymbol{\chi}} & \boldsymbol{U}. \end{array} \tag{4.9}$$

The adjoint quotient can also be twisted by a non-trivial G-bundle E or rather its adjoint bundle $\mathrm{ad}(E)$. Indeed, the adjoint bundle is obtained by the adjoint action of G on $\mathfrak{g}$. By the G-invariance and $\mathbb{C}^*$-equivariance of χ, it follows that χ glues to a morphism

$$\boldsymbol{\chi}_E : K_\Sigma \otimes_{\mathbb{C}} \mathrm{ad}(E) \cong K_\Sigma \times_{\mathbb{C}^*} \mathrm{ad}(E) \to K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W. \tag{4.10}$$

Clearly, if E is the trivial bundle then $\boldsymbol{\chi}_E = \boldsymbol{\chi}$.

[4]Here we identify a line bundle L with its $\mathbb{C}^*$-bundle $L^\times$ obtained by removing its zero section.

Remark 4.14. It is clear that any other $\mathbb{C}^*$-equivariant object related to $\mathfrak{t}$ and $\mathfrak{t}/W$ (cf. Chapter 1) glues as well. For example, each root $\alpha \in R$ gives rise to a section

$$\boldsymbol{\alpha} : \tilde{\boldsymbol{U}} \to \tilde{u}^* K_\Sigma$$

and a reflection $\boldsymbol{s}_\alpha : \tilde{\boldsymbol{U}} \to \tilde{\boldsymbol{U}}$ as well as their fixed point sets $\tilde{\boldsymbol{U}}_\alpha = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}_\alpha$. Further one can glue the stratifications of $\mathfrak{t}$ and $\mathfrak{t}/W$ to obtain similar stratifications for $\tilde{\boldsymbol{U}}$ and $\boldsymbol{U}$ respectively. In particular, we have global analogues of (1.40), (1.41), namely

$$\tilde{\boldsymbol{U}}^1 := K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}^1, \tag{4.11}$$

$$\boldsymbol{U}^1 := K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}^1/W. \tag{4.12}$$

Moreover, $s_{br} = \prod_{\alpha \in R} \alpha \in \mathbb{C}[\mathfrak{t}]^W$ induces a section

$$s_{\boldsymbol{br}} : \boldsymbol{U} \to u^* K_\Sigma^{|R|}.$$

Its vanishing locus (with its reduced structure) is in analogy to Corollary 1.62 the discriminant $\mathrm{discr}(\boldsymbol{q}) \subset \boldsymbol{U}$, or branch locus, of $\boldsymbol{q}$.

Before constructing the Hitchin map we need to recall the definition of the *Hitchin base*

$$\mathbf{B}(\Sigma, G) := H^0(\Sigma, \boldsymbol{U}) \tag{4.13}$$

It only depends on the Lie algebra of $\mathfrak{g} = \mathrm{Lie}(G)$, in particular *not* on the type of G. Note that, a priori, it is only an affine variety with a $\mathbb{C}^*$-action coming from the $\mathbb{C}^*$-action on $\mathfrak{t}/W$. The *Hitchin map* is then given by

$$\boldsymbol{h} : \mathbf{Higgs}(\Sigma, G) \to \mathbf{B}(\Sigma, G), \quad [E, \theta] \mapsto \boldsymbol{\chi}_E(\theta). \tag{4.14}$$

Even though it is well-known, we make the following explicit (cf. [Sim94b]):

Lemma 4.15. *The Hitchin map $\boldsymbol{h}$ is a well-defined morphism. It is $\mathbb{C}^*$-equivariant with respect to the $\mathbb{C}^*$-action on* **Higgs** *induced by $(\lambda, (E, \theta)) \mapsto (E, \lambda\theta)$ and the natural $\mathbb{C}^*$-action on* **B**.

Proof. First of all, the map is well-defined by the G-equivariance of χ. To see that it is a morphism, observe that **Higgs** is a coarse moduli space for semistable Higgs bundles. It therefore universally corepresents the functor ([Sim94a])

$$\mathbf{Higgs}^\natural(S) = \{(\mathcal{E}, \Theta) \to \Sigma \times S \ G\text{-Higgs bundle}\}/\sim.$$

Let $\mathbf{B}^\natural$ be the functor of points of $\mathbf{B}(\Sigma, G)$, i.e. $\mathrm{Hom}(-, \mathbf{B}(\Sigma, G))$. For each family $(\mathcal{E}, \Theta) \to \Sigma \times S$ of G-Higgs bundles, we obtain a morphism

$$s \mapsto \boldsymbol{\chi}_{E_s}(\Theta_s) \in \mathrm{Hom}(S, \mathbf{B}(\Sigma, G))$$

via pullback[5]. It descends to isomorphism classes and since $f^*(\mathcal{E}, \Theta)_t \cong (\mathcal{E}, \Theta)_{f(t)}$ it defines a morphism $\mathbf{Higgs}^\natural \to \mathbf{B}^\natural$ of functors which is represented by $\boldsymbol{h}$ as a morphism.
Finally, the $\mathbb{C}^*$-equivariance follows immediately from that of $\chi : \mathfrak{g} \to \mathfrak{t}/W$. □

The choice of generators $\chi_1, \dots, \chi_r \in \mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{t}]^W$ of degree $d_j = \deg(\chi_j)$ gives isomorphisms

$$\boldsymbol{U} \cong \bigoplus_{j=1}^r K_\Sigma^{d_j}, \quad \mathbf{B}(\Sigma, G) \cong \bigoplus_{j=1}^r H^0(\Sigma, K^{d_j}).$$

[5]Formally, we actually consider $\boldsymbol{\chi}_{\mathcal{E}}(\Theta)$ and then pull back, which gives a morphism to $\Sigma \times \{s\}$.

As in the case of $\mathfrak{t}/W$, we therefore obtain a vector bundle structure on $\boldsymbol{U}$ and a vector space structure on $\mathbf{B}(\Sigma, G)$. The Hitchin map can then be expressed non-canonically as

$$\boldsymbol{h} = (\boldsymbol{h}_1, \dots, \boldsymbol{h}_r) : \mathbf{Higgs}(\Sigma, G) \to \bigoplus_{j=1}^{r} H^0(\Sigma, K^{d_j}).$$

Theorem 4.16 ([Hit87a], [Fal93], [Sim94b]). *Let G be a connected semisimple complex Lie group and Σ a compact Riemann surface of genus $g \geq 2$. Then the Hitchin map*

$$\boldsymbol{h}_d : \mathbf{Higgs}_d(\Sigma, G) \to \mathbf{B}(\Sigma, G)$$

restricted to a connected component of $\mathbf{Higgs}(\Sigma, G)$ *is proper and surjective. Moreover, it is an algebraically completely integrable system in the sense of Chapter 2.*

Remark 4.17. If one considers $\boldsymbol{h} : \mathbf{Higgs}(\Sigma, G) \to \mathbf{B}(\Sigma, G)$ instead, then its generic fibers have $|\pi_1(G)|$ connected components ([DP12]). In particular, $\boldsymbol{h}$ is still an algebraically completely integrable system in a more general sense (than our definition), i.e. allowing for disconnected fibers. The generic fibers might even have infinitely many connected components which we already see in the next example.
The restriction $\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}(\Sigma, G)$ to the *neutral component*, is special: Hitchin was the first one ([Hit92]) to construct sections for $\boldsymbol{h}_1$, which are often referred to as *Hitchin sections*. They map into the regular parts of the fibers and are Lagrangian. The restriction $\boldsymbol{h}_1$ was denoted by $\mathbf{M}_{Hit}(\Sigma, G) \to \mathbf{B}_{Hit}(\Sigma, G)$ in the introduction.

Example 4.18. Let us continue the toy Example 4.11 which gives an example of a Hitchin system for reductive G at the same time. In that example we have seen that $\mathbf{Higgs}_d(\Sigma, \mathbb{C}^*) = T^*\mathrm{Pic}_d(\Sigma) \cong \mathrm{Pic}_d(\Sigma) \times H^0(\Sigma, K)$. The Hitchin map is simply the projection (since clearly $\mathfrak{g} = \mathfrak{t} = \mathbb{C}$)

$$\boldsymbol{h}_d : T^*\mathrm{Pic}_d(\Sigma) \to \mathbf{B}(\Sigma, \mathbb{C}^*) = H^0(\Sigma, K).$$

Its fibers $\boldsymbol{h}_d^{-1}(b) = \mathrm{Pic}_d(\Sigma)$ are Lagrangian with respect to the canonical symplectic structure. Hence we end up with an ACIS, the Hitchin system for the simplest case $G = \mathbb{C}^*$.

4.3.1 Generic Hitchin fibers

The description of generic Hitchin fibers has a long history starting with Hitchin's original paper ([Hit87a]) which was later generalized ([Don95],[DG02], [DP12],[Fal93],[Sco98]). Suffice it to say that to describe the isomorphism class of the connected components of $\boldsymbol{h}^{-1}(b)$ for generic $b \in \mathbf{B}$ is in general much more subtle than to describe its isogeny class[6]. We begin our treatment in the general case, where G is any semisimple complex Lie group, following the comprehensive treatments in [DG02], [DP12]. After that we restrict to the case where G is simply connected or of adjoint type, which is enough for our purposes.

Cameral curves

The basic idea to describe the fibers is rather simple but elegant and best illustrated for $G \subset GL(n, \mathbb{C})$. To make it precise takes some care and we only confine ourselves to sketch it here. A $GL(n, \mathbb{C})$-Higgs bundle is a pair (V, ϕ) consisting of a vector bundle V and a section $\phi \in H^0(\Sigma, \mathrm{End}(V) \otimes K_\Sigma)$ together with extra structure determining $G \subset GL(n, \mathbb{C})$. If ϕ is generic, then ϕ_x is regular semisimple for generic $x \in \Sigma$. Hence it can be reconstructed

[6] Note that this makes sense because the connected components of $\boldsymbol{h}^{-1}(b)$ are torsors for abelian varieties/

from its (unordered) eigenvalues $\chi(\phi_x) \in \mathfrak{t}/W$. Going to the branched covering $\hat{p}_b : \hat{\Sigma}_b \to \Sigma$ parametrizing its eigenvalues, the *spectral curve*, the Higgs bundle can be reconstructed as

$$(\hat{p}_{b,*}L, \psi) \cong (E, \phi)$$

where ψ is multiplication by the corresponding eigenvalue ([Hit87a], [Hit87b], [BNR89])[7]. In particular, the Higgs bundle is already determined by abelian data (namely a line bundle) so that this process is often referred to as *abelianization*.
This approach does work for the classical groups $G \subset GL(n, \mathbb{C})$ ([Hit87b]). It has the drawback that the spectral curves might be singular even for generic $b \in \mathbf{B}$ and that it does not readily generalize to arbitrary reductive G. Donagi and Faltings took another another approach by, loosely speaking, remembering the ordering of the eigenvalues. This leads to *cameral curves* ([Don95], [Fal93]) and it turns out that they can be used to describe generic Hitchin fibers for an arbitrary complex reductive Lie group G (see also [DG02]).

Let us first define the *universal cameral curve* $\tilde{\boldsymbol{\Sigma}} \to \Sigma \times \mathbf{B}$ via the cartesian square[8]

$$\begin{tikzcd} \tilde{\boldsymbol{\Sigma}} \arrow[r] \arrow[d, "\boldsymbol{p}_1"] \arrow[dd, bend right=60, "\boldsymbol{p}"'] & \tilde{\boldsymbol{U}} \arrow[d, "\boldsymbol{q}"] \\ \Sigma \times \mathbf{B} \arrow[r, "ev"] \arrow[d, "\boldsymbol{p}_2 = pr"] & \boldsymbol{U} \\ \mathbf{B}. & \end{tikzcd} \tag{4.15}$$

By construction $\tilde{\boldsymbol{\Sigma}}$ inherits a W-action and all morphisms in this diagram are W-equivariant. The pullback $\tilde{\Sigma}_b := i_b^* \tilde{\boldsymbol{\Sigma}}$ via the inclusion $i_b : \Sigma \to \{b\} \times \Sigma$ is the cameral curve $\tilde{\Sigma}_b \hookrightarrow \tilde{\boldsymbol{U}}$ corresponding to $b \in \mathbf{B}$ and we denote by

$$p_b := \boldsymbol{p}_{1,b} : \tilde{\Sigma}_b \to \Sigma$$

the induced map. These curves can be singular but for generic $b \in \mathbf{B}$ they are non-singular and $p_b : \tilde{\Sigma}_b \to \Sigma$ is a simply ramified Galois covering. More precisely, let

$$\mathbf{B}^\circ := \{b \in \mathbf{B} \mid b \text{ transversal to } \mathrm{discr}(\boldsymbol{q})^{sm}\}. \tag{4.16}$$

It can be shown that $\mathbf{B}^\circ$ is Zariski-open and dense in $\mathbf{B}$ ([Sco98]). From Section 1.4.4 it follows that this is precisely the locus of smooth cameral curves with simple Galois ramification. Let us describe the branch and ramification loci of $p_b : \tilde{\Sigma}_b \to \Sigma$ more explicitly. First of all, its ramification locus $D_b \subset \tilde{\Sigma}_b$ is the fixed point locus of the W-action and can be decomposed into

$$D_b = \coprod_{\alpha \in R^+} D_b^\alpha = \coprod_{\alpha \in R_s^+} D_b^\alpha \cup \coprod_{\beta \in R_l^+} D_b^\beta,$$

cf. [Sco98]. Note that each D_b^α is non-empty for every $\alpha \in R^+$. Indeed, a root $\alpha \in R$ gives rise to a non-zero section of $p_b^* K_\Sigma$ so that

$$|D_b^\alpha| = |W| \cdot \deg K_\Sigma > 0.$$

[7] To make this work, one has to pay attention to how the eigenvalues/-lines of ϕ degenerate at non-generic $x \in \Sigma$, compare with (4.16).
[8] All the objects appearing in this diagram are algebraic and we take the fiber product in the algebraic category. Its analytification is the fiber product of the analytified objects. Hence we can work either within the algebraic or the analytic category via GAGA.

The branch locus of p_b is of course $Br_b := b^*\mathrm{discr}(\boldsymbol{q})$, which equals $p_b(D_b)$ as a set. By definition of $\mathbf{B}^\circ$, it is a reduced divisor. Its global version is given by $Br := ev^*\mathrm{discr}(\boldsymbol{q})$. Alternatively, one can describe it via the section $s_{\boldsymbol{br}} : \boldsymbol{U} \to u^*K_\Sigma^{|R|}$:

$$Br = V(ev^*s_{\boldsymbol{br}}) \subset \Sigma \times \mathbf{B}, \quad ev^*s_{\boldsymbol{br}} \in H^0(\Sigma \times \mathbf{B}, \mathrm{pr}_1{}^*K_\Sigma^{|R|}).$$

Clearly, $i_b^*Br = Br_b$ which in particular implies that

$$Br_b \cong K_\Sigma^{|R|}$$

as divisors so that Br_b consists of $|R|\cdot\deg K_\Sigma$ points. These can be divided into two classes $Br_b = Br_b^s + Br_b^l$ depending whether their preimage corresponds to short and long roots respectively. By the Riemann-Hurwitz formula, we conclude that the genus $g(\tilde{\Sigma}_b)$ of $\tilde{\Sigma}_b$, $b \in \mathbf{B}^\circ$, is given by

$$g(\tilde{\Sigma}_b) = (g-1)|W|\left(1 + \frac{1}{2}|R|\right) + 1.$$

The next lemma will be important in the next chapter. We emphasize that it is crucial to restrict to $\mathbf{B}^\circ$.

Lemma 4.19. *The divisor $Br \cap (\Sigma \times \mathbf{B}^\circ) \subset \Sigma \times \mathbf{B}^\circ$ is smooth.*

Clearly, this divisor is algebraic.

Proof. This is intuitively clear because the branch points of the cameral curves do not collide when we move in the Zariski-open $\mathbf{B}^\circ$. To make this precise let $b_0 \in \mathbf{B}^\circ$ and choose (the germ of) a neighborhood $T \subset \mathbf{B}^\circ$ of b_0. The preimage $\boldsymbol{p_2}^{-1}(b_0)$ consists of $|R| \cdot \deg K_\Sigma$ points and we fix one of them, say $x_0 \in Br_{b_0}$. Around $(x_0, b_0) \in \Sigma \times \mathbf{B}^\circ$ the divisor $Br \cap (\Sigma \times \mathbf{B}^\circ)$ is given by

$$\{(x,b) \mid s_{\boldsymbol{br}}(ev(x,b)) = 0\} \subset S \times T$$

where S is (the germ of) a neighborhood of x_0 that does not contain any other branch point of $b \in T$. This is possible because we work within $\mathbf{B}^\circ$. Now use local trivializations $\boldsymbol{U}_{|S} = u^{-1}(S) \cong S \times \mathfrak{t}/W$ and $u^*_{|u^{-1}(S)}K_\Sigma \cong S \times \mathfrak{t}/W \times \mathbb{C}$. In these terms $s_{\boldsymbol{br}} \circ ev$ can be expressed as

$$(x, f_b) \mapsto (x, f_b(x), s_{br}(f_b(x))).$$

Here $f_b : S \to \mathfrak{t}/W$ corresponds to a (global) section $b \in \mathbf{B}^\circ$ in the local trivialization. Since f_b and s_{br} are transversal to each other by definition of $\mathbf{B}^\circ$, it follows that $s_{\boldsymbol{br}} \circ ev$ is transversal to the zero section at (x_0, b_0). In other words, the divisor is smooth. $\square$

Remark 4.20. From now on, we almost exclusively work in the locus $\Sigma \times \mathbf{B}^\circ$ and $\mathbf{B}^\circ$. Therefore we will often denote $Br \cap (\Sigma \times \mathbf{B}^\circ)$ simply by Br.

Finally, we observe that the evaluation map $ev : \Sigma \times \mathbf{B}^\circ \to \boldsymbol{U}$ factors over $\boldsymbol{U}^1$ (cf. (4.12)) yielding the following commutative diagram

$$\begin{tikzcd} \Sigma \times \mathbf{B}^\circ \arrow[r, "ev"] & \boldsymbol{U}^1 \\ Br \arrow[u, hook] \arrow[r, "ev"] & \mathrm{discr}(\boldsymbol{q}) \cap \boldsymbol{U}^1. \arrow[u, hook] \end{tikzcd} \tag{4.17}$$

Generalized Prym varieties

To describe the isomorphism classes of generic Hitchin fibers ([DG02],[DP12]), also known as *generalized Prym varieties*, we need to define the following sheaves on Σ,

$$\overline{\mathcal{T}}(b) := p_{b,*}(\mathbf{\Lambda}_G \otimes \mathcal{O}^*_{\tilde{\Sigma}_b})^W, \tag{4.18}$$

$$\mathcal{T}^\circ(b) := p_{b,*}(\mathbf{\Lambda}_G)^W \otimes \mathcal{O}^*_\Sigma \tag{4.19}$$

for $b \in \mathbf{B}^\circ$. Here $\mathbf{\Lambda}_G$ stands for the cocharacter lattice $\mathrm{Hom}(\mathbb{C}^*, G)$ of the group G, often only denoted by $\mathbf{\Lambda}$. The actual sheaf $\mathcal{T}(b)$ of interest is then given by

$$\mathcal{T}(b)(U) := \{t \in \overline{\mathcal{T}}(b)(U) \mid \alpha(t)_{|D^\alpha} = +1 \; \forall \alpha \in R(T)\}. \tag{4.20}$$

Here we consider the root system (or rather root data) in the Lie group G with respect to a maximal torus $T \subset G$ with $\mathrm{Lie}(T) = \mathfrak{t}$.
Observe that by definition, they are related via $\mathcal{T}^\circ(b) \subset \mathcal{T}(b) \subset \overline{\mathcal{T}}(b)$. This last inclusion is an equality if all coroots $\alpha^\vee : \mathbb{C}^* \to T$ are *primitve*, i.e. injective as maps.

Example 4.21. Let us make this more explicit by considering the two possible (simple) A_1-cases:

$G = SL(2, \mathbb{C})$. We use the standard Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and maximal torus $T \subset G$ respectively (i.e. the diagonal ones). Since $W = \mathbb{Z}/2\mathbb{Z}$, a cameral curve $p_b = p : \tilde{\Sigma}_b \to \Sigma$ is just a branched double covering ($b \in \mathbf{B}^\circ$). The root datum with respect to T is given by $R(T) = \{\pm\alpha\}$ Here the root $\alpha : T \to \mathbb{C}^*$ is the morphism $\alpha(\mathrm{diag}(\lambda, \lambda^{-1}) = \lambda^2$ with coroot $\alpha^\vee(\lambda) = \mathrm{diag}(\lambda, \lambda^{-1})$. Let $U \subset \Sigma$ be an open set and $\tilde{U} = p^{-1}(U)$ the induced cameral cover. The W-equivariance for a morphism $t : \tilde{U} \to T$ just means $t(s_\alpha(x)) = t(x)^{-1}$. In particular, if $x \in D^\alpha$ then $t(x) = t(x)^{-1}$ so that $\alpha \circ t(x) = 1$ automatically. This shows that $\mathcal{T} = \overline{\mathcal{T}}$ in this case.

$G = PSL(2, \mathbb{C}) = SO(3, \mathbb{C})$. Again we use the standard Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and maximal torus $T \subset G$. The roots are given by $R(T) = \{\pm\alpha\}$ where

$$\alpha([\mathrm{diag}(\lambda, \mu)]) = \lambda\mu^{-1}.$$

With the previous notation it is easy to check that for $x \in D^\alpha$ and a W-equivariant $t : \tilde{U} \to T$, we must have

$$t(x) = [\mathrm{diag}(\pm 1, 1)].$$

In particular, $\alpha \circ t(x) = \pm 1$. Therefore we have $\mathcal{T}^\circ = \mathcal{T} \subsetneq \overline{\mathcal{T}}$ in this case.

These examples generalize and show that $\mathcal{T} = \overline{\mathcal{T}}$ iff G has no direct factor $SO(2n+1)$ which is the simple adjoint group of type B_n. However, we will see (as in the previous example) that in these cases the situation simplifies in the other direction, i.e. $\mathcal{T}^\circ = \mathcal{T}$.

Theorem 4.22 ([DG02])**.** *The generic Hitchin fiber $\boldsymbol{h}^{-1}(b)$, $b \in \mathbf{B}^\circ$, is a torsor for $H^1(\Sigma, \mathcal{T}(b))$.*

Hence the generic Hitchin fiber $\boldsymbol{h}^{-1}(b)$ is non-canonically isomorphic to $H^1(\Sigma, \mathcal{T}(b))$. The next lemma reflects the fact that $\boldsymbol{h}_{|\mathbf{B}^\circ}$ is an integrable system.

Lemma 4.23 ([DP12])**.** *The connected components $P^\circ(b)$, $P(b)$, $\overline{P}(b)$ of $H^1(\Sigma, \mathcal{T}^\circ(b))$, $H^1(\Sigma, \mathcal{T}(b))$, $H^1(\Sigma, \overline{\mathcal{T}}(b))$ are abelian varieties for $b \in \mathbf{B}^\circ$. They are all isogenous to each other.*
In particular, the connected components $\boldsymbol{h}_d^{-1}(b)$, $d \in \pi_1(G)$, of $\boldsymbol{h}^{-1}(b)$ are torsors for $P(b)$.

Proof. We only make some of the arguments from [DP12] more explicit: Consider the Leray spectral sequence for the composition $a_* \circ p_*^W = \tilde{a}_*^W$ where $a : \Sigma \to pt$ and $\tilde{a} : \tilde{\Sigma} = \tilde{\Sigma}_b \to pt$ are the constant maps. Note that $\tilde{a}^W(\mathcal{F}) = (.)^W \circ \tilde{a}(\mathcal{F}) = H^1(\tilde{\Sigma}, \mathcal{F})^W$ for any W-sheaf $\mathcal{F}$ on $\tilde{\Sigma}$.
The corresponding five-term exact sequence of the Grothendieck spectral sequence for the composition $a_* \circ p_*^W$ reads as

$$\begin{array}{c} 0 \longrightarrow H^1(\Sigma, p_*^W\mathcal{F}) \xrightarrow{\gamma} H^1(\tilde{\Sigma}, \mathcal{F})^W \longrightarrow H^0(\Sigma, R^1p_*^W\mathcal{F}) \longrightarrow \\ \longrightarrow H^2(\Sigma, p_*^W\mathcal{F}) \longrightarrow H^2(\tilde{\Sigma}, \mathcal{F})^W. \end{array} \tag{4.21}$$

Since p is a finite map, it follows that $R^1p_*^W\mathcal{F} \cong \mathcal{H}^1(W, p_*\mathcal{F})$ (see [Gro57]). The latter sheaf has stalks $H^1(W, (p_*\mathcal{F})_x)$ which is finite because $H^k(W, M)$ is finite for $k \geq 1$ and any W-module M, cf. [Wei94]. Since $\mathcal{H}^1(W, p_*\mathcal{F})$ is a local system on $\Sigma^\circ = \Sigma - Br_b$, it follows that $H^0(\Sigma, R^1p_*^W\mathcal{F})$ is finite.
We can use this to see that $\overline{P} = H^1(\Sigma, p_*^W\mathcal{F})^\circ$, $\mathcal{F} = \mathbf{\Lambda} \otimes \mathcal{O}_{\tilde{\Sigma}}$, is an abelian variety. Indeed, it is classical that the connected component $\tilde{P}$ of $H^1(\tilde{\Sigma}, \mathbf{\Lambda} \otimes \mathcal{O}_{\tilde{\Sigma}}^*)^W$ is an abelian variety. Now restricting γ to the connected components in (4.21) shows that $\gamma^\circ : \overline{P} \to \tilde{P}$ is injective with finite cokernel, i.e. is an isogeny. In particular, $\overline{P}$ carries the structure of an abelian variety as well.
To prove the statement for P° and P we observe that there are short exact sequences

$$0 \longrightarrow \mathcal{T}^\circ \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{T}^\circ \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{T} \longrightarrow \overline{\mathcal{T}} \longrightarrow \overline{\mathcal{T}}/\mathcal{T} \longrightarrow 0$$

by construction (cf. [DP12]). Note that the quotients are supported on the branch locus of $\tilde{\Sigma} \to \Sigma$, i.e. they are (sums of) skyscrapers. The corresponding long exact sequences show that each of the natural maps $H^1(\Sigma, \mathcal{T}^\circ) \to H^1(\Sigma, \mathcal{T}) \to H^1(\Sigma, \overline{\mathcal{T}})$ is surjective with finite kernel. Hence the restrictions $P^\circ \to P \to \overline{P}$ are isogenies. It follows that P and P° are abelian varieties as well. ☐

Remark 4.24. The proof in particular shows that the $\mathbb{Z}$-Hodge structures $H_1(Q, \mathbb{Z})$ (for $Q = P^\circ, P, \overline{P}$) of weight -1 become isomorphic to $H^1(\tilde{\Sigma}, \mathbf{\Lambda} \otimes_{\mathbb{Z}} \mathbb{Q})^W(1)$ after tensoring with $\mathbb{Q}$. Hence the complex structures on these complex tori are determined by the (Tate twisted) Hodge filtration $F^\bullet H^1(\tilde{\Sigma}, \mathfrak{t})^W(1)$.

Each of the sheaves $\mathcal{T}^\circ$, $\mathcal{T}$ and $\overline{\mathcal{T}}$ has a real version denoted by $\mathcal{T}_{\mathbb{R}}^\circ$, $\mathcal{T}_{\mathbb{R}}$ and $\overline{\mathcal{T}}_{\mathbb{R}}$ respectively. These are defined by replacing $\mathcal{O}_{\tilde{\Sigma}}^*$ with the constant sheaf $S_{\tilde{\Sigma}}^1$. If $d \in \Sigma$ is a branch point corresponding to a W-orbit $W \cdot \alpha$, $\alpha \in R$, then the stalks are given by (writing $\mathbf{\Lambda}_G$ multiplicatively)

$$\begin{aligned} \overline{\mathcal{T}}_{\mathbb{R},d} &= \{\Pi_j\, \lambda_j \otimes z_j \in \mathbf{\Lambda}_G \otimes S^1 \mid \alpha^\vee(\Pi_j\, z_j^{\langle \alpha, \lambda_j \rangle}) = 1 \in \mathbb{C}^*\}, \\ \mathcal{T}_{\mathbb{R},d} &= \{\Pi_j\, \lambda_j \otimes z_j \in \mathbf{\Lambda}_G \otimes S^1 \mid \Pi_j\, z_j^{\langle \alpha, \lambda_j \rangle} = 1 \in S^1\}, \\ \mathcal{T}_{\mathbb{R},d}^\circ &= \{\Pi_j\, \lambda_j \otimes z_j \in \mathbf{\Lambda}_G \otimes S^1 \mid \Sigma_j\, \langle \alpha_j, \lambda_j \rangle = 0 \in \mathbb{Z}\}, \end{aligned} \tag{4.22}$$

cf. [DP12]. It is important to note that this description is actually independent of the chosen root in the W-orbit. The real versions already contain all of the cohomological information:

Lemma 4.25 ([DP12]). *Let $b \in \mathbf{B}^\circ$ and let $\mathcal{F}$ be one of the sheaves $\mathcal{T}^\circ(b)$, $\mathcal{T}(b)$ or $\overline{\mathcal{T}}(b)$ and denote by $\mathcal{F}_\mathbb{R}$ the corresponding real version. Then the natural inclusion $\mathcal{F}_\mathbb{R} \hookrightarrow \mathcal{F}$ induces an isomorphism of abelian groups*

$$H^1(\Sigma, \mathcal{F}_\mathbb{R}) \cong H^1(\Sigma, \mathcal{F}).$$

Remark 4.26. As mentioned earlier, cameral curves have been introduced by Donagi ([Don93], [Don95]) after Hitchin's original papers ([Hit87a], [Hit87b]) to study generic Hitchin fibers. Hitchin used spectral curves instead which parametrize eigenvalues of the corresponding Higgs fields. As Donagi has shown ([Don93]) cameral curves give rise to spectral curves by choosing parabolic Weyl subgroups $W_P \subset W$. Spectral curves are very useful, for example their genus (resp. covering degree) is usually by far lower than that of the corresponding cameral curves. However, we will not discuss them here because cameral curves are better suited for our purposes.

Example 4.27. In the A_1-case the notion of spectral and cameral curves coincide and we can compare the results from [Hit87a] and [DG02]. Of course, from the general results it follows that they coincide but we consider it worthwhile to compare them here directly, since the approaches of [Hit87a] and [DG02] are very different.
It follows immediately for $G = SL(2, \mathbb{C})$ that $\mathbf{B} \cong H^0(\Sigma, K^2)$ and

$$\mathbf{B}^\circ = \{b \in H^0(\Sigma, K^2) \mid b \text{ has simple zeros}\}.$$

If $b \in \mathbf{B}^\circ$, then $p : \tilde{\Sigma}_b \to \Sigma$ is a branched double covering which is branched at the zeros of b. Hitchin has shown in [Hit87a] that

$$\boldsymbol{h}_1^{-1}(b) \cong \mathrm{Prym}(\tilde{\Sigma}_b/\Sigma) := \{L \in \mathrm{Jac}(\tilde{\Sigma}_b) \mid \tau^* L = L^{-1}\}$$

for the natural involution $\tau : \tilde{\Sigma}_b \to \tilde{\Sigma}_b$.
Let us compare this with P_b, the connected component of $H^1(\Sigma, \mathcal{T}(b))$. From Example 4.21, we know that $\mathcal{T} = \overline{\mathcal{T}} = p_*^W \mathcal{F}$ for $\mathcal{F} = \mathcal{O}^*_{\tilde{\Sigma}_b}$. In the proof of Lemma 4.23 we have seen that $H^1(\Sigma, \overline{\mathcal{T}})$ is at least isogenous to

$$(H^1(\tilde{\Sigma}, \mathcal{O}^*_{\tilde{\Sigma}_b}) \otimes \mathbf{\Lambda}_G)^W = H^1(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}_b})^- = \mathrm{Prym}(\tilde{\Sigma}_b/\Sigma), \quad W = \mathbb{Z}/2\mathbb{Z},$$

where the superscript $-$ are the anti-invariants under the $W = \mathbb{Z}/2\mathbb{Z}$-action (also see Remark 4.36). To show that it is even an isomorphism, it is enough to compute $H^1(W, (p_*\mathcal{F})_x)$ for all $x \in \Sigma$ (cf. (4.21)). First assume that $x \notin Br_b$ is not a branch point. Then $M_x := (p_*\mathcal{F})_x$ is the W-module

$$\mathcal{F}_x \oplus \mathcal{F}_x, \quad w \cdot (f_1, f_2) = (f_2^{-1}, f_1^{-1}),$$

for $f_i \in \mathcal{F}_x$, $w = -1$. Using group cohomology for cyclic groups ([Wei94]), we easily compute (writing everything additively)

$$H^1(W, M_x) = \{m \in M_x \mid (1 + w) \cdot x = 0\} / ((1 - w) \cdot M_x) = 0, \quad x \notin Br_b.$$

If $x \in Br_b$ is a branch point, then M_x is the W-module

$$\mathcal{F}_x, \quad w \cdot f = f^{-1}, \quad f \in \mathcal{F}_x.$$

Observing that $\mathcal{F}_x$ is divisible, we obtain again $H^1(W, M_x) = 0$. Altogether, the exact sequence (4.21) yields an isomorphism

$$\mathrm{Prym}(\tilde{\Sigma}_b/\Sigma) \cong H^1(\Sigma, \mathcal{T}(b))$$

as expected.

4.3.2 Case of adjoint and simply connected groups

We begin by specializing our discussion to simple Lie groups $G = G_{ad}$ of adjoint type. This is one of the relevant cases for us roughly because the cohomology of resolutions of singularities of type Δ are closely related to these groups, see Section 1.1.1. In this case we can be a bit more explicit.

Proposition 4.28 ([DP12]). *Let G be a simple* adjoint *complex Lie group and $\mathbf{B} = \mathbf{B}(\Sigma, G)$ its Hitchin base. Then the inclusion*

$$\mathcal{T}_{\mathbb{R}}^{\circ}(b) \hookrightarrow \mathcal{T}_{\mathbb{R}}(b)$$

is in fact an equality for any $b \in \mathbf{B}^{\circ}$. Moreover, the cocharacter lattice $\mathrm{cochar}(P_b)$ *of the abelian variety $P_b^{\circ} = P_b \subset H^1(\Sigma, \mathcal{T}(b))$ is given by*

$$\mathrm{cochar}(P_b) = H^1(\Sigma, (p_{b,*}\mathbf{\Lambda}_G)^W).$$

Proof. This first statement is contained in Lemma 3.4 of [DP12]. Let $d \in \Sigma$ be a branch point corresponding to a W-orbit $W \cdot \alpha$ for a root $\alpha \in R$. According to (4.22) we have to show

$$z^{\langle \alpha, \lambda \rangle} = 1 \in S^1 \quad \Longrightarrow \quad \langle \alpha, \lambda \rangle = 0 \in \mathbb{Z} \tag{4.23}$$

for $\lambda \otimes z \in \mathbf{\Lambda}_G \otimes S^1$. Let $\epsilon_{G,\alpha} \in \mathbb{Z}_+$ be the positive generator of the image of $\langle \alpha, \bullet \rangle : \mathbf{\Lambda}_G \to \mathbb{Z}$. Then (4.23) follows if[9] $\epsilon_{G,\alpha} = 1$. But G is adjoint so that $\mathbf{\Lambda}_G = \mathrm{coweights}(\mathfrak{g})$, i.e. we can find $\lambda \in \mathbf{\Lambda}_G$ such that $\langle \alpha, \lambda \rangle = 1 = \epsilon_{G,\alpha}$. Altogether we obtain $\mathcal{T}_{\mathbb{R}}^{\circ} = \mathcal{T}_{\mathbb{R}}$.
For the second statement (cf. Claim 3.6 in [DP12]), consider the exponential sequence

$$0 \longrightarrow \mathbb{Z}_\Sigma \longrightarrow \mathbb{R}_\Sigma \longrightarrow S^1_\Sigma \longrightarrow 0$$

on Σ. Tensoring (over $\mathbb{Z}$) with $(p_*\mathbf{\Lambda}_G)^W$ yields the exact $\mathscr{T}\!or$-sequence

$$\mathscr{T}\!or_1((p_*\mathbf{\Lambda}_G)^W, S^1_\Sigma) \xrightarrow{\delta} (p_*\mathbf{\Lambda}_G)^W \longrightarrow (p_*\mathbf{\Lambda}_G)^W \otimes \mathbb{R}_\Sigma \longrightarrow (p_*\mathbf{\Lambda}_G)^W \otimes S^1_\Sigma \longrightarrow 0. \tag{4.24}$$

Since the stalks of $(p_*\mathbf{\Lambda}_G)^W$ are free, it follows that $\mathscr{T}\!or_1((p_*\mathbf{\Lambda}_G)^W, S^1_\Sigma) = 0$. The monodromy group of $(p_*^{\circ}\mathbf{\Lambda}_G)^W$ is all of W which yields that $H^0(\Sigma, \mathcal{T}^{\circ}) = \mathbf{\Lambda}_G^W = 0$. Moreover, $H^2(\Sigma, (p_*\mathbf{\Lambda}_G)^W)$ is torsion (cf. Lemma 6.3 in [DP12]) implying that

$$H^2(\Sigma, (p_*\mathbf{\Lambda}_G)^W \otimes \mathbb{R}) \cong H^2(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \otimes \mathbb{R} = 0$$

by the projection formula together with flatness of $\mathbb{R}$. The latter also shows us that

$$H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W \otimes \mathbb{R}) / H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \cong H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \otimes S^1$$

the connected component of $H^1(\Sigma, \mathcal{T}_{\mathbb{R}}^{\circ})$. Altogether we obtain from (4.24) the short exact sequence

$$0 \longrightarrow H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \otimes S^1 \longrightarrow H^1(\Sigma, \mathcal{T}_{\mathbb{R}}^{\circ}) \longrightarrow H^2(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \longrightarrow 0.$$

But $H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \otimes S^1$ is connected and $H^2(\Sigma, (p_*\mathbf{\Lambda}_G)^W)$ is finite which implies that $P^{\circ} = H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W) \otimes S^1$ as real tori and therefore

$$\mathrm{cochar}(P^{\circ}) = H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W))_{\mathrm{tf}}.$$

Since G is of adjoint type, it follows that $H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W))_{\mathrm{tor}} = 0$, see Remark 4.38, which concludes the proof. $\square$

[9] Note that $\lambda \otimes 1 = 0 \in \mathbf{\Lambda}_G \otimes_{\mathbb{Z}} S^1$ - one of the dangers when forming the tensor product of a multiplicative with an additive abelian group.

Example 4.29. Let us give an explicit computation for the first statement of Proposition 4.28 when $G = SO(2n+1)$ is a simple adjoint complex Lie group of type B_n. Let $d \in \Sigma$ be a branch point corresponding to a W-orbit $W \cdot \alpha$ for a root $\alpha \in R$. Then we have to show that the inclusion

$$(\mathbf{\Lambda}_G)^{\rho_\alpha} \otimes S^1 \hookrightarrow \mathcal{T}_{\mathbb{R},d} = \{\Pi_j \lambda_j \otimes z_j \mid \Pi_j z_j^{\langle \alpha, \lambda_j \rangle} = 1\} \subset \mathbf{\Lambda}_G \otimes S^1$$

is an equality for the reflection $\rho_\alpha = s_\alpha^\vee : \mathbf{\Lambda}_G \to \mathbf{\Lambda}_G$, $\rho_\alpha(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee$. The irreducible root and coroot system R and $R^\vee$ respectively are given by (see e.g. [Spr09])

$$R = \{\pm e_i, \pm e_j \pm e_k \ (j \neq k)\} \subset \mathbb{Z}^n,$$
$$R^\vee = \{\pm 2e_i, \pm e_j \pm e_k \ (j \neq k)\} \subset \mathbb{Z}^n$$

where the e_i are the canonical basis of $\mathbb{Z}^n$. Let $Q^\vee = \langle R^\vee \rangle_{\mathbb{Z}}$, $V^\vee = Q^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and

$$P^\vee = \{v \in V^\vee \mid \langle R, v \rangle \subset \mathbb{Z}\}$$

as in Section 1.2 so that $P^\vee = \mathbb{Z}^n \subset V^\vee = \mathbb{R}^n$. Since G is of adjoint type the cocharacters of G satisfy $\mathbf{\Lambda}_G = X^\vee(G) = P^\vee = \mathbb{Z}^n$. The reflection ρ_α for a short root $\alpha = e_i \in R$ (so $\alpha^\vee = 2e_i \in R^\vee$ is a long root) is given by

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, -a_i, \dots, a_n)$$

implying that

$$(\mathbf{\Lambda}_G)^{\rho_\alpha} \otimes S^1 \cong \{(z_1, \dots, z_n) \in (S^1)^n \mid z_i = 1\}$$

To match this with the stalk $\mathcal{T}_{\mathbb{R},d}$, $d \in D_\alpha$, observe that

$$\mathcal{T}_{\mathbb{R},d} = \{(z_1, \dots, z_n) \in (S^1)^n \mid 1 = \Pi_j z_j^{\langle \alpha, e_j \rangle} = \Pi_j z_j^{\delta_{ij}} = z_i\}.$$

Hence the equality follows for short roots α.
The reflection ρ_α for a long root $\alpha = e_i - e_j \in R$ for $i < j$ (so $\alpha^\vee = e_i - e_j \in R^\vee$) acts as

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_j, \dots, a_i, \dots, a_n).$$

Since this is simply a permutation, we immediately see that

$$(\mathbf{\Lambda}_G)^{\rho_\alpha} \otimes S^1 \cong (\mathbf{\Lambda}_G \otimes S^1)^{\rho_\alpha} = \overline{\mathcal{T}}_{\mathbb{R},d}.$$

But the coroot $\alpha^\vee = e_i - e_j$ is primitive so that $\overline{\mathcal{T}}_{\mathbb{R},d} = \mathcal{T}_{\mathbb{R},d}$.

Together with Remark 4.24, this gives a complete description of the polarizable $\mathbb{Z}$-Hodge structure of weight 1 corresponding to $P_b = P_b(G_{ad})$, namely

$$(H^1(\Sigma, p_{b,*}^W \mathbf{\Lambda}), F^\bullet H^1(\tilde{\Sigma}_b, \mathfrak{t})^W).$$

Corollary 4.30. *Let $G = G_{ad}$ be a simple adjoint complex Lie group and*

$$\boldsymbol{h}_{1,ad}^\circ : \mathbf{Higgs}_1(\Sigma, G_{ad})^\circ \to \mathbf{B}^\circ$$

the restriction of the Hitchin map to the neutral component and away from singular fibers. Then $\boldsymbol{h}_1^\circ$ is isomorphic as a family of abelian varieties to the family $\mathcal{J}(\mathsf{V}_{ad}^H) \to \mathbf{B}^\circ$ determined by the polarizable $\mathbb{Z}$-VHS

$$\mathsf{V}_{ad}^H := \left(R^1 \boldsymbol{p}_{2,*}(\boldsymbol{p}_{1,*}^W \mathbf{\Lambda}), \mathcal{F}^\bullet (R^1 \boldsymbol{p}_*^W \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ}) \right)_{|\mathbf{B}^\circ} \cong \mathsf{V}^*(\boldsymbol{h}_{1,ad}^\circ)(-1)$$

of weight 1 over $\mathbf{B}^\circ$, *where $\mathbf{\Lambda} = \mathbf{\Lambda}_{G_{ad}}$ is the cocharacter lattice of G_{ad}.*

Proof. It is not difficult to see that $R^1\boldsymbol{p}_{2,*}(\boldsymbol{p}_{1,*}^W\boldsymbol{\Lambda})$ is a local system (e.g. see the proof of Theorem 5.15), so the statement makes sense. From [DG02], [DP12] it is known that $\boldsymbol{h}_{1,ad}^\circ$: $\mathbf{Higgs}_1(\Sigma, G)^\circ \to \mathbf{B}^\circ$ is a torsor for $\mathcal{J}(\mathsf{V}_{ad}^H) \to \mathbf{B}^\circ$. But the former has sections, namely Hitchin sections, see Remark 4.17, so that the claim follows. □

Remark 4.31. The VHS V_{ad}^H differs from another VHS that can be found in the literature (e.g. [HHP10]), namely

$$(\mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \boldsymbol{\Lambda}), F^1\mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \mathfrak{t})) = \big(R^1\boldsymbol{p}_*^W\boldsymbol{\Lambda}, \mathcal{F}^\bullet(R^1\boldsymbol{p}^W\mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ})\big)_{|\mathbf{B}^\circ}$$

over $\mathbf{B}^\circ$. As we have seen in the fiberwise case, this is in general only an isogenous VHS. Due to its simpler description, this VHS is particularly useful when the underlying integral structure is not important. This is for example the case, when one wants to compute the cubic of the Hitchin system ([HHP10], [Bal06]). However, since the integral structure is important for us, we work with V_{ad}^H when necessary.

We next describe the Lagrangian structure on $\boldsymbol{h}_1^\circ$ in terms of V_{ad}^H by giving an abstract Seiberg-Witten differential. Needless to say that it is defined via the $\mathfrak{t}$-valued (holomorphic) Seiberg-Witten differentials $\lambda_b \in H^0(\tilde{\Sigma}_b, K_{\tilde{\Sigma}} \otimes \mathfrak{t})^W$, $b \in \mathbf{B}^\circ$ (e.g. [HHP10]). By construction, they are defined via the tautological section $\boldsymbol{\tau} : \boldsymbol{U} \to u^*\boldsymbol{U}$ and hence give a section

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_{SW} : \mathbf{B}^\circ \to \mathcal{F}^1\mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \mathfrak{t})^W, \tag{4.25}$$

which we often call Seiberg-Witten differential as well. We can now strengthen Corollary 4.30.

Corollary 4.32. *The section $\boldsymbol{\lambda} \in H^0(\mathbf{B}^\circ, \mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \mathfrak{t})^W)$ is an abstract Seiberg-Witten differential. It defines a Lagrangian structure on $\mathcal{J}(\mathsf{V}_{ad}^H) \to \mathbf{B}^\circ$ such that it becomes isomorphic as an integrable system to the Hitchin system $\boldsymbol{h}_1^\circ : \mathbf{Higgs}_1^\circ(\Sigma, G_{ad}) \to \mathbf{B}^\circ$ over $\mathbf{B}^\circ$.*

Proof. It is proven in Proposition 8.2. of [HHP10] that $\boldsymbol{\lambda}$ is an abstract Seiberg-Witten differential, i.e.

$$\phi_{\boldsymbol{\lambda}} : T\mathbf{B}^\circ \to \mathcal{F}^1\mathsf{V}_{ad}^H, \quad X \mapsto \nabla_X \boldsymbol{\lambda},$$

is an isomorphism. Hence $\mathcal{J}(\mathsf{V}_{ad}) \to \mathbf{B}^\circ$ carries a Lagrangian structure $\omega_{\boldsymbol{\lambda}}$ by Proposition 2.36 where we use the natural polarization on $\mathsf{V}_{ad} = \mathsf{V}_{ad}^H$. By construction of $\omega_{\boldsymbol{\lambda}}$ (see the proof of Proposition 2.36), $\phi_{\boldsymbol{\lambda}}$ induces a symplectomorphism

$$(T^*\mathbf{B}^\circ/\Lambda, \hat{\eta}) \cong (\mathcal{J}(\mathsf{V}_{ad}), \omega_{\boldsymbol{\lambda}}).$$

Here $\Lambda \subset T^*\mathbf{B}^\circ$ is the corresponding bundle of lattices (cf. 2.3), not to be confused with the cocharacter lattice $\boldsymbol{\Lambda}_G$ of G. Any choice of a Lagrangian section $s : \mathbf{B}^\circ \to \mathbf{Higgs}_1^\circ(\Sigma, G)$, say a Hitchin section, in turn yields a symplectomorphism $T^*\mathbf{B}^\circ/\Lambda \cong \mathbf{Higgs}_1^\circ(\Sigma, G)$ over $\mathbf{B}^\circ$ by Proposition 2.9. Altogether this yields the claim. □

Before turning to the simply-connected case, let us outline another but equivalent way to endow $P_b = P_b^\circ$ with the structure of an abelian variety. This is probably well-known to experts, but since this point of view will be important later on, we explicitly mention it here. The point is that there is another description of the Hodge filtration on

$$H^1(\Sigma, (p_*\boldsymbol{\Lambda})^W)) \otimes \mathbb{C}$$

without appealing to the previous arguments. One way to do so is provided by the following result due to Zucker:

Theorem 4.33 ([Zuc79]). *Let Σ be a compact Riemann surface and $j : \Sigma^\circ = \Sigma - S \hookrightarrow \Sigma$ the complement of a finite subset $S \subset \Sigma$. Further let V be a polarized $\mathbb{Z}$-VHS of weight m over Σ°. Then the sheaf cohomology groups $H^k(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ ($k = 0, 1, 2$) carry a polarized $\mathbb{Z}$-Hodge structure of weight $k+m$ which is functorial with respect to pullbacks and morphisms of VHS. Moreover, these Hodge structures are compatible with Tate twists and the Leray spectral sequence for projective morphisms $f : X \to \Sigma$.*

We emphasize that even though Zucker works with polarized $\mathbb{R}$-VHS throughout [Zuc79], his polarized Hodge structures can be refined to $\mathbb{Z}$ as long as the VHS carries a $\mathbb{Z}$-structure, cf. Section 2 in [Zuc79]. His theory also works for Σ° directly. More precisely, if V is a VHS, then the cohomology groups $H^k_{(c)}(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}}$ carry a functorial mixed Hodge structure. They are compatible in the sense that the natural map $H^k_c(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}} \to H^k(\Sigma^\circ, \mathsf{V})_{\mathrm{tf}}$ is a morphism of MHS. In particular, the above Hodge structure on

$$H^1(\Sigma, j_*\mathsf{V}) = \mathrm{im}[H^1_c(\Sigma^\circ, \mathsf{V}) \to H^1(\Sigma^\circ, \mathsf{V})] \tag{4.26}$$

(see [Loo97]) coincides with the induced one. Our next application of Zucker's theorem is precisely our case of interest.

Lemma 4.34. *Let $j : \Sigma^\circ = \Sigma - S \to \Sigma$ be as before and V a polarized $\mathbb{Z}$-VHS of weight $m = 2k$ and Tate type over Σ°. Then there exists a commutative diagram*

$$\begin{tikzcd} \hat{S} \arrow[r, hook] \arrow[d] & \hat{\Sigma} \arrow[d, "f"] & \hat{\Sigma}^\circ \arrow[l, hook', "\hat{j}"'] \arrow[d, "f^\circ"] \\ S \arrow[r, hook] & \Sigma & \Sigma^\circ \arrow[l, hook', "j"'] \end{tikzcd} \tag{4.27}$$

such that f° is a Galois covering and f is branched. Zucker's Hodge structure on $H^1(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ is isogenous to[10] *$H^1(\hat{\Sigma}^\circ, \hat{j}_*\mathsf{V}_0)^W = H^1(\hat{\Sigma}, \mathsf{V}_0)^W$ where W is the covering group of f° and V_0 the typical stalk of $\mathsf{V}_\mathbb{Z}$. In particular, $H^1(\Sigma, j_*\mathsf{V})_{\mathrm{tf}}$ only has types $(k+1, k)$ and $(k, k+1)$.*

Proof. Up to a Tate twist, the V only consists of a local system $\mathsf{V}_\mathbb{Z}$ of positive definite lattices so that we only write $\mathsf{V} = \mathsf{V}_\mathbb{Z}$. This implies that its monodromy group W has to be finite and we obtain an unbranched Galois covering $f^\circ : \hat{\Sigma}^\circ \to \Sigma^\circ$ with covering group W. Since f° is locally given by $z \mapsto z^k$, it follows that f° can be completed to a branched covering $f : \hat{\Sigma} \to \Sigma$. This yields the diagram (4.27) as claimed. By construction we have $(f^\circ)^*\mathsf{V} \cong \mathsf{V}_0$, i.e. $\mathsf{V} \cong (f^\circ_*\mathsf{V}_0)^W$ by (1.44). Now the inclusion $i : (f^\circ_*\mathsf{V}_0)^W \hookrightarrow f^\circ_*\mathsf{V}_0$ is obviously a morphism of VHS. Note that this makes sense because $f^\circ_*\mathsf{V}_0$ is again a polarized $\mathbb{Z}$-VHS of Tate type. Moreover, the natural morphism $\phi : H^1(\Sigma^\circ, f^\circ_*\mathsf{V}_0)_{\mathrm{tf}} \to H^1(\hat{\Sigma}^\circ, \mathsf{V}_0)_{\mathrm{tf}}$, induced by the Leray spectral sequence, is a morphism of Hodge structures (cf. Section 15 in [Zuc79]). As f° is finite, ϕ is an isomorphism. By the W-equivariance of f°, these morphisms fit into the commutative diagram

$$\begin{tikzcd} H^1(\Sigma^\circ, f^\circ_*\mathsf{V}_0) \arrow[r, "\phi"] & H^1(\hat{\Sigma}^\circ, \mathsf{V}_0) \\ H^1(\Sigma^\circ, \mathsf{V}) \arrow[u, "i"] \arrow[r, "\phi^W"] & H^1(\hat{\Sigma}^\circ, \mathsf{V}_0)^W \arrow[u, hook] \\ H^1_c(\Sigma^\circ, \mathsf{V}) \arrow[u] \arrow[r, "\psi^W"] & H^1_c(\hat{\Sigma}^\circ, \mathsf{V}_0)^W . \arrow[u] \end{tikzcd} \qquad \mathsf{V} = (f^\circ_*\mathsf{V}_0)^W \tag{4.28}$$

[10] Note that these cohomology groups are torsion-free.

Here ψ^W is induced from the natural morphism $\psi : H^1_c(\Sigma^\circ, \mathsf{V}) \to H^1_c(\hat{\Sigma}^\circ, \mathsf{V}_0)^W$. Arguing as above, we see that it is compatible with Hodge structures. Thus (4.28) is a commutative diagram of Hodge structures. Further ψ^W and ϕ^W are isomorphisms over $\mathbb{Q}$ because we can then split off $(f^\circ_* \mathsf{V})^W$. But the lower square in (4.28) factorizes over (cf. (4.26))

$$H^1(\Sigma, j_*\mathsf{V}) \to H^1(\hat{\Sigma}^\circ, j_*\mathsf{V}_0)^W = H^1(\hat{\Sigma}, \mathsf{V}_0)^W,$$

which thus has to be an isogeny as well. □

This lemma fits precisely into the previous context. Indeed, the sheaf $(p_*\mathbf{\Lambda})^W$ is a polarizable $\mathbb{Z}$-VHS V away from the branch locus $Br_b \subset \Sigma$ of weight 0 and Tate type. On $\Sigma^\circ = \Sigma - Br_b$ we obviously have $\mathsf{V} = (p^\circ_*\mathbf{\Lambda})^W$. Moreover, the adjunction morphism

$$(p_*\mathbf{\Lambda})^W \longrightarrow j_*j^*(p_*\mathbf{\Lambda})^W = j_*\mathsf{V}$$

is an isomorphism (cf. proof of Corollary 1.46).

Proposition 4.35. *The $\mathbb{Z}$-Hodge structure of weight 1 corresponding to P_b (whose underlying real torus is $H^1(\Sigma, (p_*\mathbf{\Lambda})^W) \otimes S^1$) coincides with Zucker's $\mathbb{Z}$-Hodge structure on $H^1(\Sigma, (p_*\mathbf{\Lambda})^W)$. Both are isogenous to $H^1(\tilde{\Sigma}, \mathbf{\Lambda})^W$.*

Proof. Recall that $H^1(\Sigma, (p_*\mathbf{\Lambda})^W)$ is torsion-free. By construction, we further know that $\hat{\Sigma}$ of Lemma 4.34 coincides with the cameral curve $\tilde{\Sigma}$. Hence the claim follows from that lemma together with the previous remarks. □

Remark 4.36. Even though it is somewhat obvious, we still point out that one has to be careful with the notation $H^1(\tilde{\Sigma}, \mathbf{\Lambda})^W$. The possible confusion stems from the fact that the W-action on $H^1(\tilde{\Sigma}, \mathbf{\Lambda})$ depends on the W-structure on the constant sheaves $\mathcal{F} = \mathbf{\Lambda}_{\tilde{\Sigma}}$ or $\mathfrak{t}_{\tilde{\Sigma}}$. In the case at hand, there are two natural W-structures. The trivial one is given by the natural isomorphisms $w^*\mathcal{F} \cong \mathcal{F}$. The other W-structure is the 'diagonal' W-structure,

$$\varphi_w : w^*\mathcal{F} \to \mathcal{F}, \quad f \mapsto w \cdot f \circ w^{-1},$$

where we consider sections f of $\mathcal{F}$ as functions. In general these structures are very different and give other W-actions on the cohomology groups $H^k(\tilde{\Sigma}, \mathcal{F})$. To see some of the differences, consider the example $k = 1$, $\mathcal{F} = \mathfrak{t}_{\tilde{\Sigma}}$ and denote by $H^1(\tilde{\Sigma}, \mathcal{F})^{W_{triv}}$ and $H^1(\tilde{\Sigma}, \mathcal{F})^W$ the W-invariants for the trivial and the diagonal W-structure respectively. Then it follows that

$$\dim H^1(\tilde{\Sigma}, \mathfrak{t})^{W_{triv}} = \dim H^1(\tilde{\Sigma}, \mathfrak{t}) = 2g \cdot \dim \mathfrak{t},$$
$$\dim H^1(\tilde{\Sigma}, \mathfrak{t})^W = 2(g-1)\dim G,$$

where $g = g(\Sigma)$ is the genus of Σ and where we used $\dim Z(G) = 0$. It is not hard to see that the latter dimension is always larger than the former. For example if $G = SL(m, \mathbb{C})$, $g = 2$, then

$$\dim H^1(\tilde{\Sigma}, \mathfrak{t})^W = 2(m^2 - 1) > 2(m-1) = \dim H^1(\tilde{\Sigma}, \mathfrak{t})^{W_{triv}}$$

(of course $m \geq 2$).

We now briefly summarize the simply-connected case, i.e. $G = G_{sc}$ of type Δ. As before we define by $\mathbf{\Lambda}_{sc} := \mathbf{\Lambda}(G_{sc})$ the cocharacter lattice of G_{sc}. Since the Hitchin base only sees the Lie algebra, it follows that $\mathbf{B}(\Sigma, G_{sc}) = \mathbf{B}(\Sigma, G_{ad})$ naturally. Then the analogue of Proposition 4.28 is

Proposition 4.37. *Let G_{sc} be a simple simply-connected complex Lie group of type Δ and $\mathbf{B}$ the corresponding Hitchin base.*

i) ([DP12]) If $b \in \mathbf{B}^\circ$, then $\mathcal{T}(b) = \mathcal{T}^\circ(b)$ and

$$\mathrm{cochar}(P_b) \cong H^1(\Sigma, (p_*\mathbf{\Lambda}_{sc})^W)_{\mathrm{tf}}.$$

ii) The VHS of weight -1 corresponding to the neutral component $\boldsymbol{h}^\circ_{1,sc} : \mathbf{Higgs}^\circ_1(\Sigma, G_{sc}) \to \mathbf{B}^\circ$ is given by

$$\mathsf{V}^H_{sc} = ((R^1\boldsymbol{p}_{2,*}\boldsymbol{p}^W_{1,*}\mathbf{\Lambda}_{sc})_{\mathrm{tf}}, \mathcal{F}^\bullet(R^1\boldsymbol{p}^W_* \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ}))_{|\mathbf{B}^\circ} \cong \mathsf{V}^*_H(\boldsymbol{h}^\circ_{1,sc})(-1)$$

The analogue of Proposition 4.35 is still valid by using Lemma 4.34.

Remark 4.38.

a) Even though the adjoint and the simply-connected case are very similar in nature, the cohomology groups $H^1(\Sigma, \mathcal{T})$ behave differently. In fact it can be shown (cf. the proof of Lemma 4.2 in [DP12]) that

$$H^1(\Sigma, (p_*\mathbf{\Lambda}(G))^W)_{\mathrm{tor}} \cong \begin{cases} 0, & G = Sp(2r, \mathbb{C}), \\ Z(G), & \text{else.} \end{cases}$$

Hence this is always zero for $G = G_{ad}$ but is non-vanishing e.g. for $G = SL(r, \mathbb{C})$, $r \geq 2$.

b) Let ${}^LG_{ad}$ be the Langlands dual group of the simple adjoint complex Lie group G_{ad}, so that $\mathbf{\Lambda}({}^LG_{ad}) = \mathbf{\Lambda}^\vee_{ad}$ by definition. Moreover, ${}^LG_{ad}$ is a simple simply-connected group. Let $\boldsymbol{h}^\circ_1 : \mathbf{Higgs}^\circ_1(\Sigma, G_{ad}) \to \mathbf{B}^\circ$ and ${}^L\boldsymbol{h}^\circ_1 : \mathbf{Higgs}^\circ_1(\Sigma, {}^LG_{ad})$ be the corresponding neutral component of the Hitchin system. Applying Proposition 4.37, we see that $\mathsf{V}(\boldsymbol{h}^\circ_1)$ and $\mathsf{V}({}^L\boldsymbol{h}^\circ_1)$ are (up to a Tate twist) dual VHS. This is a very simple instance of Langlands duality for Hitchin systems ([DP12]).
Observe that if G_{ad} is of type ADE, then ${}^LG_{ad}$ is just the simple simply-connected group of the same type as G_{ad}.

Chapter 5

BCFG-Hitchin systems and Calabi-Yau threefolds

This chapter contains the main results of our study, in particular a proof of Theorem 0.1 from the introduction (Corollary 5.56). We already pointed out that the Calabi-Yau integrable system associated with a complete family of *compact* CY3s (*compact CY integrable system* for short) cannot be isomorphic to *any* Hitchin system. There are at least three reasons for this:

a) Most importantly, we have seen that the intermediate Jacobian $J^2(X)$ of a cCY3 X is only a non-degenerate complex torus, but not an abelian variety. But the (generic) fibers of Hitchin systems, i.e. generalized Prym varieties, are abelian varieties.

b) Even if $J^2(X)$ was an abelian variety, it is self-dual which is in general false for Hitchin systems. In fact, *Langlands duality* is an important feature of Hitchin systems,

$$\mathbf{Higgs}^{\circ}(\Sigma, G)^{\vee} \simeq \mathbf{Higgs}^{\circ}(\Sigma, {}^{L}G)$$

over $\mathbf{B}^{\circ}(\Sigma, G) \cong \mathbf{B}^{\circ}(\Sigma, {}^{L}G)$ (see [DP12]). Here $^{\vee}$ stands for taking the dual torus fibration and ${}^{L}G$ is the Langlands dual group.

c) Hitchin systems underlie a special Kähler geometry, but compact CY integrable systems the richer structure of a projective special Kähler geometry. In some sense, this could be remedied though, because it was shown in [HHP10] that the special Kähler geometry of Hitchin systems can be enhanced to projective special Kähler geometry.

Especially b) suggests that one could try to work with *non-compact* CY3s instead of compact ones. Even though this was not the way how Diaconescu, Donagi and Pantev ([DDP07] and the earlier work [DDD$^+$06] on large N duality and geometric transitions) discovered the relation between (non-compact) Calabi-Yau integrable systems and ADE-Hitchin systems, it makes plausible why we work with non-compact CY3s in this chapter. Note however, that there is so far no general theory of non-compact CY integrable systems (but see Remark 5.14). At least for our constructed families, we can construct non-compact CY integrable systems 'by hand'. We emphasize that they can be constructed without relying on Hitchin systems.
The first half of this chapter mainly follows the three steps of the introduction. We begin with the simply-laced case though. This has three reasons: First, it makes the relation between the constructions of [DDP07] and ours clearer. Second, some of the arguments to construct an isomorphism between the corresponding variations of (mixed) Hodge structures (V(M)HS) become

more transparent, when they are first discussed without incorporating graph automorphisms. And third, it is not difficult to see that these arguments also work for homology intermediate Jacobians and simple simply-connected complex Lie groups (Theorem 5.33).

5.1 Calabi-Yau threefolds associated with Δ_h

The aim of this section is to give an amplified outline of the construction of the family $\mathcal{X} \to \mathbf{B}$ of non-compact Calabi-Yau threefolds over the Hitchin base $\mathbf{B}$ of type Δ_h, which will be a connected Dynkin diagram of type ADE throughout this section. The rough idea is to pull back a family $\mathcal{S} \to \boldsymbol{U}$ of surfaces to $\mathbf{B}$ in order to obtain $\mathcal{X} \to \mathbf{B}$. Our outline is amplified in the sense that we apply the results from Section 1.5 to the family $\mathcal{S} \to \boldsymbol{U}$ to obtain additional results.
Furthermore, we will give two constructions of $\mathcal{X}$, the first one is more local in nature and goes back to Szendröi [Sze04]. The other one, which is in fact a special case of the former, is more global and uses a $\mathbb{C}^*$-invariant Slodowy slice. This construction was already suggested in [DDP07].

5.1.1 Local construction

Let $\Gamma \subset SL(2,\mathbb{C})$ be a finite subgroup, which corresponds to an ADE-Dynkin diagram Δ_h under the correspondence of Chapter 1. The basic idea will be to glue the quotient $\mathbb{C}^2/\Gamma$ over the Riemann surface Σ to obtain a Calabi-Yau threefold X_0, which has a (isolated) curve of singularities of type Δ_h and then to deform X_0 in an appropriate way. To do so, take a vector bundle $V \to \Sigma$ of rank 2 whose structure group reduces to $C(\Gamma) = C_{GL(2,\mathbb{C})}(\Gamma)$, the centralizer of Γ in $GL(2,\mathbb{C})$. Equivalently, V is a Γ-equivariant bundle, where Γ acts trivially on Σ. For the construction of Calabi-Yaus, we further require $\wedge^2 V \cong K_\Sigma$.

Lemma 5.1. *Let $\Gamma \subset SL(2,\mathbb{C})$ correspond to the Dynkin diagram Δ_h of type* ADE *and $V \to \Sigma$ be a Γ-equivariant vector bundle. Then there are the following possibilities for V depending on the type of Δ_h:*

A_1	V *unobstructed*
$\mathrm{A}_{\geq 2}$, D_4	$V \cong L_1 \oplus L_2$, $L_1 \otimes L_2 \cong K_\Sigma$
$\mathrm{D}_{\geq 5}$ *and* E_k	$V = L \oplus L$, $L^2 \cong K_\Sigma$.

Proof. This follows immediately from Lemma 1.6. □

Remark 5.2.

a) In the following, we do not consider the somewhat exceptional case A_1 because it has been extensively discussed in [DDD$^+$06]. It is also irrelevant for our later constructions because A_1 does not have non-trivial graph automorphisms.

b) The isomorphism classes of bundles for type A_n ($n \geq 2$), D_4, are given by $\mathrm{Pic}(\Sigma)$ because $L_2 = K_\Sigma \otimes L_1^{-1}$. However, the polystable V are only the ones with $L_1 \in \mathrm{Pic}^{g-1}(\Sigma)$.
In contrast, the moduli for type $\mathrm{D}_{\geq 5}$ and E are given by the spin bundles of Σ, so there is only a discrete moduli. All the corresponding vector bundles are polystable.

So far, we have not specified, whether the bundle V is supposed to be algebraic or holomorphic. By GAGA, we know that if V is holomorphic, it is at least isomorphic to (the analytification of) an algebraic one. Moreover, we have the following:

Lemma 5.3. *Let $H^1_{an}(\Sigma, \mathcal{O}_{an}(C(\Gamma)))$ and $H^1_{Zar}(\Sigma, \mathcal{O}(C(\Gamma)))$ be the first cohomology groups of the sheaf $\mathcal{O}_{an}(C(\Gamma))$ and $\mathcal{O}(C(\Gamma))$ respectively[1]. Then there is an isomorphism of groups*

$$H^1_{an}(\Sigma, \mathcal{O}_{an}(C(\Gamma))) \cong H^1_{Zar}(\Sigma, \mathcal{O}(C(\Gamma)))\,.$$

In particular, the set (actually group) of isomorphisms classes of $C(\Gamma)$-bundles is the same in the analytic and the Zariski topology respectively.

Proof. Since $C(\Gamma) \cong \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$, the isomorphism follows from the classical isomorphism $H^1_{an}(\Sigma, \mathcal{O}^*_{an}) \cong H^1_{Zar}(\Sigma, \mathcal{O}^*)$ (noting that Σ is projective). The second claim follows from the fact that $C(\Gamma)$ is connected and reductive. In particular, algebraic $C(\Gamma)$-bundles are Zariski-locally trivial (cf. introduction to Chapter 4) and therefore classified by $H^1_{Zar}(\Sigma, \mathcal{O}(C(\Gamma))$. $\square$

Remark 5.4. In the following, we will therefore drop the subscripts and only write $H^1(\Sigma, \mathcal{O}(C(\Gamma)))$. However, we emphasize that it is important to know that the gluing procedures below work algebraically, i.e. yield complex algebraic varieties and morphisms. The main reason for this is that (mixed) Hodge modules have (full) functoriality only in the algebraic setting. But this is crucial for relating Hitchin systems with Calabi-Yau integrable systems (Step III) from the introduction, cf. Section 5.2.2).

Since all the constructions below only depend on the class $\alpha_V \in H^1(\Sigma, \mathcal{O}(C(\Gamma)))$ corresponding to the $C(\Gamma)$-bundle V, i.e. only its isomorphism class, we can further assume that V is in fact algebraically defined.

Each Γ-equivariant bundle V has a Γ-action on its total space, such that each fiber of

$$\pi_0 : X_0 := \mathrm{tot}(V)/\Gamma \to \Sigma$$

is isomorphic to $\mathbb{C}^2/\Gamma$. In particular, X_0 has a curve of singularities of type Δ_h which corresponds to the image of the zero section under $\mathrm{tot}(V) \to X_0$.

Proposition 5.5. *Let $\alpha = \alpha_V \in H^1(\Sigma, \mathcal{O}(C(\Gamma)))$ be determined by Γ-equivariant vector bundle $V \to \Sigma$ with $\det V = K_\Sigma$. There are flat families $\boldsymbol{\sigma}_\alpha : \mathcal{S}_\alpha \to \boldsymbol{U}$ and $\boldsymbol{\theta}_\alpha : \tilde{\mathcal{S}}_\alpha \to \tilde{\boldsymbol{U}}$ of surfaces with the following properties:*

i) ([Sze04],[DDP07]) The restriction of $\boldsymbol{\sigma}_\alpha$ to the zero section of $\boldsymbol{U}$ is isomorphic to X_0. Moreover, the restriction to any fiber $\boldsymbol{U}_x \cong \mathfrak{t}/W$, $x \in \Sigma$, is isomorphic to $S \to \mathfrak{t}/W$, the semi-universal deformation of $\mathbb{C}^2/\Gamma$.

ii) ([Sze04],[DDP07]) After base change along $\boldsymbol{q} : \tilde{\boldsymbol{U}} \to \boldsymbol{U}$ the family $\tilde{\mathcal{S}}_\alpha$ gives a simultaneous resolution of $\boldsymbol{q}^*\mathcal{S}_\alpha$:

$$\begin{tikzcd}
\tilde{\mathcal{S}}_\alpha \arrow[r] \arrow[d, "\tilde{\boldsymbol{\sigma}}_\alpha"'] \arrow[rr, bend left, "\boldsymbol{\psi}_\alpha"] & \boldsymbol{q}^*\mathcal{S}_\alpha \arrow[r] \arrow[d] & \mathcal{S}_\alpha \arrow[d, "\boldsymbol{\sigma}_\alpha"] \\
\tilde{\boldsymbol{U}} \arrow[r, equal] & \tilde{\boldsymbol{U}} \arrow[r] & \boldsymbol{U}.
\end{tikzcd}$$

iii) There is a non-trivial section $\hat{\boldsymbol{\omega}}_\alpha \in H^0(\tilde{\mathcal{S}}_\alpha, \Omega^2_{\tilde{\boldsymbol{\sigma}}} \otimes (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma)$ where $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$ is the sheaf of relative differential forms of degree 2 for $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}_\alpha$. It induces a (fiberwise) period map

$$\boldsymbol{\eta} : \tilde{\boldsymbol{U}} \to \tilde{u}^*\tilde{\boldsymbol{U}},$$

which coincides with the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$.

[1]We denote by $\mathcal{O}(C(\Gamma))$ and $\mathcal{O}_{an}(C(\Gamma))$ the sheaf of regular and holomorphic functions with values in $C(\Gamma)$ respectively (i.e. we consider the projective variety Σ and its analytification).

iv) *Similarly, there exists a section* $\hat{\boldsymbol{\nu}}_\alpha \in H^0(\mathcal{S}_\alpha, K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma)$, *where* $K_{\boldsymbol{\sigma}}$ *is the relative canonical sheaf of* $\boldsymbol{\sigma} = \boldsymbol{\sigma}_\alpha$, *such that*

$$\boldsymbol{\psi}_\alpha^* \hat{\boldsymbol{\nu}}_\alpha = \hat{\boldsymbol{\omega}}_\alpha$$

under the natural isomorphism $\boldsymbol{\psi}_\alpha^* K_{\boldsymbol{\sigma}} \cong K_{\tilde{\boldsymbol{\sigma}}} = \Omega^2_{\tilde{\boldsymbol{\sigma}}}$.

Proof. Let $\alpha = \alpha_V \in H^1(\Sigma, \mathcal{O}(C(\Gamma)))$ be the cohomology class corresponding to V. Further let $(D_i)_{i\in I}$ be an open covering of Σ that trivializes V and $(\alpha_{ij} : D_{ij} \to C(\Gamma))_{ij}$ be a cocycle representing α for $D_{ij} := D_i \cap D_j$. The group $C(\Gamma)$ acts on the semi-universal deformation space S of $\mathbb{C}^2/\Gamma$ so that we can glue $U_{ij} \times S$ to obtain a complex variety $\mathcal{S}_\alpha$. In Lemma 1.48, we have seen that the flat morphism $\sigma : S \to \mathfrak{t}/W$ is equivariant with respect to the $C(\Gamma)$-action on S and the natural $\mathbb{C}^*$-action on $\mathfrak{t}/W$ induced by taking determinant. But as $\det V \cong K_\Sigma$, i.e. $\det(\alpha_{ij})$ is a cocycle for K_Σ, we conclude that we can glue $\sigma : S \to \mathfrak{t}/W$ to obtain a morphism

$$\boldsymbol{\sigma}_\alpha : \mathcal{S}_\alpha \to \boldsymbol{U}.$$

Now the statements of i) are clear by noticing that $S_0 \cong \mathbb{C}^2/\Gamma$ and the construction of X_0.

Analogously, we can glue $D_{ij} \times \tilde{S}$ via the cocycle α_{ij} to obtain a complex variety $\tilde{\mathcal{S}}_\alpha$ and a morphism $\tilde{\boldsymbol{\sigma}}_\alpha : \tilde{\mathcal{S}}_\alpha \to \tilde{\boldsymbol{U}}$. Since the simultaneous resolution $\tilde{S} \to q^*S$ is $\mathbb{C}^*$-equivariant (cf. Remark 1.53), it glues to a morphism $\tilde{\mathcal{S}}_\alpha \to \boldsymbol{q}^*\mathcal{S}_\alpha$ yielding a simultaneous resolution. This shows ii).

To construct the section of iii) we need the gluing data for the sheaf $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$. Here and in the rest of this proof, we drop the subscript α from the notation. By construction, the following commutative square exists

$$\begin{tikzcd}
 & \tilde{\mathcal{S}}_{ij} \arrow[dl, "\tilde{\boldsymbol{\sigma}}_{ij}"'] \arrow[r, "\psi_i"] \arrow[dd, equal] & D_{ij} \times \tilde{S} \arrow[r, "\mathrm{id}\times\tilde{\sigma}"] & D_{ij} \times \mathfrak{t} \\
\tilde{\boldsymbol{U}} & & & \\
 & \tilde{\mathcal{S}}_{ij} \arrow[ul, "\tilde{\boldsymbol{\sigma}}_{ij}"] \arrow[r, "\psi_j"] & D_{ij} \times \tilde{S} \arrow[uu, "g_{ij}"'] \arrow[r, "\mathrm{id}\times\tilde{\sigma}"] & D_{ij} \times \mathfrak{t} \arrow[uu, "h_{ij}"']
\end{tikzcd}$$

for $\tilde{\mathcal{S}}_{ij} := \tilde{\mathcal{S}}_{|D_{ij}}$ and where $\psi_i : \tilde{\mathcal{S}}_{D_i} \to D_i \times \tilde{S}$ are the trivializations coming from the construction[2]. Observe that

$$\begin{aligned} g_{ij}(x,s) &= (x, \alpha_{ij}(x) \cdot s) = (x, \mu(\alpha_{ij}(x), s)), \\ h_{ij}(x,t) &= (x, \alpha_{ij}(x) \cdot t) = (x, \det \alpha_{ij}(x) t), \end{aligned}$$

for $(x,s) \in D_{ij} \times \tilde{S}$ and the action map $\mu : C(\Gamma) \times \tilde{S} \to \tilde{S}$.
On each $D_i \times \tilde{S}$ we have the sheaves $\mathcal{E}_i := \Omega^2_{\mathrm{id}\times\tilde{\sigma}} \cong \mathrm{pr}_{2,i}^* \Omega^2_{\tilde{\sigma}}$ together with the sections $\mathrm{pr}_{2,i}^* \hat{\omega}$ where $\hat{\omega} \in \Gamma(\tilde{S}, \Omega^2_{\tilde{\sigma}})$ is the natural relative symplectic form. Clearly, $\mathcal{E}_i$ and $\mathcal{E}_j$ are canonically isomorphic over D_{ij}. Now $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$ is glued from the[3] $\psi_i^*\mathcal{E}_i$ on D_{ij} via the isomorphisms

$$\varphi_{ij} := \psi_i^* dg_{ji}^t : \psi_j^* \mathcal{E}_j = \psi_i^* g_{ji}^* \mathcal{E}_j \longrightarrow \psi_i^* \mathcal{E}_i$$

[2]Here we assume that $\tilde{\boldsymbol{U}}$ also trivializes over D_i.
[3]Note that $\psi_i^*\mathcal{E}_i \cong \Omega^2_{\tilde{\boldsymbol{\sigma}}|\tilde{\mathcal{S}}_i}$ which one can think of as a local trivialization of $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$.

over D_{ij}. Here we denote by $dg^t_{ji} : g^*_{ij}\mathcal{E}_j \to \mathcal{E}_i = \mathcal{E}_j$ the natural morphism (over D_{ij}). Observe that we have $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ and (φ_{ij}) is the gluing (or descent) datum for $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$. Indeed, we can write this composition as

$$\begin{tikzcd}
\psi_k^*\mathcal{E}_k \arrow[rr, "\varphi_{kj}"] \arrow[d, "\cong"] & & \psi_j^*\mathcal{E}_j \arrow[rr, "\varphi_{ji}"] \arrow[d, "\cong"] & & \psi_i^*\mathcal{E}_i \arrow[d, "\cong"] \\
\psi_i^*(g^*_{ji})g^*_{kj}\mathcal{E}_j \arrow[rr, "\psi_i^* g^*_{ji} dg^t_{kj}"] & & \psi_i^* g^*_{ji}\mathcal{E}_j \arrow[rr, "\psi_i^* dg^t_{ji}"] & & \psi_i^*\mathcal{E}_j.
\end{tikzcd}$$

The lower line is φ_{ik} by the chain rule, showing the cocycle condition for (φ_{ij}).

Claim. Define the local sections

$$\hat{\omega}_i := \psi_i^* \mathrm{pr}^*_{2,i}\hat{\omega} \in \Gamma(\mathcal{S}_i, \psi_i^*\mathcal{E}_i)$$

where $\hat{\omega}$ is as in Remark 1.105 c) of Section 1.5. On the overlaps $\tilde{\mathcal{S}}_{ij}$, they transform as follows

$$\varphi_{ij}(\hat{\omega}_j) = ((\mathrm{pr}_{1,i} \circ \psi_i)^* \det \alpha_{ji})\, \hat{\omega}_i. \tag{5.1}$$

Before we prove this claim, let us see how it yields the desired section. Observe that $(\mathrm{pr}_{1,i} \circ \psi_i)^* \det \alpha_{ji}$ is a cocycle for $(\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma^{-1}$. Hence in order to obtain a well-defined global section on $\tilde{\mathcal{S}}$, we have to tensor with $(\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma$. More precisely, let $\zeta_i \in \Gamma(D_i, K_\Sigma)$ be the local frames of $K_\Sigma = \wedge^2 V$ induced from the trivializations of V over D_i so that $\zeta_i = (\det g_{ij})\ \zeta_j$ on D_{ij}. Letting $\hat{\zeta}_i := \psi_i^* \mathrm{pr}^*_{1,i}\zeta_i$, we see that the local sections

$$\hat{\omega}_i \otimes \hat{\zeta}_i \in \Gamma(\tilde{\mathcal{S}}_i, \Omega^2_{\tilde{\boldsymbol{\sigma}}} \otimes (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma)$$

glue to give a global section $\hat{\boldsymbol{\omega}} \in \Gamma(\tilde{\mathcal{S}}, \Omega^2_{\tilde{\boldsymbol{\sigma}}} \otimes (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma)$.

We still have to give a proof of (5.1), which might seem obvious but is in fact a bit subtle. To simplify notation, we drop the subscript ij and only write $g : D \times \tilde{S} \to D \times \tilde{S}$ etc. Then the second component of $dg : TD \oplus T\tilde{S} \to TD \oplus T\tilde{S}$ at $(x, s) \in D \times \tilde{S}$ is given by

$$d\mu_{\alpha(x)\cdot s}(d\alpha_x(v), w) = d\mu_{\alpha(x)\cdot s}(d\alpha_x(v), 0) + d\mu_{\alpha(x)\cdot s}(0, w). \tag{5.2}$$

Note that $d\mu_{\alpha(x)\cdot s}(0, w) = d(\mu_{\alpha(x)})_s(w)$. Here $\mu : \mathbb{C}^* \times \tilde{S} \to \tilde{S}$ is the $\mathbb{C}^*$-action and we denote $\mu_{\alpha(x)} = \mu(\alpha(x), -)$.
Two sections of $\Omega^2_{\mathrm{id}\times\tilde{\sigma}}$ coincide iff they take the same values on vertical tangent vectors. Let $p \in \tilde{\mathcal{S}}_{ij}$ and $\psi_i(p) = (x, s) \in D_{ij} \times \tilde{S}$. Then we clearly have

$$\ker d_{(x,s)}(\mathrm{id} \times \tilde{\sigma}) = 0 \oplus \ker d_s\tilde{\sigma} \subset T_x D_{ij} \oplus T_s\tilde{S}.$$

In particular, the first summand in (5.2) plays no role for our discussion. For $w_k \in \ker d_s\tilde{\sigma}$ $(k = 1, 2)$ one computes

$$\begin{aligned}
&\varphi_{ij}(\hat{\omega}_j)_p\Big((0, w_1), (0, w_2)\Big) \\
&= \mathrm{pr}^*_{2,j}\hat{\omega}_{g_{ji}(x,s)} \circ dg_{ji,(x,s)}\Big((0, w_1), (0, w_2)\Big) && (\psi_j \circ \psi_i^{-1} = g_{ji}) \\
&= \det \alpha_{ji}(x)\, \hat{\omega}_s(w_1, w_2) && (\mathbb{C}^*\text{-equivariance and } (5.2)) \\
&= (\mathrm{pr}_{1,i} \circ \psi_i)^* \det \alpha_{ji}(p)\, (\mathrm{pr}^*_{2,i}\hat{\omega})_{\psi(p)}\Big((0, w_1), (0, w_2)\Big) && (\psi_i(p) = (x, s)) \\
&= (\mathrm{pr}_{1,i} \circ \psi_i)^* \det \alpha_{ji}(p)\, (\hat{\omega}_i)_p\Big((0, w_1), (0, w_2)\Big).
\end{aligned}$$

Here we have used the $\mathbb{C}^*$-equivariance of the relative form $\hat{\omega}$ (cf. Lemma 1.100 and Lemma 1.48). This shows (5.1).
It remains to give the period map obtained from $\hat{\boldsymbol{\omega}}$. The construction of $\hat{\boldsymbol{\omega}}$ shows that it yields a morphism

$$(x,t) \mapsto \nu(t) \otimes ((x,t), \zeta_i(x))$$

in each trivialization $\tilde{\boldsymbol{U}}_{D_i} \cong D_i \times \mathfrak{t}$. By the $\mathbb{C}^*$-equivariance of the (local) period map $P_{\tilde{S}}$, it follows that these morphisms glue to give a morphism

$$\boldsymbol{\eta} : \tilde{\boldsymbol{U}} \to H^2(\tilde{S}_0, \mathbb{C}) \otimes \tilde{u}^* K_\Sigma.$$

After the identification $\mathfrak{t} \cong H^2(\tilde{S}_0, \mathbb{C})$ via $P_{\tilde{S}}$ (which is compatible with all the group actions, see Corollary 1.98 in Section 1.5), $\boldsymbol{\eta}$ is just the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$.
The existence of a global section $\hat{\boldsymbol{\nu}} \in H^0(\mathcal{S}, K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma)$ works similarly as in iii) by gluing the section $\hat{\nu} \in \Gamma(S, K_\sigma)$ from Section 1.5.2. More precisely, there exists a section $\hat{\boldsymbol{\nu}}^{reg} \in \Gamma(\mathcal{S}^{reg}, K_{\boldsymbol{\sigma}})$, which can be constructed as $\hat{\boldsymbol{\omega}}$. Here $\mathcal{S}^{reg} \subset \mathcal{S}$ is the locus which is glued from $S^{reg} \subset S$. This works because S^{reg} is $\mathbb{C}^*$-invariant. Using a codimension argument as in the proof of Corollary 1.104, we see that it uniquely extends to a section $\hat{\boldsymbol{\nu}} \in \Gamma(\mathcal{S}, K_{\boldsymbol{\sigma}})$. It satisfies $\boldsymbol{\psi}^*\hat{\boldsymbol{\nu}} = \hat{\boldsymbol{\omega}}$ under the isomorphism $\boldsymbol{\psi}^* K_{\boldsymbol{\sigma}} \cong K_{\tilde{\boldsymbol{\sigma}}}$ by construction, since this holds for the corresponding local sections. ☐

Corollary 5.6. *Let $\tilde{\boldsymbol{\sigma}}_\alpha = \tilde{\boldsymbol{\sigma}} : \tilde{\mathcal{S}}_\alpha \to \tilde{\boldsymbol{U}}$ and $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma} : \mathcal{S}_\alpha \to \boldsymbol{U}$ be the projections as in Proposition 5.5. Then the sheaves $\Omega^2_{\tilde{\boldsymbol{\sigma}}}$ and $K_{\boldsymbol{\sigma}}$ satisfy (dropping α from the notation)*

$$\Omega^2_{\tilde{\boldsymbol{\sigma}}} \cong (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma^{-1}, \quad K_{\boldsymbol{\sigma}} \cong (u \circ \boldsymbol{\sigma})^* K_\Sigma^{-1}$$

where $\tilde{u} : \tilde{\boldsymbol{U}} \to \Sigma$ and $u : \boldsymbol{U} \to \Sigma$ are the natural projections.

Proof. These isomorphisms follow from the fact that $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\nu}}$ are nowhere vanishing sections (cf. Corollary 1.104) of the *line bundles* $\Omega^2_{\tilde{\boldsymbol{\sigma}}} \otimes (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma$ and $K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma$. ☐

We can now construct a family $\mathcal{X}_\alpha \to \mathbf{B} = H^0(\Sigma, \boldsymbol{U})$ over the Hitchin base $\mathbf{B}$ via base change with the evaluation map $ev : \Sigma \times \mathbf{B} \to \boldsymbol{U}$:

$$\begin{tikzcd} \mathcal{X}_\alpha \arrow[r] \arrow[d, "\boldsymbol{\pi}_{1,\alpha}"] \arrow[dd, bend right=60, "\boldsymbol{\pi}_\alpha"'] & \mathcal{S}_\alpha \arrow[d, "\boldsymbol{\sigma}_\alpha"] \\ \Sigma \times \mathbf{B} \arrow[r, "ev"] \arrow[d, "\boldsymbol{\pi}_2 = pr"] & \boldsymbol{U} \\ \mathbf{B}. & \end{tikzcd} \tag{5.3}$$

As for the universal cameral curve we first take the fiber product in the algebraic category but see Footnote 8 on page 110.

Proposition 5.7. *Let $V \to \Sigma$ be a Γ-equivariant vector bundle with* $\det(V) \cong K_\Sigma$ *and $\alpha = \alpha_V$ the corresponding class. Further let $\mathcal{X} = \mathcal{X}_\alpha \to \mathbf{B}$ the family of threefolds constructed as above. Then each member X_b, $b \in \mathbf{B}$, of this family is a quasi-projective Gorenstein threefold with trivial canonical class. If $b \in \mathbf{B}^\circ$, then X_b is smooth.*

Proof. Fix $b \in \mathbf{B}$ and consider $X_b = \boldsymbol{\pi}_\alpha^{-1}(b)$ together with its projection $\pi_b : X_b \to \Sigma$ and the inclusion $j_b : X_b \to \mathcal{S}_\alpha$ induced from base change.

Quasi-Projective: The vector bundle $\boldsymbol{U}$ over the projective variety Σ is quasi-projective. Similarly, $\mathcal{S}_\alpha$ is quasi-projective because it can be seen as an affine bundle[4] over Σ. Therefore $\mathcal{X}_\alpha$ is

[4]In fact it has a section which maps to the unique $\mathbb{C}^*$-fixed point in each fiber S.

quasi-projective as a fiber product of quasi-projective varieties. It follows that each member X_b of $\mathcal{X} = \mathcal{X}_\alpha$ is quasi-projective.
Alternatively, this fact can be seen by showing the stronger statement that $\boldsymbol{\pi}_\alpha : \mathcal{X}_\alpha \to \mathbf{B}$ is a quasi-projective morphism: The projections $u : \boldsymbol{U} \to \Sigma$ and $\tau_\alpha : \mathcal{S}_\alpha \to \Sigma$ are quasi-projective since they are (affine) bundles over the projective variety Σ. But $\boldsymbol{\sigma}_\alpha : \mathcal{S}_\alpha \to \boldsymbol{U}$ fits into the commutative diagram

$$\begin{tikzcd} \mathcal{S}_\alpha \arrow[d, "\boldsymbol{\sigma}_\alpha"'] \arrow[dr, "\tau_\alpha"] & \\ \boldsymbol{U} \arrow[r, "u"] & \Sigma. \end{tikzcd}$$

Since $\tau_\alpha = u \circ \boldsymbol{\sigma}_\alpha$ and u are quasi-projective, so is $\boldsymbol{\sigma}_\alpha$. In particular, each member of the family is quasi-projective.

Gorenstein: Since the morphism $\boldsymbol{\sigma}_\alpha : \mathcal{S}_\alpha \to \boldsymbol{U}$ is flat with Gorenstein fibers (i.e. it is a *Gorenstein morphism*), it follows that its pullback $\pi_b : X_b \to \Sigma$ under the morphism $b : \Sigma \to \boldsymbol{U}$ is again a Gorenstein morphism. But $\Sigma \to pt$ is trivially a Gorenstein morphism showing that X_b is Gorenstein.

Canonical class: The adjunction formula and base change imply

$$\begin{aligned} K_{X_b} \cong \pi_b^* K_\Sigma \otimes K_{\pi_b}, \quad j_b^* K_{\boldsymbol{\sigma}} \cong K_{\pi_b}, \\ j_b^*(K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma) \cong K_\sigma \otimes \pi_b^* K_\Sigma, \end{aligned} \tag{5.4}$$

where we abbreviated $\boldsymbol{\sigma} = \boldsymbol{\sigma}_\alpha$. In the last step we have used that $u \circ \boldsymbol{\sigma} \circ j_b = \pi_b$. Hence the section $\hat{\boldsymbol{\nu}}_\alpha \in H^0(\mathcal{S}_\alpha, K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma)$ from Proposition 5.5 pulls back to yield a nowhere vanishing section

$$s_b := j_b^* \hat{\boldsymbol{\nu}}_\alpha \in H^0(X, K_X) \tag{5.5}$$

of the locally free sheaf K_{X_b}.

Smoothness: By construction, $X = X_b$ is locally (even Zariski-locally) on Σ, given by

$$\begin{tikzcd} X_D \arrow[r] \arrow[d] & S \arrow[d, "\sigma"] \\ D \arrow[r, "b"] & \mathfrak{t}/W \end{tikzcd}$$

for an appropriate open $D \subset \Sigma$. Hence, if $b \in \mathbf{B}^\circ$, then b is transversal to σ and therefore X_D is non-singular, as we have seen in Section 1.4.4. $\square$

Let $b \in \mathbf{B}^\circ$ and $\pi_b : X_b \to \Sigma$ be the corresponding projection. By our considerations in Section 1.4.5, we see that the fibers $\pi_b^{-1}(x)$ are smoothings of $\mathbb{C}^2/\Gamma$ if $x \notin Br_b$ and have an A_1-singularity if $x \in Br_b$. Here $Br_b \subset \Sigma$ is the branch locus of the cameral covering $p_b : \tilde{\Sigma}_b \to \Sigma$ as before.

5.1.2 Global construction

What we refer to as global construction is actually a special case of the local construction. It corresponds to the 'diagonal cases', i.e.

$$\begin{array}{ll} \mathrm{A}_n,\, n \geq 2 & V = L \oplus L, \\ \text{D and E} & V = L \oplus L, \end{array}$$

where in each of these cases L is a spin bundle, i.e. $L^2 \cong K_\Sigma$. So the classes α of such vector bundles V come from $H^1(\Sigma, \mathbb{C}^*)$, where $\mathbb{C}^* \subset C(\Gamma)$ are just the diagonal matrices (which is equal to $C(\Gamma)$ in the cases DE). It now follows that for such V we have the commutative diagram

$$\begin{tikzcd} \mathcal{S}_\alpha \arrow[r, "\cong"] \arrow[d, "\boldsymbol{\sigma}"'] & L \times_{\mathbb{C}^*} S \arrow[d, "\mathrm{id}\times_{\mathbb{C}^*}\sigma"] \\ \boldsymbol{U} \arrow[r, "\cong"] & L \times_{\mathbb{C}^*} \mathfrak{t}/W. \end{tikzcd}$$

Note that on the right-hand side of the diagram $\mathbb{C}^*$ acts as described in Section 1.4.3. In particular, $\mathbb{C}^*$ acts with weights $2d_j$ on $\mathfrak{t}/W$ so that $L \times_{\mathbb{C}^*} \mathfrak{t}/W \cong K_\Sigma \times_{\mathbb{C}^*} S$. On the right-hand side of this isomorphism, $\mathbb{C}^*$ acts with weights d_j. The advantage of these cases is that $\mathcal{S}_\alpha$ then admits non-trivial $AS(\Delta)$- (resp. $\mathrm{Aut}(\Delta_h)$-)actions, cf. Remark 1.21. Of course, we again obtain a quasi-projective family $\mathcal{X}_\alpha \to \mathbf{B}$ of non-compact Gorenstein Calabi-Yau threefolds from $\mathcal{S}_\alpha$. We will treat these cases in more detail in Section 5.3, when we incorporate Dynkin graph automorphisms.

Remark 5.8. In the following, we will fix a Γ-equivariant vector bundle $V \to \Sigma$, $\det V = K_\Sigma$, and its class $\alpha = \alpha_V \in H^1(\Sigma, \mathcal{O}(C(\Gamma)))$. Therefore we will often drop the subscript α and write $\mathcal{S}$ instead of $\mathcal{S}_\alpha$ etc. However, it is an interesting and important question to understand the dependence of our constructions on this choice. We will adress this question in future work.

5.1.3 (Simultaneous) Resolutions

We have seen in Proposition 5.5 that the simultaneous resolution $\tilde{S} \to \mathfrak{t}$ of a semi-universal deformation $S \to \mathfrak{t}/W$ of a given ADE-singularity glues to give a simultaneous resolution $\tilde{\mathcal{S}} \to \tilde{\boldsymbol{U}}$ of the glued Slodowy slice $\mathcal{S}$. One can construct at least two families of smooth non-compact threefolds from this simultaneous resolution. The first one was already examined in [Sze04] and is constructed as the fiber product with respect to the evaluation map

$$ev : \Sigma \times \tilde{\mathbf{B}} \to \tilde{\boldsymbol{U}}, \quad \tilde{\mathbf{B}} := H^0(\Sigma, \tilde{\boldsymbol{U}}).$$

We denote this *smooth* family by $\tilde{\mathcal{X}} \to \tilde{\mathbf{B}}$. Via the natural quotient map[5] $\tilde{\mathbf{B}} \to \tilde{\mathbf{B}}/W \subset \mathbf{B}$ it can be considered as a family over $\mathbf{B}$.
The second one is constructed analogously but we change the base via the natural map $\tilde{\boldsymbol{\Sigma}} \to \tilde{\boldsymbol{U}}$ from the construction of the family of cameral curves. It yields the fiber product

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r] \arrow[d] & \tilde{\mathcal{S}} \arrow[d] \\ \tilde{\boldsymbol{\Sigma}} \arrow[r] & \tilde{\boldsymbol{U}} \end{tikzcd}$$

which yields a *smooth* family $\tilde{\boldsymbol{\pi}} : \tilde{\mathcal{X}} \to \mathbf{B}$ of non-compact threefolds. It can be shown, analogously to above, that these families consist of non-compact Gorenstein Calabi-Yau threefolds. The following result is a generalization of a (fiberwise) construction of [DDP07].

Lemma 5.9. *Let $\mathcal{S} \to \boldsymbol{U}$ be a glued Slodowy slice for a given* ADE*-singularity over the compact Riemann surface Σ of genus $g \geq 2$ and $\tilde{\mathcal{S}} \to \tilde{\boldsymbol{U}}$ its simultaneous resolution. Let further $\mathcal{X}$, $\tilde{\mathcal{X}}$*

[5]This locus appeared in the context of geometric transitions and large N duality in the A_1-example of [DDD+06]. For the present work, it is not that important. But we plan to examine the corresponding families over $\tilde{\mathbf{B}}/W \subset \mathbf{B}$ in the future.

and $\hat{\mathcal{X}}$ be the families of non-compact Gorenstein Calabi-Yau threefolds over $\mathbf{B}$ from above. Then they fit into the following commutative diagram (for details on the morphisms we refer to the proof):

$$
\begin{tikzcd}
\Sigma \times \tilde{\mathbf{B}} \arrow[rr, "ev"] \arrow[dd] & & \tilde{\boldsymbol{U}} \arrow[dd, equal] & \\
& \hat{\mathcal{X}} \arrow[ul] \arrow[rr] \arrow[dd] & & \tilde{\mathcal{S}} \arrow[ul] \arrow[dd, equal] \\
\tilde{\boldsymbol{\Sigma}} \arrow[rr] \arrow[dd, equal] & & \tilde{\boldsymbol{U}} \arrow[dd, equal] & \\
& \tilde{\mathcal{X}} \arrow[ul] \arrow[rr] \arrow[dd] & & \tilde{\mathcal{S}} \arrow[ul] \arrow[dd] \arrow[dddd, bend left, "\psi"] \\
\tilde{\boldsymbol{\Sigma}} \arrow[rr] \arrow[dd] & & \tilde{\boldsymbol{U}} \arrow[dd, "\boldsymbol{q}"] & \\
& \boldsymbol{q}^*\mathcal{X} \arrow[ul] \arrow[rr] \arrow[dd] & & \boldsymbol{q}^*\mathcal{S} \arrow[ul] \arrow[dd] \\
\Sigma \times \mathbf{B} \arrow[rr, "ev"] & & \boldsymbol{U} & \\
& \mathcal{X} \arrow[ul] \arrow[rr] & & \mathcal{S}. \arrow[ul]
\end{tikzcd}
\tag{5.6}
$$

In particular, for each $b \in \mathbf{B}$ the diagram

$$
\begin{tikzcd}
\tilde{X}_b \arrow[r] \arrow[d] & \boldsymbol{q}^* X_b \arrow[r] \arrow[d] & X_b \arrow[d] \\
\tilde{\Sigma} \arrow[r, equal] & \tilde{\Sigma} \arrow[r] & \Sigma
\end{tikzcd}
$$

gives a simultaneous resolution of $\boldsymbol{q}^* X_b \to \tilde{\Sigma}$ and $X_b \to \Sigma$ respectively.

The notation $\boldsymbol{q}^*\mathcal{X}$ is not quite precise here. We actually mean by $\boldsymbol{q}^*\mathcal{X}$ the fiber product of $\mathcal{X} \to \mathcal{S}$ and $\boldsymbol{q}^*\mathcal{S} \to \mathcal{S}$ in the lower front square.
Note that $\tilde{X}_b \to X_b$ is not an isomorphism, even if X_b is non-singular. This is because $Br_b \neq \emptyset$ for all $b \in \mathbf{B}$.

Proof. The first, second and fourth horizontal squares (from above) are fiber products by definition. This yields all the non-obvious morphisms and the commutativity. We only demonstrate this for the upper cube, because the rest works analogously.
Recall that $\tilde{\boldsymbol{\Sigma}}$ is constructed as the fiber product (4.15). Using the morphisms $ev : \Sigma \times \tilde{\mathbf{B}} \to \tilde{\boldsymbol{U}}$ and $\Sigma \times \tilde{\mathbf{B}} \to \Sigma \times \mathbf{B}$, we therefore obtain a morphism $f : \Sigma \times \tilde{\mathbf{B}} \to \tilde{\boldsymbol{\Sigma}}$. Applying the fiber product property for $\tilde{\mathcal{X}}$ (i.e. the second horizontal square from above), we obtain a morphism $g : \hat{\mathcal{X}} \to \tilde{\mathcal{X}}$. The fiber product property and the fact that f and g make the corresponding

diagrams commute, imply the commutativity of the square

$$\begin{array}{ccc} \hat{\mathcal{X}} & \longrightarrow & \Sigma \times \tilde{\mathbf{B}} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \tilde{\mathcal{X}} & \longrightarrow & \tilde{\boldsymbol{\Sigma}}. \end{array}$$

This is precisely the left face of the most upper cube. All its other faces commute by construction. The last statement is a direct consequence of Proposition 5.5. □

5.2 ADE-case via V(M)HS

The aim of this section is to present the main result of [DDP07]. Instead of quoting it verbatim, we present an alternative approach: We first show that the variations of (mixed) Hodge structures (V(M)HS) induced by the family $\boldsymbol{\pi}^\circ : \mathcal{X}^\circ \to \mathbf{B}^\circ$ and the Hitchin fibration $\boldsymbol{h}_1^\circ : \mathbf{Higgs}_1^\circ \to \mathbf{B}^\circ$ are isomorphic over $\mathbf{B}^\circ$. From this, we can deduce an isomorphism between the Calabi-Yau integrable system and the ADE-Hitchin system (at least over $\mathbf{B}^\circ$) as in [DDP07]. It turns out that this method applies to the BCFG-cases as well. In principle, it is hence possible to discuss all the cases at once. However, the BCFG-cases require taking invariants under graph automorphisms, so that we consider it more transparent to discuss the ADE- and BCFG-case separately.

5.2.1 Non-compact Calabi-Yau integrable systems

Since $\boldsymbol{\pi}^\circ : \mathcal{X}^\circ \to \mathbf{B}^\circ$ is a family of *non-compact* Calabi-Yau threefolds, we cannot (directly) apply the results from Chapter 3. Already the fiberwise case is quite different from the compact case: The cohomology groups $H^3(X_b, \mathbb{Z})$, $b \in \mathbf{B}^\circ$, carry a priori mixed Hodge structures. But a mixed Hodge structure has in general several different intermediate Jacobians associated with its pure graded pieces (which might only be generalized tori, see [Car80] for more details). Moreover, the crucial Lemma 3.11 uses Hodge and deformation theory specific to compact Calabi-Yau threefolds. However, we argue that in the example at hand, the families $\mathcal{X}^\circ \to \mathbf{B}^\circ$, we do obtain (algebraically completely) integrable systems, which we call *non-compact Calabi-Yau integrable systems.*
We already mentioned a fundamental difference between the two cases: In the non-compact situation, we obtain an integrable system over (an open subset of) the base $\mathbf{B}$ of the family. To obtain compact Calabi-Yau integrable systems, one has to base change the initial complete family by a $\mathbb{C}^*$-bundle. The latter corresponds to special Kähler geometry, the former to projective special Kähler geometry ([Fre99], [HHP10]).
We now investigate, how $\boldsymbol{\pi}^\circ : \mathcal{X}^\circ \to \mathbf{B}^\circ$ gives rise to a non-compact Calabi-Yau integrable system. Some of the arguments reoccur in Section 5.2.2 in the context of mixed Hodge modules. However, it is desirable to have more elementary arguments for constructing these non-compact Calabi-Yau integrable systems.

Proposition 5.10. *The cohomology sheaf* $\mathsf{V}_{\mathbb{Z}}^{CY} := R^3\boldsymbol{\pi}_*^\circ\mathbb{Z}$ *underlies a graded-polarizable* $\mathbb{Z}$-*VMHS*

$$\mathsf{V}^{CY} := (\mathsf{V}_{\mathbb{Z}}^{CY}, \mathbb{W}_\bullet^{CY}, \mathcal{F}_{CY}^\bullet).$$

Proof. This follows from Corollary 1.18. in [BEZ14] or our discussion on Saito's mixed Hodge modules in the next section. Here it is crucial that $\boldsymbol{\pi}$ is quasi-projective. □

The next commutative diagram is convenient to discuss this VMHS in more detail:

$$\begin{tikzcd}
 & \mathcal{X} \arrow[r] \arrow[d, "\boldsymbol{\pi}_1"] \arrow[dl, "\boldsymbol{\pi}"'] & \mathcal{S} \arrow[d, "\boldsymbol{\sigma}"] \\
\mathbf{B} & \Sigma \times \mathbf{B} \arrow[l, "\boldsymbol{p}_2 = \boldsymbol{\pi}_2"'] \arrow[r] & \boldsymbol{U} \\
 & \tilde{\boldsymbol{\Sigma}} \arrow[u, "\boldsymbol{p}_1"] \arrow[r] \arrow[ul, "\boldsymbol{p}"] & \tilde{\boldsymbol{U}}.
\end{tikzcd} \tag{5.7}$$

Hence $\boldsymbol{\pi}^\circ$ is the composition $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1$ where $\boldsymbol{\pi}_1^1 : \mathcal{X}^\circ \to \Sigma \times \mathbf{B}^\circ$ and $\boldsymbol{\pi}_2^\circ : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ are the obvious restrictions. Observe that $\boldsymbol{\pi}_1^1$ is in fact the pullback of $\boldsymbol{\sigma}^1 : \mathcal{S}^1 \to \boldsymbol{U}^1$ via $ev^\circ : \Sigma \times \mathbf{B}^\circ \to \boldsymbol{U}^1$. This explains why we do not use $\boldsymbol{\pi}_1^\circ$. Another reason is that $\boldsymbol{\pi}_1^1$ does have singular fibers which would not fit with our common usage of the superscript $^\circ$.

Lemma 5.11. *Let $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1 : \mathcal{X}^\circ \to \mathbf{B}^\circ$ be as before. Then the Leray spectral sequence degenerates and gives an isomorphism of abelian sheaves*

$$R^3\boldsymbol{\pi}_*^\circ\mathbb{Z} \cong R^1\boldsymbol{\pi}_{2*}^\circ R^2\boldsymbol{\pi}_{1*}^1\mathbb{Z}.$$

By using mixed Hodge modules, one can in fact obtain an isomorphism of VMHS, cf. Section 5.2.2.

Proof. Then the Leray spectral sequence for $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1$ reads as

$$E_2^{p,q} = R^p\boldsymbol{\pi}_{2*}^\circ(R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z}) \Rightarrow R^{p+q}\boldsymbol{\pi}_*^\circ\mathbb{Z}.$$

We first claim that $R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z} = 0$ for $q \notin \{0, 2\}$. To do so, we consider for each $b \in \mathbf{B}^\circ$ the commutative diagram

$$\begin{tikzcd}
Q_{t,b} \arrow[r] \arrow[d] & X_b \arrow[r] \arrow[d, "\pi_b"] & \mathcal{X}^\circ \arrow[d, "\boldsymbol{\pi}_1^1"] \\
\{t\} \times \{b\} \arrow[r, "i"] \arrow[rr, bend right, "i_{b,t}"'] & \Sigma \arrow[r, "i_b"] & \Sigma \times \mathbf{B}^\circ
\end{tikzcd}$$

where each of the squares is a fiber product. Using base change (cf. Footnote 8 on page 48), we obtain

$$i_{b,t}^*(R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z}) \cong i^*i_b^*(R^q\boldsymbol{\pi}_{1*}^1\mathbb{Z}) \cong i^*(R^q\pi_{b*}\mathbb{Z}) = (R^q\pi_{b*}\mathbb{Z})_t.$$

But from the local theory it can be seen that $(R^q\pi_{b*}\mathbb{Z})_t = 0$ if $q \notin \{0, 2\}$ for all $t \in \Sigma$ (cf. proof of Lemma 5.40 below). Hence the claim follows.
As a first consequence we see that $d_2 = 0$ on the E_2-page. To see that the higher differentials d_r, $r \geq 3$, also vanish, observe that the projection $\boldsymbol{\pi}_2^\circ = pr : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ is proper. Hence for any sheaf $\mathcal{F}$ on $\Sigma \times \mathbf{B}^\circ$ we can compute the stalks of $R^p\boldsymbol{\pi}_{2!}^\circ\mathcal{F} = R^p\boldsymbol{\pi}_{2*}^\circ\mathcal{F}$ as

$$R^p\boldsymbol{\pi}_{2*}^\circ\mathcal{F}_b \cong H^p(\Sigma, i_b^*\mathcal{F}).$$

But $\dim_c(\Sigma) = 2$, the cohomological dimension of Σ, so that $R^p\boldsymbol{\pi}_{2*}^\circ\mathcal{F} = 0$ for $p > 2$. This not only implies that $d_r = 0$ for $r \geq 3$ but also $R^3\boldsymbol{\pi}_{2*}^\circ(R^0\boldsymbol{\pi}_{1*}^1\mathbb{Z}) = 0$. Hence the Leray spectral sequence degenerates and

$$R^3\boldsymbol{\pi}_*^\circ\mathbb{Z} \cong R^1\boldsymbol{\pi}_{2*}^\circ R^2\boldsymbol{\pi}_{1*}^1\mathbb{Z}.$$

□

The second statement of the next corollary is already contained in [DDP07] but we relate it to our discussion from Section 4.3.2.

Corollary 5.12. *The cohomology group $H^3(X_b,\mathbb{Z})$, $b \in \mathbf{B}^\circ$, is torsion-free. The graded-polarizable $\mathbb{Z}$-MHS on $H^3(X_b,\mathbb{Z})$, $b \in \mathbf{B}^\circ$, is pure of weight 3 and $H^3(X_b,\mathbb{Z})(1)$ is a pure and effective Hodge structure of weight 1.*

Recall that a pure Hodge structure H is called *effective* if $H^{pq} = 0$ for $p < 0$ or $q < 0$. We make some statements about the other cohomology groups in the BCFG-case below.

Proof. The previous lemma in particular implies that

$$H^3(X,\mathbb{Z}) \cong H^1(\Sigma, R^2\pi_*\mathbb{Z}), \tag{5.8}$$

where $X = X_b$, $\pi = \pi_b$. We will see in Proposition 5.18 that $R^2\pi_*\mathbb{Z} \cong (p_{b,*}\mathbf{\Lambda})^W$. Hence the first statement follows from the fact that $\mathrm{Tors}(H^1(\Sigma,(p_{b,*}\mathbf{\Lambda})^W)) = 0$, see Remark 4.38.
Now the right-hand side of (5.8) carries a natural Hodge structure of weight 3: Let $\pi^\circ : X^\circ \to \Sigma^\circ$ be the restriction of π away from its singular fibers and $j : \Sigma^\circ \to \Sigma$. As we have seen in Chapter 1, the restriction $R^2\pi^\circ_*\mathbb{Z}$ underlies a polarized $\mathbb{Z}$-VHS of weight 2 and Tate type. Since $j_*R^2\pi^\circ_*\mathbb{Z} \cong R^2\pi_*\mathbb{Z}$ it follows from Zucker's Theorem (Theorem 4.33) that $H^1(\Sigma, R^2\pi_*\mathbb{Z})$ carries a functorial polarized $\mathbb{Z}$-Hodge structure of weight $1+2 = 3$. It turns out that the Leray spectral sequence for π lifts to mixed Hodge structures ([Ara05], [PS08] (Chapter 6) and Section 5.2.2). Hence the mixed Hodge structure on $H^3(X,\mathbb{Z})$ is in fact pure.
The second statement follows as in the proof of Lemma 4.34. In particular, its only (possibly) non-zero H^{pq} are H^{12} and H^{21}. □

Corollary 5.13. *The graded-polarizable $\mathbb{Z}$-VMHS V^{CY} is pure of weight 3, i.e. $\mathbb{W}^{CY}_\bullet = 0$, and has a second-step Hodge filtration. In particular, it is an* admissible *VMHS.*

The property of *admissibility* is rather technical but important ([SZ85], [Kas86]), not only for VMHS, but also in the theory of mixed Hodge modules. It means that the VMHS degenerates in a controlled way (at infinity). For VHS of geometric origin, this is automatically satisfied ([Sch73]), which explains the second statement of Corollary 5.13. It does in general *not* hold for VMHS of geometric origin.
The upshot of the previous discussion is that we can define the intermediate Jacobians

$$J^2(X_b) = H^3(X_b,\mathbb{C})/(F^2H^3(X_b,\mathbb{C}) + H^3(X_b,\mathbb{Z})), \quad b \in \mathbf{B}^\circ,$$

and these are even abelian varieties. This is in contrast to the compact case where the intermediate Jacobian can never be projective. Moreover, the intermediate Jacobian fibration

$$\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$$

over $\mathbf{B}^\circ$ is a family of (polarized) abelian varieties. According to Proposition 2.36, it is sufficient to give an abstract Seiberg-Witten differential to prove that this yields an integrable system. A posteriori[6] it will turn out that the period map $\rho : \mathbf{B}^\circ \to \mathsf{V}^{CY}_{\mathcal{O}} = \mathcal{H}^3(\mathcal{X}^\circ/\mathbf{B}^\circ,\mathbb{C})$ is an abstract Seiberg-Witten differential, i.e. $T\mathbf{B}^\circ \to \mathcal{F}^2\mathcal{H}^3$, $X \mapsto \nabla_X\rho$, is an isomorphism[7]. Therefore $\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$ is a non-compact Calabi-Yau integrable system.

[6] We could already prove this at this point but the argument occurs in the next section anyway.

[7] Note that we work with $\mathcal{F}^2$ (instead of $\mathcal{F}^1$ as in Section 2.2.4) because V^{CY} is an effective VHS of weight 1 up to a Tate twist.

Remark 5.14. One reason, why a general theory of non-compact CY integrable systems is out of reach, is the fact that the deformation theory of non-compact CY3s is in general not as nicely behaved as for compact CY3s. But this was crucial for the construction of compact CY3 integrable systems. However, in [KS14] Kontsevich and Soibelman gave a class of non-compact CY3s that give rise to integrable systems using deformation theory. It would be interesting to better understand the relation of their approach and ours.

Let us describe the above period map ρ in more detail. Since we also discuss the period map for the family $\tilde{\boldsymbol{\pi}}^\circ : \tilde{\mathcal{X}}^\circ \to \mathbf{B}^\circ$, it is convenient to have the following commutative diagram at hand

$$\begin{tikzcd}
\tilde{\mathcal{X}} \arrow[rr] \arrow[dr, "\tilde{\iota}"] \arrow[dd, "\tilde{\boldsymbol{\pi}}_1"] \arrow[dddd, bend right, "\tilde{\boldsymbol{\pi}}"'] & & \mathcal{X} \arrow[dr, "\iota"] \arrow[dd] & \\
& \tilde{\mathcal{S}} \arrow[rr, "\boldsymbol{\psi}"] \arrow[dd, "\boldsymbol{\theta}"] & & \mathcal{S} \arrow[dd, "\boldsymbol{\sigma}"] \\
\tilde{\boldsymbol{\Sigma}} \arrow[rr] \arrow[dr] \arrow[dd, "\boldsymbol{p}"] & & \Sigma \times \mathbf{B} \arrow[dr] & \\
& \tilde{\boldsymbol{U}} \arrow[rr] & & \boldsymbol{U} \\
\mathbf{B}. & & &
\end{tikzcd} \tag{5.9}$$

Here ι and $\tilde{\iota}$ are the natural maps from the fiber product construction.
Now recall the section $\hat{\boldsymbol{\nu}} \in H^0(\mathcal{S}, K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma)$ from Proposition 5.5. It induces the section

$$\boldsymbol{s} := \boldsymbol{\pi}^\circ_* \iota^* \hat{\boldsymbol{\nu}} \in H^0(\mathbf{B}^\circ, \boldsymbol{\pi}_*(\iota^*(K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma))).$$

As in the proof of Propositiion 5.7, it follows that $\boldsymbol{s}$ yields volume forms on each fiber X_b,

$$s_b := \boldsymbol{s}_{|X_b} \in H^0(X_b, K_{\boldsymbol{\pi}_b} \otimes \pi_b^* K_\Sigma) \cong H^0(X_b, K_{X_b}). \tag{5.10}$$

Therefore we obtain a section $\rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to \mathcal{H}^3(\mathcal{X}^\circ/\mathbf{B}^\circ, \mathbb{C})$, which we refer to as period map, even though $\rho_{\boldsymbol{s}}$ only locally induces period maps in the usual sense. A posteriori it will turn out that the section $\rho_{\boldsymbol{s}}$ is an abstract Seiberg-Witten differential, i.e. $T\mathbf{B}^\circ \to \mathcal{F}^2\mathcal{H}^3$, $X \mapsto \nabla_X \rho_{\boldsymbol{s}}$, is an isomorphism. Analogously, there is a period map $\rho_{\tilde{\boldsymbol{s}}} : \mathbf{B}^\circ \to \mathcal{H}^3(\tilde{\mathcal{X}}^\circ/\mathbf{B}^\circ, \mathbb{C})$ for the family $\tilde{\boldsymbol{\pi}}^\circ : \tilde{\mathcal{X}}^\circ \to \mathbf{B}^\circ$. Here one employs the section

$$\tilde{\boldsymbol{s}} := \tilde{\boldsymbol{\pi}}_* \tilde{\iota}^* \hat{\boldsymbol{\omega}} \in H^0(\mathbf{B}^\circ, K_{\tilde{\boldsymbol{\pi}}_1} \otimes \tilde{\iota}^* K_\Sigma).$$

The 'simultaneous resolution' $\tilde{\mathcal{X}}^\circ \to \mathcal{X}^\circ$ over $\mathbf{B}^\circ$ (cf. (5.6)) yields a natural map

$$\Psi^* : \mathcal{H}^3(\tilde{\mathcal{X}}^\circ/\mathbf{B}^\circ, \mathbb{C}) \to \mathcal{H}^3(\mathcal{X}^\circ/\mathbf{B}^\circ, \mathbb{C}),$$

which is in fact a monomorphism. Using Proposition 5.5 iv), we see that $\Psi^* \circ \rho_{\boldsymbol{s}} = \rho_{\tilde{\boldsymbol{s}}}$.

5.2.2 Isomorphism with the Hitchin system

For the next theorem, recall the polarizable (Tate-twisted) $\mathbb{Z}$-VHS of Corollary 4.30

$$\mathsf{V}^H = \mathsf{V}^H_{ad} = (\mathsf{V}^H_{\mathbb{Z}}, \mathcal{F}^\bullet_H) = \left(R^1\boldsymbol{p}_{2,*}(\boldsymbol{p}^W_{1,*}\boldsymbol{\Lambda}), \mathcal{F}^\bullet(R^1\boldsymbol{p}^W_* \mathfrak{t} \otimes \mathcal{O}_{\mathbf{B}^\circ})\right)_{|\mathbf{B}^\circ}$$

of weight 1. It is associated with the neutral component $\boldsymbol{h}^\circ_1 : \mathbf{Higgs}^\circ_1(\Sigma, G) \to \mathbf{B}^\circ$ of the Hitchin system.

Theorem 5.15. *Let Δ be a Dynkin diagram of type* ADE *with simple adjoint complex Lie group G and $\mathbf{B}^\circ \subset \mathbf{B}$ the smooth locus in the Hitchin base $\mathbf{B}$ of the same type. Then there is an isomorphism*

$$\mathsf{V}^H(-1) \cong \mathsf{V}^{CY} \tag{5.11}$$

of polarizable $\mathbb{Z}$-VHS of weight 3 over $\mathbf{B}^\circ$.

Before we prove Theorem 5.15, let us see how this implies the main result from [DDP07].

Corollary 5.16 ([DDP07])**.** *Let G be a simple adjoint complex Lie group of type* ADE *and $\boldsymbol{h}_1 : \mathbf{Higgs}_1(\Sigma, G) \to \mathbf{B}$ the neutral component of corresponding Hitchin system. Further let $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ be a family of non-compact Calabi-Yau threefolds as constructed above. Then there is an isomorphism*

$$\begin{tikzcd} \mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Higgs}_1^\circ(\Sigma, G) \arrow[dl, "\boldsymbol{h}_1^\circ"] \\ & \mathbf{B}^\circ & \end{tikzcd} \tag{5.12}$$

of integrable systems over $\mathbf{B}^\circ$ such that the cubics are intertwined.

Proof. The previous theorem implies that the two families of abelian varieties are isomorphic over $\mathbf{B}^\circ$ (see Section 2.2.4 and also Remark 4.31). Recall here that both systems have sections. It remains to prove that the cubics are exchanged.
On both sides the cubic is determined by the (abstract) Seiberg-Witten differential (cf. Proposition 2.27 and Proposition 4.32). Hence it suffices to prove that the period map $\rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to \mathcal{H}^3(\mathcal{X}^\circ/\mathbf{B}^\circ, \mathbb{C})$ corresponds to the Seiberg-Witten differential $\boldsymbol{\lambda} : \mathbf{B}^\circ \to \mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \mathfrak{t})^W$. Recall that the latter can be seen as the pullback of the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$ (cf. (4.25)).
As a first step, we pull back $\rho_{\boldsymbol{s}}$ and consider $\rho_{\tilde{\boldsymbol{s}}} = \Psi^* \circ \rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to \mathcal{H}^3(\tilde{\mathcal{X}}^\circ/\mathbf{B}^\circ, \mathbb{C})$ instead. Then we have isomorphisms

$$R^3\tilde{\boldsymbol{\pi}}^\circ_*\mathbb{C} \cong R^1\boldsymbol{p}_*R^2\tilde{\boldsymbol{\pi}}_{1*}\mathbb{C} \cong R^1\boldsymbol{p}_*\mathfrak{t}$$

of abelian sheaves. The first isomorphism follows as in Lemma 5.11. For the second isomorphism one uses the fact that $\tilde{\mathcal{S}} \to \tilde{\boldsymbol{U}}$ is C^∞-trivial. Tensoring with $\mathcal{O}_{\mathbf{B}^\circ}$ gives a bundle isomorphism $\mathcal{H}^3(\tilde{\mathcal{X}}^\circ/\mathbf{B}^\circ, \mathbb{C}) \cong \mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}^\circ/\mathbf{B}^\circ, \mathfrak{t})$. Under this isomorphism, we have $\rho_{\tilde{\boldsymbol{s}}} = \boldsymbol{\lambda}$. This is a consequence of the construction of the Leray spectral sequence for submersions ([GH94]) together with the fact that $\hat{\boldsymbol{\omega}}$ induces a (fiberwise) period map $\boldsymbol{\eta} : \tilde{\boldsymbol{U}} \to \tilde{u}^*\tilde{\boldsymbol{U}}$ that coincides with the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$ (Proposition 5.5 iii)). □

Remark 5.17. As a by-product, we see that the cohomology class $[\boldsymbol{s}_b] \in H^3(X_b, \mathbb{C})$ of the volume form sits in $H^{2,1}(X_b) \cong H^{1,0}(\tilde{\Sigma}_b, \mathbb{C})$. This is again in strong contrast to the compact case. Moreover, the previous proof justifies the earlier claim that $\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$ does give rise to an integrable system.

Proof of Theorem 5.15

In this subsection we prove Theorem 5.15. The first step is to translate a statement from the local theory to the glued setting. Thanks to our preparations this is immediate. However, it is at the heart of the relation between the V(M)HS associated with $\mathcal{X}^\circ \to \mathbf{B}^\circ$ and the VHS associated with the Hitchin system.

Proposition 5.18. *Let* $\tilde{\boldsymbol{U}}^1 \subset \tilde{\boldsymbol{U}}$ *and* $\boldsymbol{U}^1 \subset \boldsymbol{U}$ *be as in (4.11), (4.12) and* $\mathcal{S}^1 := \boldsymbol{\sigma}^{-1}(\boldsymbol{U}^1) \subset \mathcal{S}$. *Further denote by* $\boldsymbol{q}^1 : \tilde{\boldsymbol{U}}^1 \to \boldsymbol{U}^1$ *and* $\boldsymbol{\sigma}^1 : \mathcal{S}^1 \to \boldsymbol{U}^1$ *the restrictions of* $\boldsymbol{q}$ *and* $\boldsymbol{\sigma}$ *respectively. Then there is an isomorphism of constructible sheaves*

$$(\boldsymbol{q}^1_* \boldsymbol{\Lambda}_G)^W \cong R^2 \boldsymbol{\sigma}^1_* \mathbb{Z}. \tag{5.13}$$

Proof. Consider a disc $D \subset \Sigma$ such that

$$\begin{tikzcd}
\mathcal{S}_{|D} \arrow[r, "\cong"] \arrow[d, "\boldsymbol{\sigma}"'] & D \times S \arrow[d, "id\times\sigma"] \\
\boldsymbol{U}_{|D} \arrow[r, "\cong"] & D \times \mathfrak{t}/W \\
\tilde{\boldsymbol{U}}_{|D} \arrow[r, "\cong"] \arrow[u, "\boldsymbol{q}"] & D \times \mathfrak{t}. \arrow[u, "id\times q"']
\end{tikzcd}$$

It follows from Proposition 1.76 that

$$R^2(id \times \sigma^\circ)_* \mathbb{Z} \cong ((id \times q^\circ)_* \boldsymbol{\Lambda}_G)^W. \tag{5.14}$$

These are local models for $R^2\boldsymbol{\sigma}^\circ_*\mathbb{Z}$ and $(\boldsymbol{q}^\circ_*\boldsymbol{\Lambda}_G)^W$ respectively (over $\boldsymbol{U}_{|D}$). Since $\sigma : \mathcal{S} \to \boldsymbol{U}$ and $\boldsymbol{q} : \tilde{\boldsymbol{U}} \to \boldsymbol{U}$ are glued from the same class $\alpha \in H^1(\Sigma, \mathcal{O}(C(\Gamma)))$, the isomorphism (5.14) can be glued to an isomorphism

$$R^2\boldsymbol{\sigma}^\circ_*\mathbb{Z} \cong (\boldsymbol{q}^\circ_*\boldsymbol{\Lambda}_G)^W.$$

By pushing forward via $\boldsymbol{j} : \boldsymbol{U}^0 \hookrightarrow \boldsymbol{U}^1$ and arguing as in the local case (Corollary 1.46) gives (5.13). ☐

Together with Lemma 5.11, this implies Theorem 5.15 but only on the level of abelian sheaves. There are two main difficulties in lifting this isomorphism to an isomorphism of V(M)HS:

a) The Hodge filtrations $\mathcal{F}^\bullet$, i.e. holomorphic subbundles, are a datum which cannot be captured by the underlying abelian sheaves as soon as they have more than one step. However, we have to deal with two-step Hodge filtrations.

b) Let $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}^\circ_2 \circ \boldsymbol{\pi}^1_1$ (cf. 5.7). Then the fibers of $\boldsymbol{\pi}^1_1 : \mathcal{X}^\circ \to \Sigma \times \mathbf{B}^\circ$ are only generically non-singular, i.e. $R^2\boldsymbol{\pi}^1_{1*}\mathbb{Z}$ is *not* a local system (at least it is constructible). In particular, $R^2\boldsymbol{\pi}^1_{1*}\mathbb{Z}$ cannot underlie a VHS. Hence the Leray spectral sequence for $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}^\circ_2 \circ \boldsymbol{\pi}^1_1$ cannot 'live' in the category of V(M)HS. Moreover, the morphism $\boldsymbol{\pi}^1_1$ is not projective.

In order to obtain an isomorphism of VHS, we employ Morihiko Saito's powerful theory of mixed Hodge modules (MHM) ([Sai88], [Sai90]) which can deal with the above difficulties. The point is that it allows to lift the (perverse) Leray spectral sequence for the composition $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}^\circ_2 \circ \boldsymbol{\pi}^1_1$ to mixed Hodge modules and (admissible) variations of mixed Hodge structures. It turns out that our situation is so special and 'almost smooth', that we only need a very small part of this impressive theory.

It is beyond the scope of this text to give an introduction to mixed Hodge modules (for a detailed introduction see [Sch14] and [PS08] for an axiomatic account). Intuitively, they can be thought of as *perverse sheaves* with mixed Hodge structures. In particular, if the underlying perverse sheaf is a local system, then one ends up with a VHS or, more generally, an admissible VMHS (see Theorem 5.21 and Theorem 5.24). The huge advantage of mixed Hodge modules over admissible VMHS or VHS is that they admit a full six-functor formalism, at least in the *algebraic* context.

This is analogous to the relation between perverse sheaves and local systems. Saito in fact lifted the full six-functor formalism of perverse sheaves to mixed Hodge modules.

To fix notation, let X be a complex variety. Then we have the following two *abelian* categories

$\mathrm{HM}(X, w)$: algebraic polarizable pure Hodge modules of weight w on X,

$\mathrm{MHM}(X)$: algebraic polarizable mixed Hodge modules on X.

As the names suggest, Hodge modules are in particular mixed Hodge modules. Both abelian categories admit an *exact* and *faithful* functor

$$\mathrm{rat} : \mathrm{HM}(X, w) \to \mathrm{P}_{\mathbb{Q}}(X), \quad \mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{P}_{\mathbb{Q}}(X).$$

Here $\mathrm{P}_{\mathbb{Q}}(X)$ is the abelian category of perverse sheaves of $\mathbb{Q}$-vector spaces on X, often only denoted by $\mathrm{P}(X)$. It is a full subcategory of the constructible (bounded) derived category $D^b_c(X) = D^b_c(X, \mathbb{Q})$ of X.

Remark 5.19. Recall that constructiblity here means that a sheaf (resp. the cohomology sheaves of a complex) is constructible with respect to an algebraic stratification. Of course, the condition to be a local system along the strata is with respect to the analytic topology on X.
The situation is analogously for algebraic polarizable (mixed) Hodge modules because they are in fact objects on the analytification of X. Usually, algebraic (mixed) Hodge modules are polarizable by definition (cf. [Sai90], [Sch14]) but we added it here for emphasis. In the following, all (mixed) Hodge modules are assumed to be algebraic if not stated otherwise. However, we sometimes explicitly mention that they are polarizable, e.g. in relation to polarizable V(M)HS (Theorem 5.21 and Theorem 5.24).

Now we can make precise what it means that the six-functor formalism lifts to mixed Hodge modules: For example, let $f : X \to Y$ be a morphism of complex varieties. Then there exists a functor $f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ that lifts $Rf_* : D^b_c(X) \to D^b_c(Y)$,

$$\mathrm{rat} \circ f_+ \simeq Rf_* \circ \mathrm{rat}.$$

It is an important theorem that the direct image of projective morphisms (between complex varieties or manifolds) preserves pure Hodge modules, see [Sch14]. Before we proceed, let us give three basic examples.

Example 5.20 ([Sch14])**.**

a) If $X = pt$ is a point, then $\mathrm{MHM}(X)$ is the category of graded-polarizable $\mathbb{Q}$-MHS. The analogous statements holds for $\mathrm{HM}(X, w)$. If $H = (H_{\mathbb{Q}}, W_\bullet, F^\bullet H_{\mathbb{C}})$ is a graded-polarizable $\mathbb{Q}$-MHS, then
$$\mathrm{rat}(H) = H_{\mathbb{Q}} \in \mathrm{P}(pt) = \mathbb{Q} - \mathrm{mod},$$
so rat associates to a $\mathbb{Q}$-MHS its underlying $\mathbb{Q}$-vector space $H_{\mathbb{Q}}$.

b) Let X be a non-singular complex variety of dimension d_X. Then the constant $\mathbb{Q}$-Hodge module is
$$\mathbb{Q}^{Hdg} = (\mathbb{Q}_X[d_X], K_X, \mathcal{F}_\bullet K_X)$$
for the canonical sheaf K_X, seen as a filtered right[8] $\mathcal{D}$-module. The filtration is given by
$$\mathcal{F}_{-d_X-1} K_X = 0, \quad \mathcal{F}_{-d_X} K_X = K_X.$$

[8]Here we use the convention of [Sch14], which requires to tensor with K_X. See loc. cit. for further discussion and the relation between left and right $\mathcal{D}$-modules.

It satisfies $\mathrm{rat}(\mathbb{Q}^{Hdg}) = \mathbb{Q}_X[d_X]$ which is an instance of the Riemann-Hilbert correspondence. In fact, this holds in general: (Mixed) Hodge modules are special filtered $\mathcal{D}$-modules M on X and $\mathrm{rat}(M) \in \mathrm{P}(X)$ is the perverse sheaf associated to M via the Riemann-Hilbert correspondence. For our purposes, it is mostly enough to work on the level of perverse sheaves, so that we do not go into more details of the theory of (filtered) $\mathcal{D}$-modules underlying (mixed) Hodge modules.

c) The previous two examples can be combined: If $a : X \to pt$ is the constant map, then $\mathcal{H}^k a_+ \mathbb{Q}_X^{Hdg} \in \mathbb{Q}-\mathrm{MHS}$ coincide with Deligne's mixed Hodge structures on the underlying cohomology groups $H^k(X, \mathbb{Q})$.

To relate our previous discussion to (mixed) Hodge modules, we need the notion of a *smooth* (mixed) Hodge module. This is a (mixed) Hodge module M on X such that $\mathrm{rat}(M) \in \mathrm{P}(X)$ is a local system concentrated in degree $-\dim_{\mathbb{C}} X$.

Theorem 5.21 (Saito)**.** *Let X be a non-singular complex variety of dimension $d_X = \dim_{\mathbb{C}} X$. Further let $\mathsf{V} = (\mathsf{V}_{\mathbb{Q}}, \mathcal{F}^\bullet)$ be a* polarizable *$\mathbb{Q}$-VHS of weight w. Then the triple*

$$M(\mathsf{V}) := (K_X \otimes_{\mathcal{O}_X} \mathsf{V}, \mathcal{F}_\bullet, \mathsf{V}_{\mathbb{Q}}[d_X]) \in \mathrm{HM}(X, d_X + w),$$

where $\mathcal{F}_k M(\mathsf{V}) = K_X \otimes_{\mathcal{O}_X} \mathcal{F}^{-k-d_X}\mathsf{V}$ defines a polarizable Hodge module of weight $d_X + w$. This yields an equivalence between the category $\mathrm{VHS}^p_{\mathbb{Q}}(X)$ of polarizable $\mathbb{Q}$-VHS and the full subcategory $\mathrm{HM}_{sm}(X) \subset \mathrm{HM}(X)$ of smooth *polarizable Hodge modules.*

Note that this is a generalization of Example 5.20 b). It follows from the constructions that the faithful functor $\mathrm{rat} : \mathrm{HM}(X) \to \mathrm{P}(X)$ satisfies

$$\mathrm{rat}(M(\mathsf{V})) = \mathsf{V}_{\mathbb{Q}}[d_X].$$

We say that $\mathsf{V}_{\mathbb{Q}}[d_X]$ *underlies the (smooth) Hodge module* (resp. VHS) $M(\mathsf{V})$ (resp. V).
In general, Hodge modules are generically smooth and there is a way to uniquely extend a polarizable $\mathbb{Q}$-VHS from an open subset $X^\circ \subset X$ to a polarizable Hodge module on X. We only need this result in the special case where $X^\circ = X - D$ is the complement of a *smooth* divisor $D \subset X$ in the non-singular complex variety X. It is due to Saito in the general case and we only add an observation, how this result specializes in the aforementioned situation.

Theorem 5.22 (Saito)**.** *Let $D \subset X$ be a* smooth *divisor in a non-singular complex variety X and denote by $j : X^\circ \to X$ the inclusion of its complement. Assume that $M = M(\mathsf{V}) \in \mathrm{HM}(X^\circ, d_X)$ corresponds to the polarizable $\mathbb{Q}$-VHS V on X°. Then there exists a unique polarizable Hodge module $\bar{M} \in \mathrm{HM}(X^\circ, d_X)$ such that*

$$\mathrm{rat}(\bar{M}) = j_* \mathsf{V}_{\mathbb{Q}}[d_X] \in \mathrm{P}(X), \tag{5.15}$$

which is concentrated in degree $-d_X$.

Indication of proof. This works in fact more generally and we only indicate the construction. The general idea is that we can extend the filtered $\mathcal{D}$-module underlying $M(\mathsf{V})$ to all of X using Deligne's extension ([Del70]). This is possible because V has only regular singularities along D. It yields a polarizable Hodge module $\bar{M} \in \mathrm{HM}(X, d_X)$ satisfying

$$\mathrm{rat}(\bar{M}) = j_{!*} \mathsf{V}_{\mathbb{Q}}[d_X]$$

for the intermediate extension $j_{!*}$.
Hence it remains to prove that $j_{!*}\mathsf{V}_{\mathbb{Q}}[d_X] \cong j_*\mathsf{V}_{\mathbb{Q}}[d_X] \in \mathrm{P}(X)$ in case $j : X^\circ \to X$ is the inclusion of the complement of a smooth divisor. This is a local question so that we are reduced to

$$j : U^* \times U \times \cdots \times U \hookrightarrow U \times \cdots \times U$$

where $U^* \subset U \subset \mathbb{C}$ is a (punctured) disk centered around $0 \in \mathbb{C}$. Since we work with local systems, it further suffices to consider the one-dimensional case $j : U^* \hookrightarrow U$, $d_X = 1$. Using Deligne's construction of the intermediate extension (e.g. [EZ08]), the claim follows:

$$j_{!*}\mathsf{V}_{\mathbb{Q}}[1] = \tau_{\leq -1} Rj_*\mathsf{V}_{\mathbb{Q}} \simeq (R^0 j_*\mathsf{V}_{\mathbb{Q}})[1] = j_*\mathsf{V}_{\mathbb{Q}}[1].$$

□

Remark 5.23. This is closely related to Zucker's Theorem 4.33: Let $j : \Sigma^\circ \hookrightarrow \Sigma$ be the complement of a finite number of points in a Riemann surface Σ. Using the previous theorem, we obtain a Hodge module $\bar{M}(\mathsf{V}) \in \mathrm{HM}(\Sigma, w)$ from any polarizable $\mathbb{Q}$-VHS V of weight $w-1$ that we consider as a Hodge module $M(\mathsf{V}) \in \mathrm{HM}(\Sigma^\circ, w)$. The former has $\mathrm{rat}(\bar{M}(\mathsf{V}) = j_*\mathsf{V}_{\mathbb{Q}}[1]$. Since the constant map $a_\Sigma : \Sigma \to pt$ is projective, it follows that $\mathcal{H}^1 a_+\bar{M}(\mathsf{V}) \in \mathrm{HM}(pt, w+1)$ which is a pure Hodge structure of weight $w+2$. Its underlying $\mathbb{Q}$-vector space is $H^1(\Sigma, j_*\mathsf{V}_{\mathbb{Q}})$. This Hodge structure coincides with Zucker's Hodge structure on this cohomology group (see [Sch14], Section 17, for more details, especially the direct image theorem for projective morphisms).

In the end, it turns out that all the objects we work with are pure Hodge modules only. However, we need mixed Hodge modules in order to have full functoriality so that we briefly discuss them here as well. The starting point is an analogous result as for smooth polarizable Hodge modules. It extends the latter case.

Theorem 5.24 (Saito, [Sai89]). *Let X be a non-singular complex variety of dimension d_X. Moreover, let $\mathrm{VMHS}^p_{ad}(X)$ be the category of admissible graded-polarizable VMHS on X and $\mathrm{MHM}_{sm}(X) \subset \mathrm{MHM}(X)$ be the full subcategory of smooth polarizable mixed Hodge modules on X. Then there is an equivalence*

$$\mathrm{VMHS}^p_{ad}(X) \xrightarrow{\ \simeq\ } \mathrm{MHM}_{sm}(X), \quad \mathsf{V} \mapsto M(\mathsf{V}).$$

Mixed Hodge modules admit a full six-functor formalism (in the algebraic context) but we mainly need one functor. Let $f : X \to Y$ be a morphism between (non-singular) complex algebraic varieties. As mentioned above, there exists a (derived) direct image functor[9] $f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ which lifts $Rf_* : D^b_c(X) \to D^b_c(Y)$. In particular, one can construct the Leray spectral sequence for mixed Hodge modules.
More precisely, let $f : X \to Y$, $g : Y \to Z$ be morphisms between (non-singular) complex algebraic varieties and $h = g \circ f$ the composition. Then $h_+ = g_+ \circ f_+ : D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Z))$ and taking cohomology with respect to the standard t-structure on $D^b(\mathrm{MHM})$ yields the Leray spectral sequence

$$\mathcal{H}^k g_+ \mathcal{H}^m f_+ M \Rightarrow \mathcal{H}^{k+m} h_+ M, \quad M \in \mathrm{MHM}(X)$$

in the abelian category $\mathrm{MHM}(X)$. Applying the exact functor $\mathrm{rat} : \mathrm{MHM}(X) \to \mathrm{P}(X)$ gives the perverse Leray spectral sequence

$$^p\mathcal{H}^k Rg_* \, ^p\mathcal{H}^m Rf_* \mathcal{F} \ \Rightarrow \ ^p\mathcal{H}^{k+m} Rh_* \mathcal{F} \tag{5.16}$$

for $\mathcal{F} = \mathrm{rat}(M) \in \mathrm{P}(X)$.

[9]This is notation is non-standard because f_+ usually stands for the direct image of the underlying $\mathcal{D}$-module. However, we consider it useful to have a clear notational distinction between the direct image of (perverse) sheaves and (mixed) Hodge modules.

Remark 5.25. Here we have used the fact that the standard t-structure on $D^b(\mathrm{MHM}(X))$ corresponds to the perverse t-structure on $D^b_c(X)$ under rat. This is a consequence of the Riemann-Hilbert correspondence. There is also an anomalous t-structure (cf. [Sai90] (Section 4), [PS08]) on $D^b(\mathrm{MHM}(X))$ that corresponds to the standard t-structure on $D^b_c(X)$.

We can now begin with the actual proof of Theorem 5.15. To simplify notations for the rest of this subsection, we set

$$X := \mathcal{X}^\circ, \quad D := Br \overset{j}{\hookrightarrow} Y := \Sigma \times \mathbf{B}^\circ, \quad Z := \mathbf{B}^\circ. \tag{5.17}$$

Recall here that D is in fact a smooth divisor in Y, cf. Lemma 4.19. Further let $f : X \to Y$ and $g : Y \to Z$ be the natural restrictions of $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2 = pr_2$ respectively (see the remarks below (5.7)). Finally, we denote $d_X := \dim_{\mathbb{C}} X$ and analogously for Y and Z.
The next lemma is a 'decomposition theorem' for $f : X \to Y$.

Lemma 5.26. *Let $f : X \to Y$ be as before where $X = \mathcal{X}^\circ$ and $Y = \Sigma \times \mathbf{B}^\circ$. Then there is an isomorphism*

$$Rf_*A_X \simeq R^0 f_* A_X[0] \oplus R^2 f_* A_X[-2]$$

in $D^b_c(Y, A)$ where $A = \mathbb{Z}$ or $\mathbb{Q}$.

Proof. We begin with a general claim:
Claim. Let $\mathcal{A}$ be an abelian category and $K^\bullet \in C^b(\mathcal{A})$ a bounded complex such that $H^k(K^\bullet) = 0$ except for $k = 0, 2$. Then $K^\bullet \cong H^0(K^\bullet)[0] \oplus H^2(K^\bullet)[-2]$ in the bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$.

The argument for the claim is straightforward: Denote $K_2^\bullet := H^0(K^\bullet) \oplus H^2(K^\bullet)[-2]$. Then we have a quasi-isomorphism

$$\begin{tikzcd}
K_1^\bullet : \arrow[d, "\varphi"'] & 0 \arrow[r] & K^{-1} \arrow[r, "d^{-1}"] \arrow[d] & \ker d^0 \arrow[r, "0"] \arrow[d] & K^1 \arrow[r, "d_1"] \arrow[d] & \ker d^2 \arrow[r] \arrow[d] & 0 \\
K_2^\bullet : & 0 \arrow[r] & 0 \arrow[r] & H^0(K^\bullet) \arrow[r] & 0 \arrow[r] & H^2(K^\bullet) \arrow[r] & 0
\end{tikzcd}$$

with the obvious maps. On the other hand, there is a natural quasi-isomorphism $\psi : K_1^\bullet \to K^\bullet$ so that we obtain a roof

$$\begin{tikzcd}
& K_1^\bullet \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\
K^\bullet \arrow[rr] & & K_2^\bullet.
\end{tikzcd}$$

Since ψ and φ are quasi-isomorphisms, this defines an isomorphism $K^\bullet \cong K_2^\bullet$ in $D^b(\mathcal{A})$ as claimed. Observe that this argument generalizes as long as $H^{2k+1}(K^\bullet) = 0$, $k \in \mathbb{Z}$.

Now we can apply the claim as follows: Let $A_X \to \mathcal{I}^\bullet$ be an injective resolution so that $Rf_*A_X \simeq f_*\mathcal{I}^\bullet$ in $D^b(Y, A)$. Since[10] $R^k f_* A_X = \mathcal{H}^k(f_*\mathcal{I}^\bullet) = 0$ except for $k = 0, 2$ we can apply the previous claim to $K^\bullet = f_*\mathcal{I}^\bullet$. Thus we obtain an isomorphism

$$Rf_*A_X \cong \mathcal{H}^0(f_*\mathcal{I}^\bullet)[0] \oplus \mathcal{H}^2(f_*\mathcal{I}^\bullet)[-2] = R^0 f_* A[0] \oplus R^2 f_* A[-2]$$

in $D^b(Y, A)$. However, Rf_*A_X lies in $D^b_c(Y, A)$ which is a full subcategory so that the previous isomorphism is in fact an isomorphism in $D^b_c(Y, A)$. $\square$

[10]Here we follow the usual convention and write $\mathcal{H}^k$ for the ordinary k-th cohomology $\mathcal{H}^k : D^b(Y, A) \to Sh(Y, A)$.

Proof of Theorem 5.15. We first work with the constant mixed Hodge module $\mathbb{Q}_X^{Hdg}$ and show that the V(M)HS are isomorphic over $\mathbb{Q}$ (i.e. are isogenous). Further below we argue that all the arguments also work over $\mathbb{Z}$.
The Leray spectral sequence in MHM(Z) for $\mathbb{Q}_X^{Hdg}$ reads as

$$\mathcal{H}^k f_+ \mathcal{H}^m g_+ \mathbb{Q}_X^{Hdg} \Rightarrow \mathcal{H}^{k+m} h_+ \mathbb{Q}_X^{Hdg} \tag{5.18}$$

and we want to show that it degenerates on the E_2-page. Since rat : $D^b(\mathrm{MHM}(X)) \to \mathrm{P}(X)$ is faithful, it suffices to prove this for the perverse Leray spectral sequence. As it turns out, this works precisely as for the ordinary Leray spectral sequence (Lemma 5.11). The E_2-terms of the perverse Leray spectral sequence (5.16) can be computed as follows: First observe by Lemma 5.26 that

$$Rf_*(\mathbb{Q}[d_X]) \simeq R^0 f_*(\mathbb{Q}[d_X]) \oplus R^2 f_*(\mathbb{Q}[d_X - 2])$$

in $D_c^b(X)$. Next we have seen that $R^k f_*^\circ \mathbb{Q}$ is a local system[11] on $Y - D$. From the second half of Theorem 5.22, it follows that

$$j_{!*} R^k f_*^\circ [d_Y] \mathbb{Q} \simeq j_* R^k f_*^\circ \mathbb{Q}[d_Y] \simeq R^k f_* \mathbb{Q}[d_Y], \quad k = 0, 2.$$

In particular, $R^k f_* \mathbb{Q}[d_Y]$ is a perverse sheaf for every $k \in \mathbb{Z}$, hence ${}^p\mathcal{H}^m(R^k f_* \mathbb{Q}[d_Y]) = R^k f_* \mathbb{Q}[d_Y]$ for $m = 0$ and vanishes for $m \neq 0$. Noting that $d_X - d_Y = 2$ this yields

$$\begin{aligned} {}^p\mathcal{H}^m(Rf_*(\mathbb{Q}[d_X])) &= {}^p\mathcal{H}^{m+d_X}(Rf_*\mathbb{Q}) \\ &\simeq {}^p\mathcal{H}^{m+d_X}(R^0 f_* \mathbb{Q}[0]) \oplus {}^p\mathcal{H}^{m+d_X}(R^2 f_* \mathbb{Q}[-2]) \\ &= \begin{cases} (R^0 f_* \mathbb{Q})[d_Y], & m = d_Y - d_X = -2 \\ (R^2 f_* \mathbb{Q})[d_Y], & m = d_Y - d_X + 2 = 0 \\ 0, & \text{else.} \end{cases} \end{aligned}$$

In other words, the perverse cohomologies are in this special case concentrated in one degree,

$${}^p\mathcal{H}^m(Rf_*(\mathbb{Q}[d_X])) \simeq (R^{m+2} f_* \mathbb{Q})[d_Y], \quad \forall m \in \mathbb{Z}. \tag{5.19}$$

For the next step, consider a local system $\mathcal{L}$ on $Y^\circ = Y - D$. Since $D \subset Y$ is a smooth divisor over Z, it follows that $R^l g_*(j_* \mathcal{L})$ is a local system over Z with typical stalk $H^l(\Sigma, j_{b*}\mathcal{L})$. Here $j_b : D_b = D \cap \Sigma \times \{b\} \hookrightarrow \Sigma$ is the natural inclusion. More precisely, if $b \in Z$ is given, then there exists a small neighborhood $U \subset Z$ and a topological isomorphism

$$(g^{-1}(U), D \cap g^{-1}(U)) \cong (\Sigma \times U, D_b \times U)$$

as pairs. This implies the previous claim and we see from the definition of the perverse t-structure (since we can take as stratification just Z itself) that

$${}^p\mathcal{H}^l Rg_*(j_*\mathcal{L}) \simeq \mathcal{H}^{l-d_Z} Rg_*(j_*\mathcal{L})[d_Z] = R^{l-d_Z} g_*(j_*\mathcal{L})[d_Z]. \tag{5.20}$$

Since $j_* R^m f_*^\circ \mathbb{Q} \cong R^m f_* \mathbb{Q}$ the terms ${}^pE_2^{l,m}$ of the perverse Leray spectral sequence can be computed as

$$\begin{aligned} {}^pE_2^{l,m} &= {}^p\mathcal{H}^l Rg_*({}^p\mathcal{H}^m Rf_* \mathbb{Q}[d_X]) \\ &= {}^p\mathcal{H}^l Rg_*((R^{m+2} f_* \mathbb{Q})[d_Y]) && \text{by (5.19)} \\ &= \mathcal{H}^{l+d_Y-d_Z} Rg_*(R^{m+2} f_* \mathbb{Q})[d_Z] && \text{by (5.20)} \\ &= (R^{l+1} g_* R^{m+2} f_* \mathbb{Q})[d_Z]. \end{aligned}$$

[11] In fact, it can be shown that $R^0 f_* \mathbb{Q}$ is a local system on all of Y.

Hence up to an index shift with respect to the relative dimensions and a degree shift these are the terms of the ordinary Leray spectral sequence. As before, we see that this spectral sequence degenerates. Hence also the spectral sequence (5.18) degenerates and gives an isomorphism

$$\mathcal{H}^0 h_+ \mathbb{Q}_X^{Hdg} \cong \mathcal{H}^0 g_+ \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg} \tag{5.21}$$

of *smooth* polarizable mixed Hodge modules. In fact, the right-hand side is a smooth polarizable Hodge module of weight $d_X = d_Z + 3$. Indeed, $M := \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg}$ is a polarizable Hodge module of weight $d_Y + 2$ because it corresponds to a VHS away from the smooth divisor $D \subset Y$ (also cf. Section 5.4.1). But $g : Y \to X$ is a projective morphism so that $\mathcal{H}^0 g_+ M$ is a polarizable Hodge module of weight $d_Y + 2 = d_Z + 3$ on Z. From the last paragraph we know that it is even smooth. Under the equivalence of Theorem 5.21, we therefore obtain an isomorphism of VHS of weight 3 that lifts the isomorphism[12]

$$R^3 h_* \mathbb{Q} \cong R^1 g_* R^2 f_* \mathbb{Q}.$$

Finally, we observe that the isomorphisms of Proposition 5.18 lift to isomorphisms of pure Hodge modules with the help of Theorem 5.22 and 5.21. Indeed, the isomorphism (5.13) pulls back to give an isomorphism $(\boldsymbol{p}_{1*}\boldsymbol{\Lambda})^W_{|Y-D}(-1) \cong R^2 \boldsymbol{\pi}_{1*} \mathbb{Z}_{|Y-D}$ of polarizable $\mathbb{Z}$-VHS of weight 2 and Tate type over $Y - D$. This follows because not only the weight filtrations but also the Hodge filtrations are trivial. By Theorem 5.21 together with Theorem 5.22 they both can be extended over D to isomorphic pure Hodge modules M_1 and M_2 on Y. The underlying perverse sheaves are $\mathrm{rat}(M_1) = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_{\mathbb{Q}})^W[d_Y]$ and $\mathrm{rat}(M_2) = R^2 \boldsymbol{\pi}^1_{1*}\mathbb{Q}[d_Y]$ respectively as follows from (5.15) together with Proposition 5.18. But $M_2 = \mathcal{H}^0 f_+ \mathbb{Q}_X^{Hdg}$ in the notation of (5.21) so that

$$\mathcal{H}^0 h_+ \mathbb{Q}_X^{Hdg} \cong \mathcal{H}^0 g_+ M_1$$

lifts the isomorphism $R^3 h_* \mathbb{Q} \cong R^1 g_* (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W$ of local systems to (mixed) Hodge modules. Hence we obtain an isomorphism $\mathsf{V}^{CY} \cong \mathsf{V}^H(-1)$ of V(M)HS as claimed in light of Theorem 5.24. □

In the previous proof, we claimed that everything works over $\mathbb{Z}$ as well. To do so, we have to introduce integral structures on mixed Hodge modules. These are subtle because there are two natural perverse t-structures p and p_+ over $\mathbb{Z}$ that coincide with middle perversity after tensoring with $\mathbb{Q}$ ([BBD82], [Sch15], [Jut09]). Since we have to deal with both of them, we briefly recall their definitions for a general space X:

$$A \in {}^pD^{\leq 0}(X,\mathbb{Z}) \iff \mathcal{H}^n i_S^* A = 0, \text{ for all } n > -\dim S \text{ and each stratum } S.$$
$$A \in {}^pD^{\geq 0}(X,\mathbb{Z}) \iff \mathcal{H}^n i_S^! A = 0, \text{ for all } n < -\dim S \text{ and each stratum } S.$$

$$A \in {}^{p_+}D^{\leq 0}(X,\mathbb{Z}) \iff \forall \text{ stratum } S : \begin{cases} \mathcal{H}^n i_S^* A = 0 \ \forall n > 1 + \dim S, \\ \mathcal{H}^{1-\dim S} i_S^* A \text{ is torsion.} \end{cases}$$

$$A \in {}^{p_+}D^{\geq 0}(X,\mathbb{Z}) \iff \forall \text{ stratum } S : \begin{cases} \mathcal{H}^n i_S^! A = 0 \ \forall n < 0, \\ \mathcal{H}^{-\dim S} i_S^! A \text{ is torsion-free.} \end{cases}$$

Here $i_S : S \hookrightarrow X$ stands for the inclusion of a stratum $S \in X$ of a stratification with respect to

[12] Observe that morphisms between perverse sheaves, that are concentrated in one degree, are just sheaf morphisms. This follows because it is true in $D^b_c(X)$ and $\mathrm{P}(X) \subset D^b_c(X)$ is a full subcategory. In particular, this isomorphism is an isomorphism of constructible sheaves.

which A is constructible. Since both of these perversities are interchanged under Verdier duality, there is no good duality theory for perverse sheaves over $\mathbb{Z}$. At least there exists an intermediate extension $j_{!*}$ and ${}^{+}j_{!*}$ for p and p_+ respectively.

Example 5.27. Let X be a non-singular complex variety of $\dim_{\mathbb{C}} X = d_X$. Assume that $\mathcal{L}[0] \in D^b_c(X, \mathbb{Z})$ is a local system (in degree 0) with typical stalk L which is a finitely generated abelian group. As we can take the whole space as a stratum, we see that $\mathcal{L}[d_X]$ is in the heart ${}^pD^{\leq 0} \cap {}^pD^{\geq 0}$ of the perversity p. It is also in the heart ${}^{p_+}D^{\leq 0} \cap {}^{p_+}D^{\geq 0}$ iff L is torsion-free. As in the proof of Theorem 5.22, we see that ${}^{(+)}j_{!*}\mathcal{L}[d_X] \simeq j_*\mathcal{L}[d_X]$, if $j : U \to X$ is the inclusion of the complement of a smooth divisor in X.

Following Christian Schnell, we introduce integral structures for mixed Hodge modules as follows:

Definition 5.28 ([Sch15])**.** Let $M \in D^b(\mathrm{MHM}(X))$ be a complex of mixed Hodge modules. An *integral structure* on M is a constructible sheaf $M_{\mathbb{Z}} \in D^b_c(X, \mathbb{Z})$ such that

$$\mathrm{rat}(M) \simeq M_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}_X.$$

It can be shown that mixed Hodge modules with integral structure are compatible with the standard functors like cohomology, see [Sch15].

Example 5.29. Over a point and a single mixed Hodge structure $(H_{\mathbb{Q}}, W_\bullet, F^\bullet)$, every abelian group $H_{\mathbb{Z}}$ with $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_{\mathbb{Q}}$ is an integral structure. Note that $H_{\mathbb{Z}}$ is allowed to have torsion. The analogous statement applies to variations of (mixed) Hodge structures considered as (mixed) Hodge modules.

The next lemma gives another, still simple example involving integral structures.

Lemma 5.30. *Let $h : X \to Y$ be a* locally trivial *fibration such that $Rh_*\mathbb{Z}_X \in D^b_c(Y, \mathbb{Z})$. Then*

$$(Rh_*\mathbb{Z}_X[0]) \otimes \mathbb{Q}_Y[0] \simeq Rh_*(\mathbb{Q}_X[0])$$

in $D^b_c(Y, \mathbb{Z})$. In particular, $h_+\mathbb{Q}^{Hdg} \in D^b(\mathrm{MHM}(Y))$ has a natural integral structure.

Proof. The inclusion $\mathbb{Z}_X[0] \hookrightarrow \mathbb{Q}_X[0]$ gives a natural morphism $Rh_*\mathbb{Z}_X[0] \to Rh_*\mathbb{Q}_X[0]$. Using the $\mathbb{Z}$-flatness of $\mathbb{Q}$ we obtain

$$\Psi : (Rh_*\mathbb{Z}_X[0]) \otimes \mathbb{Q}_Y[0] \longrightarrow Rh_*\mathbb{Q}_X[0] \otimes \mathbb{Q}[0] \simeq Rh_*\mathbb{Q}_X[0].$$

After applying k-th cohomology and taking stalks at some $y \in Y$, we end up with the natural morphism

$$H^k(h^{-1}(y), \mathbb{Z}) \otimes \mathbb{Q} \longrightarrow H^k(h^{-1}(y), \mathbb{Q}).$$

It is an isomorphism because the cohomology groups are finitely generated by assumption. Hence Ψ is a quasi-isomorphism. □

Remark 5.31. If we work with $Rh_!$ instead of Rh_*, then this lemma clearly holds more general by the projection formula. It is not clear to us, how general the above version holds though. As long as one can compute the stalks of $R^kh_*\mathbb{Z}_X$, it seems to be fine. Also note that the finiteness condition (which is included in the definition of $D^b_c(X)$) is necessary, as the constant map $f : \mathbb{Z} \to pt$ shows ($\mathbb{Z}$ with the discrete topology).

Taking cohomology is further compatible with integral structures (again cf. [Sch15]). More precisely, let $M \in D^b(\mathrm{MHM}(X))$ which has an integral structure $M_{\mathbb{Z}} \in D^b_c(X, \mathbb{Z})$. Then we have

$$\mathrm{rat}(\mathcal{H}^k(M)) \simeq {}^p\mathcal{H}^k(\mathrm{rat}(M)) \simeq {}^p\mathcal{H}^k(\mathbb{Q} \otimes_{\mathbb{Z}} M_{\mathbb{Z}}) \simeq \mathbb{Q} \otimes_{\mathbb{Z}} {}^p\mathcal{H}^k(M_{\mathbb{Z}}).$$

In the last step, we can also use ${}^{p+}\mathcal{H}^k$ instead because both give to same results after tensoring with $\mathbb{Q}$.

Integral structure for Theorem 5.15. We take up the notation from (5.17) so that

$$\begin{tikzcd} X = \mathcal{X}^\circ \arrow[r, "f"] \arrow[rr, bend left, "h"] & Y = \Sigma \times \mathbf{B}^\circ \arrow[r, "g"] & Z = \mathbf{B}^\circ. \end{tikzcd}$$

Since h is locally trivial, we have

$$Rg_*Rf_*\mathbb{Q}_X \simeq Rh_*\mathbb{Q}_X \simeq (Rh_*\mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{Q}_X \simeq (Rg_*Rf_*\mathbb{Z}_X) \otimes_{\mathbb{Z}} \mathbb{Q}_X.$$

This in particular shows that the p_+-perverse[13] Leray spectral sequence for the composition $h = g \circ f$ for $\mathbb{Z}$-coefficients becomes the perverse spectral sequence for $\mathbb{Q}$-coefficients after tensoring with $\mathbb{Q}_X$. We can argue as before that the p_+-perverse Leray spectral sequence

$${}^{p+}\mathcal{H}^k g_* \, {}^{p+}\mathcal{H}^l f_* \mathbb{Z}_X[d_X] \Rightarrow {}^{p+}\mathcal{H}^{k+l} h_* \mathbb{Z}_X[d_X]$$

degenerates on the E_2-page. Indeed, using that Lemma 5.26 holds over $\mathbb{Z}$ and the intermediate extension for p_+ (cf. Example 5.27), we see as above that the E_2-page of the p_+-perverse Leray sequence for $h = g \circ f$ coincides with the ordinary Leray spectral sequence (up to shifts). But the latter even degenerates over $\mathbb{Z}$ (Lemma 5.11). Hence we see that the isomorphism $R^3h_*\mathbb{Q}_X \cong R^1g_*R^2f_*\mathbb{Q}_X$ is defined over $\mathbb{Z}$ and is compatible with the corresponding isomorphism of smooth mixed Hodge modules and VMHS. $\square$

5.2.3 The Langlands dual

As before let $\Gamma \subset SL(2, \mathbb{C})$ be a finite subgroup corresponding to a Dynkin diagram Δ of type ADE and $G = G_{ad}$ the simple adjoint complex Lie group of type Δ. Its Langlands dual ${}^LG$ is given by the simple simply-connected complex Lie group G_{sc} of the same type (cf. Remark 4.38). There is a way to give a relation between the Hitchin system for G_{sc} and a family of non-compact Calabi-Yau threefolds as for the adjoint group G. We only give an outline here without going into detail because it works in complete analogy with the adjoint case. The A_1-case has already been worked out in [DDD⁺06]. Even though the general case is known to experts, we give here a written account for the first time.

Fix a finite subgroup $\Gamma \subset SL(2, \mathbb{C})$ and the corresponding Lie algebraic data $\mathfrak{g} = \mathfrak{g}(\Gamma)$, $\mathfrak{t} = \mathfrak{t}(\Gamma)$ etc. The first step is the dual of Corollary 1.76.

Lemma 5.32. *Let $\sigma : S \to \mathfrak{t}/W$ be the restriction of the adjoint quotient and $q : \mathfrak{t} \to \mathfrak{t}/W$ the usual quotient. Then there are isomorphisms*

$$(R^2\sigma^\circ_*\mathbb{Z})^\vee \cong R^2\sigma^\circ_!\mathbb{Z} \cong (q^\circ_*\mathbf{\Lambda}_{sc})^W \tag{5.22}$$

of local systems over $\mathfrak{t}^\circ/W$ where $\mathbf{\Lambda}_{sc}$ is the cocharacter group of G_{sc}.

[13] We take p_+ because the stalk of $R^3h_*\mathbb{Z}$ are free.

Recall that in fact $\mathbf{\Lambda}_{sc} = \mathbf{\Lambda}_{ad}^{\vee}$ because Δ is simply-laced.

Proof. The proof of the second isomorphism is in complete analogy with Corollary 1.76, by recalling from Section 1.1.1 that

$$H_c^2(\hat{Y}, \mathbb{Z}) \cong H_2(\hat{Y}, \mathbb{Z}) \cong \mathbf{\Lambda}_{sc}.$$

Here $\hat{Y} \to Y = \mathbb{C}^2/\Gamma$ is the minimal resolution. Then the first isomorphism follows from $\mathbf{\Lambda}_{ad}^{\vee} \cong \mathbf{\Lambda}_{sc}$, since we work with local systems. □

Since $\mathbf{\Lambda}_{sc}^{\rho_\alpha} \cong (\mathbf{\Lambda}_{ad}^{\vee})^{\rho_\alpha^*}$, we also see that the isomorphisms (5.22) extend to $\mathfrak{t}^1/W$ as before. Moreover, the analogues of (5.22) also hold over $\boldsymbol{U}^\circ$ and $\boldsymbol{U}^1$.
Now let $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B} = \mathbf{B}(\Sigma, G_{ad}) = \mathbf{B}(\Sigma, G_{sc})$ be a family constructed from a Γ-equivariant vector bundle $V \to \Sigma$. Instead of the cohomology intermediate Jacobians, we can consider the compactly supported intermediate Jacobians[14], i.e.

$$J_2(X_b) := H_c^3(X_b, \mathbb{C})/(F^2 H_c^3(X_b, \mathbb{C}) + H_c^3(X_b, \mathbb{Z})), \quad b \in \mathbf{B}^\circ.$$

As in Section 5.2.1 we can conclude that $H_c^3(X_b, \mathbb{Z}) \cong H^1(\Sigma, R^2\pi_{b!}\mathbb{Z})$ and that $H_c^3(X_b, \mathbb{C}) = H_c^3(X_b, \mathbb{Z})_{\mathrm{tf}} \otimes \mathbb{C}$ carries a pure polarized Hodge structure of weight 1 (up to a Tate twist). This implies again that $J_2(X_b)$ is an abelian variety. Using the universal coefficient theorem it can be seen that $J_2(X_b) = \widehat{J^2(X_b)}$, cf. [DDP07]. In contrast to the compact case, $J_2(X_b)$ is in general not isomorphic to $J^2(X_b)$, in particular $J^2(X_b)$ is not self-dual.
The $J_2(X_b)$, $b \in \mathbf{B}^\circ$, fit together in a holomorphic family of abelian varieties, namely

$$\mathcal{J}_2(\mathcal{X}^\circ/\mathbf{B}^\circ) := R^3\boldsymbol{\pi}_!^\circ\mathbb{C}/(\mathcal{F}^2 + (R^3\boldsymbol{\pi}_!^\circ\mathbb{Z})_{\mathrm{tf}}) \to \mathbf{B}^\circ.$$

It is dual to $\mathcal{J}^2(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$. Hence it is again an integrable system by Corollary 2.36.

Theorem 5.33. *Let G_{sc} be a simple simply-connected complex Lie group of type* ADE. *Then there is an isomorphism*

$$\begin{tikzcd} \mathcal{J}_2(\mathcal{X}^\circ/\mathbf{B}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Higgs}_1^\circ(\Sigma, G_{sc}) \arrow[dl, "\boldsymbol{h}_1^\circ"] \\ & \mathbf{B}^\circ & \end{tikzcd} \tag{5.23}$$

of integrable systems over $\mathbf{B}^\circ$ that respects the cubics.

Proof. We keep the notation (5.17). The local system $R^3h_!\mathbb{Q}$ over $\mathbf{B}^\circ$ underlies a graded-polarizable $\mathbb{Q}$-VMHS $\mathsf{V}_{CY} = (\mathsf{V}_{CY}^{\mathbb{Z}}, \mathbb{W}_\bullet, \mathcal{F}^\bullet)$ which is in fact pure. This again follows from the isomorphism $H^3(X_b, \mathbb{Q}) \cong H^1(\Sigma, R^2\pi_{b!}\mathbb{Q})$ of mixed Hodge structures. The corresponding mixed Hodge module has $R^3h_!\mathbb{Z}$ as an integral structure. Note however, that the stalks have torsion (cf. Remark 4.38). Hence we work with the perversity p instead. The same argument as in the previous section (cf. proof of Theorem 5.15) shows that the p-perverse Leray spectral sequence yields an isomorphism

$$R^3h_!\mathbb{Z}_X \cong R^1g_!R^2f_!\mathbb{Z}_X.$$

[14]By Poincaré duality we could work with homology intermediate Jacobians instead. However, from a sheaf-theoretic point of view it is more natural to work with compactly supported cohomology.

Clearly, this induces an isomorphism $R^3 h_! \mathbb{Z}_{\mathrm{tf}} \cong R^1 g_! R^2 f_! \mathbb{Z}_{\mathrm{tf}}$, which can be used as an integral structure $\mathsf{V}_{CY}^{\mathbb{Z}}$ as well. Since proper direct images can be lifted to mixed Hodge modules, it follows as before that

$$\mathsf{V}_{CY} = (\mathsf{V}_{CY}^{\mathbb{Z}}, \mathbb{W}_\bullet, \mathcal{F}^\bullet) \cong \mathsf{V}^H(G_{sc})(-1) = ((R^1 \boldsymbol{p}_{2*} \boldsymbol{p}_{1*}^W \boldsymbol{\Lambda}_{sc})_{\mathrm{tf}}, 0, \mathcal{F}^\bullet)_{|\mathbf{B}^\circ}(-1)$$

as graded-polarizable $\mathbb{Z}$-VMHS of weight 3, where $\boldsymbol{p}^\circ = \boldsymbol{p}_2^\circ \circ \boldsymbol{p}_1^1 : \tilde{\boldsymbol{\Sigma}}^\circ \to \mathbf{B}^\circ$ is (the restriction of) the universal cameral curve. Then the claim follows as Corollary 5.16 together with Proposition 4.37. □

Remark 5.34. It should be possible to deduce the above statement directly from Theorem 5.15 using Verdier duality. For example, the fact that we had to work with p_+ and p, seems to be a 'shadow' of Verdier duality (which interchanges p_+ with p).

5.3 Families with global $AS(\Delta)$-symmetry

As we have seen in Chapter 1, folding provides a useful tool to pass from ADE-Dynkin diagrams Δ_h to BCFG-Dynkin diagrams Δ. We now want to use this simple but elegant method to obtain similar statements as above for the BCFG-cases. By our preparatory work, some of these results are almost immediate, e.g. we will easily obtain a family $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ of Gorenstein Calabi-Yau threefolds over the Hitchin base $\mathbf{B}$ of type BCFG with fiber-preserving $AS(\Delta)$-action.

Let Δ be a BCFG-Dynkin diagram and $(\Delta_h, AS(\Delta))$ the associated pair consisting of an ADE-Dynkin diagram Δ_h and $AS(\Delta) \subset \mathrm{Aut}(\Delta_h)$ such that $\Delta_h^{AS} = \Delta$. Then we can consider Slodowy slices $S \subset \mathfrak{g} = \mathfrak{g}(\Delta)$ and $S_h \subset \mathfrak{g}_h = \mathfrak{g}(\Delta_h)$. Recall that there are groups $\mathbf{C} \cong AS(\Delta)$ and $\mathbf{CA} \cong AS(\Delta)$ that naturally act on S and S_h respectively[15]. To globalize them over Σ, we fix a 'diagonal' vector bundle $V = L \oplus L$ as in Section 5.1.2, in particular $L^2 = K_\Sigma$. Then we have the glued Slodowy slices[16]

$$\mathcal{S} = L \times_{\mathbb{C}^*} S, \quad \mathcal{S}_h = L \times_{\mathbb{C}^*} S_h, \tag{5.24}$$

together with the simultaneous resolutions $\boldsymbol{\psi} : \tilde{\mathcal{S}} \to \mathcal{S}$ and $\boldsymbol{\psi}_h : \tilde{\mathcal{S}}_h \to \mathcal{S}_h$. Since the $AS(\Delta)$-action commutes with the $\mathbb{C}^*$-action, $\mathcal{S}$ and $\mathcal{S}_h$ carry $AS(\Delta)$-actions. This works analogously for the bundles $\boldsymbol{U} = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W$ and $\boldsymbol{U}_h = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}_h/W_h$. The natural morphisms $\boldsymbol{\sigma} : \mathcal{S} \to \boldsymbol{U}$ and $\boldsymbol{\sigma}_h : \mathcal{S}_h \to \boldsymbol{U}_h$ are $AS(\Delta)$-equivariant. It follows from the local theory, Corollary 1.56 ii), that there is a *cartesian* square

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\cong} & \mathcal{S}_{h,\mathbf{CA}} \\ {\scriptstyle \boldsymbol{\sigma}}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{\sigma}_h} \\ \boldsymbol{U} & \xrightarrow{\cong} & \boldsymbol{U}_h^{\mathbf{CA}}, \end{array} \tag{5.25}$$

where all the morphisms are $AS(\Delta)$-equivariant and $\mathcal{S}_{h,\mathbf{CA}} = \boldsymbol{\sigma}_h^{-1}(\boldsymbol{U}_h^{\mathbf{CA}})$. The lower isomorphism in (5.25) is induced by an isomorphism $\mathfrak{t}/W \cong (\mathfrak{t}_h/W_h)^{\mathbf{CA}}$. Hence the family $\mathcal{S} \to \boldsymbol{U}$ of surfaces inherits all the properties from $\mathcal{S}_h \to \boldsymbol{U}_h$ that are stable under fiber products. However, since each fiber carries an $AS(\Delta)$-action, it has some further special properties.

[15] Even though $\mathbf{C} \cong AS(\Delta) \cong \mathbf{CA}$, we often write $\mathbf{C}$ or $\mathbf{CA}$ instead, to emphasize how the $AS(\Delta)$-action is constructed (cf. Section 1.4.3).

[16] Remark 5.8 applies here as well, i.e. we already drop the dependency on V, or rather L, from the notation but we will address this question in the future. Let us at least comment that there are now only finitely many choices (more precisely 2^{2g} many).

Lemma 5.35. *There exist nowhere vanishing sections* $\hat{\boldsymbol{\omega}} \in H^0(\tilde{\mathcal{S}}, K_{\tilde{\boldsymbol{\sigma}}} \otimes (\tilde{u} \circ \tilde{\boldsymbol{\sigma}})^* K_\Sigma)$ *and* $\hat{\boldsymbol{\nu}} \in H^0(\mathcal{S}, K_{\boldsymbol{\sigma}} \otimes (u \circ \boldsymbol{\sigma})^* K_\Sigma)$. *They are* $AS(\Delta)$*-invariant,*

$$a^* \hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}, \quad a^* \hat{\boldsymbol{\nu}} = \hat{\boldsymbol{\nu}}, \quad a \in AS(\Delta)$$

and satisfy $\boldsymbol{\psi}^* \hat{\boldsymbol{\nu}} = \hat{\boldsymbol{\omega}}$ *for* $\boldsymbol{\psi} : \tilde{\mathcal{S}} \to \mathcal{S}$. *Moreover,* $\hat{\boldsymbol{\omega}}$ *induces a period map*

$$\boldsymbol{\eta} : \tilde{\boldsymbol{U}} \to \tilde{u}^* \tilde{\boldsymbol{U}} = \tilde{u}^* K_\Sigma \otimes \mathfrak{t} \tag{5.26}$$

for the Cartan subalgebra $\mathfrak{t}$ *of type* BCFG *which coincides with the tautological section* $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^* \tilde{\boldsymbol{U}})$.

Proof. The existence of the sections follows from Proposition 5.5 via pullback, cf. (5.25). By construction, the invariance property of $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{\nu}}$ follow from the invariance property of the corresponding local sections $\hat{\omega}$ and $\hat{\nu}$ respectively, see Corollary 1.104.
Finally, we see as in the proof of Proposition 5.5 iii) that $\hat{\boldsymbol{\omega}}$ induces a morphism

$$\boldsymbol{\eta} : \tilde{\boldsymbol{U}} \to H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}} \otimes \tilde{u}^* K_\Sigma$$

because $\hat{\boldsymbol{\omega}}$ is $\mathbf{C}$-invariant. It coincides with the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^* \tilde{\boldsymbol{U}})$ after identifying $H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}} = \mathfrak{t}$ via the local period map $P_{\tilde{S}} : \mathfrak{t} \cong H^2(\tilde{S}_0, \mathbb{C})^{\mathbf{C}}$ of Corollary 1.98. ☐

Using the evaluation map $ev : \Sigma \times \mathbf{B} \to \boldsymbol{U}$, we obtain a family $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ of threefolds as in (5.3).

Proposition 5.36. *The morphism* $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ *is a family of quasi-projective Gorenstein Calabi-Yau threefolds that carry actions by* $\mathbf{C} \cong AS(\Delta)$. *Each member* X_b *admits a* $\mathbf{C}$*-invariant nowhere vanishing section* $s_b \in H^0(X_b, K_{X_b})^{\mathbf{C}}$ *and is non-singular for* $b \in \mathbf{B}^\circ$.

Proof. The cartesian square (5.25) yields the cartesian square

$$\begin{CD} \mathcal{X} @>>> \mathcal{X}_{h,\mathbf{CA}} \\ @V{\boldsymbol{\pi}}VV @VV{\boldsymbol{\pi}_h}V \\ \mathbf{B} @>{\cong}>> \mathbf{B}_h^{\mathbf{CA}} \end{CD} \tag{5.27}$$

of $AS(\Delta)$-spaces with $\mathcal{X}_{h,\mathbf{CA}} = \boldsymbol{\pi}^{-1}(\mathbf{B}_h^{\mathbf{CA}})$. Here we endow $\mathbf{B}_h = H^0(\Sigma, \boldsymbol{U}_h)$ with the natural action by $\mathbf{CA} \cong AS(\Delta)$. The first statement now follows from (5.27).
Denote by $j : \mathcal{X} \to \mathcal{S}$ the induced map from the fiber product construction and $j_b : X_b \to \mathcal{S}$ the corresponding restriction. As before, we see that

$$s_b := j_b^* \hat{\boldsymbol{\nu}} \in H^0(X_b, K_{X_b})$$

is a nowhere vanishing section. Since j_b is $\mathbf{C}$-equivariant by construction, it follows from Lemma 5.35 that s_b is $\mathbf{C}$-invariant. ☐

Remark 5.37. The cartesian square (5.27) explains the relation between the family $\mathcal{X} \to \mathbf{B}$ and $\mathcal{X}_h \to \mathbf{B}_h$, i.e. the former is just a subfamily of the latter. So far, we mainly worked with the restricted family $\mathcal{X}_h^\circ \to \mathbf{B}_h^\circ$ and we focus on the restriction

$$\mathcal{X}^\circ = \mathcal{X}_{h|\mathbf{B}^\circ} \to \mathbf{B}^\circ \subset \mathbf{B}_h$$

in the following. The next lemma shows that these two restrictions are disjoint.

Lemma 5.38. *If $\mathbf{B} \hookrightarrow \mathbf{B}_h$ are as before, then $\mathbf{B}^\circ \cap \mathbf{B}_h^\circ = \emptyset$. Moreover, the fibers $X_{h,b}$ of $\boldsymbol{\pi}_h$ are non-singular for $b \in \mathbf{B}^\circ \coprod \mathbf{B}_h^\circ \subset \mathbf{B}_h$.*

Proof. This is a direct consequence of the local theory: Let $\alpha \in \Delta$ be a long root. Under folding, it corresponds to an $AS(\Delta)$-orbit $O(\beta)$ of length ≥ 2 for some $\beta \in \Delta_h$. Now let $b \in \mathbf{B}^\circ$ be given. Since $D_b^\alpha \subset \tilde{\Sigma}_b$ is non-empty, $b : \Sigma \to \boldsymbol{U} \subset \boldsymbol{U}_h$ does not map into $\boldsymbol{U}_h^1$. In particular, it cannot lie in $\mathbf{B}_h^\circ$. The last statement is now immediate from Corollary 5.36. □

Remark 5.39. Even though the BCFG-Hitchin base $\mathbf{B}$ is naturally contained in the ADE-Hitchin base as $\mathbf{B}_h^{AS} \subset \mathbf{B}_h$, the cameral curves do not behave as nicely under pullback as the above threefolds. The simple reason begin that the commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r, hook] \arrow[d, "q"] & \mathfrak{t}_h \arrow[d, "q_h"] \\ \mathfrak{t}/W \arrow[r, hook] & \mathfrak{t}_h/W_h \end{tikzcd}$$

is *not* cartesian (simply because q_h is a branched W_h-covering whereas q is a branched $W = W_{h,AS}$-covering). Therefore a BCFG-cameral curve cannot be the pullback of an ADE-cameral curve, also cf. Lemma 5.38. However, it would be interesting to know, if it is possible to 'fold' ADE-Hitchin fibers to BCFG-Hitchin fibers. As we have seen, this would require to understand the (possibly singular) ADE-Hitchin fibers over the locus $\mathbf{B}^\circ \subset \mathbf{B}_h$.

Even though many of the properties of $\mathcal{X}_h \to \mathbf{B}_h$ directly carry over to $\mathcal{X} \to \mathbf{B}$ we still have to be careful, as the previous lemma shows. For example, the fibers of $\pi_b : X_b \to \Sigma$, $b \in \mathbf{B}^\circ$, are slightly different from those of $\pi_{h,b} : X_{h,b} \to \Sigma$, $b \in \mathbf{B}_h^\circ$, as we see in the proof of the next lemma. It is the analogue of Lemma 5.11 over BCFG-Hitchin bases.

Lemma 5.40. *Let $\boldsymbol{\pi}^\circ = \boldsymbol{\pi}_2^\circ \circ \boldsymbol{\pi}_1^1 : \mathcal{X}^\circ \to \mathbf{B}^\circ$ be the family of non-singular, non-compact CY3s with a nontrivial $AS(\Delta)$-action. Then the Leray spectral sequence*

$$R^p \boldsymbol{\pi}_{2*}^\circ R^q \boldsymbol{\pi}_{1*}^1 \mathbb{Z} \Rightarrow R^{p+q} \boldsymbol{\pi}_*^\circ \mathbb{Z}$$

degenerates and gives an isomorphism $R^3 \boldsymbol{\pi}_*^\circ \mathbb{Z} \cong R^1 \boldsymbol{\pi}_{2*}^\circ R^2 \boldsymbol{\pi}_{1*}^1 \mathbb{Z}$.

Proof. Since this is very close to Lemma 5.11, we restrict ourselves to the fiberwise degeneration of the Leray spectral sequence. As before, it follows that the differential d_r on the E_r-page, $r \geq 3$, vanishes. Hence it remains to prove that the differential on the E_2-page,

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1},$$

for $E_2^{r,s} = H^r(\Sigma, R^s \pi_* \mathbb{Z})$, vanishes as well. We will prove this by showing that

$$R^q \pi_* \mathbb{Z} = 0, \quad q \neq 0, 2, \tag{5.28}$$

because then d_2 either maps from or to 0.
Fix $b \in \mathbf{B}^\circ$ and let $D \subset \Sigma$ denote the branch points of the cameral cover $\tilde{\Sigma}_b \to \Sigma$ as well as $\Sigma^\circ = \Sigma - D$. If $x \in \Sigma^\circ$, then the restriction $X_U = \pi^{-1}(U)$ of $X = X_b$ is given by the fiber product

$$\begin{tikzcd} X_U \arrow[r] \arrow[d, "\pi_U"] & S^\circ \arrow[d, "\sigma^\circ"] \\ U \arrow[r, "b_U"] & \mathfrak{t}^\circ/W, \end{tikzcd}$$

where $U \subset \Sigma^\circ$ is a small enough open neighborhood of x. From Section 1.4.6 we know that $S^\circ \to \mathfrak{t}^\circ/W$ is a C^∞-locally trivial fiber bundle with typical fiber $\tilde{Y}_h$, the minimal resolution of Y_h. Thus we obtain for $\bar{t} \in \mathfrak{t}^\circ/W$,

$$(R^q\sigma^\circ_*\mathbb{Z})_{\bar{t}} = H^q(S_{\bar{t}}, \mathbb{Z}) = 0, \quad q \neq 0, 2.$$

This implies the claim for $x \in \Sigma^\circ$ by base change (for locally trivial fibrations).
It remains to consider the case $x \in D$. Again, we find an open neighborhood U around x such X_U is given by a fiber product similar as above. The only difference is that $\bar{t} := b_U(x) \in \mathfrak{q}^{sm} \subset \mathfrak{t}^1/W$, the non-singular part of the discriminant of $q : \mathfrak{t} \to \mathfrak{t}/W$. By Example 1.75, $S_{\bar{t}}$ is therefore isomorphic to $\tilde{Y}_h$ with up to three exceptional curves contracted[17]. Therefore $S_{\bar{t}}$ has the homotopy type of a tree of 2-spheres. If U is small enough, we can deformation retract X_U onto $\pi^{-1}(x) \cong S_{\bar{t}}$ to conclude that $(R^q\pi_*\mathbb{Z})_x = 0$ for $q \neq 0, 2$. So we have proven (5.28) also in this case and the Leray spectral sequence degenerates. □

The next lemma is in complete analogy with the important Proposition 5.18 (also cf. Proposition 1.76). Therefore its proof is omitted.

Lemma 5.41. *Let $\tilde{\boldsymbol{U}}^1 \subset \tilde{\boldsymbol{U}}$ and $\boldsymbol{U}^1 \subset \boldsymbol{U}$ be as in (4.11), (4.12) and $\mathcal{S}^1 := \boldsymbol{\sigma}^{-1}(\boldsymbol{U}^1) \subset \mathcal{S}$. Moreover, let $\boldsymbol{q}^1 : \tilde{\boldsymbol{U}}^1 \to \boldsymbol{U}^1$ and $\boldsymbol{\sigma}^1 : \mathcal{S}^1 \to \boldsymbol{U}^1$ be the restrictions of $\boldsymbol{q}$ and $\boldsymbol{\sigma}$ respectively. Then there is an isomorphism*

$$R^2\boldsymbol{\sigma}^1_*\mathbb{Z} \cong (\boldsymbol{q}^1_*\boldsymbol{\Lambda}_h)^W, \quad (R^2\boldsymbol{\sigma}^1_*\mathbb{Z})^{\mathbf{C}} \cong (\boldsymbol{q}^1_*\boldsymbol{\Lambda})^W$$

of constructible sheaves.

These considerations can be used to make some statements about the cohomology groups $H^*(X) = H^*(X, \mathbb{Z})$, where $X = X_b$, $b \in \mathbf{B}^\circ$ is a fixed CY3 with projection $\pi : X \to \Sigma$. They all apply to the ADE-cases as well but since they need some extra care for the BCFG-cases, we discuss them only now. From Lemma 5.40 we can conclude:

$$H^0(X) \cong \mathbb{Z}, \quad H^1(X) \cong H^1(\Sigma, \mathbb{Z}),$$
$$H^3(X) \cong H^1(\Sigma, R^2\pi_*\mathbb{Z}), \quad H^4(X) \cong H^2(\Sigma, R^2\pi_*\mathbb{Z}),$$
$$H^k(X) = 0, \quad k \geq 5.$$

Here we have used that $\pi_*\mathbb{Z} \cong \mathbb{Z}$. The remaining interesting cohomology group is $H^2(X, \mathbb{Z})$. Its graded pieces (where the filtration comes from the Leray spectral sequence) are given by $H^0(\Sigma, R^2\pi_*\mathbb{Z})$ and $H^2(\Sigma, \pi_*\mathbb{Z})$, as we see from the previous proof. However, $H^0(\Sigma, R^2\pi_*\mathbb{Z}) = 0$ (cf. Lemma 5.44 below) so that in fact

$$H^2(X, \mathbb{Z}) \cong H^2(\Sigma, \pi_*\mathbb{Z}) \cong H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

Finally, we have an analogous result for the mixed Hodge structure on third cohomology.

Lemma 5.42. *Let $b \in \mathbf{B}^\circ$. Then $H^3(X_b, \mathbb{Z})$ is torsion-free and underlies a pure Hodge structure of weight 3 which is effective of weight 1 up to a Tate twist.*

Proof. The second statement follows precisely as before, but the first statement needs a bit more care. Lemma 5.41 implies that $R^2\pi_{b*}\mathbb{Z} \cong (p_{b*}\boldsymbol{\Lambda}_h)^W$. Since the stalk is *not* $\boldsymbol{\Lambda}$ we cannot directly

[17] These are not connected to each other in the homogeneous Dynkin diagram Δ_h, so $S_{\bar{t}}$ has at most three A_1-singularitites.

apply Remark 4.38 to conclude the vanishing of torsion. However, the argument for $(p_{b*}\mathbf{\Lambda})^W$ also works for $(p_{b*}\mathbf{\Lambda}_h)^W$ (cf. Proposition 5.47 below) so that

$$\mathrm{Tors}(H^1(\Sigma, R^2\pi_{b*}\mathbb{Z})) = \mathrm{Tors}(H^1(\Sigma, (p_{b*}\mathbf{\Lambda}_{G_h})^W)) = 0.$$

□

This argument applies to the other cohomology groups $H^k(X_b, \mathbb{Z})$, $b \in \mathbf{B}^\circ$ as well. However, $H^4(X_b, \mathbb{Z})$ might have non-zero torsion.

5.4 The BCFG-case

The underlying real torus of the (cohomology) intermediate Jacobian $J^2(X)$ of a member $X = X_b$ for $b \in \mathbf{B}^\circ$ is given by

$$J^2(X) = H^3(X, \mathbb{Z}) \otimes S^1.$$

The complex structure on $J^2(X)$ is specified by the pure Hodge structure on $H^3(X, \mathbb{Z})$. By the $\mathbf{C}$-equivariance of $\boldsymbol{\pi}$, each member X of the family $\mathcal{X}$ inherits a $\mathbf{C}$-action. In particular, $\mathbf{C}$ acts on $H^3(X, \mathbb{Z})$ by Hodge morphisms. Since the category of pure (or mixed) Hodge structures is abelian, the $\mathbf{C}$-invariants $H^3(X, \mathbb{Z})^{\mathbf{C}}$ carry a natural pure Hodge structure for $b \in \mathbf{B}^\circ$. It follows that

$$J^2_{\mathbf{C}}(X) := H^3(X, \mathbb{Z})^{\mathbf{C}} \otimes S^1$$

is not only a real subtorus of $J^2(X)$ but is in fact an abelian subvariety. Note that this is a priori not $J^2(X)^{\mathbf{C}}$, since the fixed point set might have several connected components. In the following we want to relate $J^2_{\mathbf{C}}(X) = J^2_{\mathbf{C}}(X_b)$ to the generalized Prym variety $P_b = H^1(\Sigma, \mathcal{T}(b))$ for $b \in \mathbf{B}^\circ$ and eventually prove a global result as in the ADE-case.

Theorem 5.43. *Let Δ be a connected Dynkin diagram of type* BCFG*, G the associated simple adjoint complex Lie group and $\mathbf{B} = \mathbf{B}(\Sigma, G)$ the corresponding Hitchin base. If $b \in \mathbf{B}^\circ$, then*

$$P_b \cong J^2_{\mathbf{C}}(X_b)$$

as abelian varieties.

Isomorphic as real tori

We prove Theorem 5.43 in several steps starting on the real level. To this end, we follow a direct approach and consider the exact sequence

$$0 \longrightarrow \mathcal{F}^{\mathbf{C}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{5.29}$$

where $\mathcal{F} := R^2\pi_*\mathbb{Z}$ and $\mathcal{G}$ is the quotient. It induces the long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^0(\Sigma, \mathcal{F}) \longrightarrow H^0(\Sigma, \mathcal{G}) \\ \hookrightarrow H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^1(\Sigma, \mathcal{F}) \longrightarrow H^1(\Sigma, \mathcal{G}) \\ \hookrightarrow H^2(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^2(\Sigma, \mathcal{F}) \longrightarrow H^2(\Sigma, \mathcal{G}). \end{array} \tag{5.30}$$

Since $\mathcal{F}^{\mathbf{C}} \cong (p_*\mathbf{\Lambda})^W$, we can use earlier results to conclude that $H^0(\Sigma, \mathcal{F}^{\mathbf{C}}) = 0$ and $H^2(\Sigma, \mathcal{F}^{\mathbf{C}})$ is torsion.

Lemma 5.44. *Let $\mathcal{F} = R^2\pi_*\mathbb{Z}$ be as before. Then it has no global sections, $H^0(\Sigma, \mathcal{F}) = 0$.*

Proof. Recall that $\mathcal{F} \cong j_*\mathcal{F}^\circ$ where $\mathcal{F}^\circ = R^2\pi^\circ\mathbb{Z}$ so that $H^0(\Sigma, \mathcal{F}) = \mathbf{\Lambda}_h^{mon}$ for the monodromy group mon of $\mathcal{F}^\circ$. Since $R^2\pi^\circ\mathbb{Z} \cong (p_*^\circ\mathbf{\Lambda}_h)^W$ by Lemma 5.41, the monodromy group is

$$mon = W = \langle \prod_{\beta \in \mathbf{C}\cdot\alpha} \rho_\beta \mid \alpha \in R_h \rangle \subset W_h, \quad \rho_\beta = s_\beta^\vee.$$

Note that the elements of a $\mathbf{C}$-orbit are orthogonal to each other so that the ordering in the above product is irrelevant. This also implies that for $\rho_\alpha = \prod_{\beta \in \mathbf{C}\cdot\alpha} \rho_\beta \in W^\vee$, $\alpha \in R_h$, we have

$$\mathbf{\Lambda}_h^{\rho_\alpha} = \bigcap_{\beta \in \mathbf{C}\cdot\alpha} \mathbf{\Lambda}_h^{\rho_\beta}$$

for the hyperplane $\mathbf{\Lambda}_h^{\rho_\alpha} \subset \mathbf{\Lambda}_h$ fixed by ρ_α. Therefore we obtain

$$H^0(\Sigma, R^2\pi_*\mathbb{Z}) \cong \mathbf{\Lambda}_h^W = \mathbf{\Lambda}_h^{W_h} = 0.$$

□

Consequently, the long exact sequence (5.30) simplifies to the left. At this point, we could go on to further reduce it, e.g. by showing that $H^0(\Sigma, \mathcal{G}) = 0$. Instead we choose the most direct way possible by explicitly showing that the induced map

$$H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^1(\Sigma, \mathcal{F})^{\mathbf{C}} \subset H^1(\Sigma, \mathcal{F})$$

is an isomorphism. To do so, we recall and slightly extend some results from [DP12], Section 6, to describe these cohomology groups. We work out the alternative approach in an example (Example 5.48) to give another point of view on this result and to obtain additional information (at least in the example).

Let $\mathcal{L}$ be a local system with typical stalk L over $\Sigma^\circ \xhookrightarrow{j} \Sigma$. Then we have the following well-known lemma.

Lemma 5.45. *Let $\mathcal{L}$ and $j : \Sigma^\circ \hookrightarrow \Sigma$ as before. Then the following holds*

$$\begin{aligned} H^1(\Sigma, j_*\mathcal{L}) &\cong \ker[H^1(\Sigma^\circ, \mathcal{L}) \to H^0(\Sigma, R^1 j_*\mathcal{L})] \\ &\cong \mathrm{im}[H^1_c(\Sigma^\circ, \mathcal{L}) \to H^1(\Sigma^\circ, \mathcal{L})]. \end{aligned}$$

Proof. The first description is a consequence of the five-term exact sequence coming from the Leray spectral for the open inclusion $j : \Sigma^\circ \hookrightarrow \Sigma$. Its first three (non-trivial) terms are given by

$$0 \longrightarrow H^1(\Sigma, j_*\mathcal{L}) \longrightarrow H^1(\Sigma^\circ, \mathcal{L}) \xrightarrow{\beta} H^0(\Sigma, R^1 j_*\mathcal{L}) \tag{5.31}$$

yielding the first description. For the second description see [Loo97]. □

Hence as a first step we have to describe $H^1(\Sigma^\circ, \mathcal{L})$. In fact this will be sufficient for our purposes. Let $Br = \{y_1, \ldots, y_n\} \xhookrightarrow{i} \Sigma$ be the branch points, i.e. the complement of Σ°. As in [DP12] it is convenient to add an extra point y_0 to Br, since it simplifies some of the arguments. Now we can describe the fundamental group of $\Sigma^\circ - \{y_0\}$ as follows: Choose an arc system

$\delta_1, \ldots, \delta_{2g}, \gamma_0, \gamma_1, \ldots, \gamma_m$ where the γ_j generate the (local) fundamental group around the puncture y_j. Then we have the well-known description(s)

$$\pi_1(\Sigma^\circ - \{y_0\}, \mathfrak{o}) = \left\langle \delta_1, \ldots, \delta_{2g}, \gamma_0, \ldots, \gamma_m \;\middle|\; \gamma_0 = \prod_{i=1}^{g} [\delta_i, \delta_{i+g}] \prod_{j=0}^{m} \gamma_j \right\rangle$$
$$= \langle \delta_1, \ldots, \delta_{2g}, \gamma_1, \ldots, \gamma_m \rangle ,$$

where $\mathfrak{o} \in \Sigma^\circ - \{y_0\}$ is a fixed base point. The second description is reminiscent of the fact that $\Sigma^\circ - \{y_0\}$ is homotopy equivalent to the bouquet of $2g + m$ circles, all attached to the point $\mathfrak{o}$. We now fix an isomorphism $\mathcal{L}_{\mathfrak{o}} \cong L$ once and for all and denote by $\rho_i = mon(\gamma_i), w_j = mon(\delta_j) \in \mathrm{Aut}(L)$ the monodromy transformation corresponding to γ_i and δ_j respectively. Clearly, since $\mathcal{L}$ is a local system on Σ°, we must have $\rho_0 = mon(\gamma_0) = id_L$.

Remark 5.46. Clearly, $R^1 j_* \mathcal{L}$ is a skyscraper sheaf supported on $Br = \Sigma - \Sigma^\circ$. By a local computation, it can be shown that $H^1(D_j, \mathcal{L}) = L_{\rho_j}$ are the coinvariants in L where $D_j \subset \Sigma$ is a small disc around $b_j \in Br$. Taking the limit over all such discs yields

$$R^1 j_* \mathcal{L} = \bigoplus_{k=1}^{m} (R^1 j_* \mathcal{L})_{y_k} \cong \bigoplus_{k=1}^{m} L_{\rho_k}.$$

The morphism $\beta : H^1(\Sigma^\circ, \mathcal{L}) \to \bigoplus_k L_{\rho_k}$ in (5.31) associates to a class its values at the stalks. In particular, β is $\mathbf{C}$-equivariant so that the $\mathbf{C}$-action on $H^1(\Sigma, j_* \mathcal{L}) = \ker \beta$ is induced by that on $H^1(\Sigma^\circ, \mathcal{L})$.

The next proposition is essentially contained in [DP12] where the case $\mathcal{L} = (p^\circ_* \mathbf{\Lambda})^W$ is discussed. It turns out that the method of proof works more generally. We need a more general statement because we work with $\mathcal{L} = R^2 \pi^\circ_* \mathbb{Z} \cong (p^\circ_* \mathbf{\Lambda}_h)^W$ as well.

Proposition 5.47. *Let $\mathcal{L}$ be a local system over Σ° and $L \cong \mathcal{L}_{\mathfrak{o}}$ its typical stalk. Further let $p : \tilde{\Sigma} \to \Sigma$ be a smooth cameral curve.*

i) There is a natural isomorphism

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong \frac{L^{2g+m}}{(1 - w_1, \ldots, 1 - w_{2g}, 1 - \rho_1, \ldots, 1 - \rho_m)L}.$$

ii) Assume additionally that $(p^\circ)^* \mathcal{L} \cong L_{\tilde{\Sigma}^\circ}$ as abelian sheaves. Then there is a non-canonical isomorphism

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong H^1(\Sigma, L) \oplus \frac{L^m}{(1 - \rho_1, \ldots, 1 - \rho_m)L}.$$

Proof. The first part can be proven as Proposition 6.5. in [DP12] by using that

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}) \cong H^1(\pi_1, L), \quad \pi_1 = \pi_1(\Sigma^\circ - \{y_0\}, \mathfrak{o}),$$

still holds. Note again that $\rho_0 = id_L$ gives no contribution.
For the second part, we first observe that

$$\mathcal{L} \cong (p^\circ_* L)^W \tag{5.32}$$

by the additional assumption, cf. Section 1.4.6. In the proof of Proposition 6.5. of [DP12] it was shown that topologically one can assume the following situation: There exists a disc $D \subset \Sigma$ such that $Br \subset D$ and all the γ_j's are contained in D. Moreover, one can assume

$$p^{-1}(\Sigma - D) = \coprod_{w \in W} [\Sigma - D]_w$$

where $[\Sigma - D]_w$ are the connected components which are all isomorphic to $\Sigma - D$ (via p). Then (5.32) implies that $w_i = id_L$ giving the second (non-canonical) isomorphism. $\square$

Proof of Proposition 5.43. Lemma 5.45 and Remark 5.46 imply that it is sufficient to show that the map

$$H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}^{\mathbf{C}}) \to H^1(\Sigma^\circ - \{y_0\}, \mathcal{L})^{\mathbf{C}} \subset H^1(\Sigma^\circ - \{y_0\}, \mathcal{L}),$$

induced from the inclusion $\mathcal{L}^{\mathbf{C}} = (R^2\pi_*^\circ\mathbb{Z})^{\mathbf{C}} \hookrightarrow \mathcal{L} = R^2\pi_*^\circ\mathbb{Z}$, is an isomorphism. By Proposition 5.47 this amounts to showing that the natural map

$$\iota : H^1(\Sigma, \mathbf{\Lambda}) \oplus \frac{\mathbf{\Lambda}^m}{(1-\rho_1, \dots, 1-\rho_m)\mathbf{\Lambda}} \longrightarrow \left(H^1(\Sigma, \mathbf{\Lambda}_h) \oplus \frac{\mathbf{\Lambda}_h^m}{(1-\rho_1, \dots, 1-\rho_m)\mathbf{\Lambda}_h} \right)^{\mathbf{C}} \tag{5.33}$$

is an isomorphism. Here we have fixed isomorphisms $\mathcal{L}_{\mathfrak{o}}^{\mathbf{C}} \cong \mathbf{\Lambda}$ and $\mathcal{L}_{\mathfrak{o}} \cong \mathbf{\Lambda}_h$ as before. Also note that $\rho_j = \rho_{\alpha_j}$ for roots α_j which correspond to the monodromy around y_j.
Of course, ι preserves the respective first factors in (5.33) giving an isomorphism $H^1(\Sigma, \mathbf{\Lambda}) \cong H^1(\Sigma, \mathbf{\Lambda}_h)^{\mathbf{C}}$. So it remains to check the second factors. For the injectivity, assume that $\iota([\lambda_1, \dots, \lambda_m]) = 0$. This happens iff there exists $\mu \in \mathbf{\Lambda}_h$ such that

$$\lambda_i = (1-\rho_i)\mu = \langle \alpha_i, \mu \rangle \alpha_i^\vee \in \mathbf{\Lambda} \subset \mathbf{\Lambda}_h, \quad \forall i = 1, \dots, m.$$

So we have to exclude the case that $\langle \alpha_i, \mu \rangle \notin \langle \alpha_i, \mathbf{\Lambda} \rangle \subset \mathbb{Z}$. However, this is impossible because $1 \in \langle \alpha_i, \mathbf{\Lambda} \rangle$ (or $\epsilon_{G,\alpha_i} = 1$ in the notation of the proof of Proposition 4.28), since G is adjoint.
For the surjectivity of ι, assume $[\lambda_1, \dots, \lambda_m]_h \in \mathbf{\Lambda}_h^m/(1-\rho_1, \dots, 1-\rho_m)\mathbf{\Lambda}_h$ such that

$$\begin{aligned} &\sigma \cdot [\lambda_1, \dots, \lambda_m]_h = [\lambda_1, \dots, \lambda_m]_h \\ \Leftrightarrow\ &\sigma \cdot \lambda_i - \lambda_i = \langle \alpha_i, \mu \rangle \alpha_i^\vee, \quad \forall i = 1, \dots, m \end{aligned}$$

for some $\mu \in \mathbf{\Lambda}_h$ and $\sigma \in \mathbf{C}$ is a generator for the cyclic group $\mathbf{C}$. For the moment assume $\mathbf{C} = \mathbb{Z}/2\mathbb{Z}$. Then using $\alpha_i^\vee \in \mathbf{\Lambda} = \mathbf{\Lambda}_h^{\mathbf{C}}$ we have

$$\sigma \cdot (\sigma \cdot \lambda_i - \lambda_i) = \lambda_i - \sigma \cdot \lambda_i = \sigma \cdot \lambda_i - \lambda_i \ \Leftrightarrow\ 2(\sigma \cdot \lambda_i - \lambda_i) = 0.$$

Hence $\lambda_i = \sigma \cdot \lambda_i$ for all $i = 1, \dots, m$ so that $\lambda_i \in \mathbf{\Lambda}_h^{\mathbf{C}}$. In other words, $[\lambda_1, \dots, \lambda_m]_h$ is in the image of ι. In case $\mathbf{C} = S_3$ one can argue similarly by taking generators of order 2 and 3. Therefore ι is an isomorphism in all cases. $\square$

Together with Lemma 5.41, this yields the isomorphism $H^1(\Sigma, R^2\pi_*\mathbb{Z})^{\mathbf{C}}) \cong H^1(\Sigma, (p_*\mathbf{\Lambda})^W)$ and hence $J^2_{\mathbf{C}}(X_b) \cong P_b$ as *real* tori.

Example 5.48. We take up the exact sequences (5.29) and (5.30) in the special case $\Delta_h = \mathrm{A}_3$ and $\Delta = \mathrm{B}_2$. It also gives another opportunity to give an application of Proposition 5.47 and provides more information. However, it is not necessary for the rest of our discussion.

Let $G = SO(5,\mathbb{C})$ and $G_h = PSL(4,\mathbb{C})$ be the corresponding simple adjoint Lie groups. The root data $(X(G_h), R_h, X^\vee(G_h, R_h^\vee)) = (\mathbf{\Lambda}_h^\vee, R_h, \mathbf{\Lambda}_h, R_h^\vee)$ can be described as

$$\mathbf{\Lambda}_h^\vee = \{(x_1,\dots,x_4) \in \mathbb{Z}^4 \mid \sum_{i=1}^4 x_i = 0\}, \quad R_h = \{e_i - e_j \mid i \neq j\},$$
$$\mathbf{\Lambda}_h = \mathbb{Z}^4/\mathbb{Z}(e_1 + \dots e_4), \quad R_h^\vee = \{\overline{e_i - e_j} \mid i \neq j\},$$

with the obvious pairing $\langle \bullet, \bullet \rangle : \mathbf{\Lambda}_h^\vee \otimes \mathbf{\Lambda}_h \to \mathbb{Z}$ ([Spr09]).
Now $\omega_1^\vee = \overline{e_1}, \omega_2^\vee = \overline{e_1 + e_2}, \omega_3^\vee = \overline{e_1 + e_2 + e_3} = -\overline{e_4} \in \mathbf{\Lambda}_h \cong \mathbb{Z}^3$ form a basis which is dual to the choice $\alpha_i = e_i - e_{i+1}$, $i = 1,2,3$ of simple roots in R. The generator $\sigma \in \mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ acts on the dual basis via $\sigma \cdot \omega_1^\vee = \omega_3^\vee$, $\sigma \cdot \omega_2^\vee = \omega_2^\vee$. It follows that

$$\mathbf{\Lambda}_h^{\mathbf{C}} = \{(a,b,a) \mid a,b \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

in this basis. Now the short exact sequence (5.29) corresponds to the short exact sequence

$$0 \longrightarrow \mathbf{\Lambda}_h^{\mathbf{C}} \longrightarrow \mathbf{\Lambda}_h \longrightarrow Q \longrightarrow 0$$

of W-modules on the locus Σ°. Using the previous results, this sequence is isomorphic to

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\ i\ } \mathbb{Z}^3 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $i(a,b) = (a,b,0)$.
Even though $j_*\mathcal{F}^\circ \cong \mathcal{F}$, hence analogously for $\mathcal{F}^{\mathbf{C}}$, we cannot conclude directly that $j_*\mathcal{G}^\circ \cong \mathcal{G}$ (which is very convenient to compute $H^k(\Sigma,\mathcal{G})$). The problem is that taking invariants $(.)^\rho$ is in general not right-exact, where ρ is the monodromy around some branch point. To address these issues we continue the above computations.

Lemma 5.49. *Let $\rho_1\rho_3, \rho_2 \in W \subset W_h$ be the generators for the Weyl group of $\mathbf{\Lambda} = \mathbf{\Lambda}_h^{\mathbf{C}}$ and $\sigma \in \mathbf{C} = \mathbb{Z}/2\mathbb{Z}$ the non-trivial generator. Then they act on $Q \cong \mathbb{Z}$ as follows:*

$$\rho_1\rho_3 \cdot q = -q, \quad \rho_2 \cdot q = q, \quad \sigma \cdot q = -q, \quad q \in \mathbb{Z}.$$

In particular, $Q^W = 0$ and $Q^{\mathbf{C}} = 0$.

Proof. In the basis $\omega_i^\vee$ the following hold true

$$\rho_1(a,b,c) = (-a, a+b, c), \quad \rho_2(a,b,c) = (a+b, -b, b+c), \quad \rho_3(a,b,c) = (a, b+c, -c).$$

Now we have for $[a,b,c] = [a,0,c] \in Q$:

$$\rho_1\rho_3[a,0,c] = [-a, -a+c, -c] = -[a,0,c]$$

so that $\rho_1\rho_3$ acts via -1 on $Q \cong \mathbb{Z}$. Similarly, we see that

$$\rho_2 \cdot [a,0,c] = [a,0,c]$$

i.e. ρ_2 acts trivially on Q. Finally, we compute

$$\sigma \cdot [a,0,c] = [c,0,a] = -[a,0,c]$$

because $[a,0,c] + [c,0,a] = 0$. $\square$

Lemma 5.50. *There is a natural isomorphism $j_*\mathcal{G}^\circ \cong \mathcal{G}$ so that $H^0(\Sigma, \mathcal{G}) = Q^W = 0$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow (\mathcal{F}^{\mathbf{C}})^\circ \longrightarrow \mathcal{F}^\circ \longrightarrow \mathcal{G}^\circ \longrightarrow 0$$

over Σ°. Localizing the long exact sequence for j_* at a branch point yields the long exact sequence

$$0 \longrightarrow \mathbf{\Lambda}^\rho \longrightarrow \mathbf{\Lambda}_h^\rho \xrightarrow{\beta} Q^\rho \xrightarrow{\delta} \mathbf{\Lambda}_\rho \xrightarrow{\gamma} \mathbf{\Lambda}_{h,\rho} \longrightarrow Q_\rho \longrightarrow 0,$$

where ρ is the local monodromy and we can assume $\rho \in \{\rho_1\rho_3, \rho_2\}$. In the case $\rho = \rho_2$ it is immediate to check (since $Q^{\rho_2} = Q$) that β is surjective and γ is injective. Each of these facts implies that $\delta = 0$. This is trivially true for $\rho = \rho_1\rho_3$ because $Q^{\rho_1\rho_3} = 0$. Altogether we see that the sequence

$$0 \longrightarrow j_*(\mathcal{F}^{\mathbf{C}})^\circ \longrightarrow j_*\mathcal{F}^\circ \longrightarrow j_*\mathcal{G}^\circ \longrightarrow 0$$

is in fact exact. Since $j_*\mathcal{F}^\circ \cong \mathcal{F}$, it follows that $j_*\mathcal{G}^\circ \cong \mathcal{G}$ naturally. □

In particular, we see that the long exact sequence (5.30) gives rise to the short exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \longrightarrow H^1(\Sigma, \mathcal{F}) \longrightarrow \operatorname{im}(\delta) \longrightarrow 0, \tag{5.34}$$

where $\delta : H^1(\Sigma, \mathcal{F}) \to H^2(\Sigma, \mathcal{F}^{\mathbf{C}})$ is the connecting homomorphism. By construction, it is $\mathbf{C}$-equivariant with respect to the natural $\mathbf{C}$-actions. Hence (5.34) is a short exact sequence of $\mathbf{C}$-modules. If we can show that $\operatorname{im}(\delta)^{\mathbf{C}} \subset H^1(\Sigma, \mathcal{G})^{\mathbf{C}} = 0$, then it follows again that $H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \cong H^1(\Sigma, \mathcal{F})^{\mathbf{C}}$ via the natural morphism. To do so, we use Proposition 5.47. Since $Q^{\mathbf{C}} = 0$, the next result implies this claim.

Lemma 5.51. *If $\beta_1, \ldots, \beta_m \in R^\vee$ denote roots corresponding to the branch points, then*

$$\left(\frac{Q^{2g+m}}{(1-\rho_{\beta_1}, \ldots, 1-\rho_{\beta_m})Q} \right)^{\mathbf{C}} = 0.$$

Proof. As $\rho_1\rho_3, \rho_2$ generate $W \subset W_h$, ρ_{β_j} either acts as -1 or $+1$ on $Q = \mathbb{Z}$. Assume that $\beta_1, \ldots, \beta_k$ act by -1 and $\beta_{k+1}, \ldots, \beta_{k+l}$ by $+1$ such that $k + l = m$. It follows that

$$\frac{Q^{2g+m}}{(1-\rho_{\beta_1}, \ldots, 1-\rho_{\beta_m})Q} \cong (\mathbb{Z}/2\mathbb{Z})^k \oplus \mathbb{Z}^l.$$

Moreover, $\mathbf{C}$ acts by -1 on each of these factors so that the claim follows. □

Even though this approach takes longer than the previous one, it gives additional information, e.g. $H^0(\Sigma, \mathcal{G}) = 0$ in this example. They share the crucial ingredient that only $W \subset W_h$ acts on $\mathbf{\Lambda}_h$ and not on all of W_h.

Isomorphic as abelian varieties

We next need to show that the natural isomorphism $H^1(\Sigma, \mathcal{F}^{\mathbf{C}}) \to H^1(\Sigma, \mathcal{F})^{\mathbf{C}}$, $\mathcal{F} = R^2\pi_*\mathbb{Z}$, actually is an isomorphism of polarized $\mathbb{Z}$-Hodge structures. This follows from the functoriality of Zucker's Hodge structure: We have the inclusion $(\mathcal{F}^{\mathbf{C}})^\circ \hookrightarrow \mathcal{F}^\circ$ of polarized $\mathbb{Z}$-VHS of weight 2 over Σ°. The induced morphism from above,

$$H^1(\Sigma, j_*(\mathcal{F}^{\mathbf{C}})^\circ) \longrightarrow H^1(\Sigma, j_*\mathcal{F}^\circ)^{\mathbf{C}} \hookrightarrow H^1(\Sigma, j_*\mathcal{F}^\circ)$$

is therefore a morphism of polarized $\mathbb{Z}$-Hodge structures of weight $2+1=3$. Note that $H^1(\Sigma, j_*\mathcal{F}^\circ)^{\mathbf{C}}$ is a Hodge substructure because $\mathbf{C}$ acts on $\mathcal{F}^\circ$ by Hodge morphisms. In total, we see that

$$H^1(\Sigma,(p_*\boldsymbol{\Lambda})^W) \cong H^1(\Sigma,(R^2\pi_*\mathbb{Z})^{\mathbf{C}})(1) \cong H^1(\Sigma, R^2\pi_*\mathbb{Z})^{\mathbf{C}}(1)$$

as polarizable $\mathbb{Z}$-Hodge structures of weight 1. Therefore the previous isomorphism $P_b \cong J^2_{\mathbf{C}}(X_b)$ of real tori is in fact an isomorphism of abelian varieties. This concludes the proof of Theorem 5.43.

5.4.1 Global isomorphism

Using the methods from Section 5.2 we can globalize the previous isomorphisms. As in the ADE-case we introduce the shorthand notation

$$X := \mathcal{X}^\circ, \quad D := Br \xhookrightarrow{j} Y := \Sigma \times \mathbf{B}^\circ, \quad Z := \mathbf{B}^\circ \subset \mathbf{B}.$$

Let $f = \boldsymbol{\pi}_1^1 : X \to Y$ and $g = \boldsymbol{\pi}_2^\circ : Y \to Z$ be the restrictions of $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2 = pr_2$ respectively as well as $h = \boldsymbol{\pi}^\circ = g \circ f$ their composition. Further recall the universal (BCFG-)cameral curve $\boldsymbol{p} : \tilde{\boldsymbol{\Sigma}} \to \mathbf{B}$. It factorizes through the projection $\boldsymbol{p}_1 : \tilde{\boldsymbol{\Sigma}} \to \Sigma \times \mathbf{B}$ and we denote by $\boldsymbol{p}_1^1 : \tilde{\boldsymbol{\Sigma}}^\circ \to \Sigma \times \mathbf{B}^\circ$ the corresponding restriction. Then we have an isomorphism (Proposition 5.41)

$$(\boldsymbol{p}^\circ_{1*}\boldsymbol{\Lambda}_h)^W(-1) \cong R^2 f^\circ_* \mathbb{Z} \tag{5.35}$$

of polarizable $\mathbb{Z}$-VHS of weight 2 and Tate type on $Y - D$. Denote by $M^1, M^2 \in \mathrm{HM}(Y)$ the functorial intermediate extensions of these VHS to pure polarizable Hodge modules over Y of weight $d_Y + 2$ (cf. Theorem 5.21). Recall that

$$\begin{aligned}\mathrm{rat}(M^1) &= j_*(\boldsymbol{p}^\circ_{1*}\boldsymbol{\Lambda}_{h,\mathbb{Q}})^W(-1)[d_Y] \cong (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_{h,\mathbb{Q}})^W(-1)[d_Y],\\ \mathrm{rat}(M^2) &= j_* R^2 f^\circ_* \mathbb{Q}[d_Y] \cong R^2 f_* \mathbb{Q}[d_Y],\end{aligned}$$

since $D \subset Y$ is a smooth divisor. By construction, $M^1_{\mathbb{Z}} = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_h)^W(-1)[d_Y]$ and $M^2_{\mathbb{Z}} = R^2 f_*\mathbb{Z}[d_Y]$ are integral structures for M^1 and M^2 respectively. Moreover, the isomorphism (5.35) extends to give an isomorphism $M_1 \cong M_2$ of pure Hodge modules that is compatible with the integral structures.

Proposition 5.52. *Let $h : \mathcal{X}^\circ \to \mathbf{B}^\circ$ be the family of non-compact, non-singular CY3s and $g : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ the projection. Then there are natural isomorphisms*

$$\mathcal{H}^0 h_+ \mathbb{Q}^{Hdg} \cong \mathcal{H}^0 g_+ M_2 \cong \mathcal{H}^0 g_+ M_1$$

in $\mathrm{HM}(Z, d_Z{+}3)$ which is compatible with the integral structures. In particular, the corresponding $\mathbb{Z}$-VHS of weight 3 are isomorphic.

Observe that the last statement makes sense because all the involved Hodge modules are smooth.

Proof. The second isomorphism is immediate, so we are left with the first one. It can be seen as in the ADE-case (Theorem 5.15) via the Leray spectral sequence. From the perspective of perverse and constructible sheaves, the only difference is that in the BCFG-case the fibers of $\boldsymbol{\pi}_1 : X \to Y$ can have singularities of type A_1, $\mathrm{A}_1 \times \mathrm{A}_1$ or $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, cf. Example 1.75. But the (perverse) Leray spectral sequence still degenerates because these fibers again only have cohomology in degree 0 and 2. $\square$

Abbreviate $\mathcal{G} = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_h)^W$ so that $\mathcal{G}^{\mathbf{C}} = (\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W$. We next give an analogue of Theorem 5.15 for the BCFG-case which globalizes Theorem 5.43. To this end, it remains to relate $R^1g_*((\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda}_h)^W)^{\mathbf{C}}$ with $R^1g_*((\boldsymbol{p}^1_{1*}\boldsymbol{\Lambda})^W)$ which is compatible with the structures as $\mathbb{Z}$-VHS (equivalently smooth Hodge modules). As a warmup, we consider it on the sheaf level:

Lemma 5.53. *The morphism*

$$\iota : R^1g_*(\mathcal{G}^{\mathbf{C}}) \to R^1g_*(\mathcal{G})^{\mathbf{C}}$$

induced from the inclusion $\mathcal{G}^{\mathbf{C}} \hookrightarrow \mathcal{G}$ is an isomorphism.

Proof. Since $g : \Sigma \times \mathbf{B}^\circ \to \mathbf{B}^\circ$ is proper, the above morphism gives

$$H^1(\Sigma, (p_{b*}\boldsymbol{\Lambda})^W) \to H^1(\Sigma, (p_{b*}\boldsymbol{\Lambda}_h)^W)^{\mathbf{C}}$$

on stalks at $b \in \mathbf{B}^\circ$. But this coincides with the morphism from Theorem 5.43 which is an isomorphism. $\square$

To lift this isomorphism to smooth Hodge modules (equivalently VHS) we observe that $M^1 \in \mathrm{HM}(Y)$ carries a $\mathbf{C}$-action that commutes with rat. As $\mathrm{HM}(Y)$ is an abelian category, $(M^1)^{\mathbf{C}} \subset M^1$ is a Hodge submodule and

$$\mathrm{rat}((M^1)^{\mathbf{C}}) = \mathrm{rat}(M^1)^{\mathbf{C}} = \mathcal{G}^{\mathbf{C}}[d_Z]$$

by construction. The inclusion $(M^1)^{\mathbf{C}} \hookrightarrow M^1$ induces the morphism

$$\iota^{Hdg} : \mathcal{H}^1g_+((M^1)^{\mathbf{C}}) \to \mathcal{H}^1g_+(M^1)^{\mathbf{C}},$$

such that $\mathrm{rat} \circ \iota^{Hdg} = \iota \circ \mathrm{rat}$ (up to a shift).

Proposition 5.54. *ι^{Hdg} is an isomorphism.*

Proof. Since $\mathrm{HM}(Z)$ is an abelian category, we obtain an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{H}^1g_+((M^1)^{\mathbf{C}}) \xrightarrow{\iota^{Hdg}} \mathcal{H}^1g_+(M^1)^{\mathbf{C}} \longrightarrow coK, \longrightarrow 0$$

where $K = \ker(\iota^{Hdg})$ and $coK = \mathrm{coker}(\iota^{Hdg})$. Note that they are itself smooth Hodge modules. Applying the exact functor $\mathrm{rat} : \mathrm{HM}(Z) \to \mathrm{P}(Z)$ yields

$$0 \longrightarrow \mathrm{rat}(K) \longrightarrow R^1g_*(\mathcal{G}^{\mathbf{C}})[d_Z] \xrightarrow{\iota} (R^1g_*\mathcal{G})^{\mathbf{C}}[d_Z] \longrightarrow \mathrm{rat}(coK) \longrightarrow 0,$$

an exact sequence in $\mathrm{P}(Z)$. In particular, this implies $\mathrm{rat}(K) \cong \ker(\iota[d_Z]) = 0$ and $\mathrm{rat}(coK) \cong \mathrm{coker}(\iota[d_Z]) = 0$. But a smooth Hodge module $M \in \mathrm{HM}_{sm}(Z)$ is already zero iff $\mathrm{rat}(M) = 0$. This follows from the equivalence $\mathrm{HM}_{sm}(Z) \simeq \mathrm{VHS}^p_{\mathbb{Q}}(Z)$ because a VHS V is zero iff the underlying local system is zero. Therefore ι^{Hdg} is an isomorphism. $\square$

Summarizing, we obtain the analogue of Theorem 5.15 in the BCFG-case.

Theorem 5.55. *Let Δ be a Dynkin diagram of type* BCFG *with simple adjoint complex Lie group G and $\mathbf{B}^\circ \subset \mathbf{B}$ the locus of smooth cameral curves in the Hitchin base $\mathbf{B}$ of the same type. Let V^{CY} be the graded-polarizable $\mathbb{Z}$-VMHS of weight 3 of the family $\mathcal{X}^\circ \to \mathbf{B}^\circ$ of non-compact CY3s with $\mathbf{C}$-action and $\mathsf{V}^H = \mathsf{V}^H_{ad}$ the polarizable $\mathbb{Z}$-VHS of weight 1 of the Hitchin system $\mathbf{Higgs}^\circ_1(\Sigma, G) \to \mathbf{B}^\circ$. Then there is an isomorphism*

$$(\mathsf{V}^{CY})^{\mathbf{C}} \cong \mathsf{V}^H(-1) \tag{5.36}$$

of graded-polarizable $\mathbb{Z}$-VMHS of weight 3 over $\mathbf{B}^\circ$ so that $(\mathsf{V}^{CY})^{\mathbf{C}}$ is pure. Moreover, V^{CY} is pure itself.

Note that this does hold with $\mathbb{Z}$-coefficients because all the above isomorphisms are compatible with the $\mathbb{Z}$-structures. This can be seen as in the proof of Theorem 5.15.
After all our preparations, we can finally prove the existence of an isomorphism between the BCFG-Hitchin system and the $\mathbf{C}$-invariant part of the Calabi-Yau integrable system $\mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \to \mathbf{B}^\circ$.

Corollary 5.56. *Let Δ be an irreducible Dynkin diagram of type* BCFG*, Δ_h its corresponding* ADE*-Dynkin diagram and $\mathbf{C} \subset \mathrm{Aut}(\Delta_h)$ such that $(\Delta_h)^{\mathbf{C}} = \Delta$. Denote by G_h and G the simple adjoint complex Lie groups of type Δ_h and Δ respectively. Further let $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ be a family of non-compact Calabi-Yau threefolds with $\mathbf{C}$-action as constructed above. Then there is an isomorphism*

$$\begin{tikzcd} \mathcal{J}^2_{\mathbf{C}}(\mathcal{X}^\circ/\mathbf{B}^\circ) \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Higgs}^\circ_1(\Sigma, G) \arrow[dl, "\boldsymbol{h}^\circ_1"] \\ & \mathbf{B}^\circ & \end{tikzcd} \tag{5.37}$$

of integrable systems over $\mathbf{B}^\circ$ such that the cubics are intertwined.

Proof. The proof works in complete analogy as the one of Corollary 5.16 so that we only highlight the main difference, namely that we need to invoke $\mathbf{C}$-invariance. By Lemma 5.35, the period maps $\rho_{\boldsymbol{s}}$ and $\rho_{\tilde{\boldsymbol{s}}}$ map to the $\mathbf{C}$-invariant parts, more precisely

$$\rho_{\boldsymbol{s}} : \mathbf{B}^\circ \to \mathcal{H}^3(\mathcal{X}^\circ/\mathbf{B}^\circ, \mathbb{C})^{\mathbf{C}}, \quad \rho_{\tilde{\boldsymbol{s}}} : \mathbf{B}^\circ \to \mathcal{H}^3(\tilde{\mathcal{X}}^\circ/\mathbf{B}^\circ, \mathbb{C})^{\mathbf{C}}.$$

The same lemma further implies that $\boldsymbol{\psi}^* \circ \rho_{\boldsymbol{s}} = \rho_{\tilde{\boldsymbol{s}}}$. Finally, it remains to be seen that $\rho_{\tilde{\boldsymbol{s}}}$ equals to the tautological section $\boldsymbol{\tau} \in H^0(\tilde{\boldsymbol{U}}, \tilde{u}^*\tilde{\boldsymbol{U}})$ under the isomorphism $\mathcal{H}^3(\tilde{X}^\circ/\mathbf{B}^\circ, \mathbb{C}) \cong \mathcal{H}^1(\tilde{\boldsymbol{\Sigma}}/\mathbf{B}^\circ, \mathfrak{t})$ induced by the Leray spectral sequence. But this follows as in the proof of Corollary 5.16 by Lemma 5.35, which contains the $\mathbf{C}$-invariant analogue of Proposition 5.5 iii). $\square$

Remark 5.57.

a) Unfortunately, this result does not yield a family of (non-compact) CY3s over $\mathbf{B}^\circ$, whose intermediate Jacobian fibration is isomorphic to the BCFG-Hitchin system (see next section for further discussion). Though it is remarkable that the family $\mathcal{X}_h \to \mathbf{B}_h$ can be used to obtain both the Hitchin system of type Δ_h and of type $\Delta = \Delta_h^{AS}$, $AS = AS(\Delta)$, cf. Remark 5.37.

b) The Langlands dual case, i.e. via the homology intermediate Jacobian fibrations, works in analogy to Section 5.2.3.

5.4.2 Equivariant cohomology

The above result differs in nature from the ADE-case of [DDP07]. Namely, the intermediate Jacobian fibration associated with the family $\mathcal{X}^\circ \to \mathbf{B}^\circ \subset \mathbf{B}$ of non-singular, non-compact CY3s over the Hitchin base $\mathbf{B}$ is *not* isomorphic to the corresponding Prym fibration. Instead we have to take invariants under graph automorphisms to obtain the desired result. This is in analogy with the definition of BCFG-singularities though. Therefore a reformulation of Corollary 5.56 along the lines of Section 1.3.1 would be desirable: Find a family $\mathcal{Z} \to \mathbf{B}^\circ$ of geometric objects that yields the VHS of the corresponding BCFG-Hitchin system. There are at least two natural possibilities. The first one is the quotient family $\mathcal{X}^\circ/\mathbf{C} \to \mathbf{B}^\circ$. The second one is the family $[\mathcal{X}^\circ/\mathbf{C}] \to \mathbf{B}^\circ$ of quotient stacks. In this section we only look at $[X_b/\mathbf{C}]$, $b \in \mathbf{B}^\circ$ and its third cohomology $H^3([X_b/\mathbf{C}], \mathbb{Z}) = H^3_{\mathbf{C}}(X_b, \mathbb{Z})$. Unfortunately, we only have partial results but we

plan to come back and discuss the ordinary quotients $X_b/\mathbf{C}$ as well as the respective family cases in the future.
We will then also take up the fixed point locus $X_b^{\mathbf{C}} \subset X_b$. For the moment, let us at least mention that the fixed point locus in each X_b is of codimension 2. One way to see this, is to consider the $\mathbf{C}$-action on a semi-universal deformation $S \to \mathfrak{t}/W$ of a BCFG-singularity. Using the explicit construction of Section 1.3.2, one computes that $S^{\mathbf{C}} \to \mathfrak{t}/W$ is finite over $\mathfrak{t}/W$ (and still surjective). Hence the projection $X_b^{\mathbf{C}} \to \Sigma$ is finite and surjective as well so that $X_b^{\mathbf{C}} \subset X_b$ is of codimension 2.

Let $b \in \mathbf{B}^\circ$ and $X = X_b$ be the corresponding CY3 with projection $\pi : X \to \Sigma$. It is known that $H^i_{\mathbf{C}}(X, \mathbb{Z})_{\mathrm{tf}}$ carries a natural mixed Hodge structure ([Del74]) and we have

$$H^i_{\mathbf{C}}(X, \mathbb{Q}) \cong H^i(X, \mathbb{Q})^{\mathbf{C}}.$$

This only tells us that the mixed Hodge structures on $H^i_{\mathbf{C}}(X, \mathbb{Z})_{\mathrm{tf}}$ and $H^i(X, \mathbb{Z})^{\mathbf{C}}_{\mathrm{tf}}$ are in general isogenous but not isomorphic. In order to study the case $i = 3$ in more detail, we use the (Serre) spectral sequence from Section 1.3.1. We claim that its E_2-page is given by ($H^i = H^i(X, \mathbb{Z})$, $b = b_1(\Sigma)$)

$$\begin{array}{llllll}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H^4(X,\mathbb{Z})^{\mathbf{C}} & H^1(\mathbf{C}, H^4) & H^2(\mathbf{C}, H^4) & H^3(\mathbf{C}, H^4) & H^4(\mathbf{C}, H^4) & \cdots \\
H^3(X,\mathbb{Z})^{\mathbf{C}} & H^1(\mathbf{C}, H^3) & H^2(\mathbf{C}, H^3) & H^3(\mathbf{C}, H^3) & H^4(\mathbf{C}, H^3) & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}/N_2\mathbb{Z} & 0 & \mathbb{Z}/N_4\mathbb{Z} & \cdots \\
\mathbb{Z}^b & 0 & (\mathbb{Z}/N_2\mathbb{Z})^b & 0 & (\mathbb{Z}/N_4\mathbb{Z})^b & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}/N_2\mathbb{Z} & 0 & \mathbb{Z}/N_4\mathbb{Z} & \cdots
\end{array}$$

Here we have similarly as in Section 1.3.1

$$\begin{aligned}
&\mathbf{C} = \mathbb{Z}/2\mathbb{Z} : N_{2k} = 2, \quad \forall k \in \mathbb{N}, \\
&\mathbf{C} = S_3 : N_{2(2l-1)} = 2, \quad N_{4m} = 6, \quad \forall l, m \in \mathbb{N}.
\end{aligned}$$

Indeed, we already know that $H^i(X, k) = 0$ for $i \geq 5$. The terms E_2^{pq}, $0 \leq q \leq 2$, follow from (1.20), if we can show that

$$H^0(X, \mathbb{Z}) \cong \mathbb{Z}, \quad H^1(X, \mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^b, \quad H^2(X, \mathbb{Z}) \cong \mathbb{Z} \tag{5.38}$$

are all trivial $\mathbf{C}$-modules. For H^0 this is clear because $\mathbf{C}$ acts via biholomorphisms. Since $\mathbf{C}$ acts trivially on Σ, it follows that $\mathbf{C}$ acts trivially on $H^i(X, \mathbb{Z}) \cong H^i(\Sigma, \pi_*\mathbb{Z}) = H^i(\Sigma, \mathbb{Z})$, $i = 1, 2$, as well.
From the above, we can only conclude that $d_r = 0$ for $r \geq 6$. If $2 \leq r \leq 5$, the differentials d_r^{pq} are potentially non-zero. For example, the differentials $d_r^{1,4} : E_r^{1,4} \to E_r^{1+r, 4-r+1}$, $2 \leq r \leq 5$, are a priori zero only for $r = 4$. However, we can still say a bit more about $H = H^3_{\mathbf{C}}(X, \mathbb{Z})$: From the E_2-page, it follows that $E^3_\infty = H$ carries a filtration of the form

$$0 = F^4 H = F^3 H \subsetneq F^2 H = F^1 H \subsetneq F^0 H = H^3_{\mathbf{C}}(X, \mathbb{Z}).$$

Thus its non-trivial graded pieces are $E_\infty^{2,1} = F^2 H / F^3 H = F^2 H$ and

$$\iota : E_\infty^{0,3} = F^0 H / F^1 H \hookrightarrow E_2^{0,3} = H^3(X, \mathbb{Z})^{\mathbf{C}}.$$

The latter is torsion-free so that $E_\infty^{0,3}$ is torsion-free as well. It follows that $E_\infty^{0,3} = H^3_{\mathbf{C}}(X,\mathbb{Z})_{\mathrm{tf}}$ for rank reasons. The differentials

$$d_2^{0,3} : E_2^{0,3} \to E_2^{2,2} = \mathbb{Z}/2\mathbb{Z}, \quad d_4^{0,3} : E_4^{0,3} \to E_4^{2,2},$$

are a priori non-zero. This implies that the monomorphism $\iota : H^3_{\mathbf{C}}(X,\mathbb{Z})_{\mathrm{tf}} \to H^3(X,\mathbb{Z})^{\mathbf{C}}$ is either an isomorphism (iff $d_2^{0,3} = 0 = d_4^{0,3}$) or has cokernel $\mathbb{Z}/2\mathbb{Z}$ (one of the differentials is zero, the other non-zero), $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ (if both are non-zero). We leave it for future work to examine these differentials further and to incorporate mixed Hodge structures.

5.5 Monodromy along the fibers

In the previous sections, we have studied non-singular CY3s X_b with a projection $\pi_b : X_b \to \Sigma$. We have seen that the monodromy group of $R^2\pi^\circ_{b*}\mathbb{Z}$ does not have contributions from the corresponding graph automorphisms. The aim of this section is to study an approach from the physics literature to incorporate graph automorphisms and eventually obtain non-simply-laced Dynkin diagrams. It has been formulated in mathematical terms in [Sze04] and often operates under the name 'monodromy along the fibers'[18] Our motivation to study this approach is due to the fact that it is another natural anstatz to relate Calabi-Yau integrable systems with BCFG-Hitchin systems. However, we argue in this section that it does not produce BCFG-Hitchin systems but presumably other types of non-compact Calabi-Yau integrable systems.

Remark 5.58. Our discussion differs from the one in [Sze04] in at least two ways. First of all, we work with $\mathfrak{t}/W$ instead of $\mathfrak{t}$. This makes a huge difference concerning the corresponding $\mathbb{C}^*$-actions. We explain the second dstinction in Remark 5.59 below after some further preparation.

Let $S \subset \mathfrak{g}$ be a Slodowy slice in an arbitrary simple Lie algebra $\mathfrak{g}$ and denote by Δ its Dynkin diagram. Then we have seen in Section 1.4.3 that there is a commutative square

$$\begin{tikzcd} \tilde{S} \arrow[r, "\psi"] \arrow[d, "\tilde{\sigma}"'] & S \arrow[d, "\sigma"] \\ \mathfrak{t} \arrow[r, "q"] & \mathfrak{t}/W \end{tikzcd} \tag{5.39}$$

of $\mathbb{C}^* \times H$-spaces. Here we define

$$H := \begin{cases} \mathbf{C} \cong AS(\Delta), & \mathfrak{g} \text{ of type BCFG}, \\ \mathbf{CA} \cong \mathrm{Aut}(\Delta), & \mathfrak{g} \text{ of type } \mathrm{A}_{2k+1},\, \mathrm{D}_k,\, \mathrm{E}_6, \end{cases}$$

so that we do not have to distinguish between the two cases[19]. Recall that $\mathbf{C}$ is defined via inner automorphisms and $\mathbf{CA}$ via outer automorphisms of $\mathfrak{g}$. To glue (5.39) over Σ with a non-trivial H-action, we choose an unbranched $\boldsymbol{A}$-covering $f : C \to \Sigma$, where $1 \neq \boldsymbol{A} \subset H$ is a non-trivial subgroup. We make the following two assumptions, which are not strictly necessary but simplify the discussion:

i) $\boldsymbol{A}$ is cyclic, hence abelian;

ii) C is connected.

The covering $f : C \to \Sigma$ uniquely determines a class $\beta \in H^1(\Sigma, \boldsymbol{A})$[20] and vice versa. Since the C is connected, the structure group $\boldsymbol{A}$ of the covering cannot be further reduced (cf. [Sze04]). Additionally, choose a class $\alpha \in H^1(\Sigma, \mathcal{O}^*)$ that corresponds to a spin bundle L on Σ so that we obtain an element

$$(\alpha, \beta) \in H^1(\Sigma, \mathcal{O}^*) \oplus H^1(\Sigma, \boldsymbol{A}) \cong H^1(\Sigma, \mathcal{O}^* \times \boldsymbol{A}).$$

Since $\mathbb{C}^* \times \boldsymbol{A}$ acts on (5.39), we can glue it to obtain the commutative square

$$\begin{tikzcd} \tilde{\mathscr{S}}_{\alpha,\beta} \arrow[r, "\underline{\psi}"] \arrow[d, "\underline{\tilde{\sigma}}"'] & \mathscr{S}_{\alpha,\beta} \arrow[d, "\underline{\sigma}"] \\ \tilde{\mathscr{U}}_{\alpha,\beta} \arrow[r, "\underline{q}"] & \mathscr{U}_{\alpha,\beta} \end{tikzcd} \tag{5.40}$$

[18] It would be more precise to call it 'monodromy by graph automorphisms along the fibers'.

[19] Note here our convention that $AS(\Delta) = 1$ if $\Delta = \Delta_h$ is of type ADE.

[20] When working algebraically, we consider these cohomology groups in the étale topology, i.e. $H^1_{et}(X, \boldsymbol{A})$, because coverings are in general not Zariski-locally trivial.

of complex algebraic varieties. From now on, we fix $(\alpha,\beta) \in H^1(\Sigma, \mathcal{O}^* \times \boldsymbol{A})$ and drop it from the notation. We leave it for future work to investigate the dependence on these choices.

Remark 5.59. The exact sequence (1.18) from Section 1.3 induces an exact sequence of pointed sets

$$(H^1(\Sigma, C_\Gamma), \star) \longrightarrow (H^1(\Sigma, N_\Gamma), \star) \xrightarrow{\delta} (H^1(\Sigma, \mathrm{Aut}(\Delta_h)), \star). \tag{5.41}$$

In [Sze04] a class $\gamma \in H^1(\Sigma, N_\Gamma)$ with $\delta(\gamma) \neq \star$ is used for gluing $\tilde{S} \to \mathfrak{t}$ over Σ. We work instead with a class $(\alpha,\beta) \in H^1(\Sigma, \mathcal{O}^* \times \boldsymbol{A})$. One reason for this is that (5.41) might not be surjective on the right, so that it is not clear to us, what the image of δ is. This is more transparent in our approach. Another reason is that the N_Γ-action factorizes via the morphism $(\det, p) : N_\Gamma \to \mathbb{C}^* \times \mathrm{Aut}(\Delta_h)$ anyway, where $p : N_\Gamma \to \mathrm{Aut}(\Delta_h)$ is as in (1.18).

Lemma 5.60. *With the previous notation, there are sections*

$$\underline{\hat{\omega}} \in H^0(\tilde{\mathscr{S}}, \Omega^2_{\underline{\tilde{\sigma}}} \otimes (\underline{\tilde{u}} \circ \underline{\tilde{\sigma}})^* K_\Sigma),$$
$$\underline{\hat{\nu}} \in H^0(\tilde{\mathcal{S}}, K_{\underline{\sigma}} \otimes (\underline{u} \circ \underline{\sigma})^* K_\Sigma),$$

which are glued from $\hat{\omega}$ and $\hat{\nu}$ respectively.
Moreover, pulling back (5.40) via $f : C \to \Sigma$ yields the commutative square

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{f^*\alpha} & \xrightarrow{\boldsymbol{\psi}} & \mathcal{S}_{f^*\alpha} \\ {\scriptstyle\tilde{\boldsymbol{\sigma}}}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\sigma}} \\ \tilde{\boldsymbol{U}}_C & \xrightarrow{\boldsymbol{q}} & \boldsymbol{U}_C \end{array} \tag{5.42}$$

over C, where $\mathcal{S}_{f^*\alpha} = f^*L \times_{\mathbb{C}^*} S$, $\boldsymbol{U}_C = K_C \times_{\mathbb{C}^*} \mathfrak{t}/W$ etc. are constructed via $f^*\alpha \in H^1(\Sigma, \mathcal{O}_C^*)$ as in Section 5.1.2.

Note that $(f^*L)^2 = K_C$ so that we are indeed in the situation of Section 5.1.2.

Proof. The construction of the sections $\underline{\hat{\omega}}$ and $\underline{\hat{\nu}}$ works analogously as in the proof of Proposition 5.5. The main difference is that we need the $\boldsymbol{A}$-invariance of $\hat{\omega}$ and $\hat{\nu}$ (Corollary 1.104). Therefore the analogue of (5.1) reads as[21]

$$\varphi_{ij}(\hat{\omega}_j) = ((\mathrm{pr}_{1,i} \circ \psi_i)^* (\alpha_{ji})^2)\, \hat{\omega}_i. \tag{5.43}$$

But again $(\mathrm{pr}_{1,i} \circ \psi_i^*(\alpha_{ji})^2)_{ij}$ corresponds to $(\underline{\tilde{u}} \circ \underline{\tilde{\sigma}})^* K_\Sigma^{-1}$, so that we can glue to obtain a global section $\underline{\hat{\omega}} \in H^0(\tilde{\mathscr{S}}, \Omega^2_{\underline{\tilde{\sigma}}} \otimes (\underline{\tilde{u}} \circ \underline{\tilde{\sigma}})^* K_\Sigma)$. The construction of $\underline{\hat{\nu}} \in H^0(\tilde{\mathcal{S}}, K_{\underline{\sigma}} \otimes (\underline{u} \circ \underline{\sigma})^* K_\Sigma)$ works analogously.
The second claim is immediate because $f : C \to \Sigma$ trivializes the class β by construction and $(f^*L)^2 = f^*K_\Sigma = K_C$. □

Before we construct threefolds, we make a digression to study $\mathscr{B} := H^0(\Sigma, \mathscr{U})$ in more detail. First of all, we observe that pullback f^* induces isomorphisms

$$\mathscr{B} = H^0(\Sigma, \mathscr{U}) \cong H^0(C, f^*\mathscr{U})^{\boldsymbol{A}} = H^0(C, \boldsymbol{U}_C)^{\boldsymbol{A}}, \quad \boldsymbol{U}_C = K_C \times_{\mathbb{C}^*} \mathfrak{t}/W. \tag{5.44}$$

[21] Also recall that $\mathbb{C}^* \subset C(\Gamma)$ acts by weight 2 on $\mathfrak{t}$. This explains the difference between (5.1) and (5.43). In (5.1 we worked with more general cocycles, but it specializes to (5.43), when it takes values in $\mathbb{C}^* \subset C(\Gamma)$, cf. Lemma 1.48.

Here it is important that $\boldsymbol{A}$ only acts via pullback and *not* by acting on the fibers. In particular, there is no folding process involved.
If $\mathfrak{t}/W$ is of type BCFG, it follows directly from the construction (or (5.44)) that

$$\mathscr{B} = H^0(\Sigma, \boldsymbol{U}_\Sigma), \quad \boldsymbol{U}_\Sigma = K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W,$$

i.e. $\mathscr{B}$ coincides with the Hitchin base of the corresponding type. Indeed, β acts trivially on $\mathfrak{t}/W$ in these cases. The next lemma is convenient to make $\mathscr{B}$ more explicit in the ADE-cases.

Lemma 5.61 ([Slo80b], Section 8.8)**.** *Let $\mathfrak{t} \subset \mathfrak{g}$ be of type Δ for a Dynkin diagram Δ of type* $\mathrm{A}_{2k+1}, \mathrm{D}_{\geq 4}, \mathrm{E}_6$*. Then there is a vector space structure on $\mathfrak{t}/W$, induced by an appropriate choice of independent generators of $\mathbb{C}[\mathfrak{t}]^W$, such that the natural* $\mathrm{Aut}(\Delta)$*-action is linear. It commutes with the $\mathbb{C}^*$-action on $\mathfrak{t}/W$ induced by multiplication on $\mathfrak{t}$.*

Here we let $\mathrm{Aut}(\Delta)$ act via the split exact sequence

$$0 \longrightarrow N(T) \longrightarrow \mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \longrightarrow \mathrm{Aut}(\Delta) \longrightarrow 0.$$

The group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{t}) \subset \mathrm{Aut}(\mathfrak{g})$ is the subgroup of automorphisms of $\mathfrak{g}$ fixing $\mathfrak{t}$ (as a Cartan subalgebra, not necessarily pointwise). Moreover, $N(T) \subset G$ is the normalizer of the maximal torus $T \subset G$ in the simple adjoint Lie group G corresponding to $\mathfrak{g}$. It acts by conjugation so that $N(T) \hookrightarrow \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ makes sense.
Let us fix generators as in Lemma 5.61. Then $\boldsymbol{A}$ acts linearly on $\mathfrak{t}/W \cong \mathbb{C}^r$. Twisting with the class $\beta \in H^1(\Sigma, \boldsymbol{A})$ gives rise to a bundle $\hat{\mathscr{U}} \to \Sigma$ with fiber $\mathfrak{t}/W$. We can decompose

$$\mathfrak{t}/W = \bigoplus_{i=1}^{r} \mathfrak{t}/W[d_i]$$

with respect to the $\mathbb{C}^*$-action (i.e. $\mathfrak{t}/W[d_i]$ carries weight d_i). Since the two actions commute, the bundle $\hat{\mathscr{U}}$ is graded as well,

$$\hat{\mathscr{U}} = \bigoplus_{i=1}^{r} \hat{\mathscr{U}}[d_i].$$

Taking into account the $\mathbb{C}^*$-action, we therefore see that

$$\mathscr{U} \cong \bigoplus_{i=1}^{r} \hat{\mathscr{U}}[d_i] \otimes K_\Sigma^{d_i}.$$

Example 5.62. The previous discussion can be made more explicit by considering the different cases separately. The simplest cases are those, when $-id \notin W$ so that $-id$ induces a graph automorphism. This happens for the cases $\Delta = \mathrm{A}_{2k+1}, \mathrm{D}_{2n+1} (2n+1 \geq 5), \mathrm{E}_6$ ([Bou02]) so that $\boldsymbol{A} = \mathbb{Z}/2\mathbb{Z}$. Let $M \to \Sigma$ be the line bundle corresponding to $\beta \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})$. Here $\boldsymbol{A}$ acts on $\mathbb{C}$ by its non-trivial action. By construction, it is a non-trivial line bundle of torsion 2. Now we have the following degrees ([Bou02])

Δ	degrees
A_{2k+1}	$d_1 = 2, \ldots, d_{2k+1} = 2k+2$
D_r	$d_1 = 2, \ldots, d_{r-1} = 2(r-1), d_r = r$
E_6	$d_1 = 2, d_2 = 5, d_3 = 6, d_4 = 8, d_5 = 9, d_6 = 12.$

Therefore (5.5) specializes to ($K = K_\Sigma = L^2$)

$$\begin{aligned} \mathrm{A}_{2k+1}: &\quad \mathscr{U} \cong K^2 \oplus K^3 M \oplus \cdots \oplus K^{2k+1} M \oplus K^{2k+2} \\ \mathrm{D}_{2n+1}: &\quad \mathscr{U} \cong \left(\bigoplus_{m=1}^{4n} K^m \right) \oplus K^{2n+1} M \\ \mathrm{E}_6: &\quad \mathscr{U} \cong K^2 \oplus K^5 M \oplus K^6 \oplus K^8 \oplus K^9 M \oplus K^{12}. \end{aligned}$$

The remaining cases are D_{2n}, $n \geq 2$. For $n \geq 2$ we have the W-invariants

$$t_1 = \sum_{i=1}^{2n} x_i^2, \quad t_k = \sum_{\sigma \in S_{2n}} x_{\sigma(1)}^2 \cdots x_{\sigma(k-1)}^2 \quad (k = 2, \ldots, 2n-1), \quad t_{2n} = x_1 \cdots x_{2n},$$

cf. [Bou02]. They have degree $\deg(t_k) = 2k$, $k = 1, \ldots, 2n-1$, and $\deg(t_{2n}) = 2n$. For $n \geq 3$ there is only one non-trivial graph automorphism φ and we have

$$\varphi^*(t_k) = t_k, \quad k = 1, \ldots, 2n-1, \quad \varphi^*(t_{2n}) = -t_{2n}.$$

In these cases we hence obtain

$$\mathscr{U} \cong \left(\bigoplus_{m=1}^{2n-1} K^{2m} \right) \oplus K^{2n} M.$$

The remaining case is D_4 where

$$\mathfrak{t}/W = \mathfrak{t}/W[2] \oplus \mathfrak{t}/W[4] \oplus \mathfrak{t}/W[6].$$

Note that $\mathfrak{t}/W[4]$ is in fact two-dimensional because the weight 4 occurs twice. Of course, this also happens in the other D_{2n}-cases where $\mathfrak{t}/W[2n]$ is two-dimensional as well. The difference is that $\boldsymbol{A}$ acts a priori non-trivially on all of $\mathfrak{t}/W[4]$ and not just on a one-dimensional subspace thereof. However, it can be shown that it acts trivially on $\mathfrak{t}/W[2]$ and $\mathfrak{t}/W[6]$ (the first one is a direct calculation and for the second one see [Slo80b], Section 8.8). This implies that

$$\mathscr{U} \cong K^2 \oplus \mathscr{U}_2 \oplus K^6$$

for a subbundle $\mathscr{U}_2 \subset \mathscr{U}$ of rank 2.

Remark 5.63. The A_{2k+1}-cases can also be written as

$$\mathscr{U} \cong \bigoplus_{m=2}^{2k+2} (KM)^m$$

because $M^2 = \underline{\mathbb{C}}$. This is closely related to *M-twisted Higgs bundles* in the sense of [GPR00], where K is replaced by KM. However, the other cases behave differently so that $\mathscr{B}$ is in general not the base of an 'M-twisted Hitchin fibration'.

We now come to the construction of a family $\mathscr{Y} \to \mathscr{B}$ of threefolds, which works precisely as in the previous sections. More precisely, it is defined via the fiber product and projection

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\boldsymbol{\pi'}"'] & \mathscr{S} \arrow[d] \\ \Sigma \times \mathscr{B} \arrow[r, "ev"] \arrow[d] & \mathscr{U} \\ \mathscr{B}. & \end{tikzcd}$$

Before we study its properties in more detail, we define

$$\mathscr{B}^\circ := \{s \in \mathscr{B} \mid s \text{ transversal to } \operatorname{discr}(\underline{q})\},$$

where $\operatorname{discr}(\underline{q})$ denotes the discriminant of $\underline{q} : \tilde{\mathscr{U}} \to \mathscr{U}$. The discriminant can be described as before: Recall the function $s_{br} = \prod_{\gamma \in R} \gamma \in \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}/W]$. It is obviously even $\operatorname{Aut}(R)$-invariant so that it gives rise to a section

$$\underline{s}_{br} : \mathscr{U} \to \pi_{\mathscr{U}}^* K_\Sigma^{|R|}.$$

Then the discriminant $\operatorname{discr}(\underline{q})$ coincides with its vanishing locus $V(\underline{s}_{br})^{red}$ with its reduced structure. Moreover, we see that the divisor

$$Br' := V(ev^* \underline{s}_{br})^{red} \subset \Sigma \times \mathscr{B}$$

describes the discriminant of $\mathscr{Y} \to \Sigma \times \mathscr{B}$. In particular, singular fibers of $\pi_b' : Y_b \to \Sigma$ precisely lie over $Br_b' = i_b^* Br'$ for the inclusion $i_b : \Sigma \times \mathscr{B}$. So this is analogous to the previous (untwisted) cases. There is a difference though: The reflections $s_\gamma : \mathfrak{t} \to \mathfrak{t}$ for a root γ do *not* glue to morphisms $\tilde{\mathscr{U}} \to \tilde{\mathscr{U}}$. The hyperplanes $\mathfrak{t}_\gamma \subset \mathfrak{t}$ are rather glued together as dictated by the class $\beta \in H^1(\Sigma, \boldsymbol{A})$. In particular, $\tilde{\mathscr{U}}$ does not carry a natural (non-trivial) W-action. However, we can still glue $q^1 : \mathfrak{t}^1 \to \mathfrak{t}^1/W$ (Section 1.4.5) to obtain $\underline{q}^1 : \tilde{\mathscr{U}}^1 \to \mathscr{U}^1$. We now further investigate the locus $\mathscr{B}^\circ \subset \mathscr{B}$.

Lemma 5.64. *Let $f : C \to \Sigma$ be the étale $\boldsymbol{A}$-covering and $\mathscr{B} = H^0(\Sigma, \mathscr{U})$, $\mathbf{B}_C = H^0(C, \boldsymbol{U}_C)$ as before. Then $\mathscr{B} \neq 0$ and*

$$\dim \mathscr{B} = \dim H^0(\Sigma, K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W), \tag{5.45}$$

possibly except for the D_4*-case. Furthermore, the isomorphism $f^* : \mathscr{B} \to \mathbf{B}_C^{\boldsymbol{A}}$ satisfies*

$$f^*(\mathscr{B}^\circ) = (\mathbf{B}_C^\circ)^{\boldsymbol{A}} \neq \emptyset.$$

Proof. Since $g_\Sigma \geq 2$, it follows that $H^0(\Sigma, K^d) \neq 0$ ($K = K_\Sigma$) for every integer $d \geq 1$. Hence Example 5.62 shows that $\mathscr{B} \neq 0$. The same example shows that it is enough to prove that $h^0(\Sigma, K^d M) = h^0(\Sigma, K^d)$ for $d \geq 2$ in order to arrive at (5.45). Riemann-Roch tells us that

$$h^0(\Sigma, K^d M) - h^0(K^{-d+1} M^{-1}) = (2d-1)(g-1).$$

But $\deg(K^{-d+1}M) = \deg(K^{-d+1}) < 0$ so that

$$h^0(\Sigma, K^d M) = (2d-1)(g-1) = h^0(\Sigma, K^d).$$

Now let $b \in \mathscr{B}^\circ$. Locally it is given by a morphism $\Sigma \supset U \to \mathfrak{t}/W$ which is transversal to the branch locus of $\mathfrak{t} \to \mathfrak{t}/W$. In particular, this is a local condition. Since $f : C \to \Sigma$ is étale, it follows that $f^*(b)$ is transversal to the branch locus of $\tilde{\boldsymbol{U}}_C \to \boldsymbol{U}$, i.e. $f^*(b) \in \mathbf{B}_C^\circ$. Hence $f^*(\mathscr{B}^\circ) = (\mathbf{B}_C^\circ)^{\boldsymbol{A}}$ which shows that $\mathscr{B}^\circ \neq \emptyset$ as well. □

This lemma in particular says that the dimension of $\mathscr{B}$ coincides with that of the Hitchin base $\mathbf{B}_\Sigma = H^0(\Sigma, K_\Sigma \times_{\mathbb{C}^*} \mathfrak{t}/W)$ (except for possibly D_4). This confirms our comment from above (cf. (5.44)): Twisting with a class with values in $\boldsymbol{A}$ ('monodromy along fibers') does not fold $\mathfrak{t}/W$ to $(\mathfrak{t}/W)^{\boldsymbol{A}}$.

We can exploit this relation to give another description of Br' tying it to cameral curves over C. Let $\tilde{\boldsymbol{C}} \to \mathbf{B}_C = H^0(C, K_C \times_{\mathbb{C}^*} \mathfrak{t}/W)$ be the universal cameral curve. We have seen that

$f^* : \mathscr{B} \to \mathbf{B}_C^{\boldsymbol{A}}$ is an isomorphism, which preserves the loci $\mathscr{B}^\circ$ and $(\mathbf{B}_C^\circ)^{\boldsymbol{A}}$. Hence we can associate to $b \in \mathscr{B}^\circ$ the smooth cameral curve $p_b : \tilde{C}_{f^*(b)} \to C$. By construction, it follows that $Br(p_b) = f^{-1}(Br'_b)$ as divisors, where $Br(p_b) \subset C$ is the branch locus of p_b. More generally, we have

$$(f \times (f^*)^{-1})^{-1}(Br') = Br \cap \mathbf{B}_C^{\boldsymbol{A}}.$$

As it turns out, the family $\boldsymbol{\pi'} : \mathscr{Y} \to \mathscr{B}$ has similar properties as the families $\boldsymbol{\pi} : \mathcal{X} \to \mathbf{B}$ that we constructed earlier.

Proposition 5.65. *The morphism $\boldsymbol{\pi'} : \mathscr{Y} \to \mathscr{B}$ is quasi-projective and its fibers are quasi-projective Gorenstein Calabi-Yau threefolds. If $b \in \mathscr{B}^\circ$, then Y_b is non-singular and $H^3(Y_b, \mathbb{Z})$ carries a pure $\mathbb{Z}$-Hodge structure of weight 3 of types $(1,2)+(2,1)$. In particular, $J^2(Y_b)$ carries the structure of an abelian variety.*
Finally, the base change of $\mathscr{Y} \to \Sigma \times \mathscr{B}$ via $f \times (f^*)^{-1} : C \times \mathbf{B}_C^{\boldsymbol{A}} \to \Sigma \times \mathscr{B}$ is isomorphic to $\mathcal{X}_{f^*\alpha} \to C \times \mathbf{B}_C^{\boldsymbol{A}}$, where $\mathcal{X}_{f^*\alpha}$ is constructed from (5.42).

Proof. Quasi-projectivity, the Gorenstein and Calabi-Yau property can be seen as in the proof of Proposition 5.7 together with Lemma 5.60.
To see that Y_b is a non-singular complex algebraic variety, observe that we have a cartesian square

$$\begin{tikzcd} X_{f^*b} \arrow[r, "g"] \arrow[d, "\pi"'] & Y_b \arrow[d, "\pi'"] \\ C \arrow[r, "f"] & \Sigma, \end{tikzcd} \tag{5.46}$$

where X_{f^*b}, $b \in \mathscr{B}^\circ$ is glued from the class $f^*\alpha$. By Lemma 5.64 and Proposition 5.7, we know that X_{f^*b} is non-singular. Since $f : C \to \Sigma$ is étale and (5.46) is cartesian, $g : X_{f^*b} \to Y_b$ is étale. This implies that Y_b is non-singular as well.
The statement about $H^3(Y_b, \mathbb{Z})$ can be seen as Corollary 5.12. In particular, we have an isomorphism induced from the (perverse) Leray spectral sequence

$$H^1(\Sigma, R^2\pi'_{b*}\mathbb{Z}) \cong H^3(Y_b, \mathbb{Z}),$$

where $\pi' = \pi'_b \colon Y_b \to \Sigma$ is the natural map. Away from the finite subset $Br'_b \subset \Sigma$, this map is locally trivial (in the analytic topology). We denote $\Sigma^\circ = \Sigma - Br'_b$ its complement and by $\pi'^\circ \colon Y_b \to \Sigma^\circ$ the restriction. Then $R^2\pi'^\circ_*\mathbb{Z}$ is a polarizable $\mathbb{Z}$-VHS of weight 2. Applying Zucker's theorem 4.33, it follows that $H^3(Y_b, \mathbb{Z})$ is of the claimed type. □

Corollary 5.66. *The intermediate Jacobian fibration $\mathcal{J}^2(\mathscr{Y}^\circ/\mathscr{B}^\circ) \to \mathscr{B}^\circ$ associated with $\mathscr{Y}^\circ \to \mathscr{B}^\circ$ is a family of abelian varieties.*

Proof. This is similar to our discussion in Section 5.2.1: The family $\pi'^\circ : \mathscr{Y}^\circ \to \mathscr{B}^\circ$ yields a graded-polarizable $\mathbb{Z}$-VMHS with underlying local system $R^3\pi'^\circ_*\mathbb{Z}$ (either by [BEZ14] or [Sai90]). The previous proposition implies that it is pure of weight 3 and only has a two-step Hodge filtration, which implies the claim. □

As is clear from the construction, the main difference between the families $\mathscr{Y}^\circ \to \mathscr{B}^\circ$ and the families $\mathcal{X}^\circ_{f^*\alpha} \to \mathbf{B}_C^\circ$ lies in the monodromy (groups) of $R^2\pi'^\circ_{b*}\mathbb{Z}$, where $\pi'_b : Y_b \to \Sigma$ are the natural projections. The next lemma shows that we obtain contributions by graph automorphisms, i.e. that we indeed obtain 'monodromy along the fibers'.

Proposition 5.67. *Let $b \in \mathscr{B}^\circ$, $\pi' = \pi'_b \colon Y_b \to \Sigma$ and $p_b : \tilde{C}_{f^*(b)} \to C$ be the corresponding cameral curve (over C). Then the branched covering $f \circ p_b \colon \tilde{C}_{f^*b} \to \Sigma$ is a simply ramified* Galois covering *of C. Its Galois group H_{mon} is determined by an extension*

$$1 \longrightarrow W \longrightarrow H_{mon} \longrightarrow \boldsymbol{A} \longrightarrow 1,$$

where W is the Weyl group from before. It coincides with the monodromy group of the local system $R^2\pi'^\circ_*\mathbb{Z}$.

Proof. Note that both $f \colon C \to \Sigma$ and $p = p_b \colon \tilde{C} = \tilde{C}_{f^*b} \to C$ are Galois coverings. However, this does not imply that the composition $\tilde{p} := f \circ p : \tilde{C} \to \Sigma$ is a simply ramified Galois covering (note that $\tilde{p}$ is simply ramified because f is unramified[22]). To see that $\tilde{p}$ is Galois, let $q \colon C_\mu \to \Sigma$ be the simply ramified Galois covering determined by the monodromy $\mu \colon \pi_1(\Sigma^\circ, x_0) \to \mathrm{Aut}(\boldsymbol{\Lambda}_h)$ of $R^2\pi'^\circ\mathbb{Z}$, i.e. $\mathrm{im}(q_*) = \ker(\mu) \subset \pi_1(\Sigma^\circ, x_0)$. By construction, we must have

$$\mathrm{im}\, \tilde{p}^\circ_* = \mathrm{im}\, f_* \circ p^\circ_* \subset \ker \mu = \mathrm{im}\, q^\circ, \tag{5.47}$$

so that there exists a covering map $\tilde{C}^\circ \to C^\circ_\mu$. Indeed, we know that

$$\begin{array}{ccc} X_{f^*b} & \longrightarrow & Y_b \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ C & \xrightarrow{f} & \Sigma \end{array}$$

is a fiber product. Note that these are precisely the threefolds that we considered in Section 5.1. It follows that $f^*R^2\pi_*\mathbb{Z} \cong R^2\pi'_*\mathbb{Z}$ over C° and therefore

$$p^*f^*R^2\pi_*\mathbb{Z} \cong \boldsymbol{\Lambda}_{h,\tilde{C}^\circ}$$

over $\tilde{C}^\circ$. This implies (5.47) and further $\ker \mu = \mathrm{im}\, q^\circ_* \subsetneq \mathrm{im}\, f_*$. Again from covering theory, there exists a covering map $h^\circ : C^\circ_\mu \to C^\circ$ fitting into the commutative diagram of covering spaces

$$\begin{tikzcd} & \tilde{C}^\circ \arrow[d, "p^\circ"] \arrow[dd, bend left=60, "\tilde{p}^\circ"] \\ C^\circ_\mu \arrow[r, "h^\circ"] \arrow[dr, "q^\circ"'] & C^\circ \arrow[d, "f"] \\ & \Sigma^\circ. \end{tikzcd}$$

From the above, it follows that $\mathrm{im}\, p^\circ_* = \ker f^*\mu \supset \mathrm{im}\, h^\circ_*$. Thus we have a covering map $C^\circ_\mu \to \tilde{C}^\circ$ over C° and Σ°. Altogether C°_μ and $\tilde{C}^\circ$ are coverings over C° and Σ° that cover each other. Therefore C°_μ and $\tilde{C}^\circ$ are isomorphic over C° and Σ°. In particular, $\tilde{p} \colon \tilde{C} \to \Sigma$ is a simply ramified Galois covering.
Since $f \colon C \to \Sigma$ is an unramified $\boldsymbol{A}$-Galois covering and $p \colon \tilde{C} \to C$ a simply ramified W-Galois covering, the Galois group H_{mon} of $\tilde{p} = f \circ p \colon \tilde{C} \to \Sigma$ is an extension

$$1 \longrightarrow W \longrightarrow H_{mon} \longrightarrow \boldsymbol{A} \longrightarrow 1.$$

By construction, H_{mon} coincides with the monodromy group $\pi_1(\Sigma^\circ, x_0)/\ker \mu \cong \mathrm{im}\, \mu \subset \mathrm{Aut}(\boldsymbol{\Lambda}_h)$ of $R^2\pi'^\circ_*\mathbb{Z}$. □

[22]Since $f \colon C \to \Sigma$ is unramified, we drop the superscript $\circ$, even if we restrict to C°.

Remark 5.68. This lemma shows that the monodromy group of $R^2\pi'^{\circ}_*\mathbb{Z}$ is larger than in the case without monodromy along fibers. In particular, it cannot come from a simply ramified W-covering $\tilde{\Sigma} \to \Sigma$, i.e. a cameral curve, over Σ.
Furthermore, it is plausible that the extension H_{mon} is the trivial one, so that H_{mon} is the automorphism group of the corresponding root systems.

The above discussion suggests that the intermediate Jacobian fibration $\mathcal{J}^2(\mathscr{Y}^\circ/\mathscr{B}^\circ) \to \mathscr{B}^\circ$, a family of abelian varieties, is not (directly) related to Hitchin systems associated with Σ. However, we do not know yet, if it is an integrable system or not and we leave it for the future to investigate this question in more detail. It true, 'monodromy along fibers' would provide new examples of non-compact Calabi-Yau integrable systems.

Appendix A

Appendix

A.1 Non-degenerate complex tori

The main reference for this section is [BL99], in which all the subsequent statements and definitions can be found.
Let V be a complex vector space of dimension g and $(V_{\mathbb{R}}, J)$ the underlying real vector space with complex structure. Further let $\Lambda \subset V_{\mathbb{R}}$ be a full sublattice, i.e. $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V_{\mathbb{R}}$. Then we obtain a complex torus

$$X := V/\Lambda = (V_{\mathbb{R}}, J)/\Lambda.$$

We have canonical isomorphisms

$$H_1(X, \mathbb{Z}) = \Lambda, \quad T_0 X = V, \quad T_0^{\mathbb{C}} X = V \oplus \overline{V}.$$

The Néron-Severi lattice $NS(X)$ of X can be described as follows.

Theorem A.1. *Let $X = V/\Lambda$ be a complex torus. Then*

$$\begin{aligned} NS(X) \cong& \{E \in \mathrm{Alt}^2(\Lambda, \mathbb{Z}) \mid E_{\mathbb{R}}(Jv, Jw) = E_{\mathbb{R}}(v, w)\} \\ \cong& \{E' \in \mathrm{Alt}^2(\Lambda, \mathbb{R}) \mid E'(\Lambda, \Lambda) \subset \mathbb{Z}, \quad E'(Jv, w) = E'(Jw, v)\}. \end{aligned}$$

Here $E_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear extension of $E \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$.

Another description of the data in Theorem A.1 is in terms of Hermitian forms on V.

Lemma A.2. *Let $E' \in \mathrm{Alt}^2(\Lambda, \mathbb{R})$ and define*

$$H = H_{E'} : V \times V \to \mathbb{C}, \quad H(v, w) = E'(Jv, w) + iE'(v, w).$$

Then $E' \in NS(X)$ (under the isomorphism of Theorem A.1) iff

$$\mathrm{im}(H)(\Lambda, \Lambda) \subset \mathbb{Z}, \quad H(v, w) = \overline{H(v, w)}.$$

Conversely, a hermitian form $H : V \times V \to \mathbb{C}$, i.e. it is $\mathbb{C}$-linear in the first variable with $H(v, w) = \overline{H(w, v)}$, satisfies $\mathrm{im}(H) \in NS(X) \subset \mathrm{Alt}^2(\Lambda, \mathbb{R})$ iff

$$\mathrm{im}(H)(\Lambda, \Lambda) \subset \mathbb{Z}.$$

By the elementary divisor theorem, we can find a *symplectic basis* of Λ for any $E \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$, such that it is given by the matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

Here $D = \mathrm{diag}(d_1, \ldots, d_g)$ such that d_i is a divisor of d_{i+1} and $d_i \geq 0$. Note that $d_i > 0$ iff E is non-degenerate iff $H_{E_\mathbb{R}}(v, w) = E_\mathbb{R}(Jv, w) + iE_\mathbb{R}(v, w)$ is non-degenerate. Indeed, let $v \in V$ be such that $H(v, w) = 0$ for all $w \in V$, then

$$\begin{aligned} & E_\mathbb{R}(Jv, w) + iE_\mathbb{R}(v, w) = 0, \quad \forall w \in V, \\ \Leftrightarrow \quad & E_\mathbb{R}(v, Jw) = 0, \ E_\mathbb{R}(v, w) = 0, \quad \forall w \in V, \\ \Leftrightarrow \quad & E_\mathbb{R}(v, w) = 0, \quad \forall w \in V. \end{aligned}$$

Definition A.3 (Polarization on a complex torus)**.** Let V/Λ be a complex torus. A polarization of index k is an alternating bilinear form $E \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ such that

i) $E_\mathbb{R}(Jv, Jw) = E_\mathbb{R}(v, w)$,

ii) $H_{E_\mathbb{R}}(v, w) := E_\mathbb{R}(Jv, w) + iE_\mathbb{R}(v, w)$ is a non-degenerate hermitian form of index k.

A polarized complex torus of index k is a pair (X, E) consisting of a complex torus X and a polarization E of index k.

In the light of Theorem A.1 a polarized complex torus is a pair (X, L) consisting of a complex torus X and a non-degenerate line bundle L of index k, i.e. $c_1(L)$ corresponds to a polarization of index k in the sense of our definition.

Example A.4. The torus $X = V/\Lambda$ is an abelian variety iff Λ admits a polarization E of index 0. In that case, $E = c_1(L)$ for an ample line bundle $L \in \mathrm{Pic}(X)$.

A polarization of index 0 in particular satisfies the two *Riemann bilinear relations*:

(I) $E \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ with $E_\mathbb{R}(Jv, Jw) = E_\mathbb{R}(v, w)$ and non-degenerate,

(II) $H_{E_\mathbb{R}}(v, v) > 0$ for all $v \neq 0$.

Of course, if E is of index $k > 0$, then it only satisfies the first Riemann bilinear relation.

A.2 Variations of (mixed) Hodge structures

This section is mainly based on [PS08] and [SZ85]. We fix throughout a complex manifold B and denote by R either $\mathbb{Z}$ or $\mathbb{Q}$.

Definition A.5. An *R-variation of Hodge structures* (R-VHS) of weight k is a pair $\mathsf{V} = (\mathsf{V}_R, \mathcal{F}^\bullet\mathsf{V})$ consisting of a locally constant sheaf V_R of finitely generated R-modules and a decreasing (Hodge) filtration $\mathcal{F}^\bullet = \mathcal{F}^\bullet\mathsf{V}$ of $\mathsf{V}_\mathcal{O} := \mathsf{V}_R \otimes_\mathbb{Z} \mathcal{O}_B$ by holomorphic subbundles. They satisfy the following conditions:

- The pair $\mathsf{V}_b = (\mathsf{V}_{R,b}, \mathcal{F}^\bullet_{|b})$[1] is an R-Hodge structure of weight k for each $b \in B$, where $\mathcal{F}^\bullet_{|b}$ denotes the fiber of $\mathcal{F}^\bullet$ at b.

[1]Here we denote, as in Section 2.2.2, the fiber of a vector bundle V at b by $V_{|b}$.

- *Griffiths' transversality*: $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_B$ for the natural connection ∇ on $\mathsf{V}_{\mathcal{O}}$.

A morphism $\varphi : \mathsf{V} = (\mathsf{V}_R, \mathcal{F}^\bullet) \to \mathsf{V}' = (\mathsf{V}'_R, \mathcal{F}'^\bullet)$ of R-VHS is a morphism $\varphi_R : \mathsf{V}_R \to \mathsf{V}'_R$ that is compatible (after tensoring with $\mathcal{O}_B$) with the Hodge filtrations.

If V is an R-VHS of weight k, we denote by $\mathsf{V}(m)$ its m-th *Tate twist*, which is an R-VHS of weight $k - 2m$. A *polarization* Q on V is a morphism $Q : \mathsf{V} \otimes \mathsf{V} \to R(-k)_B$ of VHS that induces a polarization on the R-Hodge structure V_b for every $b \in B$.
If $\pi : \mathcal{X} \to B$ is a proper holomorphic submersion, then its higher direct images $R^k\pi_*\mathbb{Z}$ yield VHS. We often denote the corresponding bundles as

$$\mathcal{H}^k(\mathcal{X}/B, \mathbb{C}) := \mathcal{H}^k(\pi, \mathbb{C}) := R^k\pi_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_B. \tag{A.1}$$

The next example is closely related to Chapter 2 and the previous Section A.1.

Example A.6. Let B be a connected complex manifold and $\pi : \mathcal{X} \to B$ a family of polarized abelian varieties with vertical bundle $\mathcal{V} \to B$. Then $\mathsf{V}_{\mathbb{Z}} := R^1\pi_*\mathbb{Z}$ underlies a VHS V of weight 1. The relative polarization of π induces a polarization Q on V. It has the following useful implication.

Lemma A.7. *The polarization Q on V induces a natural isomorphism*

$$R^1\pi_*\mathcal{O}_{\mathcal{X}} \cong (\pi_*\Omega^1_{\mathcal{X}/B})^* = \mathcal{V}.$$

Here $\Omega^1_{\mathcal{X}/B} = \Omega^1_{\mathcal{X}}/\pi^*\Omega^1_B$ is the sheaf of relative 1-forms which is dual to the relative tangent sheaf $T_{\mathcal{X}/B}$.

Proof. Intuitively, this isomorphism is clear because for each fiber one has

$$H^1(X_b, \mathcal{O}_{X_b}) = H^{1,0}(X_b) \cong H^{0,1}(X_b)^*$$

via the polarization Q_b on $H^1(X_b, \mathbb{C})$. To make this precise in the relative case, we need the relative holomorphic de Rham complex $\Omega^\bullet_{\mathcal{X}/B}$, where by convention $\Omega^0_{\mathcal{X}/B} = \mathcal{O}_{\mathcal{X}}$. This complex is a resolution of $\pi^*\mathcal{O}_B$ and is filtered by

$$F^p\Omega^\bullet_{\mathcal{X}/B} = (\Omega_{\mathcal{X}/B})^{\geq p}.$$

The relative Hodge to de Rham spectral sequence [2] degenerates at the E_1-term (as a consequence of the fiberwise degeneration), which implies

$$R^q\pi_*\Omega^p_{\mathcal{X}/B} \cong \frac{F^p\mathcal{H}^k(\mathcal{X}/B)}{F^{p+1}\mathcal{H}^k(\mathcal{X}/B)}, \quad k = p + q,$$

where $\mathcal{H}^k(\mathcal{X}/B) = R^k\pi_*\mathbb{C} \otimes \mathcal{O}_B$ as usual. In particular, the sheaves $R^q\pi_*\Omega^p_{\mathcal{X}/B}$ are locally free. Let us consider the special case $q = 1$, $p = 0$, so that

$$R^1\pi_*\mathcal{O}_{\mathcal{X}} \cong F^0\mathcal{H}^1/F^1\mathcal{H}^1 = \mathcal{H}^1/\pi_*\Omega^1_{\mathcal{X}/B}.$$

Since $Q(F^0\mathcal{H}^1, F^1\mathcal{H}^1) = 0$ and Q is non-degenerate, we obtain an isomorphism $R^1\pi_*\mathcal{O}_{\mathcal{X}} \cong (\pi_*\Omega^1_{\mathcal{X}/B})^* = \pi_*T_{\mathcal{X}/B} = \mathcal{V}$.

□

[2] This is the spectral sequence which is associated to the above finite filtration and the functor π_*.

Remark A.8. The previous proof only needed the fact that Q satisfies the first Riemann bilinear relation. Therefore the above statement also holds for non-degenerate torus fibrations over B, i.e. when the fibers are non-degenerate tori of index $k > 0$.

There is also a relative version of mixed Hodge structures over a base B.

Definition A.9. Let B be a complex manifold and $R = \mathbb{Z}$ or $\mathbb{Q}$. An *R-variation of mixed Hodge structures* (R-VMHS) is a triple $\mathsf{V} = (\mathsf{V}_R, \mathbb{W}_\bullet, \mathcal{F}^\bullet)$ consisting of

- a locally constant sheaf V_R of finitely generated R-modules;
- an increasing (weight) filtration $\mathbb{W}_\bullet$ of $\mathsf{V}_\mathbb{Q} = \mathsf{V}_R \otimes_\mathbb{Z} \mathbb{Q}_B$ by locally constant subsheaves;
- a decreasing (Hodge) filtration $\mathcal{F}^\bullet$ of $\mathsf{V}_R \otimes_\mathbb{Z} \mathcal{O}_B$ by holomorphic subbundles satisfying Griffiths' transversality condition.

Moreover, the filtrations $\mathbb{W}_{\bullet,b}$ and $\mathcal{F}^\bullet_{|b}$ of $V_\mathbb{Q}$ and $V_\mathbb{C}$ respectively are an R-mixed Hodge structure on $V_{R,b}$ for each $b \in B$.

If V is an R-VMHS, then its graded pieces $Gr_k^W \mathsf{V}$ are $\mathbb{Q}$-VHS of weight k by definition. An R-VMHS V is *graded-polarizable*, if the $Gr_k^W \mathsf{V}$ are polarizable $\mathbb{Q}$-VHS.

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